Radiative damping in plasma-based accelerators

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The electrons accelerated in a plasma-based accelerator undergo betatron oscillations and emit synchrotron radiation. The energy loss to synchrotron radiation may seriously affect electron acceleration. The electron dynamics under combined influence of the constant accelerating force and the classical radiation reaction force is studied. It is shown that electron acceleration cannot be limited by radiation reaction. If initially the accelerating force was stronger than the radiation reaction force, then the electron acceleration is unlimited. Otherwise the electron is decelerated by radiative damping up to a certain instant of time and then accelerated without limits. It is shown that regardless of the initial conditions the infinite-time asymptotic behavior of an electron is governed by a self-similar solution providing that the radiative damping becomes exactly equal to $2/3$ of the accelerating force. The relative energy spread induced by the radiative damping decreases with time in the infinite-time limit. The multistage schemes operating in the asymptotic acceleration regime when electron dynamics is determined by the radiation reaction are discussed.

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I. INTRODUCTION

The laser-plasma concept for charged particle acceleration has been proposed more than 30 years ago [1]. Since then the plasma-based methods of electron acceleration demonstrate an impressive progress in the past ten years. The quasimonoenergetic electron bunches are generated in laser-plasma acceleration experiments [2–4]. The electron energy in laser wakefield acceleration experiments exceeds 1 GeV for cm-scale acceleration length [5] and energy doubling of 42 GeV electrons in a meter-scale plasma wakefield accelerator is demonstrated [6]. Recently, the physics of linear colliders based on laser-plasma accelerators have been discussed [7–9].

It is generally believed that the next generation of lepton colliders should achieve center-of-mass energy around 1 TeV. Today the conventional accelerator technology provides a acceleration gradient which is not high enough for the TeV lepton collider with reasonable size and cost. Yet the plasma-based methods have attracted much attention because of the possibility of attaining a very high acceleration gradient. The accelerating plasma structure is a plasma wave generated behind the driver which can be the laser pulse or the electron bunch. For plasma density $n = 10^{15}$–$10^{18}\ \mathrm{cm}^{-3}$ the accelerating field is of the order 0.3–10 GV/m that is much stronger than that in the conventional accelerating structures.

The physics considerations for optimal laser-plasma accelerator parameters have been recently discussed [7,8]. It has been shown that the quasilinear regime when $a_0 \sim 1$ and plasma density range $n = 10^{15}$–$10^{18}\ \mathrm{cm}^{-3}$ are more favorable for ultrahigh energy laser-plasma acceleration, where $a_0 = eA_L/(mc^2) \approx 8.5 \times 10^{-10} \lambda_L[\mu\mathrm{m}] \times I_L^{1/2}[\mathrm{W/cm}^2]$ is the peak amplitude of the normalized vector potential of the laser field, A_L is the amplitude of the laser vector potential, I_L is the laser intensity, λ_L is the laser wavelength, and m, e, and c are the electron mass, the electron charge, and the speed of light, respectively. a_0 is proportional to the transverse momentum of the electron in the laser field. The laser pulse interacts with a plasma in the bubble regime at $a \gg 1$ [10,11]. The plasma bubble, which is almost free from the plasma electrons, is formed behind the laser pulse in the bubble regime instead of plasma wave excited behind of the laser pulse in the quasilinear regime. It is generally believed now [7,8] that the bubble regime of acceleration is less suited for lepton colliders than the quasilinear regime, for example, because of unstable positron acceleration.

The strength of the accelerating plasma field excited by the laser pulse is proportional to the laser ponderomotive potential. It can be estimated as $E_0 = F_{\mathrm{acc}}/e = fmc\omega_p/e$, where F_{acc} is the accelerating force, $\omega_p = (4\pi e^2 n/m)^{1/2}$ is the plasma frequency, n is the density of the background plasma, and f is the numerical factor determined by the parameters of the driver and the plasma. For example, if the driver is the linearly polarized Gaussian laser pulse with duration T_L then $f = \pi^{1/2} 2^{-3/2} a_0^2 \omega_p T_L \exp(-\omega_p^2 T_L^2/4)$ [12]. The accelerating force peaks at the resonant laser pulse duration $T_{L,\mathrm{res}} = 2^{1/2}/\omega_p$ so that $f_{\max} \approx 0.35 a_0^2$ [8]. To demonstrate the radiation reaction effect we will use in

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our paper the laser-plasma parameters, which are close to optimal and derived in Refs. [7,8]: resonant laser pulse duration, $a_0 \approx 2^{1/2}$, $f \approx 0.7$, and $n = 10^{15}$–10^{18} cm^{-3}.

There are a number of effects which limit the energy gain in the plasma-based accelerators [12]. One of the main limitations comes from the dephasing. The velocity of the relativistic electrons becomes slightly higher than the plasma wave phase velocity, which is determined by the driver velocity. The accelerated electrons slowly outrun the plasma wave and leave the accelerating phase. This problem can be partially solved by the use of proper longitudinal gradient of plasma density [13,14]. Another limitation is caused by the driver depletion as the driver energy converts into the energy of the plasma wave. The driver evolution during acceleration (e.g., laser pulse diffraction or electron bunch expansion because of Coulomb repulsion) also imposes certain restrictions on the electron energy gain. In the case of laser-plasma accelerators, the laser pulse can be guided over long distances in the preformed plasma density channel [15] or with relativistic optical guiding when diffraction is compensated by relativistic self-focusing [16]. In order to accelerate electrons far beyond the energy limited, for example, by the laser depletion the multistage schemes can be used [7,8]. Recently, proton-driven acceleration schemes have been proposed due to extremely large dephasing and depletion length [17].

The electron acceleration in the plasma wave is accompanied with the transverse betatron oscillations caused by the action of the focusing force on the electron from the plasma wakefield. The focusing force acting on the relativistic electron near the driver axis can be approximated as follows: $F_\perp \simeq -m\kappa^2\omega_p^2 r$, where r is the transverse displacement of the electron from the driver axis, κ are the focusing constant determined by the parameters of the driver and the plasma. Physically, κ characterizes the degree of the electron evacuation in the plasma wave. $\kappa^2 \simeq 1/2$ in the bubble regime when the plasma electrons are completely evacuated in the accelerating region [11]. If the driver is the linearly polarized Gaussian laser pulse with resonant pulse duration and $a_0^2 = 2$, then $\kappa^2 \simeq 0.11$ [8]. The frequency of the betatron oscillations is $\omega_\beta = \omega_p \kappa \gamma^{-1/2}$, where γ is the relativistic gamma factor of the electron.

The accelerated electrons undergoing betatron oscillations emit synchrotron radiation [18–20]. The radiated power can be estimated as follows: $P_{\rm rad} \simeq 2r_e\gamma^2F_\perp^2/(3mc)$, where $r_e = e^2/(mc^2) \simeq 3\times 10^{-13}$ cm is the classical electron radius, c is the speed of light. Since the power is proportional to the square of the electron energy, the radiation losses can stop electron acceleration at some threshold value of the electron energy. The threshold energy can be estimated by balancing the accelerating force and the radiation reaction force, $F_{\rm rrf} \simeq P_{\rm rad}/c$, so that $\gamma_{\rm th}^2 \simeq f/(\epsilon\kappa^4R_\beta^2)$, where $R_\beta = k_p r$ is the normalized amplitude of betatron oscillations, $\epsilon = 2r_e\omega_p/(3c)$ and $k_p = \omega_p/c$. The threshold energy is $\sim$100 GeV for $f = 0.7$, $n = 10^{19}$ cm^{-3} and $R_\beta = 1$ and $\kappa^2 = 0.11$. Therefore the radiative damping may be a serious limitation of electron acceleration in the high-energy regime. However, the self-consistent treatment is needed to study electron dynamics more accurately since the betatron oscillation amplitude determining radiation damping may evolve significantly during acceleration.

The electron acceleration in plasma with the radiation reaction effect has been studied theoretically [7,8,21,22]. The radiation reaction has been treated as a perturbation [21]. The first-order radiative correction to the energy gain of the accelerated electron bunch and the energy spread induced by radiation emission have been derived for the constant accelerating force. The dependence of the electron energy on time has been calculated in the plasma channel without the accelerating force and with the radiation reaction force [22]. Here we study the electron acceleration treating the radiation damping unperturbatively and analyzing the infinite-time limit. We show that regardless of the initial conditions the infinite-time asymptotic behavior of an electron is governed by a self-similar solution providing that the radiative damping becomes exactly equal to 2/3 of the accelerating force.

II. BASIC EQUATIONS

We start from the relativistic equation for electron motion in an electromagnetic field with the radiative reaction force in Landau-Lifshitz form [23]

$$\gamma\frac{du^i}{dt} = \frac{cr_e}{e}F^{ik}u_k + \frac{2r_e^2}{3mc}(F_1^i + F_2^i + F_3^i), \qquad (1)$$

$$F_1^i = (e/r_e)\frac{\partial F^{ik}}{\partial x^l}u_k u^l, \qquad (2)$$

$$F_2^i = -F^{il}F_{kl}u^k, \qquad (3)$$

$$F_3^i = (F_{kl}u^l)(F^{km}u_m)u^i, \qquad (4)$$

where F_{ik} is the electromagnetic field tensor, u_k is the 4-velocity of the electron. The first term in the right-hand side (rhs) of Eq. (1) corresponds to the Lorentz 4-force and the second one corresponds to the radiation reaction 4-force. Equation (1) is derived under the assumption that the absolute value of the first term is larger than that of the second term. However, some spatial components of radiation reaction force can be larger than that of the Lorentz force. Therefore the radiative damping can dominate over acceleration.

We assume that the ultrarelativistic electrons ($\gamma \gg 1$) are accelerated along the x axis by the force $F_{\rm acc} \gg F_\perp v_\perp/c$ and undergo betatron oscillations driven by the focusing force $F_\perp \simeq -m\kappa^2\omega_p^2 y$ along the y axis. Under our assumptions, $F_3 \gg F_1, F_2$ and the focusing forces make a major contribution to the energy losses through radiation. Therefore Eq. (1) can be reduced to the form

$$\frac{dp_y}{dt} = -m\kappa^2\omega_p^2 y - \frac{2r_e}{3c}\frac{\kappa^4\omega_p^4 y^2}{c^2} p_y\gamma, \tag{5}$$

$$\frac{dy}{dt} = \frac{p_y}{m\gamma}, \tag{6}$$

$$\frac{d\gamma}{dt} = f\omega_p - \frac{2r_e}{3c}\frac{\kappa^4\omega_p^4 y^2}{c^2}\gamma^2. \tag{7}$$

The first term on the rhs of Eq. (7) describes the action of the longitudinal component of the Lorentz force that provides electron acceleration, while the second term describes the radiation reaction force, $F_{\rm rrf}$. The obtained equations describe the betatron oscillations with radiative damping. When the force of radiative friction is disregarded ($r_e = 0$), the first two equations are equivalent to the equation of linear oscillator with a slowly varying frequency. The solution of the equation in WKB approximation is [21]

$$y \simeq C\sqrt{\omega_\beta(t)}\sin\left(\int \omega_\beta dt\right), \tag{8}$$

$$p_y \simeq \frac{Cm\kappa^2\omega_p^2}{\sqrt{\omega_\beta(t)}}\cos\left(\int \omega_\beta dt\right), \tag{9}$$

$$\omega_\beta = \frac{\omega_p}{\sqrt{2\gamma}}, \qquad \gamma = \gamma_0 + f\omega_p t. \tag{10}$$

It follows from the solution that the amplitude and ω_β decreases in the course of acceleration.

It is convenient to introduce new variables as follows:

$$P = \frac{p_y}{mc}\epsilon^{1/2} f^{1/2}, \tag{11}$$

$$Y = yk_p f^{3/2}\epsilon^{1/2}, \tag{12}$$

$$T = \frac{\omega_p t\kappa^2}{f}, \tag{13}$$

$$G = \gamma\frac{\kappa^2}{f^2}. \tag{14}$$

Then Eqs. (5)–(7) take a form

$$\frac{dP}{dT} = -Y - Y^2 PG, \tag{15}$$

$$\frac{dY}{dT} = \frac{P}{G}, \tag{16}$$

$$\frac{dG}{dT} = 1 - Y^2G^2. \tag{17}$$

When the number of betatron oscillations is large, we can use the averaging method [24]. To do this let us introduce a new variable,

$$U\exp\left(i\int G^{-1/2}dT\right) = Y - iG^{-1/2}P. \tag{18}$$

Substituting Eq. (18) in Eqs. (15)–(17) and averaging over the fast time related to the betatron oscillations yields the averaged equations:

$$\frac{dU}{dT} = -\frac{U}{4G} - \frac{1}{16}G|U|^2U, \tag{19}$$

$$\frac{dG}{dT} = 1 - \frac{1}{2}|U|^2G^2. \tag{20}$$

To derive Eq. (19) we have to use Eq. (7) before averaging in order to exclude dG/dT. Assuming that $2S = |U|^2 = Y^2 + P^2/G = R_\beta^2 f^3\epsilon \simeq 2\langle Y^2\rangle$ we can rewrite Eqs. (19) and (20) as follows:

$$\frac{dS}{dT} = -\frac{1}{2}\frac{S}{G} - \frac{1}{4}GS^2, \tag{21}$$

$$\frac{dG}{dT} = 1 - SG^2. \tag{22}$$

In the next sections we will analyze the obtained system of equations.

III. DYNAMICS OF THE ACCELERATED ELECTRONS

As $G > 0$ and $S > 0$ then $dS/dt < 0$ and the amplitude of the betatron oscillations always decreases with time. This means that for arbitrary electron energy the betatron oscillation amplitude will be small enough at a certain instance of time to be radiation reaction force less than the accelerating force.

Let us consider the solutions of Eqs. (21) and (22) in some limiting cases. At the absence of the accelerating force ($f = 0$), it follows from Eqs. (21) and (22) that $SG^{-1/4} = \text{const}$ and

$$\gamma = \gamma_0\left(1 + \frac{5\epsilon R_{\beta,0}^2\gamma_0}{16}\omega_p t\right)^{-4/5}, \tag{23}$$

which is an agreement with the solution calculated in Ref. [22], where $R_{\beta,0} = R_\beta(t = 0)$.

At the absence of the radiation reaction [the last terms in the rhs of Eqs. (21) and (22) are absent] we obtain

$$G = G_0 + T, \qquad \sqrt{G}S = \text{const}, \tag{24}$$

which is in agreement with Eqs. (8) and (9). If the radiation reaction force is much weaker than the accelerating one, then to the first order in the radiation reaction force the normalized electron energy is

$$G = G_0 + T - \tfrac{2}{5}[1 - (G_0 + T)^{5/2}], \qquad (25)$$

which is an agreement with the result obtained in Ref. [21].

The closed-form solutions of Eqs. (21) and (22) are rather complicated and can be implicitly expressed through hypergeometric functions (see the Appendix). However, the main properties of the solution can be analyzed without the closed form. The system of Eqs. (21) and (22) has integral of motion

$$I = \frac{1 - 3SG^2/2}{S^{9/4}(SG^2)^{3/4}} = \text{const.} \qquad (26)$$

Physically, G, S, and I mean the following. G is the normalized electron energy, $S^{1/2}$ is the normalized amplitude of the betatron oscillations. $I \approx S^2 G$ in the limit $SG^2 \ll 1$ and the conservation of I implies that the product of betatron oscillation amplitude ($\sim S^{1/2}$) and the transverse momentum amplitude ($\sim S^{1/2}G^{1/4}$) is conserved.

The electron trajectories in the phase space $S - G$ are the integral lines determined by Eq. (26). The phase portrait of the system governed by Eqs. (21) and (22) is shown in Fig. 1. It is seen from Fig. 1 that if initially the accelerating force is stronger than the radiation reaction force ($SG^2 < 1$) then the electron energy monotonically increases with time. Otherwise the electron energy decays up to the time instance when $F_{\rm acc} = F_{\rm rrf}$ (that corresponds to $SG^2 = 1$) and then it monotonically increases with time.

We verify our analytical results by numerical simulations. The exact equation (1) and the averaged equations of motions (21) and (22) are integrated numerically for test electrons for $f = 0.1$ and $n = 10^{15}$ cm^{-3}. For simplicity, we consider the structure of the transverse electromagnetic field similar to the bubble regime: $\kappa^2 = 0.5$ and $E_\perp \approx H_\perp$. The dependence of the normalized integral of motion $I_n = I^{-1}(\epsilon\kappa f^2)^{-3}$ and γ on $\omega_p t$ for initial condition $\gamma_0 = 2000$ and $R_{\beta,0} = 0.8$, $p_{y,0} = 0$ is shown on Fig. 2. It is seen from Fig. 2 that the solution of the exact equations and that of the approximate averaged are in a good agreement. Moreover, the integral I is almost constant for the exact equations (1) [see Fig. 2(b)].

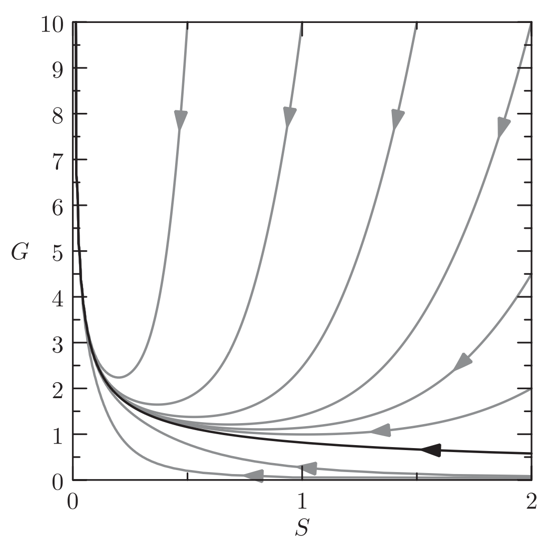

FIG. 1. The phase portrait of the system governed by Eqs. (21) and (22).

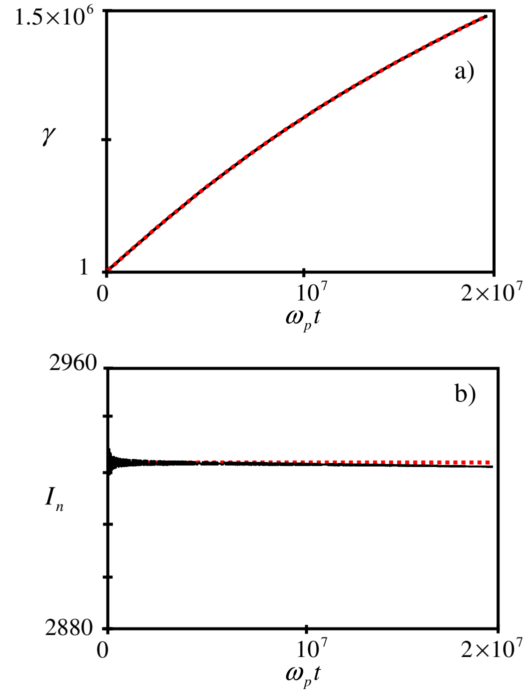

FIG. 2. The dependence of (a) γ and (b) I_n on $\omega_p t$ calculated by solving the exact Eq. (1) (black solid lines) and by solving the approximate Eqs. (21) and (22) (red dashed lines) for $f = 0.1$, $\kappa^2 = 0.5$, $n = 10^{15}$ cm^{-3} and for initial conditions $\gamma_0 = 2000$, $R_{\beta,0} = 0.8$, $p_{y,0} = 0$.

IV. ASYMPTOTIC ACCELERATION

It is seen from Fig. 1 that all electron trajectories merge in the limit $t \to \infty$ so that $G \to \infty$ and $S \to 0$. It follows from Eq. (26) that $S = 2G^{-2}/3$ in this limit. We will call the electron acceleration in this limit an asymptotic acceleration regime (AAR). To find asymptotic trajectory we can solve Eqs. (21) and (22) by the perturbation method with the assumption that I is small. To the first order in I the solution is

$$S \approx \tfrac{2}{3}G^{-2}, \qquad (27)$$

$$G \propto \tfrac{1}{3}T. \qquad (28)$$

We can conclude that in the asymptotic acceleration regime the radiation reaction force is equal to two-thirds of the accelerating force:

$$F_{\mathrm{rrf}} = \tfrac{2}{3} F_{\mathrm{acc}}, \tag{29}$$

so that the electron energy increases linearly with time while the betatron amplitude is reversely proportional to the time.

Equations (21) and (22) can be solved exactly (see the Appendix). It follows from the exact solution that the asymptotic acceleration regime is determined by the parameters $G_{\mathrm{tr}} = T_{\mathrm{tr}} = I^{2/9}$ and $S_{\mathrm{tr}} = I^{-1/9}$. The characteristic time of transition to asymptotic acceleration is T_{tr}. To derive the asymptotic solution the initial condition should be applied. We assume that $S_0 G_0^2 \ll 1$ which is typical for the initial parameters of the electron beam. For example, this condition is fulfilled for the initial parameters $\gamma_0 mc^2 < 0.1$ TeV, $n < 10^{18}$ cm^{-3}, $R_{\beta,0} = 1$, $f = 0.7$, $\kappa^2 = 0.11$. Therefore the normalized electron energy is in the limit $T \gg T_{\mathrm{tr}}$,

$$G = \frac{\delta}{3} G_{\mathrm{tr}} + \frac{1}{3} T, \tag{30}$$

where $\delta \approx 1.85$ (see the Appendix).

The averaged equations of motions (21) and (22) are integrated numerically for the test electrons with the same parameters as for Fig. 2 for three values of the initial betatron amplitude $R_{\beta,0} = 0.8, 0.2, 0.1$. It is seen from Fig. 3 that the asymptotic solution (29) is in good agreement with the numerical results.

The radiation damping rate varies for the electrons with different betatron oscillation amplitudes. This causes the energy spread in the electron bunch accelerated in the plasma wave. We assume that the betatron oscillation amplitudes of the electrons in the bunch are initially uniformly distributed in the range $R_{\min} < R_{\beta,0} < R_{\max}$ and $R_{\max} \gg R_{\min}$. We also again assume that $S_0 G_0^2 \ll 1$. Then the normalized mean energy and the normalized square of the energy spread are in AAR:

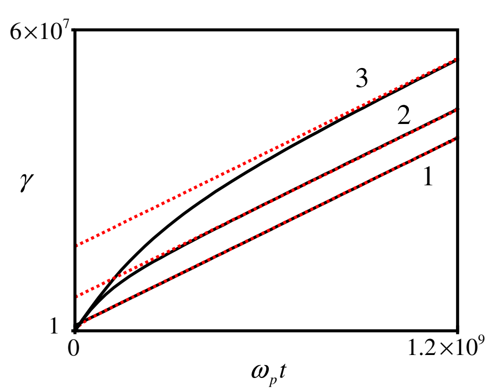

FIG. 3. The dependence of γ on $\omega_p t$ in AAR: analytic solution (red dashed lines) and numerical solution (black solid line) for $R_{\beta,0} = 0.8$ (lines 1), $R_{\beta,0} = 0.2$ (lines 2) and $R_{\beta,0} = 0.1$ (lines 3). The other parameters are the same as in Fig. 2.

$$\langle G \rangle \simeq \frac{2}{R_{\max}^2} \int_{R_{\min}}^{R_{\max}} G R_{\beta,0} dR_{\beta,0} \simeq G_{\max} \delta + \frac{T}{3}, \tag{31}$$

$$\sigma_G^2 = \langle G^2 \rangle - \langle G \rangle^2 \simeq G_{\max}^2 \frac{\delta^2}{3} \left(\frac{R_{\max}}{R_{\min}} \right)^{2/3}, \tag{32}$$

where $G_{\max} = G_{\mathrm{tr}}(R_{\beta,0} = R_{\max})$. It follows from Eqs. (31) and (32) that the relative energy spread, $\sigma_G / \langle G \rangle$, decreases with time in AAR.

V. DISCUSSION

Our model is derived under the conditions that $F_\perp$ gives the main contribution to the radiative damping and $F_3 \gg F_1, F_2$. However, $F_\perp$ goes to zero in the limit $t \to \infty$. Therefore we should check: should the accelerating force and terms F_1, F_2 be taken into account in the expression for radiation reaction force in this limit? First it is significant that the radiation reaction force remains constant in AAR because $F_\perp \sim R_\beta \to 0$ and $\gamma \to \infty$ for $t \to \infty$ in such way that $R_\beta^2 \gamma^2 = \mathrm{const}$. Making use of Eq. (29) and relation $v_y \sim \omega_\beta y$ we get $F_2/F_3 \sim f\epsilon \ll 1$ and $F_1/F_3 \sim (3/4)\kappa^2 f \gamma^{-1/2} \epsilon^{1/2} \ll 1$, where we assume that $\kappa \sim f \sim 1$. The contribution from the accelerating force (or from E_x) to F_3 is of the order $F_2/F_3 \ll 1$. Therefore our model defined by Eqs. (21) and (22) is valid in AAR. The typical values of the ratios of F_1/F_3 and F_2/F_3 obtained by numerical integration of Eq. (1) for parameters used in Fig. 2 are less than 10^{-7} and 10^{-10}, respectively, which agrees with the estimations derived above.

For high-energy electrons, quantum electrodynamics (QED) effects can be important. The energy of the photon emitted by the accelerated electron can be so high that the quantum recoil becomes strong. The photon emission can be treated in a classical approach if QED parameter $\chi = [(mc\gamma \mathbf{E} + \mathbf{p} \times \mathbf{H})^2 - (\mathbf{p} \cdot \mathbf{E})^2]^{1/2}/(mcE_{\mathrm{cr}}) \simeq \gamma F_\perp/(eE_{\mathrm{cr}})$ is much less than unity, where $E_{\mathrm{cr}} = m^2c^3/(e\hbar) \approx 1.32 \times 10^{16}$ V/cm is the QED critical field [25]. χ can be estimated in AAR as follows: $\chi \approx [(2f/\alpha) \times (\hbar\omega_p/mc^2)]^{1/2} \ll 1$, where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant. Therefore the classical approach for the radiation reaction force is valid in the limit $t \to \infty$ because, like for the corrections to the radiation reaction force, the growth of γ in χ is compensated by decreasing of $F_\perp$.

Estimates show that the length passed by the laser pulse to reach AAR is very large and much larger than dephasing and depletion lengths for modern laser systems. However, the multistage acceleration schemes for TeV lepton colliders are now discussed [7,8]. Each electron stage is equipped with its own laser system. The stage length should be smaller than the dephasing and depletion lengths. The final stages where the energy of the accelerated lepton reaches TeV level may operate in AAR.

We estimate the transition length to AAR for the accelerator parameters close to that discussed in [8]. The distance passed by the electron before reaching AAR is $k_p l_{\mathrm{tr}} \simeq (f/\kappa^2) T_{\mathrm{tr}} \simeq 1.6(\epsilon^2 \gamma_0 R_{\beta,0}^4 f \kappa^8)^{-1/3}$. For the initial parameters $n = 10^{18}\ \mathrm{cm}^{-3}$, $R_{\beta,0} = 1$, $\gamma_0 = 2 \times 10^3$, $f = 0.7$, $\kappa^2 = 0.11$ the electron comes into AAR after passing 7800 laser-driven acceleration stages with total distance $l_{\mathrm{tr}} \simeq 73$ m, achieving the energy $\gamma mc^2 \simeq 5$ TeV and $R_\beta \simeq 0.008$, where the stage distance is chosen to be equal to the half dephasing length [8] and the distance between the acceleration stages is neglected. For the rarefied plasma $n = 10^{15}\ \mathrm{cm}^{-3}$, AAR is achieved in 78 stages with $l_{tr} \simeq 23$ km, $\gamma mc^2 \simeq 48$ TeV, and $R_\beta \simeq 0.005$. AAR may be achieved within one acceleration stage in the proton-driven acceleration schemes because of the very large dephasing length [17].

In conclusion, we have shown that the electron acceleration is not limited by the radiative damping in plasma-based accelerators. Even if the radiation reaction force is stronger than the accelerating force at the beginning, then acceleration eventually succeeds deceleration with time. The damping of the betatron oscillations leads to the transition to the self-similar asymptotic acceleration regime in the infinite-time limit when the radiation reaction force becomes equal to 2/3 of the accelerating force. The relative energy spread induced by the radiative damping in the accelerated electron bunch decreases with time in this regime. This opens the possibility to use high density plasma at the late stages of multistage plasma-based accelerators despite the fact that the radiative damping is enhanced as density increases. The high density plasma can be favorable because it provides high accelerating gradient and, thus, reduces the length of the acceleration stages. The obtained results can be also applied to any other accelerating systems with the linear focusing forces.

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APPENDIX: THE EXACT SOLUTION OF THE AVERAGED EQUATIONS

We introduce new variables $g = G/G_{\mathrm{tr}}$, $\tau = T/T_{\mathrm{tr}}$, and $s = (S/S_{\mathrm{tr}})(G/G_{\mathrm{tr}})^{-1/4}$, where

$$G_{\mathrm{tr}} = T_{\mathrm{tr}} = I^{2/9}, \tag{A1}$$

$$S_{\mathrm{tr}} = I^{-1/9}. \tag{A2}$$

Then Eqs. (21), (22), and (26) take a form

$$\frac{ds}{d\tau} = -\frac{3}{4}\frac{s}{g}, \tag{A3}$$

$$\frac{dg}{d\tau} = 1 - sg^{9/4}, \tag{A4}$$

$$s^{-3}g^{-9/4} - \frac{3}{2}s^{-2} = 1. \tag{A5}$$

The obtained equations do not depend on any parameters. Therefore the characteristic time of transition to asymptotic acceleration is $T \approx T_{\mathrm{tr}}$ and the electron is accelerated in the asymptotic regime when $T \gg T_{\mathrm{tr}}$. Expressing g through s by Eq. (A5) and substituting it in Eq. (A3), we have

$$\frac{ds}{d\tau} = -\frac{3}{4}s^{13/9}\left(s^2 + \frac{3}{2}\right)^{4/9}. \tag{A6}$$

The solution of the equation is

$$\varphi(s) - \varphi(s_0) = -\tau, \tag{A7}$$

$$\varphi(x) = 2^{4/9}(3 + 2s^2)^{5/9}s^{-4/9} - 2^{13/9}3^{5/9}s^{14/9}{}_2F_1\left(\frac{7}{9}, \frac{4}{9}; \frac{16}{9}; -\frac{2s^2}{3}\right), \tag{A8}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [26] and $s_0 = s(\tau = 0)$. The asymptotic expansions of function $\varphi(s)$ are

$$\varphi(s) \approx 3\left(\frac{3s}{2}\right)^{-4/9}, \qquad s \ll 1, \tag{A9}$$

$$\varphi(s) \approx \delta + s^{-4/3}, \qquad s \gg 1, \tag{A10}$$

$$\delta = -\frac{27\Gamma(-\frac{1}{3})\Gamma(\frac{16}{9})}{28\Gamma(\frac{4}{9})} \approx 1.85. \tag{A11}$$

Thus, $s \sim \tau^{-9/4} \ll 1$ and $g \sim \tau \gg 1$ in the limit $\tau \gg 1$. To derive the asymptotic solution the initial condition should be applied. We assume that $S_0 G_0^2 \ll 1$ which is typical for the initial parameters of the electron beam. Making use of Eq. (A10), we obtain

$$\tfrac{9}{4}s^{-4/9} \approx \tau + \delta. \tag{A12}$$

Therefore the normalized electron energy and the square of the normalized betatron amplitude are in the limit $T \gg T_{\mathrm{tr}}$,

$$G = \frac{\delta}{3}G_{\mathrm{tr}} + \frac{1}{3}T, \tag{A13}$$

$$S = \tfrac{2}{3}G^{-2}. \tag{A14}$$

In AAR the electron energy increases linearly with time while the betatron amplitude is reversely proportional to the time.

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