Effect of undulator harmonics field on free-electron laser harmonic generation

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The harmonics field effect of a planar undulator on free-electron laser (FEL) harmonic generation has been analyzed. For both the linear case and the nonlinear case, the harmonic fraction of the radiation can be characterized by the coupling coefficients. The modification of the coupling coefficients is given when the third harmonics magnetic field component exists, thus the enhancement of the harmonic radiation can be predicted. The numerical results show that with the third harmonics magnetic field component that has the opposite sign to the fundamental, the intensity of third-harmonic radiation can be increased distinctly for both the small signal gain and the nonlinear harmonic generation. The increase is larger for the smaller undulator deflecting parameter.

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I. INTRODUCTION

Using the higher harmonic is a way of free-electron laser (FEL) developing towards the shorter wavelength ranges [1–4]. For a planar undulator with an ideal sinusoidal periodic magnetic field, the electrons also radiate at odd harmonics on axis due to their nonuniform axial motion. In actual planar undulators, the magnetic field is nonsinusoidal and with harmonics field components, that has effects on the harmonic radiation. Normally the harmonics fields are very weak. For example, in the hybrid permanent magnet undulator of the Shanghai deep ultraviolet free-electron laser source [5], the third-harmonic field is less than 1% of the fundamental field. For a standard Halbach-type pure permanent magnet undulator, which has four magnetic blocks per period, the third-harmonic field may be even weaker. By increasing the harmonic field component aptly, the harmonic radiation can be enhanced [6]. Therefore some methods for this purpose were proposed, such as putting high permeability shims inside the undulator [7], optimizing the magnetic blocks size in a permanent magnet undulator [8]. Ratios of the third harmonics magnetic field to the fundamental magnetic field approaching 30% have been measured experimentally by Halbach [9]. In this paper, we analyze the effect of undulator harmonics field on FEL harmonic generation, the case of third harmonics magnetic field is considered specially.

II. ANALYSIS

In a planar undulator with a sinusoidal periodic magnetic field, the electrons oscillate at odd harmonics frequency in the transverse direction, thus leading to the odd harmonics radiations in the forward direction [10]. For a FEL utilizing such an undulator, the nth harmonic optical field equation and the phase equation in one-dimensional mode are [11]

$$\frac{d}{dz}\tilde{a}_{sn} \simeq \frac{r_e n_e a_u [J, J]_n \lambda_{sn}}{\gamma} \langle e^{-in\phi} \rangle \tag{1}$$

$$\frac{d^2\phi}{dz^2} = -\frac{2a_u k_u}{\gamma^2} \mathrm{Re} \sum_n [J, J]_n k_{sn} \tilde{a}_{sn} e^{in\phi}, \tag{2}$$

where $\tilde{a}_{sn} = a_{sn} e^{-i\varphi_{sn}}$, $a_{sn} = eE_{sn}/(mc^2 k_{sn})$, and $a_u = eB_u/(mc^2 k_u)$ are dimensionless vector potential of the rms nth harmonic radiation field E_{sn} and undulator field B_u, respectively; $k_{sn} = 2\pi/\lambda_{sn}$, $k_u = 2\pi/\lambda_u$ are the corresponding wave number; ϕ_{sn} is the phase of the radiation field; r_e is the classical electron radius; n_e and γ is the density and energy of electrons; φ is the ponderomotive phase of electron $\varphi = (k_s + k_u)z - \omega_s t$. The angular bracket represents the average over the electron's initial phases and initial phase velocities $\varphi_0{}'$. $[J, J]_n$ is the coupling coefficient

$$[J, J]_n = (-1)^{(n-1)/2}\left[J_{(n-1)/2}\left(\frac{n a_u^2}{2(1 + a_u^2)}\right) - J_{(n+1)/2}\left(\frac{n a_u^2}{2(1 + a_u^2)}\right)\right]. \tag{3}$$

J is integer order Bessel function. From Eqs. (1) and (2) the small signal gain of the nth harmonic optical field in the low gain FEL is given by

$$g_n = -n\left(\frac{[J, J]_n}{[J, J]_1}\right)^2 (4\pi N\rho)^3 \left\langle \frac{\partial}{\partial x} \mathrm{sinc}^2 \frac{x}{2} \right\rangle_{\phi_0{}'}, \tag{4}$$

where ρ is the Pierce parameter, $x = n\varphi_0{}' L$, and $L = N\lambda_u$ is the length of undulator. In high gain FEL the harmonic is generated nonlinearly. The evolution of the nth

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harmonic optical field in the exponential gain region and its saturation power can be given as [11]

$$\frac{P_n}{\rho P_e} \simeq \left(\frac{n^{n-1}[J,J]_n}{n![J,J]_1}\right)^2 \left(\frac{P_1}{\rho P_e}\right)^n = \left(\frac{n^{n-1}[J,J]_n}{n![J,J]_1}\right)^2 \left(\frac{P_{ef}}{9\rho P_e}\right)^n e^{n(z/L_g)} \tag{5}$$

$$\frac{P_{ns}}{P_{1s}} \approx \frac{(n+1)^n}{2(n*n!)^2}\left(\frac{[J,J]_n}{[J,J]_1}\right)^2, \tag{6}$$

where P_e is the power of the electron beam; P_1 and P_{1s} are the fundamental power and its saturation power, respectively; P_{ef} is the effective start-up short noise power, equal to the fraction of the spontaneous undulator radiation in one power gain length. Thus, the harmonic generation is characterized by the coupling coefficients for both the linear case and the nonlinear case. The coupling coefficients (and consequently the harmonic generation) increase with undulator deflection parameter but the increase becomes very slow after $a_u > 2$. One can expect an enhancement of the harmonics radiation by adding a harmonic field to the fundamental sinusoidal undulator field.

In actual planar undulators, the magnetic field is nonsinusoidal; when expanded in Fourier series, the field includes odd spatial harmonics due to the symmetry of the magnetic structure. Therefore the magnetic fields and corresponding dimensionless vector potential can be expressed by

$$B_u = \sum_m \sqrt{2} B_{um} \sin(mk_u z), \qquad \tilde{a}_u = \sum_m \sqrt{2} a_{um} \cos(mk_u z), \tag{7}$$

where m is for all or part odd numbers depending on the magnetic structure; B_{um} and a_{um} are the rms value of mth harmonics magnetic fields and corresponding dimensionless vector potential, respectively. Generally, all harmonics components are much smaller than the fundamental component:

$$B_{um} \ll B_{u1}, \qquad a_{um} = \frac{B_{um}}{mB_{u1}} a_{u1} \ll a_{u1}. \tag{8}$$

Using relation $\beta_\perp^2 = \tilde{a}_u^2/\gamma^2$ and only considering the lower harmonics, the electron longitudinal velocity with the harmonics undulator fields is

$$\beta_{II} \approx 1 - \frac{1}{2}\left(\frac{1}{\gamma^2} + \beta_\perp^2\right) = \bar{\beta}_\| - \frac{1}{\gamma^2}\left\{\sum_m a_{um}^2 \cos(2mk_u z) + \sum_{m\neq l} a_{um} a_{ul}\{\cos[(m+l)k_u z] + \cos[(m-l)k_u z]\}\right\}, \tag{9}$$

where $\bar{\beta}_\| = 1 - \frac{1}{2\gamma^2}(1 + \sum_m a_{um}^2)$ is the average longitudinal velocity. Accordingly the resonance condition is

$$\lambda_{sn} = \frac{\lambda_u}{2n\gamma^2}\left(1 + \sum_m a_{um}^2\right). \tag{10}$$

The longitudinal motion of the electron is

$$z = \bar{z} - \left\{\sum_m \frac{\xi_m}{k_s} \sin(2k_u \bar{z}) + \sum_{m\neq l}\left[\frac{\xi_{ml+}}{k_s} \sin[(m+l)k_u \bar{z}] + \frac{\xi_{ml-}}{k_s} \sin[(m-l)k_u \bar{z}]\right]\right\}, \tag{11}$$

where $\bar{z} = \bar{\beta}_\| ct$,

$$\xi_m = \frac{a_{um}^2}{2m(1 + \sum a_{ui}^2)} = \frac{r_m^2}{m}\xi_1, \quad \xi_{ml\pm} = \frac{a_{um} a_{ul}}{(m \pm l)(1 + \sum a_{ui}^2)} = \frac{2r_m r_l}{m \pm l}\xi_1, \quad r_m = \frac{a_{um}}{a_{u1}} = \frac{B_{um}}{mB_{u1}} \ll 1, \quad \xi_1 = \frac{a_{u1}^2}{2(1 + \sum a_{ui}^2)} < \frac{a_{u1}^2}{2(1 + a_{u1}^2)} < \frac{1}{2}. \tag{12}$$

For the case that all magnetic harmonic components are much smaller than the fundamental, all the ξ_m, $\xi_{ml\pm}$ terms that do not contain the fundamental are much smaller than 1 and can be neglected. Then Eq. (8) becomes

$$z \simeq \bar{z} - \left\{\frac{\xi_1}{k_s} \sin(2k_u \bar{z}) + \sum_{m\neq 1}\left[\frac{\xi_{m+}}{k_s} \sin[(m+1)k_u \bar{z}] + \frac{\xi_{m-}}{k_s} \sin[(m-1)k_u \bar{z}]\right]\right\}. \tag{13}$$

Including the harmonic magnetic fields [Eq. (6)] and with the optical field

$$\tilde{a}_s = \sum_n a_{sn} \sin[n(k_s z - \omega_s t) + \phi_{sn}], \tag{14}$$

the phase equation [Eq. (2)] now is

$$\phi'' = \frac{2k_u}{\gamma^2}\sum_{n,l} k_{sn} a_{sn} a_{ul}\{\cos[(nk_s + lk_u)z - n\omega_s t + \phi_n] + \cos[(nk_s - lk_u)z - n\omega_s t + \phi_n]\}. \tag{15}$$

Substituting Eq. (13) to it, we get

$$\phi'' = \frac{2k_u}{\gamma^2}\sum_n k_{sn}a_{sn}a_{u1}f_n \,\mathrm{Re}e^{-i(n\phi+\varphi_{sn})}, \tag{16}$$

where

$$f_n = \mathrm{Re}\sum_l \frac{a_{ul}}{a_{u1}}[e^{i(n-l)k_u\bar{z}} + e^{i(n+l)k_u\bar{z}}]e^{in\xi\sin(2k_u\bar{z})} \times \prod_{m\neq 1} e^{in\{\xi_{m+}\sin[(m+1)k_u\bar{z}]+\xi_{m-}\sin[(m-1)k_u\bar{z}]\}}. \tag{17}$$

In obtaining Eq. (17) the condition $k_s \gg k_u$ is used.

Similarly, when the magnetic harmonic field existed, the nth harmonic optical field equation becomes

$$\frac{d}{dz}\tilde{a}_{sn} \simeq \frac{r_e n_e a_{u1}\lambda_{sn}}{\gamma} f_n\langle e^{-in\phi}\rangle. \tag{18}$$

Comparing Eqs. (16) and (18) with Eqs. (1) and (2), it can be seen that when the magnetic harmonic fields are included the coupling coefficient is modified as

$$[J, J]_n \to f_n.$$

In the exponential of the modified coupling coefficient [Eq. (17)], many terms are small and oscillate fast; an average over the undulator period will eliminate these small contribution terms.

Among all the harmonics, the third harmonic is the most important one. In the following, we consider the case that only the third-harmonic field exists, and all other harmonics are neglected. Then Eq. (13) can be written as

$$z = \bar{z} - \frac{\zeta_1}{k_{s1}}\sin(2k_u\bar{z}) - \frac{\zeta_2}{k_{s1}}\sin(4k_u\bar{z}), \tag{19}$$

where

$$\xi_1 = \frac{k_{s1}a_{u1}^2}{4k_u\gamma^2}\left(1 + \frac{a_{u3}}{a_{u1}}\right) = \frac{a_{u1}(a_{u1} + a_{u3})}{2(1 + a_{u1}^2 + a_{u3}^2)}, \quad \xi_2 = \frac{k_{s1}a_{u3}a_{u1}}{8k_u\gamma^2} = \frac{a_{u1}a_{u3}}{4(1 + a_{u1}^2 + a_{u3}^2)}, \tag{20}$$

and it has

$$\frac{\zeta_2}{\zeta_1} = \frac{a_{u3}/a_{u1}}{2(1 + a_{u3}/a_{u1})} = \frac{B_{u3}/B_{u1}}{6 + 2B_{u3}/B_{u1}} \ll 1.$$

Then the modified coupling coefficient [Eq. (17)] is

$$f_n = \sum_{l=1,3}\frac{a_{ul}}{a_{u1}}[e^{i(n-l)k_u\bar{z}} + e^{i(n+l)k_u\bar{z}}] \times \sum_{h_1}\sum_{h_2} J_{h_1}(n\zeta_1)J_{h_2}(n\zeta_2)e^{i(h_1+2h_2)2k_u\bar{z}}. \tag{21}$$

Here $n = 1, 3$. After an average over the undulator period, the dominant product terms in the sum of Eq. (21) are those with $h_1 + 2h_2 = -(n \pm l)/2$:

$$f_n = \sum_l \frac{a_{ul}}{a_{u1}}\Bigg\{\sum_{\substack{h_1,h_2,\\ h_1+2h_2=-(n+l)/2}} J_{h_1}(n\zeta_1)J_{h_2}(n\zeta_2) + \sum_{\substack{h_1,h_2,\\ h_1+2h_2=-(n-l)/2}} J_{h_1}(n\zeta_1)J_{h_2}(n\zeta_2)\Bigg\} \quad n, l = 1, 3. \tag{22}$$

For small arguments, only Bessel functions of zero order will contribute. Because $\zeta_2 \ll \zeta_1 < 1/2$, the above equation can be further simplified by taking $h_2 = 0$. Then, at last, we give the modified coupling coefficient as

$$f_1 = J_0(\zeta_2)\left\{[J_0(\zeta_1) - J_1(\zeta_1)] + \frac{a_{u3}}{a_{u1}}[J_2(\zeta_1) + J_1(\zeta_1)]\right\} \tag{23}$$

$$f_3 = J_0(3\zeta_2)\left\{[J_2(3\zeta_1) - J_1(3\zeta_1)] + \frac{a_{u3}}{a_{u1}}[J_0(3\zeta_1) - J_3(3\zeta_1)]\right\}. \tag{24}$$

According to the above formulas, the modified harmonic coupling coefficients as a function of undulator parameter are numerically calculated for different harmonic magnetic field fraction. The results reveal that the harmonic coupling coefficients and, consequently, the harmonic emission are enhanced when B_{u3} has an opposite sign to B_{u1}, and are suppressed when the magnetic fields have the same sign (Fig. 1). The results also show that the fundamental coupling coefficient has been less affected by harmonic magnetic field (Fig. 2).

For both the small signal gain and the nonlinear harmonic generation in high gain, the harmonic FEL radiation is proportional to the square of the coupling coefficient [Eqs. (4)–(6)]. Therefore comparing with the case without the harmonic magnetic field present, the enhancement of the 3rd harmonic radiation can be given as

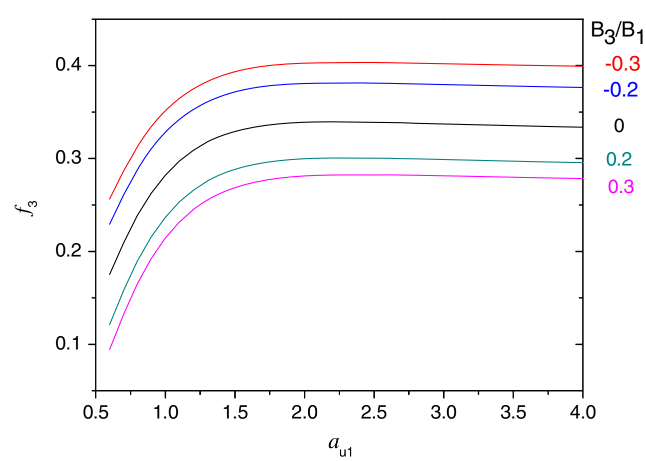

FIG. 1. Modified third-harmonic coupling coefficient due to the harmonic magnetic field.

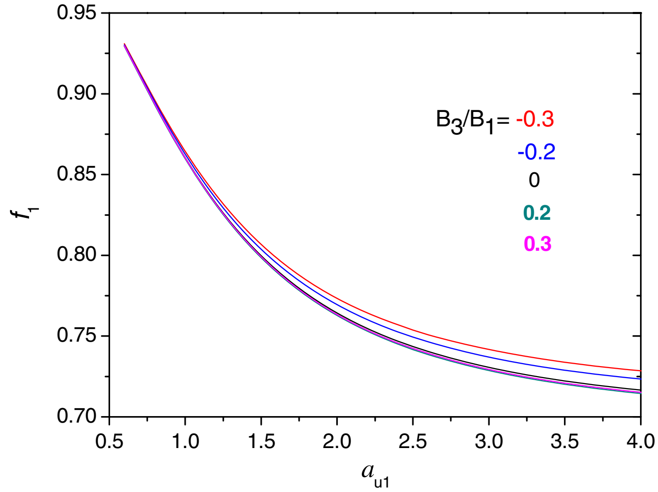

FIG. 2. Modified fundamental coupling coefficient due to the harmonic magnetic field.

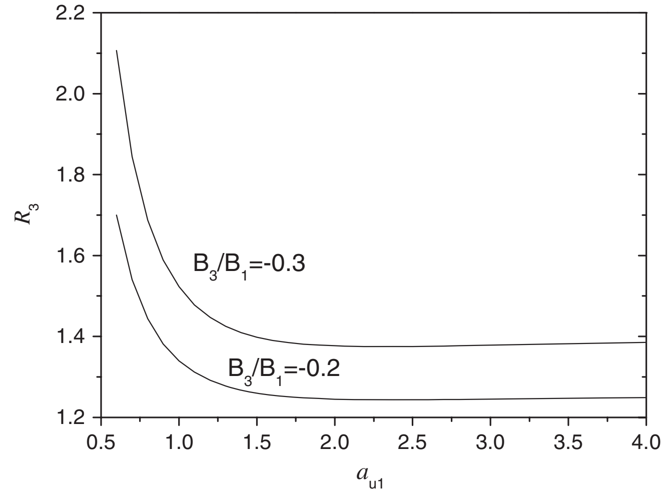

FIG. 4. The enhancement of the FEL harmonic radiation with B_3.

$$R_3 = \left(\frac{f_3/f_1}{[J,J]_3/[J,J]_1}\right)^2. \tag{25}$$

The dependence of the third-harmonic radiation on the ratio of B_{u3} to B_{u1} is shown in Fig. 3. The enhancement of the third-harmonic radiation is shown in Fig. 4. While in the calculation of Eq. (25), the arguments of the coupling coefficients and of the modified coupling coefficients are taken with a little difference to keep the same resonant wavelengths, it is a_u in the former, and a_{u1} in the latter:

$$a_u^2 = a_{u1}^2\left[1 + \left(\frac{a_{u3}}{a_{u1}}\right)^2\right]. \tag{26}$$

We can see that the harmonics radiation enhancement increases with the magnetic field ratio of the harmonics to the fundamental, and for a given harmonic magnetic field fraction, the enhancement is larger when the magnetic field is weaker. With the third harmonics magnetic field 30% of the fundamental, this field ratio translates into a vector potential ratio of 10%, the intensity increase of the third-harmonic radiation is about 40%, and becomes larger when the undulator deflecting parameter is small; for example, it is doubled for undulator deflecting parameter $K(=\sqrt{2}a_u) = 0.9$. For 75% enhancement of the third-harmonic power, the corresponding third-harmonic coupling coefficient is increased one-third, which agrees with the result of Ref. [6].

Replacing the coupling coefficients with the modified ones, Eqs. (1) and (2) were numerically solved. The parameters we used were based on that of the Hefei soft x-ray FEL proposal [12]. The electron beam parameters were energy of 800 MeV, initial energy spread of 0.01%, emittance of 2.1 nm-rad, and current of 600 A. The undulator parameter K is 1.2 with the period of 2.5 cm. Figure 5

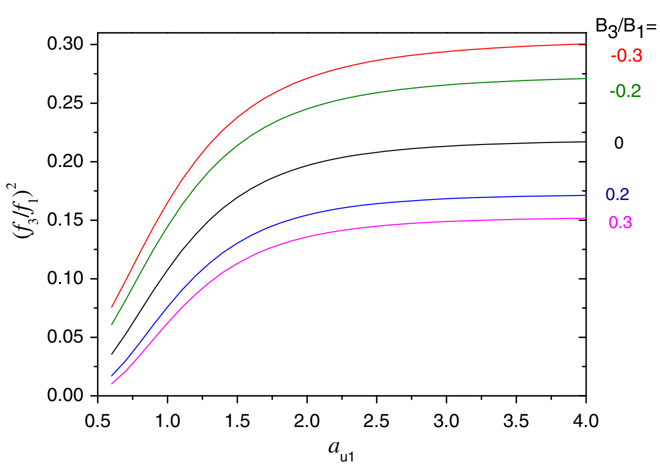

FIG. 3. The effect of B_3 on FEL harmonic generation.

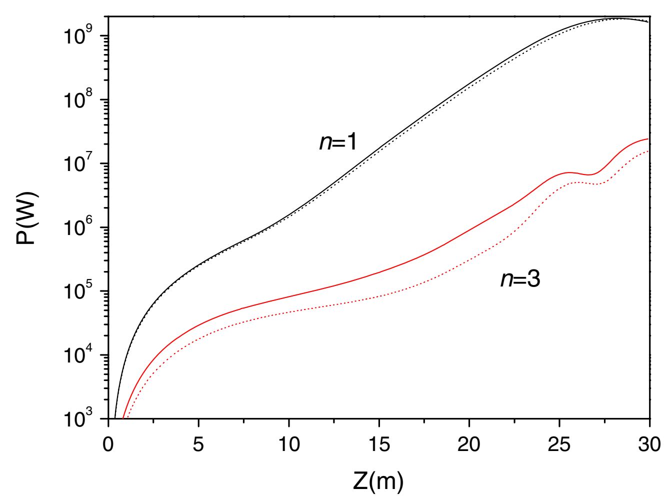

FIG. 5. The nonlinear harmonic generation in SASE FEL with (solid line) and without (dashed line) third harmonics magnetic field. The third harmonics magnetic field 30% of the fundamental and with an opposite phase to the fundamental is considered.

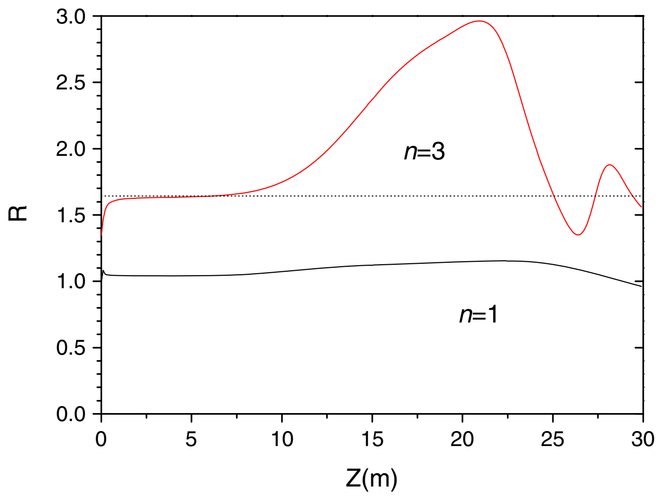

FIG. 6. The enhancement of the harmonic radiation [dashed line: result of Eq. (25)].

shows the effect of the third-harmonic magnetic field on evolution of the fundamental and the third-harmonic radiations. The case of the third-harmonic magnetic field being 30% of the fundamental is considered. It can be seen that comparing with the case without third-harmonic magnetic field present, the third-harmonic radiation was increased distinctly. The enhancement of the third harmonic radiation agrees well with our previous analytic analysis in the low gain region, is large than the analytic analysis in the exponential gain region, and waves around the analytic value in the saturation region (Fig. 6). Similar results can be obtained with other sets of parameters. The analytic results were given by Eq. (25), in the small signal region it is derived from Eq. (4), a strict formula; in the saturation region it is derived from Eq. (6), an approximate formula; in the exponential gain region it is derived from Eq. (5) but neglected that the gain length L_g also were changed with the coupling coefficient.

III. SUMMARY

In summary, we have analyzed the effects of the undulator harmonics field on the coupling coefficients and FEL harmonic generation. For the case where the third-harmonic field is present, analytical expressions of the modified coupling coefficients are given; they can be easily calculated to predict the effects of the undulator harmonics field on both the small signal gain in low gain FEL and the nonlinear harmonic generation in high gain FEL. The numerical results demonstrate that the third-harmonic emission can be distinctly enhanced by the undulator third-harmonic field that has an opposite sign to the fundamental, while the fundamental emission has been less affected. With a third-harmonic field 30% of the fundamental field, the third-harmonic emission can be enhanced about 40%, and even be doubled for the smaller value of the undulator deflecting parameter.

In addition, since the spontaneous radiation, i.e., the undulator radiation in synchrotron radiation light source, also relates to the coupling coefficients, therefore the similar analysis may be adopted and will be helpful for using the harmonic magnetic fields to enhance or suppress the harmonic undulator radiation.

ACKNOWLEDGMENTS

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