

Matrix formalism of synchrotron coupling

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In this paper we present a complete linear synchrotron coupling formalism by studying the transfer matrix which describes linear horizontal and longitudinal motions. With the technique established in the linear horizontal-vertical coupling study [D. Sagan and D. Rubin, *Phys. Rev. ST Accel. Beams* **2**, 074001 (1999)], we found a transformation to block diagonalize the transfer matrix and decouple the betatron motion and the synchrotron motion. By separating the usual dispersion term from the horizontal coordinate first, we were able to obtain analytic expressions of the transformation under reasonable approximations. We also obtained the perturbations to the betatron tune and the Courant-Snyder functions. The closed-orbit changes due to finite energy gains at rf cavities and radiation energy losses were studied by the 5×5 extended transfer matrix with the fifth column describing kicks in the 4-dimension phase space.

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I. INTRODUCTION

The synchrotron coupling (SBC) comes from dispersion at rf cavities and the path length dependence on the amplitude of betatron motion. The dispersion at an rf cavity makes the longitudinal kicks received from the cavity affect the betatron motion. Since the longitudinal kicks depend on the arrival time of the particles, the longitudinal motion is coupled to the betatron motion. On the other hand, particles with different betatron amplitudes have different path lengths which affect the arrival time. So betatron motion is also coupled to the longitudinal motion.

Traditionally many authors treated SBC with the Hamiltonian dynamics approach [1,2], which is a general and complete description and naturally covers effects of nonlinearities. It is very useful for the study of synchrotron resonances since in such cases one can focus on only the resonant term of the synchrotron potential. However, the Hamiltonian approach is cumbersome for the off-resonance cases which are most common for storage ring operations. In the linear case, a parallel approach is the matrix formalism first proposed by Chao in Ref. [3] which described the construction of the 6×6 transfer matrices and the decomposition of the coupled motion to the eigenmodes of the one-turn transfer matrix. Reference [3] also described an iterative procedure to include the nonlinear effects. Chao later applied the method to compute the beam tilt angle between x - z plane due to rf cavities [4].

Recently the study of low-alpha lattices stimulated Shoji's work on the path length effect which yielded an important result of bunch lengthening due to betatron emittance and dispersion [5]. On the other hand, Ref. [6] studied the SBC-induced closed-orbit change by considering the dipolelike kicks in the horizontal betatron phase

space due to the sudden changes of energy at a nonzero-dispersion rf cavity. The authors derived the horizontal closed-orbit changes induced by the finite energy gains at the rf cavities and verified with both simulations and experiments.

In this paper, we will study the linear synchrotron coupling under the transfer matrix framework without considering diffusion and damping due to radiation. Since the vertical motion is not essential to the SBC, we do not consider it for simplicity reasons. We then study the 4×4 horizontal-longitudinal transfer matrix in the same manner as the horizontal-vertical coupling is studied [7]. Namely, we try to decouple the horizontal and longitudinal motions by using a coordinate transformation to block diagonalize the transfer matrix and obtain the normal modes, in this case, the pure betatron mode and the pure synchrotron mode. We first study the fixed-energy case in which no rf cavity exists (or the rf gap voltage is set to zero). The transformed coordinates include the usual betatron coordinates and momentum deviation coordinate. But the longitudinal phase coordinate is modified by a term involving $Dx'_\beta - D'x_\beta$ which corresponds to the bunch lengthening effect studied by Shoji [5]. In cases with rf cavities, we first apply the previous fixed-energy transformation to separate the dispersion term. Since the synchrotron motion is usually slow, the coupling (off-diagonal) blocks of the transfer matrix for the new coordinates are small. Therefore we can perform the block diagonalization procedure proposed in Ref. [7] approximately yet with high precision. The transformation matrix is expressed analytically with the usual parameters such as the Courant-Snyder parameters, dispersion functions, and the rf voltage slope. When the normal modes are obtained, we can calculate their contributions to the beam width and bunch length. Perturbations to horizontal betatron motion due to SBC, including changes to the betatron tune and Courant-Snyder functions are also obtained.

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It is well known that SBC causes changes to the beam orbit [6,8–11]. In this study we find the closed orbit in the 4-dimension phase space in an analytical form through the 5×5 extended transfer matrix method [3], with the fifth column containing the 4-dimension kicks the beam receives from rf cavities and dipole magnets. Both finite energy gains at rf cavities and radiation energy losses in bending dipoles are considered. The radiation energy loss is random by nature. However, since usually hundreds of photons are emitted in one revolution, much more than the number of dipole magnets in which emission happens, we consider the radiation energy loss as a steady and uniform process. The radiation energy losses contribute additional terms to the horizontal closed-orbit change. The above results are verified with the accelerator modeling code AT [12].

This paper is organized as follows. Section I is this introduction. Section II describes the block diagonalization of the 4×4 transfer matrix in the fixed-energy case. Section III presents the matrix formalism of the synchrotron coupling. Section IV is the calculation of closed-orbit changes induced by finite energy gains at rf cavities and radiation energy losses. Section V shows simulation results and the comparison to the theory. Section VI gives the conclusions.

II. BLOCK DIAGONALIZATION FOR A FIXED-ENERGY RING

The 4-dimension coordinate vector is $\mathbf{X} = (\mathbf{x}^T, \mathbf{l}^T)^T$, where the horizontal coordinate vector is $\mathbf{x} = (x, x')^T$ and the longitudinal coordinate vector is $\mathbf{l} = (c\tau, \delta)^T$. The $c\tau$ coordinate instead of the phase coordinate $\phi = \phi_s - \frac{h}{R}c\tau$, where ϕ_s is the synchronous phase, h is the harmonic number, and R is the average ring radius, is used to avoid the appearance of scaling factors h/R in the transfer matrix. Note a negative $c\tau$ indicates the particle is behind the synchronous particle. The coordinates at the entrance and the exit of an accelerator component are related through its transfer matrix \mathbf{T} which can be divided into 2×2 blocks \mathbf{M} , \mathbf{E} , \mathbf{F} , and \mathbf{L} such that

$$\mathbf{T} = \begin{pmatrix} \mathbf{M} & \mathbf{E} \\ \mathbf{F} & \mathbf{L} \end{pmatrix}. \quad (1)$$

In general, the coupling blocks \mathbf{E} , \mathbf{F} of a single component are nonzero only for dipole magnets. For time-independent components (which include most common accelerator components except rf components such as rf cavity, rf dipoles, and rf quadrupoles), the coupling blocks have two zero matrix elements such that

$$\mathbf{E} = (\mathbf{0}, \mathbf{e}) \quad \text{and} \quad \mathbf{F} = \begin{pmatrix} \mathbf{f}^T \\ \mathbf{0} \end{pmatrix}, \quad (2)$$

where \mathbf{e} and \mathbf{f} are 2-component column vectors and $\mathbf{0}$'s are zero vectors of suitable sizes. The zero elements in matrix \mathbf{E} and \mathbf{F} are consequences of the fact that the horizontal

coordinates do not depend on the arrival time of the particles and the horizontal coordinates do not cause energy changes in such components. The \mathbf{L} blocks for rf cavities and other components are

$$\mathbf{L}_{\text{rf}} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{L}_{\text{other}} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, \quad (3)$$

$$w = \frac{e}{E} \frac{dV}{cd\tau} = -\frac{eV_0 \cos \phi_s}{E} \frac{h}{R}, \quad (4)$$

where V_0 is the gap voltage and E is the beam energy. The η parameter is related to the fractional phase slippage factor and is nonzero only for dipole magnets if we assume all particles have the same velocity c , the speed of light.

The transfer matrix for an accelerator section is the matrix product of the transfer matrices of the sequence of components which it consists of. For any sequence of components not containing an rf cavity, condition Eq. (2) still holds. The symplecticity requirement of the transfer matrices of such sections is equivalent to: \mathbf{M} and \mathbf{L} are symplectic and

$$\mathbf{M} \mathbf{J}_2 \mathbf{f} = \mathbf{e}, \quad \mathbf{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (5)$$

The \mathbf{e} vector and η parameter for a section from point 1 to point 2 can be written in integral forms

$$\mathbf{e}_{21} = \int_{s_1}^{s_2} \mathbf{M}(s_2|s) \begin{pmatrix} 0 \\ \frac{ds}{\rho} \end{pmatrix}, \quad (6)$$

$$\eta_{21} = - \int_{s_1}^{s_2} e^1(s_2|s) \frac{ds}{\rho}, \quad (7)$$

where s is the arclength along the reference orbit, $s_2|s$ means ‘‘from s to s_2 ,’’ ρ is the bending radius, and e^1 is the first element of \mathbf{e} .

The one-turn transfer matrix with an rf cavity in the ring can be derived in the following way. Suppose the rf cavity is located at point 2 and we want to calculate the transfer matrix at an arbitrary point 1. The transfer matrices between point 1, 2 and at the rf cavity are

$$\mathbf{T}_{21} = \begin{pmatrix} \mathbf{M}_{21} & \mathbf{E}_{21} \\ \mathbf{F}_{21} & \mathbf{L}_{21} \end{pmatrix}, \quad \mathbf{T}_{12} = \begin{pmatrix} \mathbf{M}_{12} & \mathbf{E}_{12} \\ \mathbf{F}_{12} & \mathbf{L}_{12} \end{pmatrix}, \quad (8)$$

$$\mathbf{T}_{\text{rf}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{\text{rf}} \end{pmatrix},$$

where \mathbf{M}_{21} and \mathbf{M}_{12} are the horizontal transfer matrices between the points, \mathbf{I} is the 2×2 identity matrix, and \mathbf{L}_{21} and \mathbf{L}_{12} differ from the 2×2 identity matrix by their (1, 2) elements of η_{21} and η_{12} , respectively. The one-turn transfer matrix at point 1 is

$$\mathbf{T} = \mathbf{T}_{12} \mathbf{T}_{\text{rf}} \mathbf{T}_{21} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{E}_1 \\ \mathbf{F}_1 & \mathbf{L}_1 \end{pmatrix}, \quad (9)$$

which can be expressed as

$$\mathbf{T} = \begin{pmatrix} \mathbf{M}_{12}\mathbf{M}_{21} + \mathbf{E}_{12}\mathbf{L}_{\text{rf}}\mathbf{F}_{21} & \mathbf{M}_{12}\mathbf{E}_{21} + \mathbf{E}_{12}\mathbf{L}_{\text{rf}}\mathbf{L}_{21} \\ \mathbf{F}_{12}\mathbf{M}_{21} + \mathbf{L}_{12}\mathbf{L}_{\text{rf}}\mathbf{F}_{21} & \mathbf{F}_{12}\mathbf{E}_{21} + \mathbf{L}_{12}\mathbf{L}_{\text{rf}}\mathbf{L}_{21} \end{pmatrix}. \quad (10)$$

We first consider the case when there is no rf cavity or the cavity is turned off. This corresponds to $w = 0$ and $\mathbf{L}_{\text{rf}} = \mathbf{I}$. Consequently, \mathbf{E}_1 and \mathbf{F}_1 satisfy conditions Eqs. (2) and (5). We intend to introduce a transformation $\mathbf{X} = \mathbf{U}\mathbf{X}_n$ to block diagonalize the new transfer matrix $\mathbf{T}_n = \mathbf{U}^{-1}\mathbf{T}\mathbf{U}$. It can be shown that this is achieved by

$$\mathbf{U} = \begin{pmatrix} \mathbf{I} & \mathbf{D}_1 \\ -\mathbf{D}_1^+ & \mathbf{I} \end{pmatrix}, \quad \mathbf{U}^{-1} = \begin{pmatrix} \mathbf{I} & -\mathbf{D}_1 \\ \mathbf{D}_1^+ & \mathbf{I} \end{pmatrix}, \quad (11)$$

where

$$\mathbf{D}_1 = (\mathbf{I} - \mathbf{M}_1)^{-1}\mathbf{E}_1 \quad (12)$$

and \mathbf{D}_1^+ is its symplectic conjugate [7]. We may introduce a column vector $\mathbf{d}_1 = (D_1, D_1')^T$ so that $\mathbf{D}_1 = (\mathbf{0}, \mathbf{d}_1)$. Equation (12) defines the dispersion functions D_1 and D_1' at point 1. One can also show that

$$\mathbf{E}_{21} = \mathbf{D}_2 - \mathbf{M}_{21}\mathbf{D}_1 \quad (13)$$

from $\mathbf{T}_2\mathbf{T}_{21} = \mathbf{T}_{21}\mathbf{T}_1$. The new transfer matrix is found to be

$$\mathbf{T}_n = \begin{pmatrix} \mathbf{M}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{1,n} \end{pmatrix}, \quad \text{with } \mathbf{L}_{1,n} = \begin{pmatrix} 1 & \bar{\eta} \\ 0 & 1 \end{pmatrix}, \quad (14)$$

where $\mathbf{L}_{1,n}$ is the new longitudinal transfer matrix and $\bar{\eta}$ is a constant of the ring given by

$$\bar{\eta} = - \oint \frac{Dds}{\rho}, \quad (15)$$

which is related to the usual momentum compaction factor α_0 by $\bar{\eta} = -2\pi R\alpha_0$. We have shown that

$$\bar{\eta} = \eta_1 - \mathcal{H}_1 \sin 2\pi\nu_x, \quad (16)$$

where ν_x is the betatron tune, η_1 is the (1, 2) element of \mathbf{L}_1 , and the \mathcal{H} -function along with its associated phase parameter are defined by

$$\mathcal{H} = \frac{1}{\beta} [D^2 + (\alpha D + \beta D')^2], \quad (17a)$$

$$\chi = \tan^{-1} \frac{D}{\alpha D + \beta D'}, \quad (17b)$$

where α, β are Courant-Snyder parameters. We may define the fractional phase slippage factor on a section between point 1 and point 2 by

$$\bar{\eta}_{21} = - \int_{s_1}^{s_2} \frac{Dds}{\rho}. \quad (18)$$

It has been shown that

$$\bar{\eta}_{21} = \eta_{21} - \sqrt{\mathcal{H}_1\mathcal{H}_2} \sin(\psi_{21} + \chi_1 - \chi_2), \quad (19)$$

where $\psi_{21} = \int_{s_1}^{s_2} ds/\beta$ is the betatron phase advance from point 1 to point 2.

The new coordinates at any location after transformation are related to the original coordinates by

$$\mathbf{x} = \mathbf{M}\mathbf{x}_n + \delta_n \mathbf{d}, \quad (20a)$$

$$c\tau = c\tau_n + Dx'_n - D'x_n, \quad (20b)$$

$$\delta = \delta_n. \quad (20c)$$

We recognize $\mathbf{x}_n = (x_n, x'_n)^T$ are just the betatron coordinates. The momentum deviation coordinate is not changed by the transformation. But the longitudinal phase coordinate is different by

$$Dx'_n - D'x_n = -\sqrt{2J_x\mathcal{H}} \cos(\psi - \chi), \quad (21)$$

where J_x is the horizontal betatron action variable and ψ is the phase variable [13].

III. MATRIX FORMALISM OF SYNCHROBETATRON COUPLING

When the rf cavity is turned on, its longitudinal transfer matrix deviates from the identity and is now

$$\mathbf{L}_{\text{rf}} = \mathbf{I} + \mathbf{W}, \quad \mathbf{W} = w \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (22)$$

Inserting it to Eq. (10) we get the one-turn transfer matrix

$$\mathbf{T} = \mathbf{T}^0 + w\tilde{\mathbf{T}}, \quad \mathbf{T}^0 = \begin{pmatrix} \mathbf{M}_1^0 & \mathbf{E}_1^0 \\ \mathbf{F}_1^0 & \mathbf{L}_1^0 \end{pmatrix}, \quad (23)$$

where the superscript 0 denotes quantities when the rf cavity is off and

$$\tilde{\mathbf{T}} = \begin{pmatrix} \tilde{\mathbf{M}}_1 & \tilde{\mathbf{E}}_1 \\ \tilde{\mathbf{F}}_1 & \tilde{\mathbf{L}}_1 \end{pmatrix} = \frac{1}{w} \begin{pmatrix} \mathbf{E}_{12}\mathbf{W}\mathbf{F}_{21} & \mathbf{E}_{12}\mathbf{W}\mathbf{L}_{21} \\ \mathbf{L}_{12}\mathbf{W}\mathbf{F}_{21} & \mathbf{L}_{12}\mathbf{W}\mathbf{L}_{21} \end{pmatrix}. \quad (24)$$

We may apply the procedure in Ref. [7] directly to block diagonalize matrix \mathbf{T} in Eq. (23). However, it is easier to relate the elements in the transformation matrix to the well-known parameters such as dispersion functions and rf parameters in the following way. We first apply the transformation described in the previous section and then apply the procedure of Ref. [7] to decouple the new transfer matrix. After the first transformation, the off-diagonal blocks of the transfer matrix are small because the main dispersion effect is separated. Hence, in the second transformation we can apply some approximations to derive explicit expressions for the transformation matrix and the final transfer matrix.

After the first transformation, the transfer matrix is

$$\mathbf{T}_n = \mathbf{U}^{-1}\mathbf{T}\mathbf{U} = \mathbf{T}_n^0 + w\tilde{\mathbf{T}}_n, \quad (25)$$

where

$$\mathbf{T}_n^0 = \mathbf{U}^{-1} \mathbf{T}^0 \mathbf{U} = \begin{pmatrix} \mathbf{M}_1^0 & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{1,n}^0 \end{pmatrix} \quad \text{and} \quad (26)$$

$$\tilde{\mathbf{T}}_n = \mathbf{U}^{-1} \tilde{\mathbf{T}} \mathbf{U} = \begin{pmatrix} \tilde{\mathbf{M}}_n & \tilde{\mathbf{E}}_n \\ \tilde{\mathbf{F}}_n & \tilde{\mathbf{L}}_n \end{pmatrix}.$$

Following Sagan-Rubin [7], we want to find the transformation matrix

$$\mathbf{V} = \begin{pmatrix} \gamma \mathbf{I} & \mathbf{C} \\ -\mathbf{C}^+ & \gamma \mathbf{I} \end{pmatrix} \quad (27)$$

to block diagonalize the matrix \mathbf{T}_n . According to the Sagan-Rubin procedure, we define

$$\mathbf{H} = w(\tilde{\mathbf{E}}_n + \tilde{\mathbf{F}}_n^+) \quad (28)$$

and let

$$\kappa = \frac{4\|\mathbf{H}\|}{\text{Tr}[\mathbf{M}_n - \mathbf{L}_n]^2}, \quad (29)$$

where \mathbf{M}_n and \mathbf{L}_n are diagonal blocks of \mathbf{T}_n , the γ parameter and \mathbf{C} matrix are then

$$\gamma = \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{1+\kappa}}}, \quad (30)$$

$$\mathbf{C} = -\frac{\mathbf{H}}{\gamma \sqrt{1+\kappa} \text{Tr}[\mathbf{M}_n - \mathbf{L}_n]}.$$

The elements of the \mathbf{H} matrix have been calculated to be

$$H_{11} = -2w\sqrt{\beta_1 \mathcal{H}_2} \sin \pi \nu_x \cos(\pi \nu_x - \psi_{12} - \chi_2), \quad (31a)$$

$$H_{12} = -w\sqrt{\beta_1 \mathcal{H}_2} [\tilde{\eta} \sin(\psi_{12} + \chi_2) - 2\tilde{\eta}_{12} \sin \pi \nu_x \cos(\pi \nu_x - \psi_{12} - \chi_2)], \quad (31b)$$

$$H_{21} = -2w\sqrt{\frac{\mathcal{H}_2}{\beta_1}} \sin \pi \nu_x [\sin(\pi \nu_x - \psi_{12} - \chi_2) - \alpha_1 \cos(\pi \nu_x - \psi_{12} - \chi_2)], \quad (31c)$$

$$H_{22} = w\sqrt{\frac{\mathcal{H}_2}{\beta_1}} \{ \tilde{\eta} [\alpha_1 \sin(\psi_{12} + \chi_2) - \cos(\psi_{12} + \chi_2)] - 2\tilde{\eta}_{12} \sin \pi \nu_x [\alpha_1 \cos(\pi \nu_x - \psi_{12} - \chi_2) - \sin(\pi \nu_x - \psi_{12} - \chi_2)] \}. \quad (31d)$$

It follows that

$$\|\mathbf{H}\| = w^2 \tilde{\eta} \mathcal{H}_2 \sin 2\pi \nu_x. \quad (32)$$

Equations (27) and (29)–(32) constitute an analytic form of the decoupling transformation for the general case. In the following we simplify the expressions for the off-resonance cases in which $\text{Tr}[\mathbf{M}_n - \mathbf{L}_n]$ is not close to zero. To this end we observe that w is usually a small quantity (e.g., $w = 0.008 \text{ m}^{-1}$ for SPEAR3) and κ is on the order of w^2 so that $\gamma \sqrt{1+\kappa} \approx 1 + 3\kappa/8 = 1 + O(w^2)$. To first order of w we have

$$\mathbf{C} = -\frac{\mathbf{H}}{2(\cos 2\pi \hat{\nu}_x - \cos 2\pi \nu_s)}, \quad (33)$$

where we have used $\text{Tr}[\mathbf{M}_n] = 2 \cos 2\pi \hat{\nu}_x$ and $\text{Tr}[\mathbf{L}_n] = 2 \cos 2\pi \nu_s$ by definition. The parameter $\hat{\nu}_x$ is related to the unperturbed betatron tune ν_x by $\cos 2\pi \hat{\nu}_x = \cos 2\pi \nu_x + \frac{1}{2} w \mathcal{H}_2 \sin 2\pi \nu_x$ since it has been shown that $\text{Tr}[\tilde{\mathbf{M}}_n] = \mathcal{H}_2 \sin 2\pi \nu_x$. To guarantee the symplecticity of the new transfer matrix, we must have

$$\gamma = \sqrt{1 - \|\mathbf{C}\|} = \sqrt{1 - \frac{w^2 \tilde{\eta} \mathcal{H}_2 \sin 2\pi \nu_x}{4(\cos 2\pi \hat{\nu}_x - \cos 2\pi \nu_s)^2}}. \quad (34)$$

The transfer matrix for the decoupled coordinates may be written as

$$\mathbf{T}_d = \mathbf{V}^{-1} \mathbf{T}_n \mathbf{V} = \begin{pmatrix} \mathbf{M}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_d \end{pmatrix}, \quad (35)$$

where \mathbf{M}_d and \mathbf{L}_d are given by [7]

$$\mathbf{M}_d = \gamma^2 \mathbf{M}_n - \gamma(\mathbf{C} \mathbf{F}_n + \mathbf{E}_n \mathbf{C}^+) + \mathbf{C} \mathbf{L}_n \mathbf{C}^+, \quad (36)$$

$$\mathbf{L}_d = \gamma^2 \mathbf{L}_n + \gamma(\mathbf{F}_n \mathbf{C} + \mathbf{C}^+ \mathbf{E}_n) + \mathbf{C}^+ \mathbf{M}_n \mathbf{C}. \quad (37)$$

For both of the above equations, the last three terms are on the order of w^2 . To second order of w , we have

$$\mathbf{M}_d \approx \mathbf{M}_1^0 + w \tilde{\mathbf{M}}_n + w^2 \mathbf{M}_d^{(2)}, \quad (38)$$

$$\mathbf{L}_d \approx \mathbf{L}_{1,n}^0 + w \tilde{\mathbf{L}}_n + w^2 \mathbf{L}_d^{(2)}. \quad (39)$$

Explicit expressions have been developed for $\tilde{\mathbf{M}}_n$ and $\tilde{\mathbf{L}}_n$. The elements of $\tilde{\mathbf{M}}_n$ are

$$\tilde{M}_{n,11} = \mathcal{H}_2 \sin(\psi_{12} + \chi_2) [\cos(\psi_{21} - \chi_2) + \alpha_1 \sin(\psi_{21} - \chi_2)], \quad (40a)$$

$$\tilde{M}_{n,12} = \mathcal{H}_2 \beta_1 \sin(\psi_{12} + \chi_2) \sin(\psi_{21} - \chi_2), \quad (40b)$$

$$\tilde{M}_{n,21} = \frac{\mathcal{H}_2}{\beta_1} [\cos(\psi_{21} - \chi_2) + \alpha_1 \sin(\psi_{21} - \chi_2)] [\cos(\psi_{12} + \chi_2) - \alpha_1 \sin(\psi_{12} + \chi_2)], \quad (40c)$$

$$\tilde{M}_{n,22} = \mathcal{H}_2 \sin(\psi_{21} - \chi_2) [\cos(\psi_{12} + \chi_2) - \alpha_1 \sin(\psi_{12} + \chi_2)]. \quad (40d)$$

The additional terms $w\tilde{\mathbf{M}}_n$ amount to changes to the Courant-Snyder parameters. For example, the change to β_1 to first order of w is

$$\Delta\beta_1 = \frac{w\mathcal{H}_2\beta_1}{2\sin 2\pi\nu_x} [\cos(2\pi\nu_x - 2\psi_{12} - 2\chi_2) - \cos 2\pi\nu_x]. \quad (41)$$

Also, knowing the traces $\text{Tr}[\tilde{\mathbf{M}}_d^{(2)}] = -\text{Tr}[\tilde{\mathbf{L}}_d^{(2)}] = -\tilde{\eta}\mathcal{H}_2 \cot \pi\nu_x/2$, we find the total betatron tune change to second order of w

$$\Delta\nu_x = -\frac{w\mathcal{H}_2}{4\pi} + \frac{w^2\tilde{\eta}\mathcal{H}_2}{16\pi\sin^2\pi\nu_x}. \quad (42)$$

From the expressions of \mathbf{H} and $\tilde{\mathbf{M}}_n$, it is clear that, when the rf cavity is located in a dispersion-free region and thus $\mathcal{H}_2 = 0$, there is no dynamic consequence from the coupling between horizontal and longitudinal motions.

The matrix $\tilde{\mathbf{L}}_n$ is given by

$$\tilde{\mathbf{L}}_n = \begin{pmatrix} \tilde{\eta}_{12} & \tilde{\eta}_{21}\tilde{\eta}_{12} \\ 1 & \tilde{\eta}_{21} \end{pmatrix}. \quad (43)$$

The unperturbed longitudinal transfer matrix $\mathbf{L}_n = \mathbf{L}_{1,n}^0 + w\tilde{\mathbf{L}}_n$ describes the plain longitudinal motion. The corrections due to synchrobetatron coupling is on the next higher order of w . The matrix \mathbf{L}_n can be Courant-Snyder parametrized so that

$$\mathbf{L}_n = \begin{pmatrix} \cos\Phi_s + \alpha_s \sin\Phi_s & \beta_s \sin\Phi_s \\ -\gamma_s \sin\Phi_s & \cos\Phi_s - \alpha_s \sin\Phi_s \end{pmatrix}, \quad (44)$$

with $\beta_s\gamma_s = 1 + \alpha_s^2$ and $\Phi_s = -2\pi\nu_s$, where a negative sign is chosen to make β_s always positive. The unperturbed synchrotron tune is given by

$$\nu_s = \frac{1}{2\pi} \sin^{-1} \left[\sqrt{-w\tilde{\eta} \left(1 + \frac{1}{4}w\tilde{\eta} \right)} \right]. \quad (45)$$

The unperturbed longitudinal Courant-Snyder functions are

$$\alpha_s = -\frac{1}{2} \frac{w(\tilde{\eta}_{12} - \tilde{\eta}_{21})}{\sin 2\pi\nu_s}, \quad \beta_s = -\frac{(\tilde{\eta} + w\tilde{\eta}_{12}\tilde{\eta}_{21})}{\sin 2\pi\nu_s}, \quad (46)$$

$$\gamma_s = \frac{w}{\sin 2\pi\nu_s},$$

which are equivalent forms of those found in Ref. [14]. It is noted that γ_s is a positive constant and β_s is positive but slightly varies around the ring. Without coupling, the lon-

gitudinal coordinates ($c\tau_n, \delta_n$) are related to these parameters, the longitudinal action variable J_s , and phase variable ψ_s by

$$c\tau_n = \sqrt{2J_s\beta_s} \cos\psi_s, \quad (47)$$

$$\delta_n = -\sqrt{\frac{2J_s}{\beta_s}} (\alpha_s \cos\psi_s + \sin\psi_s),$$

from which we obtain relations between the rms bunch length, rms momentum spread, and the longitudinal emittance ϵ_s :

$$\sigma_{c\tau_n}^2 = \beta_s \epsilon_s, \quad \sigma_{\delta_n}^2 = \gamma_s \epsilon_s. \quad (48)$$

Note that the bunch length varies with location because particles with different momentum deviation experience different longitudinal phase slippage. The bunch is longest at the rf cavity and can be shortened by a maximum of the $\pi^2\nu_s^2/2$ part of the original length at halfway across the ring from the cavity. For fast ramping synchrotrons, it can be as large as 5%, assuming $\nu_s = 0.1$. The synchrobetatron coupling should slightly change the synchrotron tune and the longitudinal Courant-Snyder functions given above.

The decoupled coordinates $\mathbf{X}_d = (\mathbf{x}_d^T, \mathbf{I}_d^T)^T$ are related to the original coordinates \mathbf{X} by

$$\mathbf{X} = \mathbf{U}\mathbf{V}\mathbf{X}_d = \begin{pmatrix} \gamma\mathbf{I} - \mathbf{D}_1\mathbf{C}^+ & \mathbf{C} + \gamma\mathbf{D}_1 \\ -\mathbf{C}^+ - \gamma\mathbf{D}_1^+ & \gamma\mathbf{I} - \mathbf{D}_1^+\mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{x}_d \\ \mathbf{I}_d \end{pmatrix}. \quad (49)$$

The betatron coordinates ($\mathbf{x}_n^T, \mathbf{I}_n^T$)^T are related to the normal modes by the transformation matrix \mathbf{V} . Since all four elements of the \mathbf{C} matrix in \mathbf{V} are nontrivial in general, the horizontal betatron coordinate x_n depends on the longitudinal phase, as pointed out in Ref. [6]. This is a natural consequence of the synchrobetatron coupling. In fact, the longitudinal coordinates $c\tau_n, \delta_n$ also depend on the horizontal betatron coordinates. In terms of the betatron coordinates \mathbf{X}_n , the phase space volume occupied by the beam tilts across the horizontal and longitudinal subspace. Particles flow in and out between the two subspaces. However, the total phase space volume is preserved. The reason for Ref. [6] to suggest that the total phase space volume is not preserved is because it did not fully consider the coupling effects on the horizontal betatron motion.

With Eq. (49) we can decompose the 4-dimension motion to the normal modes for any given initial condition. It also allows us to derive the effects of synchrobetatron

coupling on the beam sizes σ_x and $\sigma_{c\tau}$. Following the single mode analysis of Ref. [7], we first consider the case when only the betatron mode, or mode a , is excited so that

$$\begin{pmatrix} x \\ x' \end{pmatrix} = (\gamma \mathbf{I} - \mathbf{D}_1 \mathbf{C}^+) \begin{pmatrix} x_a \\ x'_a \end{pmatrix}, \quad (50)$$

$$\begin{pmatrix} c\tau \\ \delta \end{pmatrix} = -(\mathbf{C}^+ + \gamma \mathbf{D}_1^+) \begin{pmatrix} x_a \\ x'_a \end{pmatrix},$$

where (x_a, x'_a) are the betatron normal mode coordinates given by

$$x_a = \sqrt{2J_a \beta_a} \cos \psi_a,$$

$$x'_a = -\sqrt{\frac{2J_a}{\beta_a}} (\alpha_a \cos \psi_a + \sin \psi_a). \quad (51)$$

It follows that

$$\frac{x}{\sqrt{2J_a}} = \left[\gamma \sqrt{\beta_a} + D \left(C_{21} \sqrt{\beta_a} + C_{11} \frac{\alpha_a}{\sqrt{\beta_a}} \right) \right] \cos \psi_a + \frac{DC_{11}}{\sqrt{\beta_a}} \sin \psi_a, \quad (52a)$$

$$\frac{c\tau}{\sqrt{2J_a}} = -[\gamma(\beta_a D' + \alpha_a D) + (\beta_a C_{22} + \alpha_a C_{12})] \times \frac{1}{\sqrt{\beta_a}} \cos \psi_a - \frac{(C_{12} + \gamma D)}{\sqrt{\beta_a}} \sin \psi_a. \quad (52b)$$

The bunch width and length can be derived from the above equations by integrating over the bunch distribution. Here we consider only the off-resonance cases and assume $\cos \nu_s \approx 1$ so that $\mathbf{C} \approx \mathbf{H}/4\sin^2 \pi \nu_x$. We get

$$\sigma_{x,a}^2 = \beta_a \epsilon_a - \epsilon_a \frac{wD\sqrt{\beta_1 \mathcal{H}_2}}{\sin \pi \nu_x} \sin(\pi \nu_x - \psi_{12} - \chi_2) + \epsilon_a \frac{w^2 \mathcal{H}_2}{4\sin^2 \pi \nu_x} \left(D^2 - \frac{1}{2} \beta_1 \bar{\eta} \cot \pi \nu_x \right), \quad (53)$$

where ϵ_a is the emittance of the horizontal betatron normal mode. We see that the term on the order of w varies around the ring. The leading nonvarying correction term is on the order of w^2 . Since the $O(w^2)$ terms are very small, we will drop them in the following for brevity. Similarly for the bunch length, we obtain

$$\sigma_{c\tau,a}^2 = \epsilon_a \mathcal{H}_1^a - \epsilon_a \frac{w\sqrt{\mathcal{H}_1 \mathcal{H}_2}}{2\sin^2 \pi \nu_x} [\bar{\eta} \cos(\psi_{12} + \chi_2 - \chi_1) - 2\bar{\eta}_{12} \sin \pi \nu_x \sin(\pi \nu_x - \psi_{12} - \chi_2 + \chi_1)], \quad (54)$$

where \mathcal{H}_1^a is the \mathcal{H} -function at point 1 evaluated with the perturbed Courant-Snyder functions.

The same analysis can be applied to the synchrotron mode, or mode b , to obtain its contributions to bunch width and length, which are given by

$$\sigma_{x,b}^2 = \epsilon_b \gamma_b D^2 + \epsilon_b \gamma_b \frac{w\bar{\eta}D\sqrt{\beta_1 \mathcal{H}_2}}{2\sin^2 \pi \nu_x} \times \cos \pi \nu_x \sin(\pi \nu_x - \psi_{12} - \chi_2), \quad (55)$$

$$\sigma_{c\tau,b}^2 = \epsilon_b \beta_b - \epsilon_b \gamma_b \frac{\bar{\eta}\sqrt{\mathcal{H}_1 \mathcal{H}_2}}{\sin \pi \nu_x} \times \cos(\pi \nu_x - \psi_{12} - \chi_2 + \chi_1). \quad (56)$$

Note that γ_b is an $O(w)$ factor. The momentum spread is given by

$$\sigma_{\delta,a}^2 = \epsilon_a \frac{w^2 \mathcal{H}_2}{4\sin^2 \pi \nu_x},$$

$$\sigma_{\delta,b}^2 = \epsilon_b \gamma_b - \epsilon_b \gamma_b \frac{w^2 \bar{\eta} \mathcal{H}_2 \sin 2\pi \nu_x}{16\sin^4 \pi \nu_x}. \quad (57)$$

So to first order of w we have $\sigma_{\delta}^2 = \epsilon_b \gamma_b$.

The contributions to σ^2 from mode a and b simply add up [7], i.e.,

$$\sigma^2 = \sigma_a^2 + \sigma_b^2, \quad (58)$$

which applies to all four coordinates. Equations (53)–(57) show that, because of synchrotron coupling, the longitudinal motion affects the transverse beam size and the transverse motion affects the longitudinal beam size. The leading correction term is usually a small term on the order of w and varies with the horizontal betatron phase advance around the ring. It is noted that we have recovered Shoji's result in Ref. [5] by the first term in Eq. (54) which is independent of w and indicates that the bunch length varies from location to location according to the local \mathcal{H} -function. This is a consequence of the uneven distribution of the path length effect.

IV. CLOSED-ORBIT CHANGE DUE TO ENERGY GAIN AND LOSS

The finite energy gain at the rf cavity and the radiation energy loss around the ring are kicks to the momentum deviation coordinate. These kicks are transferred downstream and affect all other coordinates. Therefore the closed orbit of the beam is changed. To study this effect in the matrix formalism, we extend the coordinate vector to $\mathbf{X}^e = (x, x', c\tau, \delta, 1)^T$ [3]. The corresponding transfer matrix is then 5×5 . The fifth element is included to describe the kicks received by the particle, namely, the equation

$$\mathbf{X}_2 = \mathbf{T}_{21} \mathbf{X}_1 + \mathbf{g}_{21} \quad (59)$$

will now be written as

$$\mathbf{X}_2^e = \mathbf{T}_{21}^e \mathbf{X}_1^e, \quad \mathbf{T}_{21}^e = \begin{pmatrix} \mathbf{T}_{21} & \mathbf{g}_{21} \\ \mathbf{0} & 1 \end{pmatrix}, \quad (60)$$

where \mathbf{g}_{21} is a 4-vector which represents the kick-induced

shifts of phase space coordinates across the accelerator section from point 1 to point 2. The closed orbit \mathbf{X}_c is given by the fixed point $\mathbf{X}_c^e = (\mathbf{X}_c^T, 1)^T$ of the extended one-turn transfer matrix \mathbf{T}^e [3], i.e., $\mathbf{T}^e \mathbf{X}_c^e = \mathbf{X}_c^e$, which yields

$$\mathbf{X}_c = (\mathbf{I} - \mathbf{T})^{-1} \mathbf{g}, \quad (61)$$

where the \mathbf{g} vector contains the first four elements of the fifth column of \mathbf{T}^e and represents the coordinate shifts after one turn when a particle starts from a point with initial coordinates of all zeros. We will derive analytic forms for \mathbf{g} and $(\mathbf{I} - \mathbf{T})^{-1}$ below.

For an rf cavity, there is an energy kick

$$\mathbf{g}_{\text{rf}} = (0, 0, 0, \epsilon)^T, \quad \epsilon = \frac{\Delta E}{E}, \quad (62)$$

where ϵ denotes the sudden change of momentum deviation at the rf cavity. For a dipole magnet, the energy kick due to radiation energy loss will propagate to the other coordinates and cause finite changes to them. For example, the pure sector dipole with bending radius ρ and bending angle θ_L has

$$\mathbf{g}_{\text{dipole}} = \left(-\frac{\epsilon}{2\pi} \right) \begin{pmatrix} \rho(\theta_L - \sin\theta_L) \\ 1 - \cos\theta_L \\ \rho(1 - \cos\theta_L - \frac{1}{2}\theta_L^2) \\ \theta_L \end{pmatrix}, \quad (63)$$

where $-\epsilon\theta_L/2\pi$ is the momentum deviation change on this dipole magnet. Here we have assumed ϵ is the same as the momentum deviation gained at the rf cavity. It is seen that the changes of x , x' , and $c\tau$ are on the order of θ_L^3 , θ_L^2 , and θ_L^4 , respectively. Hence, when θ_L is small, we may neglect these changes for a single dipole magnet. The \mathbf{g} vectors are zeros for other accelerator components which do not cause energy gains or losses.

The one-turn extended transfer matrix can be computed as usual. For an arbitrary point 1, we have $\mathbf{T}_1^e = \mathbf{T}_{12}^e \mathbf{T}_{\text{rf}}^e \mathbf{T}_{21}^e$, from which we obtain

$$\mathbf{g}_1 = (\mathbf{T}_{12} \mathbf{g}_{21} + \mathbf{g}_{12}) + \mathbf{T}_{12} \mathbf{g}_{\text{rf}} + \mathbf{T}_{12} (\mathbf{T}_{\text{rf}} - \mathbf{I}) \mathbf{g}_{21}. \quad (64)$$

Simple calculations show that

$$\mathbf{T}_{12} \mathbf{g}_{\text{rf}} = \begin{pmatrix} \mathbf{M}_{12} & \mathbf{E}_{12} \\ \mathbf{F}_{12} & \mathbf{L}_{12} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon \end{pmatrix} = \epsilon \begin{pmatrix} \mathbf{d}_1 - \mathbf{M}_{12} \mathbf{d}_2 \\ \eta_{12} \\ 1 \end{pmatrix}. \quad (65)$$

The $\mathbf{T}_{12} \mathbf{g}_{21} + \mathbf{g}_{12}$ term has nothing to do with the rf cavity. So it does not depend on the location of point 2. In fact it represents the changes of coordinates in one turn for a particle with initial coordinate $(0, 0, 0, 0)^T$ at point 1 when the rf cavity is turned off. Since energy losses occur in dipoles, we get a summation over all dipoles in the ring

$$\begin{aligned} \mathbf{T}_{12} \mathbf{g}_{21} + \mathbf{g}_{12} &= \sum_{i=1}^{N_d} \begin{pmatrix} \mathbf{M}_{1i} & \mathbf{E}_{1i} \\ \mathbf{F}_{1i} & \mathbf{L}_{1i} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\epsilon_i \end{pmatrix} \\ &= - \begin{pmatrix} \sum_i \mathbf{e}_{1i} \epsilon_i \\ \left(\sum_i \eta_{1i} \epsilon_i \right) \\ \sum_i \epsilon_i \end{pmatrix}. \end{aligned} \quad (66)$$

Noting that $\mathbf{e}_{1i} = \mathbf{d}_1 - \mathbf{M}_{1i} \mathbf{d}_i$ and, assuming all bending radius are equal so that $\epsilon_i = \epsilon \Delta s_i / 2\pi\rho$, the summations can be turned to integrals to obtain

$$\mathbf{T}_{12} \mathbf{g}_{21} + \mathbf{g}_{12} = (-\epsilon) \begin{pmatrix} \mathbf{d}_1 - \oint_1 \mathbf{M}(s_1|s) \mathbf{d}(s) \frac{ds}{2\pi\rho} \\ \left(\oint_1 \eta(s_1|s) \frac{ds}{2\pi\rho} \right) \\ 1 \end{pmatrix}, \quad (67)$$

where we have used $\sum_i \epsilon_i = \epsilon$ and the integral \oint_1 starts from point 1, completes one revolution, and ends at point 1. We may simplify the expressions by defining new functions:

$$\mathcal{S}(s) = \int_s^{s+C} \sqrt{\mathcal{H}(s')} \sin(\psi_{ss'} + \chi_{s'}) \frac{ds'}{2\pi\rho}, \quad (68a)$$

$$\mathcal{C}(s) = \int_s^{s+C} \sqrt{\mathcal{H}(s')} \cos(\psi_{ss'} + \chi_{s'}) \frac{ds'}{2\pi\rho}, \quad (68b)$$

$$\mathcal{K}(s) = \mathcal{S}^2(s) + \mathcal{C}^2(s), \quad \xi(s) = \tan^{-1} \frac{\mathcal{S}(s)}{\mathcal{C}(s)}, \quad (68c)$$

where $\psi_{ss'}$ is the betatron phase advance from point s' to point s and C is the ring circumference. Then we can write down

$$\oint_1 \mathbf{M}(s_1|s) \mathbf{d}(s) \frac{ds}{2\pi\rho} = \begin{pmatrix} \frac{\sqrt{\beta_1}}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{C}_1 \end{pmatrix} \quad (69)$$

and

$$\begin{aligned} \oint_1 \eta(s_1|s) \frac{ds}{2\pi\rho} &= \oint_1 \bar{\eta}(s_1|s) \frac{ds}{2\pi\rho} \\ &\quad + \sqrt{\mathcal{H}_1 \mathcal{K}_1} \sin(\xi_1 - \chi_1). \end{aligned} \quad (70)$$

It can be shown that

$$\oint_1 \bar{\eta}(s_1|s) \frac{ds}{2\pi\rho} = \frac{1}{2} \bar{\eta}. \quad (71)$$

Since the first three elements of the \mathbf{g} vector have terms on the order of $O(\epsilon)$ and the contributions of the third term of Eq. (64) to these elements are on the order of $O(w\epsilon)$, we will simply drop these contributions. However, the fourth element of this term is the leading term and is not negligible. It is easy to show that

$$\begin{aligned}
g_1^4 &= wg_{21}^3 = -w\epsilon \int_{s_1}^{s_2} \eta(s_2|s) \frac{ds}{2\pi\rho} \\
&= -w\epsilon \int_{s_1}^{s_2} \bar{\eta}(s_2|s) \frac{ds}{2\pi\rho} - w\epsilon \sqrt{\mathcal{H}_2 \mathcal{K}_{21}} \sin(\xi_{21} - \chi_2)
\end{aligned} \quad (72)$$

with the definition of

$$S_{21} = \int_{s_1}^{s_2} \sqrt{\mathcal{H}(s')} \sin(\psi_{s_2 s'} + \chi_{s'}) \frac{ds}{2\pi\rho}, \quad (73a)$$

$$C_{21} = \int_{s_1}^{s_2} \sqrt{\mathcal{H}(s')} \cos(\psi_{s_2 s'} + \chi_{s'}) \frac{ds}{2\pi\rho}, \quad (73b)$$

$$\mathcal{K}_{21} = S_{21}^2 + C_{21}^2, \quad \xi_{21} = \tan^{-1} \frac{S_{21}}{C_{21}}. \quad (73c)$$

If we write down the vector \mathbf{g}_1 in a form $\mathbf{g}_1 = (\mathbf{g}_x^T, \mathbf{g}_l^T)^T$ with 2-component vector \mathbf{g}_x and \mathbf{g}_l representing the horizontal and longitudinal displacement, respectively, we have

$$\mathbf{g}_x/\epsilon = \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} S_1 - \sqrt{\mathcal{H}_2} \sin(\psi_{12} + \chi_2) \\ C_1 - \sqrt{\mathcal{H}_2} \cos(\psi_{12} + \chi_2) \end{pmatrix}, \quad (74a)$$

$$g_l^1/\epsilon = \bar{\eta}_{12} - \frac{1}{2}\bar{\eta} + \sqrt{\mathcal{H}_1 \mathcal{H}_2} \sin(\psi_{12} + \chi_2 - \chi_1) - \sqrt{\mathcal{H}_1 \mathcal{K}_1} \sin(\xi_1 - \chi_1), \quad (74b)$$

$$g_l^2/\epsilon = -w \frac{\bar{\eta}_{21}^2}{2\bar{\eta}} - w \sqrt{\mathcal{H}_2 \mathcal{K}_{21}} \sin(\xi_{21} - \chi_2), \quad (74c)$$

where g_l^1, g_l^2 are the two elements of $\mathbf{g}_l = (g_l^1, g_l^2)^T$ and in obtaining the first term of Eq. (74c) we have assumed the dipoles are distributed around the ring uniformly.

Radiation damping changes the transfer matrix \mathbf{T} by a correction term on the order of ϵ so that it is no more strictly symplectic. However, since ϵ is usually small and the correction to the closed orbit due to this effect is on the order of ϵ^2 , we will not consider the radiation damping effect. To work out the matrix inversion for $(\mathbf{T} - \mathbf{I})^{-1}$, we will make use of

$$(\mathbf{T} - \mathbf{I})^{-1} = \mathbf{U}(\mathbf{T}_n - \mathbf{I})^{-1}\mathbf{U}^{-1}, \quad (75)$$

since the off-diagonal blocks of \mathbf{T}_n are on the order of $O(w)$. With matrix \mathbf{T}_n as found in Eq. (25), we have shown that

$$(\mathbf{T}_n - \mathbf{I})^{-1} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \quad (76)$$

with

$$\mathbf{a} = (\mathbf{M}_1^0 - \mathbf{I})^{-1}, \quad (77a)$$

$$\mathbf{b} = -(\mathbf{M}_1^0 - \mathbf{I})^{-1} \begin{pmatrix} 0 & \tilde{E}_n^{11} \\ 0 & \tilde{E}_n^{21} \end{pmatrix}, \quad (77b)$$

$$\mathbf{c} = - \begin{pmatrix} \tilde{F}_n^{21} & \tilde{F}_n^{22} \\ 0 & 0 \end{pmatrix} (\mathbf{M}_1^0 - \mathbf{I})^{-1}, \quad (77c)$$

$$\mathbf{d} = (\mathbf{L}_n - \mathbf{I})^{-1}, \quad (77d)$$

where we have dropped all terms on the order of $O(w)$ or higher and we have

$$(\mathbf{L}_n - \mathbf{I})^{-1} = \frac{1}{\bar{\eta}} \begin{pmatrix} -\bar{\eta}_{21} & \frac{\bar{\eta}}{w} + \bar{\eta}_{21}\bar{\eta}_{12} \\ 1 & -\bar{\eta}_{12} \end{pmatrix}. \quad (78)$$

We then proceed to obtain the blocks of $(\mathbf{T} - \mathbf{I})^{-1}$ using Eq. (75) and finally we get the closed orbit with Eq. (61). The results are given by

$$x_c = -\frac{\epsilon\sqrt{\beta_1}\mathcal{H}_2}{2\sin\pi\nu_x} \cos(\pi\nu_x - \psi_{12} - \chi_2) + \frac{\epsilon\sqrt{\beta_1}\mathcal{K}_1}{2\sin\pi\nu_x} \cos(\pi\nu_x - \xi_1) + \epsilon D_1 \left(\frac{1}{2} - \frac{\bar{\eta}_{12}}{\bar{\eta}} \right), \quad (79a)$$

$$\begin{aligned}
x'_c &= \frac{\epsilon}{2\sin\pi\nu_x} \sqrt{\frac{\mathcal{H}_2}{\beta_1}} [\alpha_1 \cos(\pi\nu_x - \psi_{12} - \chi_2) - \sin(\pi\nu_x - \psi_{12} - \chi_2)] \\
&\quad - \frac{\epsilon}{2\sin\pi\nu_x} \sqrt{\frac{\mathcal{K}_1}{\beta_1}} [\alpha_1 \cos(\pi\nu_x - \xi_1) - \sin(\pi\nu_x - \xi_1)] + \epsilon D'_1 \left(\frac{1}{2} - \frac{\bar{\eta}_{12}}{\bar{\eta}} \right),
\end{aligned} \quad (79b)$$

$$\begin{aligned}
c\tau_c &= \epsilon \frac{\bar{\eta}_{21}\bar{\eta}_{12}}{2\bar{\eta}} + \frac{\epsilon}{2\sin\pi\nu_x} [\sqrt{\mathcal{H}_1\mathcal{H}_2} \cos(\pi\nu_x - \psi_{12} - \chi_2 + \chi_1) + \sqrt{\mathcal{K}_1\mathcal{H}_2} \cos(\pi\nu_x - \psi_{12} - \chi_2 + \xi_1) \\
&\quad - \sqrt{\mathcal{K}_1\mathcal{H}_1} \cos(\pi\nu_x + \chi_1 - \xi_1) - \mathcal{H}_2 \cos\pi\nu_x] + \epsilon \sqrt{\mathcal{H}_2\mathcal{K}_{21}} \sin(\xi_{21} - \chi_2),
\end{aligned} \quad (79c)$$

$$\delta_c = \epsilon \left(\frac{1}{2} - \frac{\bar{\eta}_{12}}{\bar{\eta}} \right). \quad (79d)$$

Note again that we have dropped all terms on the order of w or higher so that the results are valid only for off-resonance cases. Obviously we have recovered Eqs. (23) and (24) of Ref. [6] as the first term in Eq. (79a). The second term comes from the propagation of radiation energy losses. The third term comes from the energy variation around the ring. It is worth noting that the $c\tau_c$ orbit is zero at the rf cavity.

So far we have considered only one rf cavity in the ring. However, the same analysis can be easily applied to more cavities. In fact, if we neglect the interaction between the rf cavities, which corresponds to higher order terms of the w_i parameters, the one-turn transfer matrix is

$$\mathbf{T}_1 = \mathbf{T}_1^0 + \sum_i \mathbf{T}_{1i} \mathbf{W}_i \mathbf{T}_{i1}, \quad (80)$$

where the summation is over all cavities. Equation (80) means the total effect of all cavities is the linear superposition of the single cavity effects. It is then straightforward to modify the results of the one-cavity case for multiple-cavity cases.

V. SIMULATION

In this section we will verify the theory developed in the previous sections by comparing the results to simulations with the accelerator modeling code AT [12]. We use the machine model of the SPEAR Booster for the calculation. The SPEAR Booster is chosen because it has appreciable dispersion functions at the rf cavity. The model consists of 20 periods of FODO lattice over a circumference of

$2\pi R = 133.8$ m. The bending radius is $\rho = 11.82$ m for all dipoles. The extraction energy is $E = 3$ GeV. The rf frequency is $f_{\text{rf}} = 358.533$ MHz and the harmonic number is $h = 160$. At extraction, the rf gap voltage is $V_{\text{rf}} = 0.8$ MV and the one-turn radiation energy loss is $U_0 = 0.60$ MeV. In the simulation, we will consider it as a storage ring running at its extraction energy. The betatron tunes are $\nu_x = 6.16$ and $\nu_y = 4.23$ for the model. At the rf cavity, the Courant-Snyder functions are $\alpha_2 = -0.72$ and $\beta_2 = 2.09$ m and the dispersion functions are $D_2 = 0.27$ m and $D'_2 = 0.06$. The horizontal chromaticity is $C_x = -7.9$.

We first present a numerical example to verify the transformation Eq. (49). The injection point of the SPEAR Booster is located in the ring opposite to the rf cavity where we have $\alpha_1 = 0.02$, $\beta_1 = 1.54$ m, $D_1 = 0.24$ m and $D'_1 = 0.01$ and the betatron phase advance from the rf cavity to this observation point is $\psi_{12} = 0.16$ rad modulo 2π . The synchronous phase is set to $\phi_s = \pi$ and hence $w = 0.0020$ m⁻¹. The one-turn transfer matrix for betatron coordinates \mathbf{X}_n at this location is

$$\mathbf{T}_n = \begin{pmatrix} 0.550532 & 1.306266 & -0.000447 & 0.001018 \\ -0.547728 & 0.516813 & 0.000129 & -0.000294 \\ 0.000396 & -0.000910 & 0.995433 & -4.548267 \\ -0.000174 & 0.000399 & 0.002004 & 0.995433 \end{pmatrix}. \quad (81)$$

The off-diagonal blocks represents the synchrobetatron coupling effects. We then apply the second transformation \mathbf{V} as determined by Eqs. (27), (31), (33), and (34) to obtain the final transfer matrix:

$$\mathbf{T}_d = \begin{pmatrix} 0.550533 & 1.306265 & -4.6 \times 10^{-10} & 1.05 \times 10^{-9} \\ -0.547728 & 0.516813 & 1.3 \times 10^{-10} & -3.0 \times 10^{-10} \\ 4.1 \times 10^{-10} & -9.4 \times 10^{-10} & 0.995432 & -4.548266 \\ -1.8 \times 10^{-10} & 4.1 \times 10^{-10} & 0.002004 & 0.995432 \end{pmatrix}. \quad (82)$$

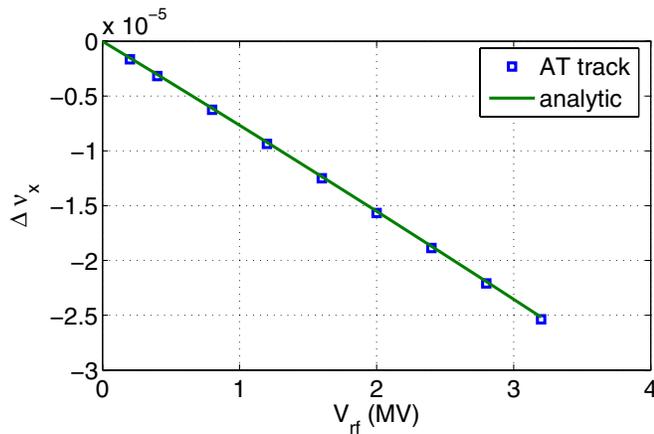


FIG. 1. (Color) The synchrobetatron coupling induced betatron detuning $\Delta\nu_x$ obtained by AT tracking is compared to analytic calculation with Eq. (42). The rf gap voltage is varied from 0.2 to 3.2 MV.

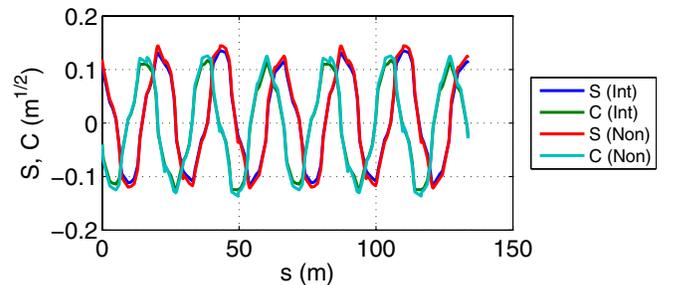


FIG. 2. (Color) Function S and C as defined in Eq. (68) for the SPEAR Booster at 3 GeV. The curves labeled “Int” are from numerical integration with the trapezoidal rule by cutting each dipole into 100 slices. The “Non” curves are obtained by calculating the coordinate shifts at the exit of each dipole, transporting to the observation point and summing up contributions of all dipoles.

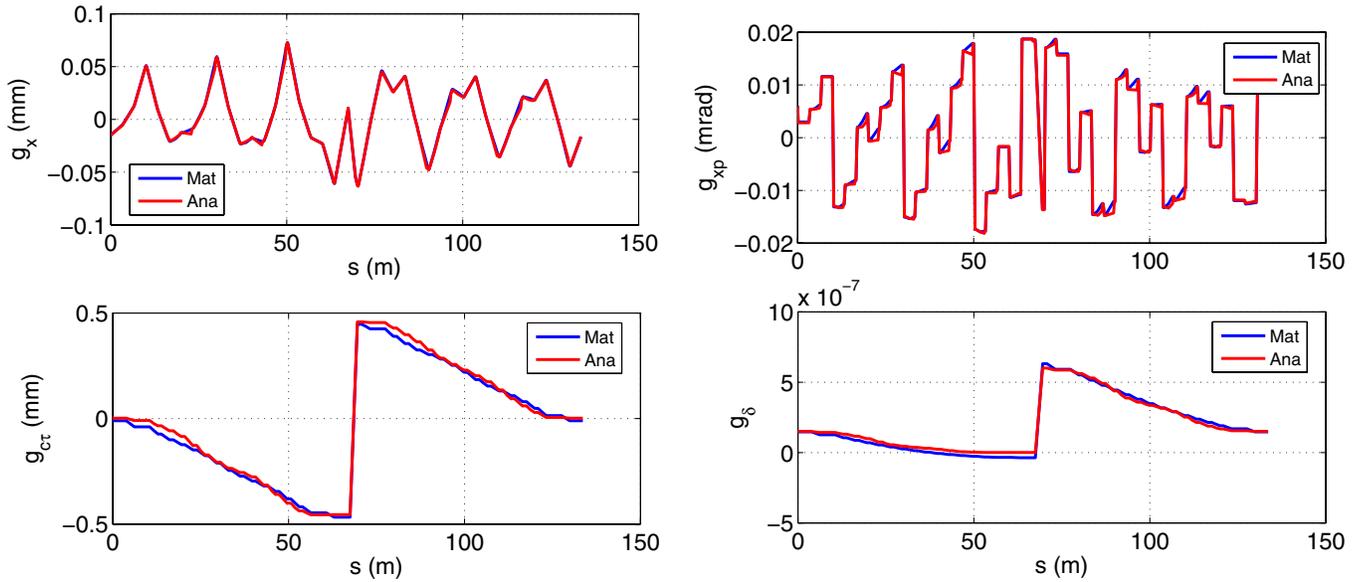


FIG. 3. (Color) The one-turn coordinate shifts due to radiation energy losses and finite energy gains at the rf cavity for the SPEAR Booster at 3 GeV. The four plots are x , x' , $c\tau$, and δ shifts, respectively.

Elements of the off-diagonal blocks are reduced to the order of magnitude of 10^{-10} from the original value of 10^{-4} , indicating the high precision of the approximations we have made.

The betatron detuning due to synchrotron coupling [Eq. (42)] is checked against AT by comparing the betatron tune from AT tracking to the analytic calculation with various rf voltage. The initial condition for AT tracking is $(1 \times 10^{-5}, 0, 0, 0, 1 \times 10^{-6}, 0)^T$. Small deviations of x

and δ are chosen to avoid significant nonlinear detuning effects. The result is shown in Fig. 1.

If radiation is turned on in simulation, the finite energy gain at the rf cavity will cause changes to the closed orbit as studied in the previous section. To calculate the closed-orbit change, we need to calculate the functions \mathcal{S} , \mathcal{C} , \mathcal{K} , and ξ defined in Eq. (68). It is found that numerical integration through a dipole does not make much difference from simply transporting the coordinate shifts at the

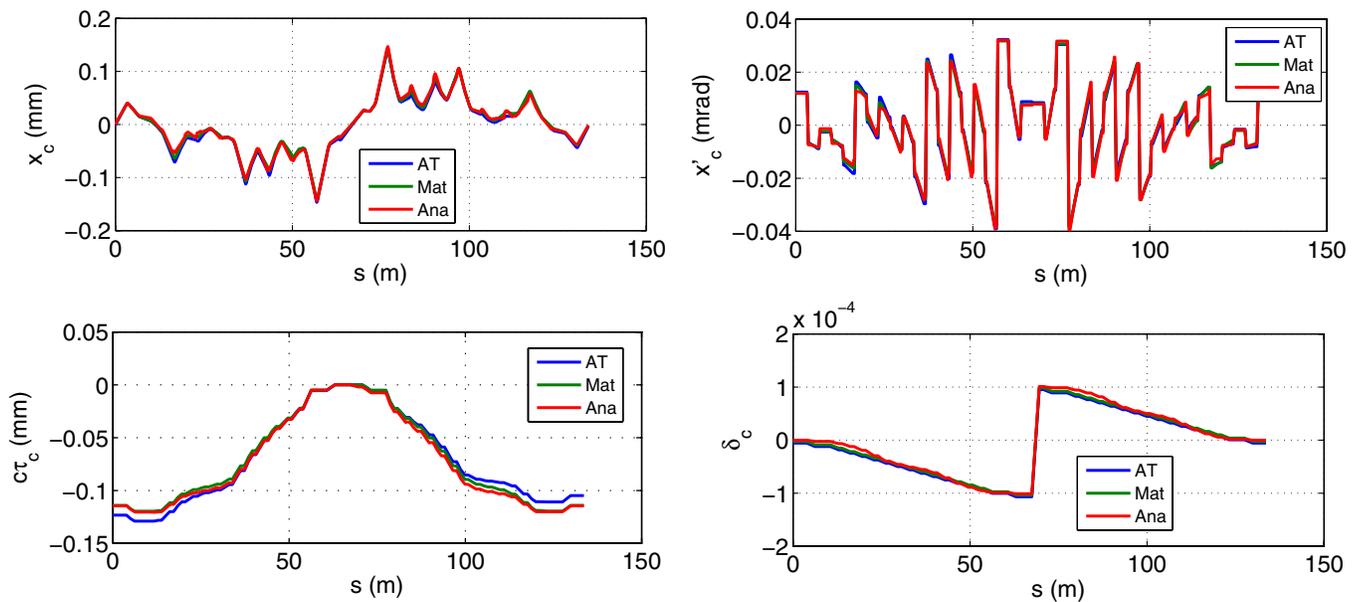


FIG. 4. (Color) The SBC-induced closed-orbit changes calculated in three different ways for the SPEAR Booster at 3 GeV. The changes to x_c , x'_c , $c\tau_c$, and δ_c are shown in the four plots. The three curves are from AT (AT), matrix inversion (Mat), and the analytic solution (Ana), respectively.

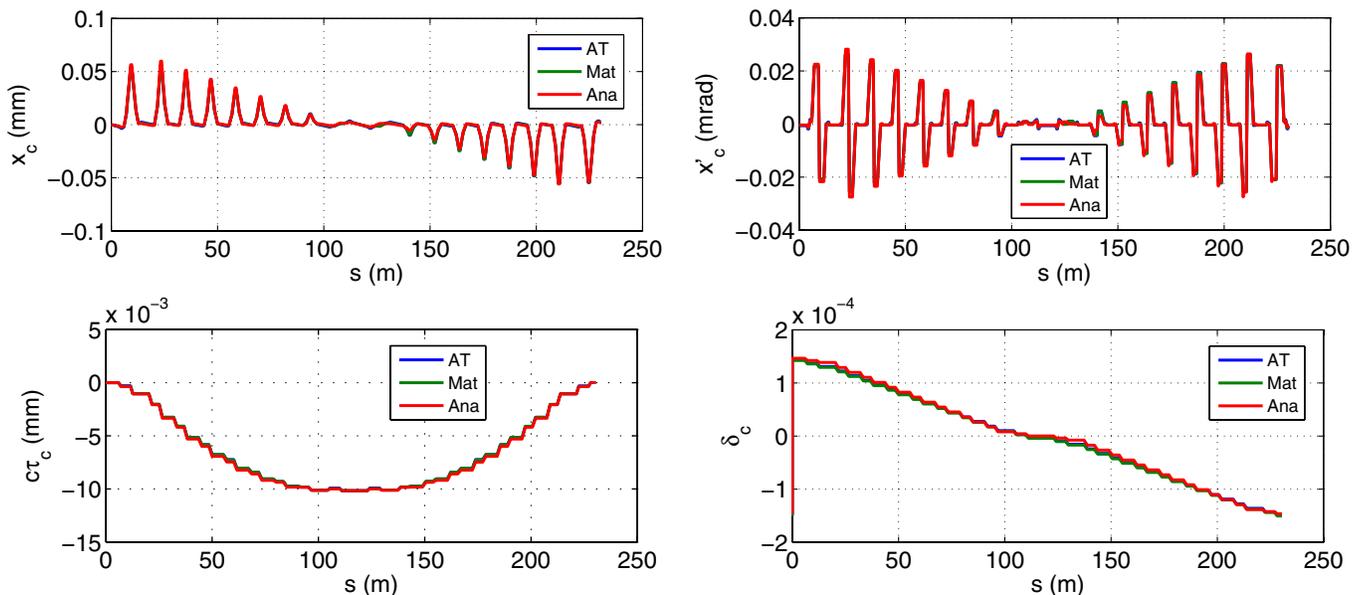


FIG. 5. (Color) The SBC-induced closed-orbit changes calculated in three different ways for the SPEAR3 ring. The changes to x_c , x'_c , $c\tau_c$, and δ_c are shown. The three curves are from AT (AT), matrix inversion (Mat), and the analytic solution (Ana), respectively.

dipole exit to the observation point. In Fig. 2 we show functions S and C for the SPEAR Booster ring calculated with both methods. The one-turn coordinate shifts, or the \mathbf{g} vector is plotted in Fig. 3. There are two curves in each plot, one is from Eq. (74), the other is from direct matrix multiplication with the transfer matrix of each accelerator element given by AT and the coordinate shifts of each element obtained by tracking through it with zero initial coordinates in AT.

We then compare the closed-orbit changes obtained with three different ways: (i) AT (using the function *findorbit6*), (ii) direct matrix calculation with Eq. (61), and (iii) the analytic solution in Eqs. (79a)–(79d). The results are shown in Fig. 4. Good agreement between the three curves are seen, verifying the analysis in the last section. The S and C functions are calculated with simple summations in the analytic approach. It is worth pointing out that all three terms in Eq. (79a) are important in determining the closed-orbit change for x_c .

The same closed-orbit calculation is done for the SPEAR3 ring. We only show the closed-orbit changes in Fig. 5. Note dispersion at the rf cavity for the SPEAR3 ring is zero. So the terms in Eqs. (79a)–(79d) involving \mathcal{H}_2 have no contribution.

VI. CONCLUSIONS

In this study we fully analyzed the linear synchrotron coupling by block diagonalizing the 4×4 horizontal-longitudinal transfer matrix. We found the transformation between the usual $(x, x', c\tau, \delta)$ coordinates and the normal modes and the transfer matrix for the normal modes in

analytic forms in terms of the Courant-Snyder functions, dispersion functions, and the rf voltage slope parameter. This enabled us to predict the 4-dimensional motion of a particle knowing only the initial condition and those common parameters. The effects of synchrotron coupling on the horizontal betatron motion, including changes to the Courant-Snyder functions and the betatron tune are also presented. We then studied the beam width and the bunch length under synchrotron coupling. We readily recovered Shoji's result of dispersion-dependent bunch lengthening [5]. We found that the beam width and the bunch length slightly oscillate around the ring with the betatron phase advance measured from the rf cavity. We pointed out that the bunch length varies around the ring due to phase slippage, a fact that is often overlooked. We also pointed out that the phase space volume is preserved under SBC when not considering radiation induced diffusion and damping.

Following Ref. [6], which studied the horizontal closed-orbit changes due to finite energy gains at rf cavities, we fully explored the problem by looking for a closed-orbit in the 4-dimension phase space, considering both finite energy gains at rf cavities and radiation energy losses in dipole magnets. We recovered the horizontal closed-orbit change result in Ref. [6] and found additional terms due to energy losses.

We carried out simulations with the accelerator modeling code AT [12] to verify the analysis. We found that the block diagonalization transformation matrix had high precision despite the approximations we made to get the analytic solution. The closed-orbit results of AT also agreed well with the theory.

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