Letter

# Strict hierarchy of optimal strategies for global estimations: Linking global estimations with local ones

Zhao-Yi Zhou, Jing-Tao Qiu, and Da-Jian Zhang*
*Department of Physics, Shandong University, Jinan 250100, China*

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A crucial yet challenging issue in quantum metrology is to ascertain the ultimate precision achievable in estimation strategies. While there are two paradigms of estimations, local and global, current research is largely confined to local estimations, which are useful once the parameter of interest is approximately known. In this Letter we target a paradigm shift towards global estimations, which can operate reliably even with a few measurement data and no substantial prior knowledge about the parameter. The key innovation here is to develop a technique, dubbed virtual imaginary-time evolution, which establishes an equality between the information gained in a global estimation and the quantum Fisher information for a virtual local estimation. This offers an intriguing pathway to surmount challenges in the realm of global estimations by leveraging powerful tools tailored for local estimations. We explore our technique to reveal a strict hierarchy of achievable precision for different global estimation strategies and uncover unexpected results contrary to conventional wisdom in local estimations.

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*Introduction*. Quantum metrology lies at the heart of quantum science and technologies, aiming to design optimal strategies for precisely estimating unknown parameters with limited resources. The burgeoning capabilities of quantum sensors have opened up possibilities of harnessing quantum mechanical effects like entanglement to yield quantum metrological advantages [1–3]. This allows quantum metrology to push precision limits beyond the reach of classical methods, holding compelling promise for applications such as quantum imaging, quantum interferometry, and quantum thermometry [4–6].

The prototypical setting of quantum metrology is to estimate an unknown parameter $\theta$ carried by a quantum channel $\mathcal{E}_\theta$ with $N$ queries to $\mathcal{E}_\theta$. Various types of estimation strategies can be employed for this purpose: (i) parallel strategies [7], where these $N$ channels are applied simultaneously on a multipartite entangled state; (ii) sequential strategies [7], involving successive queries of the channels, possibly interspersed with unitary control operations; (iii) causal superposition strategies [8], where the channels are probed in a superposition of different causal orders; and (iv) general indefinite-causal-order strategies [9], encompassing the most general causal relations among the channels and including causal superposition strategies as special cases. A crucial yet challenging issue in quantum metrology is to ascertain the ultimate precision achievable in estimation strategies, which has recently motivated vibrant activity [7–19].

While there are two paradigms of estimations in quantum metrology [10], local and global, the current research on the issue is largely confined to local estimations [7–19]. This bias is partially attributed to the fact that the performance of local estimation strategies is characterized by the quantum Fisher information (QFI) [20,21] and many powerful tools are available for computing the QFI. However, unless dealing with a special class of probability models [5], local estimation strategies require the parameter to be approximately known. This severely restricts their applicability, excluding diverse situations where little is known about the parameter *a priori* [22,23]. Alternatively, the knowledge about the parameter may be acquired *a posteriori* using a sufficiently large number of measurement samples [24,25], which nevertheless demands too much experimental effort.

Here we target a paradigm shift towards global estimations. Unlike local ones, global estimation strategies can operate reliably even with a few measurement data and no substantial prior knowledge about the parameter [26–29]. This general applicability is highly valuable in realistic settings, given the limited capabilities of near-term quantum sensors [30,31]. Unfortunately, useful tools for evaluating the performance of global estimation strategies are currently lacking [32,33], raising technical obstacles in tackling the mentioned issue. Consequently, whereas significant advancements have been made towards fully understanding quantum metrological advantages in local estimations [7–19], the progress in global estimations has remained quite limited so far.

The key innovation of this Letter is to develop a technique, dubbed virtual imaginary-time evolution (ITE), which allows us to establish an equality between the information about the parameter gained in a global estimation and the QFI

*Contact author: zdj@sdu.edu.cn

associated with a virtual local estimation. We can therefore figure out the ultimate precision achievable in global estimation strategies by computing the QFI. Aided by our technique, we tackle the hierarchy problem on ultimate precision achievable in global estimation strategies, which stands out due to its vital role in understanding quantum metrological advantages but has remained open up to now [9,34]. Meanwhile, we uncover some unexpected results contrary to the conventional wisdom established for local estimations. The result of this Letter offers an intriguing pathway to surmount challenges in the realm of global estimations by leveraging powerful tools tailored in local estimations.

*Preliminaries.* Let $\rho_\theta$ denote the state produced in a strategy. To estimate $\theta$, one needs to perform a positive-operator-valued measure $\{\Pi_x\}$ on $\rho_\theta$ and then postprocessing the measurement outcome via an estimator $\hat{\theta}(x)$, where $x$ labels the outcome. The objective of a local estimation is to choose suitable $\{\Pi_x\}$ and (unbiased) $\hat{\theta}(x)$ to minimize the local variance $\mathrm{Var}[\hat{\theta}|\theta] = \sum_x p(x|\theta)[\hat{\theta}(x) - \theta]^2$, where $p(x|\theta) = \mathrm{tr}(\Pi_x \rho_\theta)$. The quantum Cramér-Rao bound reads $\mathrm{Var}[\hat{\theta}|\theta] \geqslant 1/\mathcal{I}[\rho_\theta]$ [20], where

$$\mathcal{I}[\rho_\theta] = \mathrm{tr}\big(\rho_\theta L_\theta^2\big) \tag{1}$$

is the QFI, with $L_\theta$ the symmetric logarithmic derivative defined as the Hermitian operator satisfying

$$\frac{d}{d\theta}\rho_\theta = (\rho_\theta L_\theta + L_\theta \rho_\theta)/2. \tag{2}$$

Notably, the optimal measurement saturating the bound typically depends on $\theta$, implying that local estimations are useful only when $\theta$ is approximately known. Unlike local estimations, a global estimation aims to minimize the global variance $\mathrm{Var}[\hat{\theta}] = \int d\theta \sum_x p(x,\theta)[\hat{\theta}(x) - \theta]^2$,[1] where $p(x,\theta) = p(\theta)p(x|\theta)$ is the joint probability distribution of $x$ and $\theta$, with $p(\theta)$ denoting the prior probability distribution of $\theta$. It has been shown [35–41] that $\mathrm{Var}[\hat{\theta}]$ is bounded by

$$\mathrm{Var}[\hat{\theta}] \geqslant \int d\theta\, p(\theta)\theta^2 - \mathrm{tr}(\bar{\rho}S^2). \tag{3}$$

Here $\bar{\rho} = \int d\theta\, p(\theta)\rho_\theta$ is the averaged state and $S$ is a Hermitian operator satisfying

$$\overline{\theta\rho} = (\bar{\rho}S + S\bar{\rho})/2, \tag{4}$$

with $\overline{\theta\rho} := \int d\theta\, p(\theta)\theta\rho_\theta$. The inequality (3) can be saturated by choosing $\{\Pi_x\}$ as the projective measurement of $S$ and $\hat{\theta}(x)$ as the eigenvalues of $S$ [35–41]. Crucially, the $\{\Pi_x\}$ and $\hat{\theta}(x)$ thus chosen are parameter independent, implying that global estimations are operationally meaningful even if little is known *a priori* about $\theta$. Note that the first term on the right-hand side of Eq. (3) is fixed once $p(\theta)$ is given. By contrast, the second term $\mathrm{tr}(\bar{\rho}S^2)$ depends on $\rho_\theta$ and represents the information about $\theta$ gained in a global estimation. We define

$$\mathcal{J} = \mathrm{tr}(\bar{\rho}S^2). \tag{5}$$

To ascertain the ultimate precision achievable for each of the four types of strategies (i)–(iv), we need to maximize $\mathcal{J}$ over all the allowed freedoms such as the initial state and adaptive controls (see Fig. 1). However, it is formidable to do so directly due to the lack of effective tools [33].

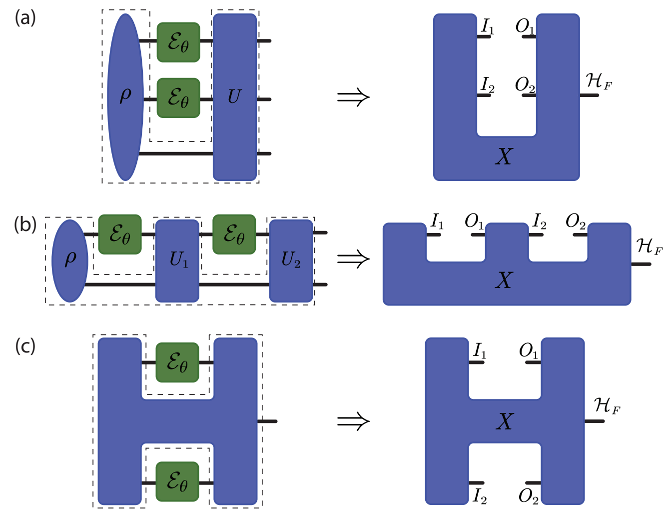

FIG. 1. Schematic of (a) parallel strategies, (b) sequential strategies, and (c) general indefinite-causal-order strategies in the $N = 2$ case. The left column illustrates that a strategy is an arrangement of physical operations such as initial-state preparations and adaptive controls. This arrangement, when concatenated with the $N$ channels, produces an output state carrying information about $\theta$. The right column depicts that a strategy amounts to a supermap which, akin to completely positive maps, can be described by a positive-semidefinite operator $X$.

*Virtual imaginary-time evolution.* To overcome this difficulty, our idea is to establish an equality between the information $\mathcal{J}$ gained in a global estimation and the QFI for a virtual local estimation so that the above maximum can be figured out indirectly by calculating the QFI.

We first specify the state $\rho_\theta$. Let $\mathcal{H}_{I_k}$ and $\mathcal{H}_{O_k}$ be the input and output Hilbert spaces of the $k$th copy of the channel $\mathcal{E}_\theta$. Denote by $\mathcal{L}(\mathcal{H})$ the set of linear operators over a Hilbert space $\mathcal{H}$. The $k$th copy of $\mathcal{E}_\theta$, as a completely positive map from $\mathcal{L}(\mathcal{H}_{I_k})$ to $\mathcal{L}(\mathcal{H}_{O_k})$, can be described by a positive-semidefinite operator in $\mathcal{L}(\mathcal{H}_{I_k} \otimes \mathcal{H}_{O_k})$, $E_\theta := \mathrm{id} \otimes \mathcal{E}_\theta(|I\rangle\rangle\langle\langle I|)$, known as the Choi-Jamiołkowski (CJ) operator [42,43]. Here id is the identity map and $|I\rangle\rangle = \sum_j |j\rangle|j\rangle$. The CJ operator of $N$ identical channels is $C_\theta := E_\theta^{\otimes N} \in \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \cdots \otimes \mathcal{H}_{I_N} \otimes \mathcal{H}_{O_N})$. A strategy is an arrangement of physical operations which, when concatenated with these $N$ channels, produces the output state $\rho_\theta$ carrying information about $\theta$ [9] (see the left column of Fig. 1). We can therefore regard a strategy as a supermap [44], taking the $N$ channels as its input and outputting the state $\rho_\theta$ (see the right column of Fig. 1). This supermap, akin to completely positive maps, can be described by a positive-semidefinite operator $X$ in $\mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \cdots \otimes \mathcal{H}_{I_N} \otimes \mathcal{H}_{O_N} \otimes \mathcal{H}_F)$ [9]. Here $\mathcal{H}_F$ is the Hilbert space upon which $\rho_\theta$ acts [45]. We have [46]

$$\rho_\theta = X \star C_\theta. \tag{6}$$

Here $\star$ denotes the link product [46], that is, $X \star C_\theta = \mathrm{tr}_{I_1 O_1 \cdots I_N O_N}[X(C_\theta^T \otimes \mathbb{I}_F)]$, where $T$ indicates the transpose operation and $\mathbb{I}_F$ represents the identity operator on $\mathcal{H}_F$.

[1] The figure of merit may be chosen to be another functional instead of the global variance [35], which is beyond the scope of this work.

We next introduce a virtual ITE. Recall that, whereas traditional time evolution is described by the operator $e^{-iHt}$ with a Hamiltonian $H$, ITE is described by the operator $e^{-H\tau}$ [47], that is, for a system undergoing ITE, its initial state $\rho(0)$ is evolved to be the state $\rho(\tau)=e^{-H\tau}\rho(0)e^{-H\tau}$ at time $\tau$. It is interesting to note that the expression $e^{-H\tau}$ is obtained from the expression $e^{-iHt}$ by setting $t$ to be purely imaginary, i.e., $t=-i\tau$ with $\tau\in\mathbb{R}$. It is also interesting to note that ITE, as a map, is completely positive but not trace preserving because of its nonunitary character. With the above knowledge, we introduce the averaged CJ operator $\bar{C}:=\int d\theta\, p(\theta)C_\theta$ and assume that $\bar{C}$ is evolved to be

$$\bar{C}(\tau):=e^{-H\tau}\bar{C}e^{-H\tau} \tag{7}$$

at time $\tau$. Here $H$ is a Hermitian operator in $\mathcal{L}(\mathcal{H}_{I_1}\otimes\mathcal{H}_{O_1}\otimes\cdots\otimes\mathcal{H}_{I_N}\otimes\mathcal{H}_{O_N})$ that satisfies

$$\overline{\theta C}+\{H,\bar{C}\}=0, \tag{8}$$

with $\overline{\theta C}:=\int d\theta\, p(\theta)\theta C_\theta$. Equations (7) and (8) specify the ITE of interest here. We clarify that the physical realization of this ITE is irrelevant in the present work, because our purpose is to devise an effective approach to computing $\mathcal{J}$. Hence we regard the ITE as a virtual process. We employ it to define the family of states

$$\sigma_\tau:=X\star\bar{C}(\tau). \tag{9}$$

Note that $\sigma_\tau$ is positive semidefinite but its trace may not equal to 1 for $\tau\neq 0$.

We now establish an equality between $\mathcal{J}$ and the QFI. To serve our purpose, we still define the QFI for $\sigma_\tau$ via Eqs. (1) and (2), although $\mathrm{tr}(\sigma_\tau)\neq 1$ when $\tau\neq 0$, that is,

$$\mathcal{I}[\sigma_\tau]=\mathrm{tr}\big(\sigma_\tau L_\tau^2\big), \tag{10}$$

where $L_\tau$ is the Hermitian operator satisfying $\frac{d}{d\tau}\sigma_\tau=(\sigma_\tau L_\tau+L_\tau\sigma_\tau)/2$. Using Eqs. (6)–(9), we have

$$\sigma_\tau|_{\tau=0}=\bar{\rho},\quad \frac{d}{d\tau}\sigma_\tau|_{\tau=0}=\overline{\theta\rho}. \tag{11}$$

Further, noting that inserting Eq. (11) into Eq. (4) yields the equality $S=L_\tau|_{\tau=0}$, we arrive at the following theorem.

*Theorem 1.* Let $\mathcal{I}[\sigma_\tau]$ be the QFI defined by Eq. (10). Then

$$\mathcal{J}=\mathcal{I}[\sigma_\tau]|_{\tau=0}, \tag{12}$$

i.e., the information $\mathcal{J}$ gained in the global estimation with $\rho_\theta$ is equal to the QFI $\mathcal{I}[\sigma_\tau]$ for the local estimation with $\sigma_\tau$ at $\tau=0$.

Thus we can alternatively compute $\mathcal{I}[\sigma_\tau]$ for obtaining $\mathcal{J}$. We emphasize that, despite $\mathrm{tr}(\sigma_\tau)\neq 1$, it is still possible to use existing tools to compute $\mathcal{I}[\sigma_\tau]$. To illustrate this point and also for later usage, we prove in the Supplemental Material (SM) [48] that the formula proposed by Fujiwara and Imai [49] remains applicable when $\mathrm{tr}(\sigma_\tau)\neq 1$.

*Theorem 2. (Fujiwara and Imai's formula).* Let $\{|\psi_j\rangle\}_{j=1}^q$ be an ensemble of pure states for $\sigma_\tau$, namely, $\sigma_\tau=\sum_{j=1}^q|\psi_j\rangle\langle\psi_j|$, where $q\geqslant\mathrm{rank}(\sigma_\tau)$ is an integer. Then

$$\mathcal{I}[\sigma_\tau]|_{\tau=0}=\min_{\{|\psi_j\rangle\}_{j=1}^q}4\,\mathrm{tr}\left(\sum_{j=1}^q|\dot{\psi}_j\rangle\langle\dot{\psi}_j|\right)\Bigg|_{\tau=0}, \tag{13}$$

where $|\dot{\psi}_j\rangle=d|\psi_j\rangle/d\tau$ and the minimum is taken over all the ensembles with fixed $q$.

*Maximal information gained in global estimation strategies.* We use index $k$ to specify the type of strategies in question, that is, $k=$ i, ii, iii, iv refer to the four types of strategies (i), (ii), (iii), and (iv), respectively. Note that the value of $\mathcal{J}$ depends on the specific strategy adopted. We are interested in the maximum of $\mathcal{J}$ over all the strategies of type $k$. Hereafter we denote this maximum by $\mathcal{J}_{\max}^{(k)}$. In addition, we define $\mathbb{X}^{(k)}$ to be the collection of $X$ that can describe all the strategies of type $k$. Using Theorem 1 as well as substituting Eq. (9) into Eq. (12), we can write $\mathcal{J}_{\max}^{(k)}$ as

$$\mathcal{J}_{\max}^{(k)}=\max_{X\in\mathbb{X}^{(k)}}\mathcal{I}[X\star\bar{C}(\tau)]|_{\tau=0}, \tag{14}$$

which is the maximal information gained in the strategies of type $k$. To ascertain the ultimate precision achievable in the strategies of type $k$, which is $\int d\theta\, p(\theta)\theta^2-\mathcal{J}_{\max}^{(k)}$ according to Eq. (3), we need to figure out $\mathcal{J}_{\max}^{(k)}$. To this end, we write the ensemble decomposition of $\bar{C}$ as $\bar{C}=\sum_{j=1}^q|\phi_j\rangle\langle\phi_j|=\Phi\Phi^\dagger$, with $\Phi:=[|\phi_1\rangle,\dots,|\phi_q\rangle]$. We introduce the set $\tilde{\mathbb{X}}^{(k)}:=\{\tilde{X}=\mathrm{tr}_F X\mid X\in\mathbb{X}^{(k)}\}$. Using Theorem 2, we show [48] that

$$\mathcal{J}_{\max}^{(k)}=\max_{\tilde{X}\in\tilde{\mathbb{X}}^{(k)}}\min_{h\in\mathbb{H}^{(q)}}\mathrm{tr}[\tilde{X}\Omega(h)], \tag{15}$$

representing a computation-friendly formula for $\mathcal{J}_{\max}^{(k)}$. Here $\mathbb{H}^{(q)}$ denotes the set of all $q\times q$ Hermitian matrices and

$$\Omega(h)=4(H^*\Phi^*-i\Phi^*h)(H^*\Phi^*-i\Phi^*h)^\dagger. \tag{16}$$

Notably, the formula (15), which is derived here for global estimations, is analogous to those for local estimations [9,15].

*Semidefinite programs for computing $\mathcal{J}_{\max}^{(k)}$.* We now convert Eq. (15) into two semidefinite programs (SDPs) for computing $\mathcal{J}_{\max}^{(k)}$. To do this, we resort to the process matrix formalism [50], which allows us to characterize $\tilde{\mathbb{X}}^{(k)}$ as

$$\tilde{\mathbb{X}}^{(k)}=\{\tilde{X}\mid\tilde{X}\geqslant 0,\ \Lambda^{(k)}(\tilde{X})=\tilde{X},\ \mathrm{tr}\tilde{X}=d_O\}, \tag{17}$$

when $k=$ i, ii, iv. Here $d_O=\dim(\mathcal{H}_{O_1}\otimes\cdots\otimes\mathcal{H}_{O_N})$ and $\Lambda^{(k)}$ is a linear map whose expression can be found in the SM [48] (see also Ref. [50]). The discussion of strategy (iii) needs to be carried out separately and is left to the SM [48]. We show that $\mathcal{J}_{\max}^{(k)}$ can be computed via the SDP [48]

$$\begin{aligned}&\max_{\tilde{X},B,C}\ -\mathrm{tr}C-4\,\mathrm{Re}[\mathrm{tr}(H^*\Phi^*B)]\\ &\text{s.t. }\Lambda^{(k)}(\tilde{X})=\tilde{X},\ \mathrm{tr}\tilde{X}=d_O,\\ &\begin{bmatrix}\tilde{X} & B^\dagger\\ B & C\end{bmatrix}\geqslant 0,\end{aligned} \tag{18}$$

referred to as the primal SDP, where $B\Phi^*$ is Hermitian. As demonstrated below, the gaps among the $\mathcal{J}_{\max}^{(k)}$ may be small, due to which numerical errors in the SDP may compromise the reliability of computed results. To overcome this issue, we propose Algorithm 1 in the SM [48]. Using Algorithm 1 to assist the primal SDP, we can obtain reliable lower bounds on $\mathcal{J}_{\max}^{(k)}$, for which numerical errors are eliminated. Further, to obtain upper bounds, we derive the dual SDP

$$\begin{aligned} &\min_{\tilde{Y},\lambda,h} \lambda \\ &\text{s.t. } \Lambda^{(k)}(\tilde{Y}) = 0, \\ &\begin{bmatrix} \frac{\lambda}{d_O} + \tilde{Y} & 2(H^*\Phi^* - i\Phi^* h) \\ 2(H^*\Phi^* - i\Phi^* h)^\dagger & \mathbb{I}_q \end{bmatrix} \geqslant 0, \end{aligned} \tag{19}$$

where $\mathbb{I}_q$ is the $q \times q$ identity matrix [48]. Likewise, Algorithm 2 is proposed in the SM [48] to eliminate numerical errors in the dual SDP [48]. Notably, compared with those in Refs. [9,15], our SDPs feature significantly fewer constraints. This enables our algorithms to yield tight bounds.

*Strict hierarchy*. Clearly, the four types of strategies (i)–(iv) form a hierarchy $\mathcal{J}_{\max}^{(\mathrm{i})} \leqslant \mathcal{J}_{\max}^{(\mathrm{ii})} \leqslant \mathcal{J}_{\max}^{(\mathrm{iii})} \leqslant \mathcal{J}_{\max}^{(\mathrm{iv})}$, since each of them is a superset of the preceding one. We show that all three inequalities can be strictly satisfied simultaneously. To this end, we examine the channel

$$\mathcal{E}_\theta = \mathcal{E}^{(\mathrm{AD})} \circ \mathcal{E}^{(\mathrm{BF})} \circ \mathcal{U}_\theta, \tag{20}$$

composed of the unitary channel $\mathcal{U}_\theta$ with the Kraus operator $e^{-i\theta\sigma_z/2}$, the bit-flip channel $\mathcal{E}^{(\mathrm{BF})}$ with the Kraus operators $K_1^{(\mathrm{BF})} = \sqrt{\eta}\mathbb{I}$ and $K_2^{(\mathrm{BF})} = \sqrt{1-\eta}\sigma_x$, and the amplitude damping channel $\mathcal{E}^{(\mathrm{AD})}$ with the Kraus operators

$$K_1^{(\mathrm{AD})} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad K_2^{(\mathrm{AD})} = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}, \tag{21}$$

where $\sigma_\alpha$ $(\alpha = x, y, z)$ denotes the Pauli matrices. Hereafter, we set $p(\theta)$ to be the uniform probability distribution, that is, $p(\theta) = 1/2\pi$ for $\theta \in [-\pi, \pi)$. Applying the two SDPs as well as the two algorithms to the channel in Eq. (20) with $\eta = 1/2$ and $\gamma = 7/10$, we can show that $\mathcal{J}_{\max}^{(\mathrm{i})} \leqslant 0.5516 < 0.5572 \leqslant \mathcal{J}_{\max}^{(\mathrm{ii})} \leqslant 0.5574 < 0.5703 \leqslant \mathcal{J}_{\max}^{(\mathrm{iii})} \leqslant 0.5705 < 0.570\,53 \leqslant \mathcal{J}_{\max}^{(\mathrm{iv})}$ when $N = 2$. We therefore reach the following theorem.

*Theorem 3.* There exist parameter estimation problems for which

$$\mathcal{J}_{\max}^{(\mathrm{i})} < \mathcal{J}_{\max}^{(\mathrm{ii})} < \mathcal{J}_{\max}^{(\mathrm{iii})} < \mathcal{J}_{\max}^{(\mathrm{iv})}, \tag{22}$$

i.e., the strict hierarchy of ultimate precision can hold for global estimation strategies (i)–(iv).

We point out that the hierarchy phenomenon reported in Theorem 3 is not exclusive to the above specific example. We have randomly generated 1000 channels and found that 780 of them obey the strict hierarchy [48].

*Unexpected results.* To illuminate distinct features of global estimations, we make reference to the work by Giovannetti *et al.* [7], where parallel and sequential strategies are examined within the local estimation framework. We need to consider the unitary channel $\mathcal{E}_\theta = \mathcal{U}_\theta$, i.e., the channel in Eq. (20) with $\eta = 1$ and $\gamma = 0$. Hereafter, we refer to the parallel strategies examined within the local (global) estimation framework as local (global) parallel strategies, and similarly for local (global) sequential strategies. It is well known that the optimal input state in local parallel strategies is the Greenberger-Horne-Zeilinger (GHZ) state $(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$ for all $\theta \in [-\pi, \pi)$ [7]. As such, an intuitive deduction may be that the optimal probe state would also be the GHZ state for global parallel strategies. However, we find that $1.5217 \leqslant \mathcal{J}_{\max}^{(\mathrm{i})} \leqslant 1.5218$ but $\mathcal{J} = 1/4$ for the GHZ state when $N = 2$, implying that the above deduction is invalid in general. Also, our result suggests that one candidate for the optimal input state in global parallel strategies is $\sqrt{3/10}|0000\rangle + \sqrt{1/5}|0101\rangle + \sqrt{1/5}|1010\rangle + \sqrt{3/10}|1111\rangle$, indicating that the optimal input state differs in structure from the GHZ state. We next switch our discussion to sequential strategies. Recall that adaptive controls are useless in improving the ultimate performance of local sequential strategies [7]. However, we find that adaptive controls are useful for global sequential strategies. Indeed, when $N = 2$, the maximal information attained in global sequential strategies without controls is 0.25, which is strictly less than the maximal information attained with controls $1.5217 \leqslant \mathcal{J}_{\max}^{(\mathrm{ii})} \leqslant 1.5219$. Finally, in contrast to the result that local parallel and sequential strategies share the same ultimate performance for every $N$ and every $\theta \in [-\pi, \pi)$ [7], we find that global sequential strategies can be superior to global parallel strategies in ultimate performance. We illustrate this point by showing that $\mathcal{J}_{\max}^{(\mathrm{i})} \leqslant 1.845\,07 < 1.845\,17 \leqslant \mathcal{J}_{\max}^{(\mathrm{ii})}$ when $N = 3$.

We present in the SM [48] more numerical results on different priors.

*Conclusion.* A bottleneck hindering the current research in global estimations is the lack of effective tools, which is in sharp contrast to the situation that many such tools are available in local estimations. In this Letter we have advocated a pathway to surmount this bottleneck. The key innovation here is the technique of the virtual ITE, in which the fictitious state $\sigma_\tau$ is constructed such that its QFI is equal to $\mathcal{J}$ at $\tau = 0$, its dependence on $X$ is linear, and its ensemble decompositions are easy to find. This opens up the exciting possibility of solving crucial problems in the realm of global estimations by leveraging powerful tools tailored for local estimations. We have demonstrated this possibility by successfully solving the hierarchy problem in global estimations with Fujiwara and Imai's formula, a useful tool in local estimations. Meanwhile, we have uncovered a number of unexpected results, highlighting that the same quantum resource may assume disparate roles when scrutinized through the lenses of local and global estimations. For instance, while the GHZ state is optimal in local estimations, its utility diminishes in global estimations. These captivating differences underscore the need for further explorations of global estimations. Finally, we remark that our technique makes it possible to construct optimal strategies in global estimations [51,52] and devise tight bounds in the nonasymptotic regime, which are two research directions for future studies.

The codes used in this article are openly available from [53].

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