Letter

Uncovering the multifractality of Lagrangian pair dispersion in shock-dominated turbulence

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Lagrangian pair dispersion provides insights into mixing in turbulent flows. By direct numerical simulations (DNSs) we show that the statistics of pair dispersion in the randomly forced two-dimensional Burgers equation, which is a typical model of shock-dominated turbulence, is very different from its incompressible counterpart because Lagrangian particles get trapped in shocks. We develop a heuristic theoretical framework that accounts for this—a generalization of the multifractal model—whose prediction of the scaling of Lagrangian exit times agrees well with our DNS.

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The statistics of the relative pair dispersion of Lagrangian particles (tracers) in turbulent flows has numerous physical applications [1–9]. Richardson's law, a pioneering work in this field, states that, in a turbulent fluid, the mean-squared displacement between a pair of tracers, $\langle R^2(t)\rangle$, at time t, scales as $\langle R^2(t)\rangle \sim t^3$ [10]. In other words, there is a dynamic exponent $z = 3/2$. Richardson's law can be viewed as a consequence of Kolmogorov's 1941 (K41) scaling theory of turbulence (see, e.g., Ref. [9]). It is now well established that the K41 theory is incomplete because it does not account for intermittency, which arises predominantly because of the most dissipative structures in the flow and leads to multifractality [11]. In brief, intermittency and multifractality in homogeneous and isotropic turbulence is characterized by the nontrivial scaling properties of moments of velocity differences, $\langle \delta v(\ell)^p\rangle$, over length scales, ℓ, i.e., $\langle \delta v(\ell)^p\rangle \sim \ell^{\zeta_p}$, with the multiscaling exponent ζ_p a nonlinear function of p (K41 yields simple scaling with $\zeta_p = p/3$).[1] Therefore, it is natural to hypothesize that Richardson's law must have multiscaling corrections too. In particular, we expect dynamic multiscaling [12–16], characterized by not one but an infinity of dynamic exponents; specifically, different moments of $R(t)$ should scale with different powers of t. However, even the most recent experimental [17–21] and numerical [5,19,22–25] measurements of $\langle R^2\rangle$ provide only limited support to Richardson's law, because the power-law behavior is observed over a range of scales that covers a decade or so. Hence, there seems to be little hope of extracting the dynamic-multiscaling behavior from experimental or numerical measurements of moments of $R(t)$.

It turns out that the Lagrangian exit time, conventionally defined as the time taken for the separation between a pair of tracers (Lagrangian interval) to exceed a given threshold (e.g., the doubling time) [26–29], provides a robust measure for dynamic multiscaling. From the statistics of these exit times it is possible to extract dynamic multiscaling exponents that match the predictions from the multifractal model of turbulence [11,30,31].

We have so far outlined Lagrangian pair dispersion in incompressible turbulence. We turn now to shock-dominated turbulence, which refers to highly compressible turbulence wherein the total energy in the irrotational modes is comparable [32] to or significantly larger than that in the solenoidal modes [33,34] depending on the Mach number; this poses formidable challenges that we must confront because such flows are widely prevalent in many astrophysical systems [35]. Over the last two decades, several groups have studied the multifractal properties of compressible turbulence, ranging from flows that are weakly compressible to those that are shock dominated [36–43]. Additionally, the dynamics of tracers and heavy inertial particles in such systems have also been investigated [44–49]. Note that Refs. [46–48] deal with surface flows which are typically weakly compressible and differ considerably from shock-dominated turbulence. However, the possible multifractal generalization of Richardson's law and exit time statistics to compressible turbulence has not been established yet.

We develop the theoretical framework that is required for this generalization to shock-dominated turbulence. It rests on the crucial observation that tracers strongly cluster at the shocks, which comprise the most dissipative structures, making Lagrangian pair dispersion in such flows qualitatively different from that in incompressible turbulence. Consequently, we define two exit times, (i) the doubling time and the (ii) halving time, as the times taken for a

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[1]The angular brackets denote the average over the turbulent, but statistically stationary, state of the fluid; and the length scale ℓ lies between the large length L_I, at which energy is injected, and the small length scale η, at which viscous dissipation becomes significant.

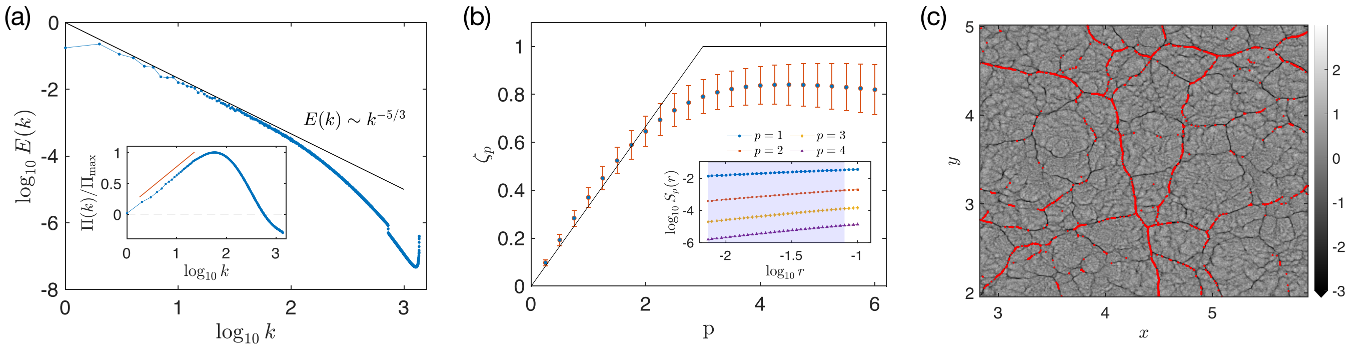

FIG. 1. (a) The NESS energy spectrum $E(k)$ on a log-log scale displaying almost two decades of K41-type inertial range scaling, $E(k) \sim k^{-5/3}$. Inset: The scale-by-scale energy flux $\Pi(k)$ on a log-lin scale implies a direct cascade of energy; in the inertial range, $\Pi(k) \sim \log k$, in agreement with earlier theoretical predictions for hydrodynamic turbulence with a similar random forcing [65,69]. (b) Scaling exponents ζ_p of equal-time structure functions $S_p(r)$; the black lines represent the bifractal scaling (2). Inset: Log-log plots of $S_p(r)$, vs r for different values of p; the shaded region denotes the regime local slope analysis for calculating ζ_p. (c) The profile of $\nabla \cdot \boldsymbol{u}$ of a part of the simulation domain overlaid with the instantaneous tracer positions (red dots); the dark filaments of large negative $\nabla \cdot \boldsymbol{u}$ are the shocks and the tracers cluster on them.

Lagrangian interval to (a) go beyond twice and (b) shrink below half its initial length, respectively. We carry out explicit direct numerical simulations (DNSs) of a simplified model of shock-dominated turbulence, namely, the randomly forced two-dimensional (2D) Burgers equation, which is not only rich enough to display the complexities of such turbulence [36,50,51], but also simple enough for us to develop a heuristic theory for our DNS results. We find that the statistics of the doubling times agree with those that follow from the conventional application of the multifractal model, but those of halving times do not. We generalize this framework to account for the clustering of tracers on shocks and obtain therefrom the scaling exponents for the moments of the distribution of halving times. The results from our heuristic theory are in excellent agreement with those from our DNS. In a similar vein, we define and investigate the scaling properties of doubling and halving frequencies of Lagrangian intervals, and try to understand them on the basis of the underlying flow intermittency.

The randomly forced 2D Burgers equation is

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x}, t), \quad \nabla \times \boldsymbol{u} = \boldsymbol{0},$$
$$\langle \hat{\boldsymbol{f}}(\boldsymbol{k}, t) \cdot \hat{\boldsymbol{f}}(\boldsymbol{k}', t') \rangle \sim |\boldsymbol{k}|^{-2} \delta(\boldsymbol{k} + \boldsymbol{k}') \delta(t - t'). \tag{1}$$

$\boldsymbol{u}(\boldsymbol{x}, t)$ is the Eulerian velocity at position $\boldsymbol{x}$ and time t, and ν is the coefficient of viscosity. $\boldsymbol{f}(\boldsymbol{x}, t)$ is a zero-mean, irrotational, Gaussian, white-in-time random force whose Fourier components $\hat{\boldsymbol{f}}(\boldsymbol{k}, t)$ (with $\boldsymbol{k}$ the wave vector) obey the constraint given in (1). The equal-time and dynamic-scaling properties of the Burgers equation, which can be mapped to the Kardar-Parisi-Zhang (KPZ) equation [52], have been studied extensively, but, most often, in one dimension (1D) through DNSs and renormalization-group (RG) methods [53–65]. We perform high-resolution (4096^2) pseudospectral DNSs of (1) on a biperiodic square domain with side $L = 2\pi$. We use two-thirds dealiasing [66,67] and the second-order exponential time-differencing Runge-Kutta scheme [68] for timestepping. After the DNS reaches a nonequilibrium statistically stationary state (NESS), we calculate the shell-integrated energy spectrum $E(k)$ which shows a power-law regime that is consistent with $E(k) \sim k^{-5/3}$, over more than one-and-half decades [see Fig. 1(a)]. This is a well-established result in one dimension, obtained first in Ref. [60], by using the dynamic RG, and confirmed by DNS in Refs. [60,63–65], and it is consistent with a straightforward scaling argument [63] that can be generalized to any dimension d. We also calculate the equal-time, longitudinal structure functions S_p and their scaling exponents ζ_p:

$$S_p(r) \equiv \langle |\delta u_\parallel(\boldsymbol{r})|^p \rangle \sim r^{\zeta_p}, \tag{2a}$$

where

$$\delta u_\parallel(\boldsymbol{r}) \equiv [\boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \left(\frac{\boldsymbol{r}}{r}\right). \tag{2b}$$

Here, the symbol $\langle \cdot \rangle$ denotes averaging over the NESS. Equation (1) exhibits biscaling, given the type of forcing we use [60,70], i.e., the scaling exponents

$$\zeta_p = \begin{cases} p/3 & \text{for} \quad p < 3, \\ 1 & \text{for} \quad p \geqslant 3. \end{cases} \tag{3}$$

Our DNS results largely agree with this [see Fig. 1(b)].[2] In the NESS, we seed the flow uniformly with tracers and track their subsequent motion. We present a pseudograyscale plot of $\nabla \cdot \boldsymbol{u}$ in Fig. 1(c), in which shocks are visible as dark filamentary structures. Tracers, shown in red, accumulate on these shocks. Henceforth, we use $\mathcal{L}_{\mathrm{I}}$ and $T_{\mathrm{L}} \equiv \mathcal{L}_{\mathrm{I}}/u_{\mathrm{rms}}$, the large-eddy-turnover time, as our characteristic length scales and timescales, respectively, with u_{rms} the root-mean-square velocity and $\mathcal{L}_{\mathrm{I}}$ the integral length scale (see Supplemental Material [71] for details).

We find that there is a small scaling range with $\langle R^2(t) \rangle \sim t^\alpha$ and $\alpha \simeq 1.3$, when $\langle R^2 \rangle$ exceeds the square of the dissipation length scale, η (see the Supplemental Material [71]).

[2]The numerical values of ζ_p, measured from our DNS, deviate slightly from (3) for $p \gtrsim 3$, possibly because of certain numerical artifacts identified in an earlier similar study in one dimension [65]. Whether this deviation is a numerical artefact or not in 2D is not our main focus here.

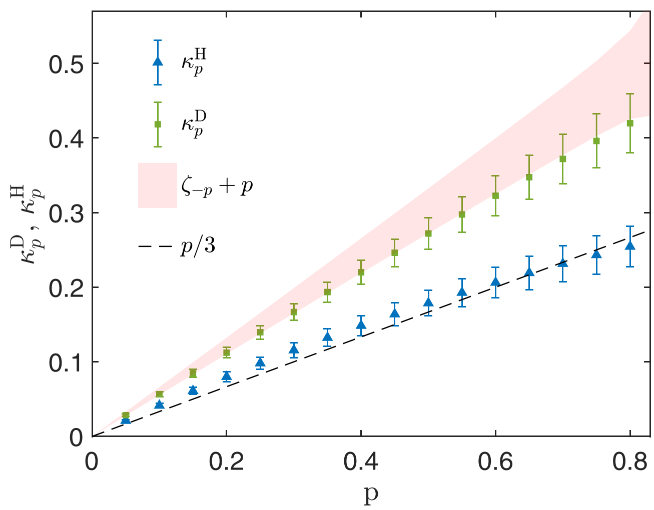

FIG. 2. Dynamic scaling exponents, of order p of the doubling times ($\kappa_{\rm p}^{\rm D}$) and halving times ($\kappa_{\rm p}^{\rm H}$) of Lagrangian intervals. Clearly, contrary to the naive expectation based on the conventional multifractal model, $\kappa_{\rm p}^{\rm D} \neq \kappa_{\rm p}^{\rm H}$. $\kappa_{\rm p}^{\rm D}$ agrees with its bridge relation prediction (4b) (shaded region) where ζ_{-p} is obtained from DNS. The dashed line is our theoretical prediction (8), which agrees very well with the numerical results.

Richardson's law is clearly violated in this shock-dominated turbulence; however, the scaling range is too small to extract an accurate value for α. Thus, we need to study Lagrangian-exit-time statistics. For a Lagrangian interval of initial length R_0, within the inertial range, applying the multifractal model of turbulence, as done previously [26–28], yields the following scaling exponents of the order-p moments of the doubling times $\tau_{\rm D}$:

$$\left\langle \tau_{\rm D}^{p}(R_0)\right\rangle \sim R_0^{\kappa_{\rm p}^{\rm D}}, \tag{4a}$$

where

$$\kappa_{\rm p}^{\rm D} = \zeta_{-p} + p. \tag{4b}$$

The second of these equations gives the bridge relation between the dynamic exponent, $\kappa_{\rm p}^{\rm D}$, and the equal-time exponent, ζ_{-p}. We probe the validity of this bridge relation only for $p < 1$, because the structure functions of negative order, $S_{-p}(r)$, exist only in this regime [72,73]. We calculate $\kappa_{\rm p}^{\rm D}$ and ζ_{-p}, from our DNS and plot them versus p in Fig. 2. Clearly, (4b) holds within error bars. By construction, the multifractal model does not distinguish between the scaling behaviors of doubling times and halving times, $\tau_{\rm H}$. This may lead us to naively expect, for compressible turbulence, that the moments of $\tau_{\rm H}$ should have the same scaling as the moments of $\tau_{\rm D}$, i.e., $\langle \tau_{\rm H}^{p}(R_0)\rangle \sim R_0^{\kappa_{\rm p}^{\rm H}}$ with $\kappa_{\rm p}^{\rm H} = \kappa_{\rm p}^{\rm D}$. Our DNS shows that this naive expectation is false, because clearly $\kappa_{\rm p}^{\rm D} \neq \kappa_{\rm p}^{\rm H}$ as shown in Fig. 2.

We must therefore generalize the multifractal model in order to construct a theory that yields the p dependence of $\kappa_{\rm p}^{\rm H}$ shown in Fig. 2. We first outline the standard argument that is used to understand Richardson's law [9]. A Lagrangian interval of length $R(t)$ undergoes a Brownian motion (we restrict ourselves to 2D) with a diffusivity $K(R)$ that depends on R itself, i.e., the probability distribution function (PDF) of R, namely, $W(R,t)$, satisfies

$$\partial_t W = \frac{1}{R}\partial_R[RK(R)\partial_R W]. \tag{5}$$

The R dependence of $K(R)$ is deduced from the following dimensional arguments, $K \sim (\delta_{\rm R} V)^2 t_{\rm cor}$, where $\delta_{\rm R} V$ is the typical velocity difference across the scale, R, and $t_{\rm cor}$ is the typical correlation time. If we now use the K41 forms, $\delta_{\rm R} V \sim R^{1/3}$ and $t_{\rm cor} \sim R/\delta_{\rm R} V \sim R^{2/3}$, we get $K \sim R^{4/3}$ which, when substituted in (5), yields Richardson's law.[3]

To carry out a similar calculation for 2D Burgers turbulence, we need an appropriate scaling form of $K(R)$. The first, straightforward, generalization is to replace the K41 result, $\delta_{\rm R} V \sim R^{1/3}$, with $\delta_{\rm R} V \sim R^{h}$, where h is the scaling exponent of the velocity field. The second key idea is to recognize that, when we consider a Lagrangian interval of length R here, we must distinguish between the following three possibilities for this interval:

Case (1) (interval along a shock): $\delta_{\rm R} V \sim R^{h}$, but the typical correlation time is determined by sweeping as both ends of the interval are trapped in the same shock, so $t_{\rm cor} \sim R$, whence $K \sim R^{2h+1}$.

Case (2) (interval straddles a shock): $\delta_{\rm R} V$ is a constant, independent of R, i.e., $h = 0$, and $t_{\rm cor} \sim R$, consequently $K \sim R$.

Case (3) (interval is away from shocks): The interval can decrease in only one of the following two ways:

(a) Case (3A): Both ends of the interval get trapped, at a later time, in the same shock, so the arguments used in case (1) apply. Therefore, at late times, $K \sim R^{2h+1}$.

(b) Case (3B): The two ends of the interval get trapped in two different shocks, which then approach each other. Hence, $\delta_{\rm R} V$ is then the velocity difference between the two shocks, which does not depend on R. Since $t_{\rm cor} \sim R$, this yields $K \sim R$ at late times.

In all of these cases, the calculation of the PDF of halving times is a first-passage problem for (5), with two absorbing boundaries, one at $R \to \infty$ and the other at $R = R_0/2$, where $R_0 \equiv R(t=0)$. Given the forms of $K(R)$ in cases (1)–(3), it is sufficient to calculate the halving-time PDFs, $\mathcal{P}_1$ and $\mathcal{P}_2$, for $K \sim R^{2h+1}$ and $K \sim R$, respectively. We obtain [74]

$$\mathcal{P}_1(\tau_{\rm H}, R_0) \sim \frac{1}{R_0^{1-2h}} \exp\left[-A_1 \frac{\tau_{\rm H}}{R_0^{1-2h}}\right] \tag{6a}$$

and

$$\mathcal{P}_2(\tau_{\rm H}, R_0) \sim \frac{1}{R_0} \exp\left[-A_2 \frac{\tau_{\rm H}}{R_0}\right], \tag{6b}$$

where A_1 and A_2 are numerical constants (see Supplemental Material [71] for details). By collecting all the cases together, we find the overall PDF of $\tau_{\rm H}$, for large $\tau_{\rm H}$,

$$\mathcal{P}(\tau_{\rm H}, R_0) \sim w_1\mathcal{P}_1 + w_2\mathcal{P}_2 + w_{3\rm A}\mathcal{P}_1 + w_{3\rm B}\mathcal{P}_2, \tag{7}$$

where w_1, w_2, $w_{3\rm A}$, and $w_{3\rm B}$ are the relative weights for the cases discussed above. As the shocks are one-dimensional

[3] An alternative argument is as follows. By K41-type arguments, the only two quantities than can appear in constructing K are ε and R, where ε is the energy dissipation rate per unit mass. Hence $K \sim \varepsilon^{1/3} R^{4/3}$.

structures, w_1 is negligible, but the other weights are not; and in the limit $R_0 \to 0$, $w_{3A}\mathcal{P}_1$ is the dominant contribution. Thereby, we obtain the following result for the moments of the PDF of τ_H:

$$\left\langle \tau_{\rm H}^p \right\rangle \sim R_0^{p(1-2h)} \sim R_0^{p/3}. \tag{8}$$

In the final step we substitute $h = 1/3$ [60,70]. Equation (8) agrees well with our numerical results (see the dashed line in Fig. 2).

Several important comments are now in order:

(a) We emphasize that, although the halving-time dynamic exponent $\kappa_{\rm p}^{\rm H}$ depends linearly on the order p, the statistics of halving times is not Gaussian—the tail of the PDF of τ_H is exponential [see (6) and (7)]. Furthermore, (8) does not imply simple dynamic scaling because we have shown explicitly that differently defined timescales have different scaling exponents.

(b) Although we discuss halving and doubling times, explicitly, our results apply to exit times in general.

(c) Our DNS study is limited to the randomly forced 2D Burgers equation (1), which is curl free. In shock-dominated turbulence, the compressive (curl-free) modes can have significantly larger energy [33,34] than the solenoidal modes or may have comparable energy [32]. We expect our theoretical arguments to be generalizable to the former in a straightforward manner. A critical problem with the multifractal description of the scaling of exit times is that the bridge relation (4), requires the calculation of negative moments of $\delta_R V$. One way to avoid the calculation of such moments is to consider the statistics of inverse exit times. In particular, we define the halving and doubling frequencies to be $\omega_H \equiv 1/\tau_H$ and $\omega_D \equiv 1/\tau_D$, respectively. Thereby we define two new sets of dynamic scaling exponents,

$$\left\langle \omega_{\rm H}^p(R_0) \right\rangle \sim R_0^{-\chi_{\rm p}^{\rm H}} \quad \text{and} \quad \left\langle \omega_{\rm D}^p(R_0) \right\rangle \sim R_0^{-\chi_{\rm p}^{\rm D}}. \tag{9}$$

As we have discussed already, the multifractal model for incompressible turbulence does not distinguish between them [26–28], so the naive expectation is $\chi_{\rm p}^{\rm H} = \chi_{\rm p}^{\rm D} = p - \zeta_p$. On the contrary, our DNS shows (within the error bars of Fig. 3)

$$\chi_{\rm p}^{\rm H} = p - \zeta_p \quad \text{and} \quad \chi_{\rm p}^{\rm D} = 2p/3; \tag{10}$$

clearly, $\chi_{\rm p}^{\rm H} \neq \chi_{\rm p}^{\rm D}$. Our extension of the multifractal model predicts the correct exponents in the following manner. The scaling of the moments of ω_D and ω_H, for $p > 1$, is determined primarily by the small-τ_D and small-τ_H behaviors of their respective PDFs. The short-time growth of R_0 can occur only if the interval is away from the shocks [case (3), but at short times]. Therefore, for all such intervals, $K \sim R^{4/3}$ and consequently $\chi_{\rm p}^{\rm D} = 2p/3$ for all p (dashed line in Fig. 3), in agreement with the results of our DNS. As for $\chi_{\rm p}^{\rm H}$, at short times, even for incompressible turbulence, intervals can decrease, hence the the prediction of the multifractal model holds.

In an earlier paper [64] we have investigated dynamic multiscaling in the 1D stochastically forced Burgers equation by using a slightly different formulation than we have used here. Instead of halving times, we have considered interval-collapse times, i.e., the time it takes for a Lagrangian interval to shrink to a point. We have shown that, in 1D, the interval-collapse-time PDF has power-law tails. In contrast, we now find that, in 2D, the PDF of halving times has exponential tails. In both 1D and 2D cases, the shocks play a key role in determining the multiscaling properties.

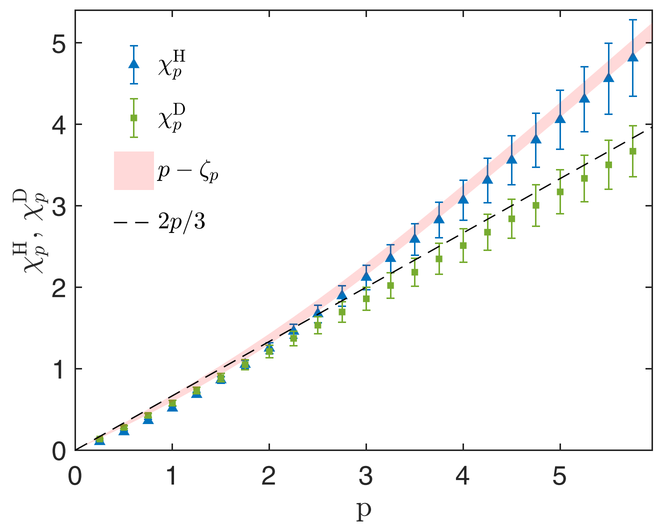

FIG. 3. Dynamic scaling exponents, of order p of the doubling frequencies ($\chi_{\rm p}^{\rm D}$) and halving frequencies ($\chi_{\rm p}^{\rm H}$) of Lagrangian intervals. Clearly, contrary to the naive expectation based on the conventional multifractal model, $\chi_{\rm p}^{\rm D} \neq \chi_{\rm p}^{\rm H}$ for $p \gtrsim 3$, up to our numerical error bars, and in agreement with our theory; $\chi_{\rm p}^{\rm H}$ agrees with its bridge relation (10) (shaded region), where ζ_p is obtained from DNS [Fig. 1(b)]; the dashed line is our prediction of $\chi_{\rm p}^{\rm D}$.

In summary, we have shown how to generalize the multifractal model for incompressible-fluid turbulence to the stochastically forced 2D Burgers equation, which is rich enough to display the complexities of shock-dominated turbulence. Our DNS demonstrates clearly that the statistics of halving times deviate starkly from those of doubling times. By generalizing the standard argument that is used to derive Richardson's law from K41 theory, we have developed a theory that leads to a natural way of understanding the statistics of halving and doubling times. The key idea in this theory is that, when we consider a Lagrangian interval of length R, we must distinguish between cases (1)–(3). Our theoretical arguments can be generalized to shock-dominated compressible turbulence, if the equal-time exponents ζ_p are known.

We expect shock-dominated turbulence in astrophysical systems at both high Mach and Reynolds numbers, e.g., in the interstellar media and molecular clouds, where the turbulence is driven by supernovae explosions. The dynamics of Lagrangian intervals in such flows provides useful insights into transport and mixing in these systems, which influence chemical kinetics and the rates of formation of stars and planetesimals [35]. We expect our generalization of Richardson's law to apply to such systems. For compressible turbulence, where the irrotational and solenoidal components of the flow have similar energies [32], our theory may require further generalization. Our work brings out the importance of calculating the statistics of both halving and doubling times of Lagrangian intervals from DNSs of compressible turbulent flows, for trans-sonic, supersonic, and hypersonic cases.

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[1] W. R. Young, Stirring and mixing, in *Summer Program in Geophysical Fluid Dynamics, 1999*, edited by J.-L. Thiffeault and C. Pasquero (Woods Hole Oceanographic Institution, Woods Hole, MA, 1999).
[2] G. Falkovich, K. Gawędzki, and M. Vergassola, Particles and fields in fluid turbulence, Rev. Mod. Phys. **73**, 913 (2001).
[3] G. K. Batchelor, Diffusion in a field of homogeneous turbulence II. The relative motion of particles, Math. Proc. Cambridge Philos. Soc. **48**, 345 (1952).
[4] B. Sawford, Turbulent relative dispersion, Annu. Rev. Fluid Mech. **33**, 289 (2001).
[5] J. P. L. C. Salazar and L. R. Collins, Two-particle dispersion in isotropic turbulent flows, Annu. Rev. Fluid Mech. **41**, 405 (2009).
[6] M. Huber, J. C. McWilliams, and M. Ghil, A climatology of turbulent dispersion in the troposphere, J. Atmos. Sci. **58**, 2377 (2001).
[7] S. Edouard, B. Legras, F. Lefèvre, and R. Eymard, The effect of small-scale inhomogeneities on ozone depletion in the arctic, Nature (London) **384**, 444 (1996).
[8] A. Griffa, A. Haza, T. M. Özgökmen, A. Molcard, V. Taillandier, K. Schroeder, Y. Chang, and P.-M. Poulain, Investigating transport pathways in the ocean, Deep-Sea Res. II: Top. Stud. Oceanogr. **85**, 81 (2013).
[9] R. Benzi and F. Toschi, Lectures on turbulence, Phys. Rep. **1021**, 1 (2023).
[10] L. F. Richardson and G. T. Walker, Atmospheric diffusion shown on a distance-neighbour graph, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[11] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, U.K., 1996).
[12] V. S. L'vov, E. Podivilov, and I. Procaccia, Temporal multiscaling in hydrodynamic turbulence, Phys. Rev. E **55**, 7030 (1997).
[13] D. Mitra and R. Pandit, Dynamic multiscaling in fluid turbulence: an overview, Physica A **318**, 179 (2003).
[14] D. Mitra and R. Pandit, Varieties of dynamic multiscaling in fluid turbulence, Phys. Rev. Lett. **93**, 024501 (2004).
[15] S. S. Ray, D. Mitra, and R. Pandit, The universality of dynamic multiscaling in homogeneous, isotropic Navier–Stokes and passive-scalar turbulence. New J. Phys. **10**, 033003 (2008).
[16] S. S. Ray, D. Mitra, P. Perlekar, and R. Pandit, Dynamic multiscaling in two-dimensional fluid turbulence, Phys. Rev. Lett. **107**, 184503 (2011).
[17] S. Ott and J. Mann, An experimental investigation of the relative diffusion of particle pairs in three-dimensional turbulent flow, J. Fluid Mech. **422**, 207 (2000).
[18] R. Shnapp and A. Liberzon (International Collaboration for Turbulence Research), Generalization of turbulent pair dispersion to large initial separations, Phys. Rev. Lett. **120**, 244502 (2018).
[19] G. E. Elsinga, T. Ishihara, and J. C. R. Hunt, Non-local dispersion and the reassessment of Richardson's t^3-scaling law, J. Fluid Mech. **932**, A17 (2022).
[20] S. Tan and R. Ni, Universality and intermittency of pair dispersion in turbulence, Phys. Rev. Lett. **128**, 114502 (2022).
[21] R. Shnapp, S. Brizzolara, M. M. Neamtu-Halic, A. Gambino, and M. Holzner, Universal alignment in turbulent pair dispersion, Nat. Commun. **14**, 4195 (2023).
[22] R. Bitane, H. Homann, and J. Bec, Time scales of turbulent relative dispersion, Phys. Rev. E **86**, 045302(R) (2012).
[23] R. Bitane, H. Homann, and J. Bec, Geometry and violent events in turbulent pair dispersion, J. Turbul. **14**, 23 (2013).
[24] A. D. Bragg, P. J. Ireland, and L. R. Collins, Forward and backward in time dispersion of fluid and inertial particles in isotropic turbulence, Phys. Fluids **28**, 013305 (2016).
[25] D. Buaria, B. L. Sawford, and P.-K. Yeung, Characteristics of backward and forward two-particle relative dispersion in turbulence at different Reynolds numbers, Phys. Fluids **27**, 105101 (2015).
[26] G. Boffetta and I. M. Sokolov, Relative dispersion in fully developed turbulence: The Richardson's law and intermittency corrections, Phys. Rev. Lett. **88**, 094501 (2002).
[27] G. Boffetta and I. M. Sokolov, Statistics of two-particle dispersion in two-dimensional turbulence, Phys. Fluids **14**, 3224 (2002).
[28] G. Boffetta, A. Celani, A. Crisanti, and A. Vulpiani, Pair dispersion in synthetic fully developed turbulence, Phys. Rev. E **60**, 6734 (1999).
[29] V. Artale, G. Boffetta, A. Celani, M. Cencini, and A. Vulpiani, Dispersion of passive tracers in closed basins: Beyond the diffusion coefficient, Phys. Fluids **9**, 3162 (1997).
[30] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, On the multifractal nature of fully developed turbulence and chaotic systems, J. Phys. A: Math. Gen. **17**, 3521 (1984).
[31] G. Parisi and U. Frisch, On the singularity structure of fully developed turbulence, in *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, *Proceedings of the International School of Physics "Enrico Fermi," Course LXXXVIII, Varenna, 1983*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), pp. 84–88.
[32] C. Federrath, J. Roman-Duval, R. S. Klessen, W. Schmidt, and M.-M. Mac Low, Comparing the statistics of interstellar turbulence in simulations and observations-solenoidal versus compressive turbulence forcing, Astron. Astrophys. **512**, A81 (2010).
[33] H. Miura, Enstrophy generation in a shock-dominated turbulence, J. Phys. Soc. Jpn. **65**, 450 (1996).
[34] C. Federrath, G. Chabrier, J. Schober, R. Banerjee, R. S. Klessen, and D. R. G. Schleicher, Mach number dependence of turbulent magnetic field amplification: Solenoidal versus compressive flows, Phys. Rev. Lett. **107**, 114504 (2011).
[35] M.-M. Mac Low and R. S. Klessen, Control of star formation by supersonic turbulence, Rev. Mod. Phys. **76**, 125 (2004).
[36] W. Schmidt, C. Federrath, and R. Klessen, Is the scaling of supersonic turbulence universal? Phys. Rev. Lett. **101**, 194505 (2008).

[37] L. Konstandin, C. Federrath, R. S. Klessen, and W. Schmidt, Statistical properties of supersonic turbulence in the Lagrangian and Eulerian frameworks, J. Fluid Mech. **692**, 183 (2012).
[38] J. Wang, Y. Yang, Y. Shi, Z. Xiao, X. T. He, and S. Chen, Cascade of kinetic energy in three-dimensional compressible turbulence, Phys. Rev. Lett. **110**, 214505 (2013).
[39] J. Wang, Y. Yang, Y. Shi, Z. Xiao, X. T. He, and S. Chen, Statistics and structures of pressure and density in compressible isotropic turbulence, J. Turbul. **14**, 21 (2013).
[40] S. Jagannathan and D. A. Donzis, Reynolds and Mach number scaling in solenoidally-forced compressible turbulence using high-resolution direct numerical simulations, J. Fluid Mech. **789**, 669 (2016).
[41] J. Wang, T. Gotoh, and T. Watanabe, Spectra and statistics in compressible isotropic turbulence, Phys. Rev. Fluids **2**, 013403 (2017).
[42] D. A. Donzis and J. P. John, Universality and scaling in homogeneous compressible turbulence, Phys. Rev. Fluids **5**, 084609 (2020).
[43] Y. Sakurai and T. Ishihara, Direct numerical simulations of compressible isothermal turbulence in a periodic box: Reynolds number and resolution-level dependence, Phys. Rev. Fluids **8**, 084606 (2023).
[44] Y. Yang, J. Wang, Y. Shi, Z. Xiao, X. T. He, and S. Chen, Intermittency caused by compressibility: A Lagrangian study, J. Fluid Mech. **786**, R6 (2016).
[45] Z. Xia, Y. Shi, Q. Zhang, and S. Chen, Modulation to compressible homogenous turbulence by heavy point particles. I. Effect of particles' density, Phys. Fluids **28**, 016103 (2016).
[46] J. R. Cressman, W. I. Goldburg, and J. Schumacher, Dispersion of tracer particles in a compressible flow, Europhys. Lett. **66**, 219 (2004).
[47] J. R. Cressman and W. I. Goldburg, Compressible flow: Turbulence at the surface, J. Stat. Phys. **113**, 875 (2003).
[48] J. R. Cressman, J. Davoudi, W. I. Goldburg, and J. Schumacher, Eulerian and Lagrangian studies in surface flow turbulence, New J. Phys. **6**, 53 (2004).
[49] D. J. Price and C. Federrath, A comparison between grid and particle methods on the statistics of driven, supersonic, isothermal turbulence, Mon. Not. R. Astron. Soc. **406**, 1659 (2010).
[50] S. Galtier and S. Banerjee, Exact relation for correlation functions in compressible isothermal turbulence, Phys. Rev. Lett. **107**, 134501 (2011).
[51] C. Federrath, On the universality of supersonic turbulence, Mon. Not. R. Astron. Soc. **436**, 1245 (2013).
[52] A. L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, U.K., 1995).
[53] J. M. Burgers, *A Mathematical Model Illustrating the Theory of Turbulence* (Elsevier, Amsterdam, 1948), pp. 171–199.
[54] U. Frisch and J. Bec, Burgulence, in *New Trends in Turbulence. Turbulence: Nouveaux Aspects*, edited by M. Lesieur, A. Yaglom, and F. David, Proceedings of the Les Houches Summer School of Theoretical Physics, LXXIV, 2000 (Springer, Berlin, 2001), pp. 341–383.
[55] J. Bec, U. Frisch, and K. Khanin, Kicked Burgers turbulence, J. Fluid Mech. **416**, 239 (2000).
[56] J. Bec and K. Khanin, Burgers Turbulence, Phys. Rep. **447**, 1 (2007).
[57] J. Bec, Universality of velocity gradients in forced Burgers turbulence, Phys. Rev. Lett. **87**, 104501 (2001).
[58] A. M. Polyakov, Turbulence without pressure, Phys. Rev. E **52**, 6183 (1995).
[59] E. Medina, T. Hwa, M. Kardar, and Y.-C. Zhang, Burgers equation with correlated noise: Renormalization-group analysis and applications to directed polymers and interface growth, Phys. Rev. A **39**, 3053 (1989).
[60] A. Chekhlov and V. Yakhot, Kolmogorov turbulence in a random-force-driven Burgers equation, Phys. Rev. E **51**, R2739 (1995).
[61] F. Hayot and C. Jayaprakash, Structure functions in the stochastic Burgers equation, Phys. Rev. E **56**, 227 (1997).
[62] F. Hayot and C. Jayaprakash, Multifractality in the stochastic Burgers equation, Phys. Rev. E **54**, 4681 (1996).
[63] F. Hayot and C. Jayaprakash, From scaling to multiscaling in the stochastic Burgers equation, Phys. Rev. E **56**, 4259 (1997).
[64] S. De, D. Mitra, and R. Pandit, Dynamic multiscaling in stochastically forced burgers turbulence, Sci. Rep. **13**, 7151 (2023).
[65] D. Mitra, J. Bec, R. Pandit, and U. Frisch, Is multiscaling an artifact in the stochastically forced Burgers equation? Phys. Rev. Lett. **94**, 194501 (2005).
[66] R. S. Rogallo, Numerical experiments in homogeneous turbulence, NASA Technical Memorandum 81315 (1981), pp. 1–91.
[67] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics* (Springer, Berlin, 1988).
[68] S. M. Cox and P. C. Matthews, Exponential time differencing for stiff systems, J. Comput. Phys. **176**, 430 (2002).
[69] A. Sain, Manu, and R. Pandit, Turbulence and multiscaling in the randomly forced Navier-Stokes equation, Phys. Rev. Lett. **81**, 4377 (1998).
[70] A. Chekhlov and V. Yakhot, Kolmogorov turbulence in a random-force-driven Burgers equation: Anomalous scaling and probability density functions, Phys. Rev. E **52**, 5681 (1995).
[71] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.6.L022032 for further details about the DNS, plot of mean-squared tracer-pair separation, scaling arguments leading to a Kolmogorov-like energy spectrum in the randomly-forced Burgers equation, and details about the heuristic theory of the scaling of the Lagrangian interval halving times.
[72] B. Castaing, Y. Gagne, and E. J. Hopfinger, Velocity probability density functions of high Reynolds number turbulence, Physica D **46**, 177 (1990).
[73] S. Y. Chen, B. Dhruva, S. Kurien, K. R. Sreenivasan, and M. A. Taylor, Anomalous scaling of low-order structure functions of turbulent velocity, J. Fluid Mech. **533**, 183 (2005).
[74] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, U.K., 2001).