Letter

Superfluid transition of a ferromagnetic Bose gas

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(Received 24 January 2024; accepted 17 April 2024; published 3 May 2024)

The strongly ferromagnetic spin-1 Bose-Einstein condensate has recently been realized with atomic ^{7}Li. It was predicted that a strong ferromagnetic interaction can drive the normal gas into a magnetized phase at a temperature above the superfluid transition, and ^{7}Li likely satisfies the criterion. We reexamine this theoretical proposal employing the two-particle-irreducible effective potential, and conclude that there exists no stable normal magnetized phase for a dilute ferromagnetic Bose gas. For ^{7}Li, we predict that the normal gas undergoes a joint first-order transition and jumps directly into a state with finite condensate density and magnetization. We estimate the size of the first-order jump and examine how a partial spin polarization in the initial sample affects the first-order transition. We propose a qualitative phase diagram at fixed temperature for the trapped gas.

DOI: 10.1103/PhysRevResearch.6.L022030

The realization of Bose-Einstein condensation (BEC) in dilute atomic gases [1–3] led to an explosion of a number of superfluid systems that can be studied experimentally, and many of them display exotic, richer physics beyond the simple spinless Bose gas model. Atomic ^{7}Li gas is such a system: its low-energy hyperfine states form a triplet with total spin 1. (For a review on spinor Bose gas, see Ref. [4].) Early experiments [3] employed magnetic trapping of a spin-polarized gas, while later all-optical techniques nonetheless relied on magnetic Feshbach resonance to produce a condensate [5,6]. Recent technology allows the trapping of unpolarized ^{7}Li [7], which enjoys an internal SO(3) symmetry of spin rotation. Notably, ^{7}Li has a spin-dependent, strongly ferromagnetic interaction [4,7–9].

The zero-temperature mean-field ground state of the spin-1 Bose gas is considered in Refs. [10,11]. Two BEC phases are predicted, depending on the interaction parameters: one has a spin dipole moment (ferromagnetic) and the other has a quadruple moment (polar). Most prior works on the finite-temperature phase diagram assume a weak spin dependence in interaction [12–14]. When the relative strength of the spin-dependent part is sufficiently large, Natu and Mueller [15] predicted a two-step process toward BEC: a ferromagnetic gas first undergoes a bosonic version of Stoner's transition to develop a spontaneous spin dipole moment, then condenses at a lower temperature. ^{7}Li likely satisfies the criterion.

The magnetization is an obvious choice for the order parameter characterizing the intermediate ferromagnetic phase (if exists.) In a second-quantized description, the Bose field itself remains normal, but certain *bilinear* of the field acquires a nonzero expectation value that spontaneously breaks the spin SO(3) but keeps the gauge $U(1)$ intact. In this regard, it is also akin to the more exotic pair condensate phase [15–18]. Phases characterized by bilinear order parameters (due to various mechanisms) have been proposed in several superfluid systems [19–22].

On the level of Gizburg-Landau theory, this normal-state magnetism is closely related to the time-reversal [23,24] and lattice rotational [25,26] symmetry breaking above unconventional superconductors, as well as the elusive quartet condensation or charge-4e superconductivity [27–32]. We want to especially highlight the similarity between the ferromagnetic Bose gas and the so-called vestigial order in the context of unconventional superconductivity [29,32,33], where the lattice symmetry plays a role similar to the spin SO(3). Both proposals share the same mechanism: the orders are mediated purely by the fluctuations of the underlying Bose fields (bosonic matter field or Ginzburg-Landau order parameter, respectively.)

In a previous paper, the present authors concluded that the vestigial order scenario cannot be realized in a weak-coupling superconductor [34]: the apparent instability toward a vestigial order, in fact, signals a joint first-order transition directly into the appropriate superconducting phase. For a pseudo-spin-$\frac{1}{2}$ Bose gas, a similar theoretical claim of normal-state magnetism was made [35,36] and refuted [37]. In this Letter, we will show that the same claim for a ferromagnetic spin-1 Bose gas is also incorrect: when the relative strength of the

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spin-dependent interaction is sufficiently large, the gas undergoes a joint firs- order transition into the BEC phase directly, just like its spin-$\frac{1}{2}$ and superconductor cousins.

We will start by reviewing our theoretical method. Next, we describe the ferromagnetic instability of a homogeneous normal gas, how it is unphysical, and the actual first-order normal-BEC transition. We discuss the experimental signature and propose a qualitative phase diagram for a trapped gas. Numerical estimations are given for ^{7}Li under experimental conditions.

Theoretical model. Let ψ_s be the annihilation field operator of a spin-1 boson in m_z spin state $s = \uparrow, \downarrow, 0$. We adopt the notation found in Refs. [10,16] for the Hamiltonian density:

$$\mathcal{H} = \psi_s^*\left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi_s + \frac{C_1}{2}\psi_s^*\psi_{s'}^*\psi_{s'}\psi_s + \frac{C_2}{2}(\vec{F})_{ss'}\cdot(\vec{F})_{tt'}\,\psi_s^*\psi_t^*\psi_{s'}\psi_{t'}. \tag{1}$$

Here $\vec{F}$ is the triplet of spin-1 matrices, and repeated indices are summed over. The effective interaction is due solely to two-body s-wave scattering, an approximation valid in the dilute limit. This Hamiltonian enjoys an SO(3) symmetry in the spin space, and the total magnetization of the system is conserved.

In terms of s-wave scattering lengths a_0 and a_2 in the channels of total spin-0 and -2, respectively, the parameters C_1 and C_2 are

$$C_1 = \frac{4\pi\hbar^2}{3m}(a_0 + 2a_2); \quad C_2 = \frac{4\pi\hbar^2}{3m}(a_2 - a_0). \tag{2}$$

Stability requires $C_1 > 0$ and $C_2 > -C_1$, and we restrict our attention to $C_2 < 0$ that favors a ferromagnetic spin moment. Numerical calculations put $C_2/C_1 = -0.46$ [4,9] for ^{7}Li.

We explore the thermodynamic of a uniform gas. Passing to the grand canonical ensemble, the gas is coupled to the spin-dependent chemical potential $\mu_\uparrow = \mu + h - q$, $\mu_\downarrow = \mu - h - q$, and $\mu_0 = \mu$. This describes an overall chemical potential μ and linear and quadratic Zeeman energies h and q, respectively. We assume that the timescale of typical experiments forbids the relaxation of total magnetization, and h is merely a corresponding Lagrange multiplier for the conserved magnetization [38]: the linear Zeeman effect of a physical magnetic field is absorbed into h. We will assume vanishing residual field and set $q = 0$. The qualitative physics of the first-order transition is unaffected by a small $q \neq 0$, and we will comment on the role of q later.

We employ the two-particle irreducible (2PI) effective potential method, essentially a nonrelativistic version of the CJT potential [39,40]. The spin-dependent self-energy of the boson is treated as the variational parameter in this formalism. Before proceeding, we adopt a dimensionless form by choosing k_BT and $\lambda_T = \sqrt{2\pi\hbar^2/mk_BT}$ as the units of energy and length, respectively. All subsequent numerical values are reported in this unit system. The dimensionless coupling constants are defined as $c_{1,2} \equiv C_{1,2}/(k_BT\lambda_T^3)$. Based on the reported parameters [7], we adopt $c_1 \approx 0.0024$ and $c_2 \approx -0.0011$ for ^{7}Li [41]. The interaction parameters being small is a direct consequence of the diluteness condition.

The 2PI potential is truncated at two-loop order, and for the normal gas the treatment is identical to a self-consistent Hartree-Fock (HF) approximation [37,42]. The HF self-energy is diagonal in the spin basis and momentum independent, leading to the ansatz that the density of spin-s atom is $\lambda_T^{-3}\mathrm{Li}_{\frac{3}{2}}(e^{-m_s})$, where the dimensionless energy gap m_s becomes the variational parameter. We introduce the shorthand $\mathrm{L}_s \equiv \mathrm{Li}_{\frac{3}{2}}(e^{-m_s})$. The dimensionless 2PI potential for normal gas is

$$\Omega_n = \sum_s\left[-\mathrm{Li}_{\frac{5}{2}}(e^{-m_s}) - (m_s + \mu_s)\mathrm{L}_s\right] + (c_1 + c_2)(\mathrm{L}_\uparrow^2 + \mathrm{L}_\downarrow^2 + \mathrm{L}_\uparrow\mathrm{L}_0 + \mathrm{L}_\downarrow\mathrm{L}_0) + (c_1 - c_2)\mathrm{L}_\uparrow\mathrm{L}_\downarrow + c_1\mathrm{L}_0^2. \tag{3}$$

(Subscript n stands for normal.) Once minimized with respect to m_s, min Ω_n is the negative of pressure in unit of k_BT/λ_T^3.

Ferromagnetic instability. We obtain the saddle point equations by taking derivatives of Ω_n. The three equations read

$$m_\uparrow + \mu_\uparrow - (c_1 + c_2)(2\mathrm{L}_\uparrow + \mathrm{L}_0) - (c_1 - c_2)\mathrm{L}_\downarrow = 0, \tag{4a}$$

$$m_0 + \mu_0 - (c_1 + c_2)(\mathrm{L}_\uparrow + \mathrm{L}_\downarrow) - 2c_1\mathrm{L}_0 = 0, \tag{4b}$$

$$m_\downarrow + \mu_\downarrow - (c_1 + c_2)(2\mathrm{L}_\downarrow + \mathrm{L}_0) - (c_1 - c_2)\mathrm{L}_\uparrow = 0. \tag{4c}$$

We first consider $h \to 0^-$. There is a symmetric branch of solution with $m_s = m(\mu)$ for all s, implicitly given by

$$m + \mu - (4c_1 + 2c_2)\,\mathrm{Li}_{\frac{3}{2}}(e^{-m}) = 0. \tag{5}$$

The solution $m(\mu)$ is positive and monotonically decreasing, ending at the mean-field critical point $\mu_c = (4c_1 + 2c_2)\,\zeta(3/2)$, where $m(\mu_c) = 0$. For ^{7}Li under experimental conditions, $\mu_c \approx 0.012$.

Take the difference of (4a) and (4c):

$$m_\uparrow - m_\downarrow = (c_1 + 3c_2)\,(\mathrm{L}_\downarrow - \mathrm{L}_\uparrow). \tag{6}$$

It becomes possible to have $m_\uparrow \neq m_\downarrow$ if $c_2/c_1 < -1/3$, a regime we dub *deep ferromagnetic*. (The ratio is -0.46 for ^{7}Li.) Equivalently, the criterion is $2a_0 > 5a_2$. Linearizing this equation, one obtains the implicit condition for ferromagnetic instability at $\mu = \mu_{\mathrm{ins}}$ along the symmetric branch:

$$\mathrm{Li}_{\frac{1}{2}}(e^{-m(\mu_{\mathrm{ins}})}) = -\left(\frac{1}{c_1 + 3c_2}\right). \tag{7}$$

The RPA spin susceptibility diverges when (7) is satisfied. By analyzing the Hessian matrix, it can be shown that the symmetric branch is a local minimum of Ω_n when $\mu < \mu_{\mathrm{ins}}$, but only a saddle point for $\mu > \mu_{\mathrm{ins}}$.

It is tempting to (incorrectly!) identify this instability as an SO(3)-breaking, second-order critical point separating symmetric and ferromagnetic phases [15]. In the usual textbook scenario, one expects to find an SO(3)-breaking minimum emerging from the critical point when $\mu > \mu_{\mathrm{ins}}$, becoming the new equilibrium state with magnetic order. [The SO(3) degeneracy is broken by the infinitesimal h, leaving a unique equilibrium state.] The usual justification is that, assuming the free energy is bounded from below, a new minimum must emerge if the symmetric branch ceases to be a minimum as μ is raised. But this innocuous conjecture turns out to be false: we will presently show that no such solution exists when

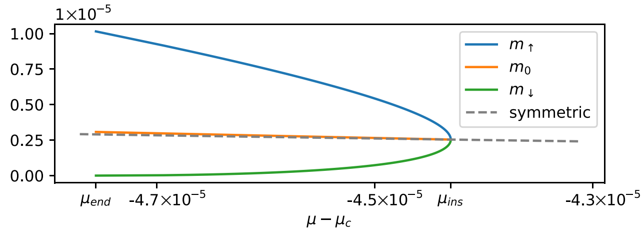

FIG. 1. The symmetric and vestigial ferromagnetic solutions for $h \to 0^-$ are plotted for ^{7}Li. The vestigial solution ends when $m_\downarrow = 0$.

$\mu > \mu_{\text{ins}}$ for a dilute gas [43]. The free energy is still bounded from below, but the true global minimum turns out to also have a superfluid order, an extra ingredient absent in the textbook scenario. We note that there is no other competing instability within the range $-c_1 < c_2 < 0$ [15].

The saddle point equations (4) can be solved numerically as-is. Given the small interaction parameters, however, the so-called critical approximation $\text{Li}_{\frac{3}{2}}(e^{-m_s}) \approx \zeta(3/2) - 2\sqrt{\pi m_s} + O(m_s)$ is appropriate (for this section only). Within this approximation, the $\sqrt{m_s}$ correction term becomes important when $\mu - \mu_c$ and m_s are of order $O(c_1^2)$, and this nonanalytic term drives the instability.

With the critical approximation, (4) admits a closed-form solution when $h \to 0^-$. The instability occurs at $\mu_{\text{ins}} - \mu_c \approx \pi(c_1 + 3c_2)(7c_1 + c_2) = -4.4 \times 10^{-5}$ for ^{7}Li. See Fig. 1. There is only one ferromagnetic solution branch with $m_\uparrow \neq m_\downarrow$, and it emerges from the instability. But this solution lies on the wrong ($\mu < \mu_{\text{ins}}$) side and consequently *cannot* be a minimum of Ω. Recall that the symmetric solution itself is a minimum for $\mu < \mu_{\text{ins}}$; given that the ferromagnetic solution can be brought arbitrarily close to the symmetric solution as $\mu \to \mu_{\text{ins}}^-$ without another saddle point coming in between, it cannot be a local minimum. The fact is also directly verified by computing the Hessiah matrix. This ferromagnetic branch therefore does not represent a physical equilibrium state. The branch ends when the down-spin is *gapless* ($m_\downarrow = 0$) at some $\mu_{\text{end}} < \mu_{\text{ins}}$. This so-called ferromagnetic end point is also not physical by any means, but will be of some importance in the discussion of the BEC branch later.

For $\mu > \mu_{\text{ins}}$, there exists no (meta)stable normal solution at all. If Ω is bounded from below, its global minimum must also have a superfluid order. Given that no other instability exists along the symmetric branch, the transition must be of first order. We thus conjecture that the gas undergoes a joint first-order transition directly into a ferromagnetic BEC state at some $\mu < \mu_{\text{ins}}$.

Our grand canonical approach makes this correct picture apparent. If one imposes a uniform density constrain instead, it seems at first that the gas enters a ferromagnetic phase as the density is raised [15]; one needs to work harder to see that the vestigial branch has a negative compressibility and is unstable, as pointed out in Ref. [37] for the spin-half case.

Before moving on to substantiate our claim of first order transition, we'd like to consider $h \neq 0$, or just $h < 0$ without loss of generality. This explicitly breaks the SO(3) symmetry. See Fig. 2(a). For sufficiently small h, a unique normal branch evolves from the combination of the $\mu < \mu_{\text{ins}}$ portion of symmetric branch and the vestigial branch. The solution is

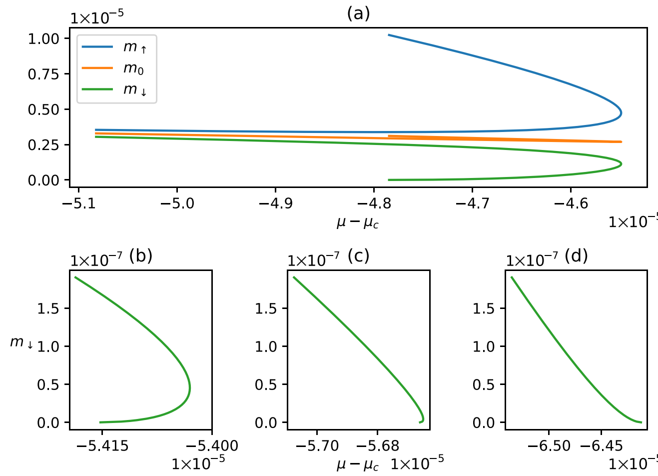

FIG. 2. We plot the solution with $h < 0$. (a) Three m_s components for a relatively small $|h| = 3 \times 10^{-8}$. The U-turn is the ferromagnetic instability. To illustrate the suppressing of instability, we plot $m_\downarrow$ for (a) $|h| = 7 \times 10^{-7} < h_t$, (b) $|h| = 1 \times 10^{-6} \lesssim h_t$, and (c) $|h| = 2 \times 10^{-6} > h_t$.

still multivalued, and this U-turn is the surviving ferromagnetic instability. One therefore still expects a first-order BEC transition preempting the instability. Above some threshold $|h| > h_t$, the instability is eliminated, and one finally expects a mean-field-like BEC transition taking place at the gapless ($m_\downarrow = 0$) end point. See Figs. 2(b)–2(d). For ^{7}Li gas under experimental conditions, we numerically find the threshold $h_t \approx 1.2 \times 10^{-6}$.

Experimentally, this picture of a joint first-order transition manifests as coexistence of normal and superfluid phases in a trapped quantum gas. To estimate the discontinuity across the phase boundary, we need to extend (3) to incorporate the superfluid order.

Superfluid solution. When $|h| > h_t$, the normal solution sees no instability until $m_\downarrow$ reaches zero. From here, we expect a continuous transition into the $U(1)$-breaking BEC state as indicated by familiar renormalization group arguments [44]; the gapless end point is also the onset of a BEC branch. By continuity, when $|h| < h_t$ we also expect the end point of the normal branch to mark the onset of a BEC branch. The ferromagnetic instability, however, prevents the gas to continuously follow the path. Before hitting the instability, the gas must makes a first-order jump from normal to BEC phase.

The original work on the CJT potential [39] already provides a general prescription to treat a BEC order. Concentrating on $h \to 0^-$, we assume the ansatz for the BEC order $\langle\psi_\uparrow\rangle = \langle\psi_0\rangle = 0$ and $\langle\psi_\downarrow\rangle = \phi$, in unit of $\lambda_T^{-3/2}$. We add to (3) the BEC part:

$$\Omega_b = -\mu\phi^2 + \tfrac{1}{2}(c_1 + c_2)\phi^4 + \phi^2[(c_1 + c_2)(2L_\downarrow + L_0) + (c_1 - c_2)L_\uparrow]. \quad (8)$$

The total 2PI potential $\Omega = \Omega_n + \Omega_b$ is required to be stationary with respect to m_s and ϕ: to the right-hand side of (4a)–(4c), one adds $(c_1 - c_2)\phi^2$, $(c_1 + c_2)\phi^2$ and $2(c_1 + c_2)\phi^2$, respectively; these goes together with the fourth

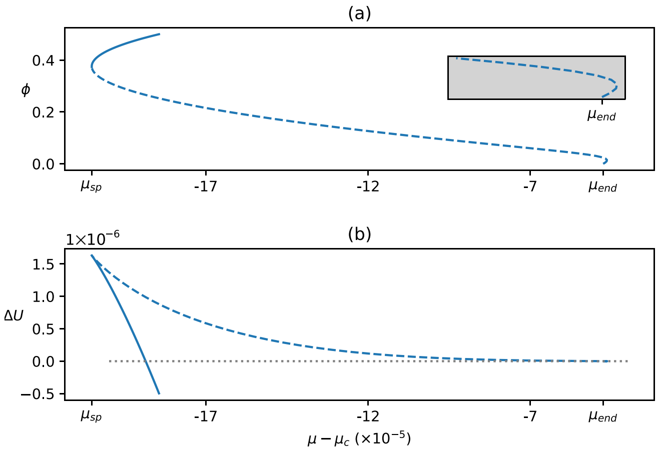

FIG. 3. (a) The solution to the saddle-point equations with $\phi \neq 0$. Only the solid part corresponds to local minimum of Ω, and the turning point $\mu_{\rm sp}$ is the spinodal point. The insert is the close-up view of the solution near $\mu_{\rm end}$: one sees the small U-turn identified as the infrared artifact of the HF approximation. (b) The plot of $\Delta U \equiv U_b - U_n$. The first-order transition occurs when $\Delta U = 0$ at $\mu = \mu_f$.

equation

$$0 = -\mu\phi + (c_1 + c_2)\phi^3 + \phi[(c_1 + c_2)(2L_\downarrow + L_0) + (c_1 - c_2)L_\uparrow]. \quad (9)$$

As $\phi = 0$ always solves (9), the symmetric and ferromagnetic branches remain stationary solutions. But now another (BEC) solution branch with $\phi \neq 0$ emerges where the ferromagnetic branch ends. Let U_b (U_n) denotes the corresponding stationary value of Ω at the BEC (symmetric) solution, respectively, and the first-order transition occurs when $U_n = U_b$. For ^{7}Li, our result is summarized in Fig. 3. The section of the BEC branch connected to the ferromagnetic end point is a unstable, as can be checked by computing the Hessian. After initially going toward the direction of larger μ, it soon turns around and extends toward the wrong ($\mu < \mu_{\rm end}$) side of the ferromagnetic end point. But the branch turns around again when it touches the spinodal point and becomes a local minimum. (On the normal side, the spinodal point is the ferromagnetic instability.) We identify the first-order transition at $\mu_f \approx 0.01914$. The scenario is reminiscent of the spin-half case explored by He *et al.* [37].

The combination of 2PI potential and HF approximation is well-known to incorrectly predict a first-order transition [34,45–47] even when one truly expects a second-order one, due to the strong infrared fluctuation when the system is almost gapless. It also weakly violates [45] the Goldstone or Hugenholtz-Pines theorem [48]. The manifestation of this infrared problem is the emergence of the nonanalytic $\sqrt{m_s}$ in the small-m_s expansion of L_s. We therefore argue that the spinodal structure found here is physical and not an artifact of the infrared problem, since the first-order transition occurs far from the region where the $\sqrt{m_s}$ terms dominate. Following He *et al.* [37], one would identify the infrared artifact with the small U-turn in the unstable portion of the BEC branch [see inset of Fig. 3(a)]. The result thus far can be qualitatively summarized in Fig. 4.

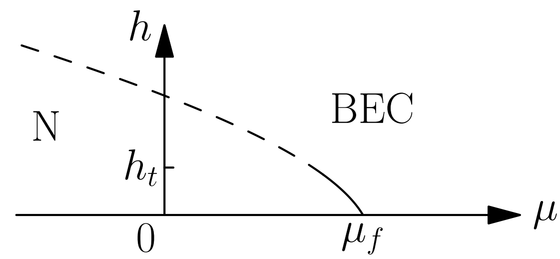

FIG. 4. The schematic phase diagram. The solid (dashed) line represents first- (second-) order phase boundary separating the normal (N) and ferromagnetic BEC phases. The first-order section of the boundary only exists if $c_1 - 3c_2 < 0$, or $2a_0 > 5a_2$ in terms of scattering lengths.

Experimental signature. Within the local density approximation (LDA), an experiment in a trap can be interpreted as a sampling across a range of μ at fixed T and h. The first-order transition shows up as a spatial discontinuities in total density, density of individual spin, magnetization, condensate density, and superfluid density. The jump is robust against a finite spin imbalance in the cloud. To estimate the size of the discontinuity, we calculate (at $h \to 0^-$) the density of each spin component on either side of the first-order transition. On the normal side, the density per component is $n_n = 2.592$. On the BEC side, the (purely spin-down) condensate density is $\phi^2 = 0.237$; the densities n_s of the thermal spin-s cloud are $n_\uparrow = 2.531$, $n_0 = 2.573$, and $n_\downarrow = 2.551$, respectively. These results are only weakly dependent on the physical density in the trap [49]. One then proceeds to work out the mismatches in various quantities. For example, total density jumps by $(n_\uparrow + n_0 + n_\downarrow + \phi^2)/3n_n - 1 = 2.3\%$. Relative magnetization is $(n_\uparrow - n_\downarrow - \phi^2)/(n_\uparrow + n_0 + n_\downarrow + \phi^2) = 3.7\%$ on the BEC side and zero on the normal side. Not surprisingly, spin-down density shows the biggest discontinuity: $(n_\downarrow + \phi^2)/n_n - 1 = 8.3\%$.

In a real experiment, the total particle number and magnetization are the external constrains rather than their conjugates μ and h. In a trap at fixed temperature, assuming $q = 0$, the $h \to 0^\pm$ solution (coexisting ferromagnetic BEC core and unpolarized normal fringe) sets the minimally allowed magnitude of magnetization: a smaller total magnetization can only be accommodated by setting $h = 0$ (hence all polarization directions are degenerate) and allowing the BEC core to have spatially varying polarization. The normal fringe remains unpolarized. (The polarization textual of the BEC core is beyond the scope of this paper.) If the magnetization is raised from zero at fixed particle number, the trapped gas exhibits three distinct phases in sequence: discontinuous coexistence of a textured BEC core and an unpolarized normal fringe (phase A), discontinuous coexistence of ferromagnetic BEC and normal fringe (phase B), and continuous coexistence of a ferromagnetic BEC and normal fringe (phase C). We propose the qualitative in-trap phase diagram Fig. 5. In phase A, the normal-BEC discontinuities at the coexistence interface are locked at the $h = 0^\pm$ values given above. Phase B has reduced discontinuities at finite h, and phase C has no discontinuity.

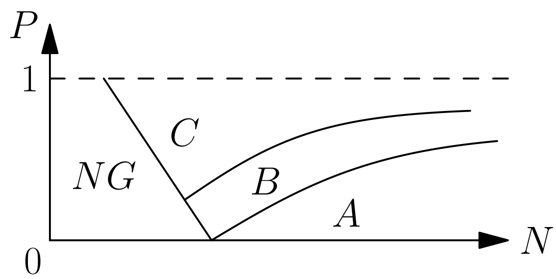

FIG. 5. Qualitative phase diagram of a trapped gas at fixed T. Let N_s be the total number of spin-s atoms, then $N = \sum_s N_s$ and $P = (N_\uparrow - N_\downarrow)/N$ here. Phases A, B, and C are given in the text, and NG stands for normal gas.

The quadratic Zeeman shift q due to the residual magnetic field is naturally positive in ^{7}Li gas. It therefore favors the spin-0 polar state, and can potentially wreck havoc on the ferromagnetic physics discussed here. The relevant energy scales are the gaps m_s on either sides, and the single particle condensation energy $\frac{c_1+c_2}{2\zeta(3/2)}\phi^4$. The smallest is the symmetric gap on the normal side $m \approx 3.5 \times 10^{-5}$. If q is positive and has a comparable magnitude, the first-order transition will be destroyed. The scale is estimated to be about 40 mG. At 10 mG, the resultant q is more than one order of magnitude smaller than m, and is but a small perturbation. Furthermore, complete experimental control over q using microwave dressing [50] has recently been demonstrated for ^{7}Li [51]: q can be tuned to zero or even a negative value independent of the magnetic field. The residual field thus poses no issue.

We assume q is a fully controllable parameter. While $q = 0$ is a quantum critical point separating easy-plane and easy-axis ferromagnetisms in a large, homogeneous system, the size of any topological defect would be extremely large with respect to interparticle distance when q is small. In a modest-sized cloud, we do not expect polarization textual in phase A to be substantially affected by a small $q \neq 0$ of either sign. Even though the first-order transition is destroyed by a large and positive q, it will survive an arbitrarily negative q. In this $q \to -\infty$ extreme easy-axis limit, the gas is effectively two-component and the first-order transition should follow the prediction made in Ref. [37]. In this limit, the phase A spin textual becomes domains of spin-up or -down separated by sharp domain walls. Experimentally, this will, in fact, be a much clearer signature compared to the transverse polarizations seen around $q \approx 0$.

Conclusion. We study the superfluid transition of a dilute ferromagnetic spin-1 Bose gas. Contrary to a previous claim [15], we find that the normal gas cannot support a ferromagnetic phase, regardless of the ratio of interaction parameters. In the deep ferromagnetic regime where the normal gas does exhibit a ferromagnetic instability upon increasing chemical potential (or density), a ferromagnetic solution exists for the self-consistent HF equation of state, but our grand canonical approach makes it apparent that the solution is thermodynamically unstable. Instead, a stable BEC solution branch emerges already at a lower chemical potential, and the gas undergoes a joint first-order transition into this BEC state *before* hitting the ferromagnetic instability. In our opinion, such vestigial order is usually not stabilized in a weakly interacting system, and this is another example.

In a trapped gas, the trap potential translates into spatial variation of the chemical potential μ within LDA, and the first-order transition shows up as the (discontinuous) coexistence of a superfluid core and a normal fringe, with discontinuities in densities and magnetization. The jump of the majority spin density, the largest of these discontinuities, is estimated to be about 8%. These discontinuities are found to be robust for a range of magnetization in the sample. We propose the constant-temperature phase diagram Fig. 5. A big positive quadratic Zeeman shift q favoring the polar state can destroy the first-order ferromagnetic BEC transition described here, but independent experimental control over q has been recently demonstrated for ^{7}Li [51] and this is a nonissue. On the $q < 0$ easy-axis side, the first-order behavior qualitatively persists even in the extreme $q \to -\infty$ limit.

Acknowledgments. We thank J.-Y. Choi for updating us on the state-of-the-art experimental capabilities. This work is supported by the National Science and Technology Council of Taiwan under Grant No. MOST 110-2112-M-001-051-MY3. Additionally, P.T.H. is supported under Grant No. NSTC 112-2811-M-001-051.

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