Letter

# Noncommutative field theory of the Tkachenko mode: Symmetries and decay rate

Yi-Hsien Du,[1] Sergej Moroz,[2,3] Dung Xuan Nguyen,[4] and Dam Thanh Son[1]
[1] *Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, IL 60637, USA*
[2] *Department of Engineering and Physics, Karlstad University, Karlstad 651 88, Sweden*
[3] *Nordita, KTH Royal Institute of Technology and Stockholm University, Stockholm 114 19, Sweden*
[4] *Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34126, Korea*

(Received 14 January 2023; revised 23 September 2023; accepted 2 February 2024; published 26 February 2024)

We construct an effective field theory describing the collective Tkachenko oscillation mode of a vortex lattice in a two-dimensional rotating Bose-Einstein condensate in the long-wavelength regime. The theory has the form of a noncommutative field theory of a Nambu-Goldstone boson, which exhibits a noncommutative version of dipole symmetry. From the effective field theory, we show that, at zero temperature, the decay width $\Gamma$ of the Tkachenko mode scales with its energy $E$ as $\Gamma \sim E^3$ in the low-energy limit. We also discuss the width of the Tkachenko mode at a small temperature.

DOI: 10.1103/PhysRevResearch.6.L012040

*Introduction.* When a superfluid rotates, a lattice of quantized vortices forms. The oscillations of the vortex lattice, the so-called Tkachenko mode [1–3] (for a recent review, see Ref. [4]), has many distinctive properties. Unlike ordinary sound waves in a solid, at low momenta, the Tkachenko wave has a quadratic dispersion relation $\omega \sim q^2$ and only one polarization [5–7]. The Tkachenko mode is a consequence of a rather intricate realization of spontaneous symmetry breaking: there are many symmetries broken by the superfluid vortex lattice, but only one Nambu-Goldstone boson (NGB) [8,9]. The Tkachenko mode should exist in rotating superfluid $^4$He, but it has been observed most conclusively in the rotating Bose-Einstein condensate of ultracold atoms [10]. At a much larger length scale, the Tkachenko mode has been suggested to be the source of an oscillation mode of the Crab pulsar [11].

As the Tkachenko mode is the only low-energy degree of freedom, one expects that it can be described by an effective field theory (EFT) which involves a single field. However, up to now, a complete understanding of the structure of such a theory has yet to be achieved. At the quadratic level, the effective Lagrangian [8] coincides with that for a Lifshitz scalar [12], but the form of the interaction terms in the Lagrangian and how they are constrained by symmetries are not known. These interaction terms are needed to calculate the decay rate of the Tkachenko mode [13].

In this Letter, we show that noncommutative field theory (see, e.g., Refs. [14,15]) provides a convenient framework for constructing the effective field theory of the Tkachenko mode. That noncommutative field theory (NCFT) may be applicable to the problem is intuitively understandable—rotating a nonrelativistic system is formally equivalent to placing it in a magnetic field, and on the lowest Landau level (LLL) the guiding-center coordinates do not commute. Because of that, NCFT has often been invoked in the context of the quantum Hall effect [16–22]. Vortex lattices can also be realized on the LLL [23–25]. As we will see, in the case of the Tkachenko mode, NCFT provides a way to organize terms in the Lagrangian consistent with symmetries. Following the formalism, we are able to determine the general structure of the interacting Lagrangian, and from there, that the decay rate of a Tkachenko mode (at zero temperature) scales like the cube of its energy

$$\Gamma \sim E^3. \quad (1)$$

This implies, in particular, that the Tkachenko mode becomes a more and more well-defined quasiparticle (i.e., $\Gamma/E \to 0$) as the energy $E$ approaches zero.

We will also establish a connection between the Tkachenko mode and the "dipole" symmetry, which recently became a popular topic (see, e.g., Refs. [26–42]). The Tkachenko mode realizes a more complex version of dipole symmetry: the magnetic translations, which form a nonabelian group.

*Tkachenko mode as a noncommutative Nambu-Goldstone boson.* One can arrive at the theory of the Tkachenko mode from microscopic considerations, taking, for example, as the starting point the microscopic theory of bosons with short-range repulsive interactions and then eliminating all redundant degrees of freedom [8]. It is instructive, however, to derive the most general form of the effective Lagrangian, relying solely on symmetries. Such an approach has the advantage of being applicable for strongly correlated rotating superfluids where microscopic calculations are not reliable, e.g., close to a quantum melting transition of the vortex crystal [43].

We first note that the lattice of vortices can be described, as a two-dimensional solid, by two ($a = 1, 2$) scalar fields $X^a(t, x^i)$; they present the coordinates frozen in the solid [44,45]. In this description, in Cartesian coordinates the lattice displacement $u^a$ is related to $X^a$ by $X^a = x^a - u^a$. The vortex current is related to $X^a$ by

$$j_v^\mu = \tfrac{1}{2} n_0 \epsilon^{\mu\nu\lambda} \epsilon^{ab} \partial_\nu X^a \partial_\lambda X^b, \tag{2}$$

where $n_0$ is the equilibrium vortex density. For a superfluid under rotation with angular velocity $\Omega$, $n_0 = \frac{1}{2\pi} B$, where the effective magnetic field $B = 2m\Omega$ with $m$ being the mass of the elementary boson.

In a superfluid, the vortices carry charge with respect to the $u(1)$ dynamical gauge field $a_\mu$ dual to the superfluid Nambu-Goldstone boson [46,47]. The boson particle number current is expressed as $j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$. The Lagrangian of the system contains terms that describe the coupling of the vortex current with the dual $u(1)$ gauge field and the kinetic and potential terms of the latter,

$$\mathcal{L} = -j_v^\mu a_\mu + \frac{m}{4\pi b} \mathbf{e}^2 - \epsilon\left(\frac{b}{2\pi}\right) + \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda, \tag{3}$$

where $e_i = \partial_0 a_i - \partial_i a_0$, and $b = \epsilon^{ij} \partial_i a_j$. In the above equation, $m\mathbf{e}^2/(4\pi b)$ represents the kinetic energy of the superfluid condensate, $\epsilon(b/2\pi)$ is the internal energy as a function of the density, and $A_\mu$ is the gauge potential of the external effective magnetic field $B$. In the lowest Landau level limit $m \to 0$, the kinetic term vanishes. In fact, this term can be dropped if one is eventually interested in the limit $\omega \sim q^2$: in this regime $\mathbf{e}^2 \ll b^2$. Without the $\mathbf{e}^2$ term, variation with respect to $a_0$ give a constraint

$$\tfrac{1}{2} \epsilon_{ab} \epsilon^{ij} \partial_i X^a \partial_j X^b = 1. \tag{4}$$

That means the map from $x^i$ to $X^a$ is area preserving. To linear order in the displacement $u^i$, Eq. (4) implies $\partial_i u^i = 0$, i.e., the displacement is divergence-free: the Tkachenko mode is a transverse sound.

The quadratic theory of this transverse sound is analyzed in the Supplement Material (SM) [9]. Here we would like to resolve the constraint Eq. (4) at the nonlinear level. This can be done iteratively, as worked out in the SM [9]. Here we take a more elegant approach: on the LLL, one expects the spatial coordinates $x$ and $y$ to become noncommutative (see, e.g., Refs. [16,20]):

$$[\hat{x},\, \hat{y}] = i\theta, \quad \theta = -\ell^2, \tag{5}$$

where $\ell = 1/\sqrt{B}$ is the magnetic length. The quantum version of Eq. (4) then can be written as

$$[\hat{X},\, \hat{Y}] = i\theta. \tag{6}$$

We then conclude that $\hat{X}^a$ and $\hat{x}^a$ are related by a unitary transformation

$$\hat{X}^a = e^{i\hat{\phi}} \hat{x}^a e^{-i\hat{\phi}}, \tag{7}$$

where the operator $\hat{\phi}$ is an arbitrary function of the two noncommuting coordinates $\hat{x}$ and $\hat{y}$. In noncommutative field theories [14,15], any operator corresponds to a Weyl symbol which is a function in space, and the above equation becomes

$$X^a(x) = e^{i\phi(x)} \star x^a \star e^{-i\phi(x)}. \tag{8}$$

Here the star product is defined as $f \star g \equiv f(x) \exp(\frac{i}{2} \theta \epsilon^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j) g(x)$. To linear order in $\phi$ the displacement $u^a$ is then

$$u^a = \{\{\phi,\, x^a\}\} = -\theta \epsilon^{ab} \partial_b \phi, \tag{9}$$

where $\{\{\cdot, \cdot\}\}$ denotes the Moyal bracket, $\{\{f,\, g\}\} = 2f \sin(\frac{1}{2} \theta \epsilon^{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j) g$. As expected, to this order, the displacement is purely transverse. To all orders in $\phi$, Eq. (8) can be written as

$$X^a = x^a + i\{\{U,\, x^a\}\} \star U^{-1} = x^a + \theta \epsilon^{ai} D_i \phi, \tag{10}$$

where $U = e^{i\phi}$, $U^{-1} = e^{-i\phi}$, and

$$D_i \phi \equiv -i \partial_i U \star U^{-1}. \tag{11}$$

Thus, we identify the Tkachenko mode with a Nambu-Goldstone boson of a noncommutative field theory. We now show that this field is a compact scalar that shifts under the particle number U(1) symmetry.

*Magnetic translations as noncommutative dipole symmetry*. On the LLL, translations are magnetic translations and do not commute:

$$[\hat{P}_x,\, \hat{P}_y] = -\frac{i}{\theta} \hat{Q}, \tag{12}$$

where $\hat{Q}$ denotes the boson particle number operator. In our case, the Tkachenko mode is the only low-energy degree of freedom, so it should provide a nontrivial representation of magnetic translations. In the noncommutative theory, translations are realized as a special class of unitary transformations that are exponents of a linear function of coordinates. Acting on $\hat{X}^a$, such a transformation changes the Weyl symbol of the latter as

$$X^a \to e^{i\alpha_i x^i} \star X^a \star e^{-i\alpha_i x^i} = X^a(\vec{x} - \vec{\xi}), \tag{13}$$

with $\xi^i = -\theta \epsilon^{ij} \alpha_j$. This is a spatial translation by $\vec{\xi}$. Viewing $X^a$ as fields in a field theory, such a translation is supposed to be generated by $\hat{X}^a \to e^{-i\vec{\xi}\cdot\hat{\vec{P}}} \hat{X}^a e^{i\vec{\xi}\cdot\hat{\vec{P}}}$. Thus, we can identify the magnetic translation as [48]

$$\hat{P}_i = \frac{1}{\theta} \epsilon_{ij} \hat{x}^j. \tag{14}$$

According to Eqs. (8) and (13) and the associativity of the star product, magnetic translations by $\vec{c}$ act on the Tkachenko field $\phi$ as multiplication on the left

$$e^{i\phi} \to \exp\left(\frac{i}{\theta} \epsilon_{ij} c^i x^j\right) \star e^{i\phi}. \tag{15}$$

This allows us to interpret the action of magnetic translations on a Tkachenko field as a noncommutative version of a dipole symmetry. Expanding in $\phi$, Eq. (15) reads

$$\phi \to \phi + \frac{1}{\theta} \epsilon_{ij} c^i x^j - \frac{1}{2} c^i \partial_i \phi + \cdots. \tag{16}$$

To leading order, these are simply a dipole symmetry transformation $\phi \to \phi + \alpha_i x^i$ with the parameter $\alpha_i = \theta^{-1} \epsilon_{ij} c^j$, but in addition, there are an infinite number of terms composed of derivatives acting on fields $\phi$. These terms make the magnetic translations noncommuting, as in Eq. (12).

Knowing the transformation law for $\phi$ under magnetic translations, we can find the transformation law for $\phi$ under particle number $U(1)$ symmetry. Apply four translations on

$e^{i\phi}$, one after another: $e^{-i\beta\hat{P}_y}e^{-i\alpha\hat{P}_x}e^{i\beta\hat{P}_y}e^{i\alpha\hat{P}_x}$, from Eq. (15) we see that $e^{i\phi}$ becomes $e^{i(\phi+\frac{\alpha\beta}{\ell^2})}$. But we also know from Eq. (12) that the product of the four translations is $e^{i\frac{\alpha\beta}{\ell^2}\hat{Q}}$. Thus, under U(1) charge, the Tkachenko field transforms exactly as the phase of the superfluid condensate: $\phi \to \phi + c$. The Tkachenko field, therefore, has a dual role: it is the condensate phase, but at the same time, its gradient is the lattice displacement. Such a dual role is possible, of course, because at low energy, the condensate phase is entirely determined by the configuration of the vortex lattice. By the "condensate phase," one should have in mind the regular part of the phase where the singular contributions from the vortices have been subtracted away.

As a condensate phase, $\phi$ then should be a compact scalar field with periodicity $2\pi$: $\phi \sim \phi + 2\pi$. The periodicity of $\phi$ can also be seen from the following argument. Let us put the system on a torus of size $L_x \times L_y$. Then the magnetic field breaks translation symmetry along the $x$ direction to a discrete group of finite translations generated by $x \to x + 2\pi\ell^2/L_y$ (which can be seen by computing the Wilson line of the gauge field along a curve wrapping the torus along the $y$ direction at fixed $x$). This discrete translation is generated by the operator $e^{2\pi i\hat{y}/L_y}$ under which $\phi \to \phi + 2\pi y/L_y + \cdots$. This is allowed only when the identification $\phi \sim \phi + 2\pi$ is valid.

*Ingredients for a Lagrangian.* We now write down the most general Lagrangian consistent with symmetries for the field $U = e^{i\phi}$. The symmetries include global $U(1)$ phase rotations $U \to e^{i\alpha}U$, global magnetic translations $U \to e^{i\vec{\alpha}\cdot\vec{x}} \star U$ (noncommutative dipole symmetry), and global rotation $U \to e^{\frac{i}{2}\omega x^2} \star U$. The structures that are covariant (i.e., transforming like $O \to e^{i\vec{\alpha}\cdot\vec{x}} \star O \star e^{-i\vec{\alpha}\cdot\vec{x}}$, etc.) under these transformations are

$$D_0\phi \equiv -i\partial_t U \star U^{-1}, \tag{17a}$$

$$D_{ab}\phi = \frac{1}{2}(\partial_a D_b\phi + \partial_b D_a\phi - \delta_{ab}\partial_c D_c\phi) + \frac{\theta}{4}[\epsilon_{ac}\partial_i D_c\phi \star \partial_i D_b\phi + (a \leftrightarrow b)], \tag{17b}$$

where $D_i\phi$ is defined as in Eq. (11). Note that $D_{ab}\phi$ is symmetric and traceless [49].

These can be expanded infinite series over $\phi$. These series have the property that, at the order $\phi^n$ with a given integer $n$, the leading terms (in derivatives) have $2n$ derivatives if one counts $\partial_t$ as two derivatives $\partial_t \sim \partial_i^2$. This counting is natural as the Tkachenko mode, which is the only low-energy degree of freedom, has a quadratic dispersion. Keeping at each power of $\phi$ only terms with the minimal number of derivatives, we have

$$D_0\phi = \dot{\phi} + \frac{\theta}{2}\epsilon^{ij}\partial_i\dot{\phi}\partial_j\phi + \cdots, \tag{18a}$$

$$D_{ab}\phi = \partial_a\partial_b\phi + \frac{\theta}{2}\epsilon^{kl}\partial_a\partial_b\partial_k\phi\partial_l\phi - \text{trace} + \frac{\theta}{4}[\epsilon_{ac}\partial_i\partial_c\phi\, \partial_i\partial_b\phi + (a \leftrightarrow b)] + \cdots. \tag{18b}$$

*Effective Lagrangian.* We can now write down the Lagrangian of the Tkachenko mode, keeping at each power of $\phi$ terms with the minimal number of derivatives, counting each occurrence of $\partial_t$ as two derivatives. This Lagrangian would allow one to compute the rate of any scattering process to leading order over the momenta of the particles involved, similarly to the nonlinear Lagrangians for superfluids [50] or solids [44,45]. In the SM [9] we explicitly derive a nonlinear effective theory of the Tkachenko field $\phi$ from the leading-order effective theory of a vortex lattice introduced in Refs. [51,52].

The most general Lagrangian consistent with the $U(1)$ and magnetic translation symmetries is a function of the invariant structures defined above:

$$L = L(D_t\phi, D_{ab}\phi). \tag{19}$$

The form of the Lagrangian can be restricted further by imposing additional symmetries. In particular, assuming the vortex lattice is a triangular lattice, one should expect the $C_6$ group of rotations by angles multiple of $\frac{2\pi}{6}$. Introducing the complex coordinate $z = x + iy$, the rotationally invariant structures are now

$$D_0\phi,\ (D_{ab}\phi)^2,\ \text{Re}(D_{zz}\phi)^3,\ \text{Im}(D_{zz}\phi)^3. \tag{20}$$

A system of particles in a magnetic field has an antiunitary $RT$ symmetry that combines spatial reflection ($R$) and time reversal ($T$):

$$x \to x, \quad y \to -y, \quad t \to -t, \quad i \to -i. \tag{21}$$

Under this symmetry $\phi \to -\phi$, which can be seen from its connection to the displacement $u^a$ in Eq. (9). Among the $C_6$ invariants in Eq. (20), $\text{Re}(D_{zz}\phi)^3$ is odd, while the rest are even. Thus the most general effective Lagrangian is a function of four arguments

$$L = L(D_0\phi,\ (D_{ab}\phi)^2,\ \text{Im}(D_{zz}\phi)^3,\ (\text{Re}(D_{zz}\phi)^3)^2). \tag{22}$$

*The Girvin-MacDonald-Platzman (GMP) algebra.* The NCFT construction realizes a key feature of the LLL—the GMP algebra [53]. Indeed, upon canonical quantization, the particle-number density $n = -\delta S/\delta(D_0\phi)$ realizes the NC U(1) gauge transformation, i.e.,

$$\left[\int d^2y\,\lambda(y)n(y),\ O(x)\right] = i\delta_\lambda O(x), \tag{23}$$

where $\delta_\lambda O$ is the infinitesimal change of $O$ under the gauge transformation, under which $e^{i\phi} \to e^{i\lambda} \star e^{i\phi}$. But the gauge transformations do not commute: $[\delta_\alpha, \delta_\beta] = \delta_{\{\{\alpha,\beta\}\}}$. From this, one derives the GMP algebra satisfied by $n(x)$. This is confirmed by explicit calculation in the SM [9].

*Quadratic Lagrangian.* The only terms that contribute to the quadratic Lagrangian are $(D_0\phi)^2$ and $(D_{ij}\phi)^2$. Modulo a total derivative, the quadratic Lagrangian is that of the quantum Lifshitz model [12]

$$\mathcal{L}_2 = \frac{c_0}{2}(\partial_0\phi)^2 - \frac{c_1}{2}(\nabla^2\phi)^2, \tag{24}$$

which corresponds to a quadratic dispersion relation $\omega \sim q^2$, see also the SM [9] for the explicit expression for the coefficients $c_0$ and $c_1$. This quadratic dispersion relation is protected by the magnetic translation symmetry [8]. From Eq. (24), one easily reproduces the power-law behavior of the correlation function of the superfluid order parameter at large distances, first found in Ref. [24] (see also Refs. [35,37,54]).

*Decay width of the Tkachenko mode.* The quadratic dispersion relation of the Tkachenko mode allows a decay of one Tkachenko quantum into two quanta. To find the rate of such decay, we need to determine the interaction vertices cubic in the field $\phi$. It is easy to see that, even as cubic terms appear when one expands the "quadratic" terms $(D_0\phi)^2$ and $(D_{ij}\phi)^2$ to cubic order in $\phi$, these terms are total derivatives. The real cubic interaction appears from the following terms in the Lagrangian: $(D_0\phi)^3$, $D_0\phi$, $(D_{ij}\phi)^2$, and $\mathrm{Im}(D_{zz}\phi)^3$. Up to a total derivative, the cubic Lagrangian has the form

$$\mathcal{L}_3 = g_1(\partial_0\phi)^3 + g_2(\partial_0\phi)(\nabla^2\phi)^2 + g_3\,\mathrm{Im}(\partial_z\partial_z\phi)^3. \quad (25)$$

From this, one easily finds the energy dependence of the decay width of the Tkachenko mode. All the cubic interaction terms scale the same way in the scaling scheme with $\partial_0 \sim \partial_i^2$. In this scheme, $\phi$ is dimensionless and the $g$'s have dimension $p^{-2} \sim E^{-1}$. The decay width $\Gamma$ is proportional to $g^2$, and to have the correct dimension, $\Gamma$ should scale as $\sim g^2E^3$. This can be confirmed by writing down the decay rate of the Tkachenko mode:

$$\Gamma_{\mathbf{q}} = \frac{1}{2\epsilon_{\mathbf{q}}}\frac{1}{2}\int \frac{d^2\mathbf{p}}{(2\pi)^2 2\epsilon_{\mathbf{p}} 2\epsilon_{\mathbf{q}-\mathbf{p}}}|\mathcal{M}(\mathbf{q}\to\mathbf{p},\mathbf{q}-\mathbf{p})|^2 \times (2\pi)\delta(\epsilon_{\mathbf{q}}-\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{q}-\mathbf{p}}). \quad (26)$$

Estimating the integral with $p \sim q$, $\mathcal{M} \sim gq^6$, we get $\Gamma_{\mathbf{q}} \sim g^2q^6 \sim g^2E^3$. The presence of an anisotropic cubic vertex means that the decay rate depends on the direction of the momentum of the decaying particle.

At small but finite temperature $T$, the $U(1)$ condensate phase disappears, but the order parameter of translation symmetry breaking $\partial_i\phi$ has a logarithmic correlation at long distances [55] (see also Refs. [35,37]). Below the Berezinskii-Kosterlitz-Thouless phase transition where the lattice melts, the Tkachenko mode should still exist. The $1 \to 2$ decay rate [Eq. (26)] is modified for modes with energy much less than $T$ by the factor $(1 + f_{\mathbf{p}} + f_{\mathbf{q}-\mathbf{p}})$ where $f_{\mathbf{p}}$ and $f_{\mathbf{q}-\mathbf{p}}$ are the occupation numbers in the final state. For $E \ll T$, this factor is of order $T/E$, hence the $1 \to 2$ rate is now $g^2TE^2$ for $E \ll T$. However, the dominant contribution to the width is now a different process: the "Landau damping" process, i.e., the absorption of the soft Tkachenko quantum by a hard thermal Tkachenko photon in the medium:

$$\Gamma_{\mathbf{q}} = \frac{1}{2\epsilon_{\mathbf{q}}}\int \frac{d^2\mathbf{p}}{(2\pi)^2 2\epsilon_{\mathbf{p}} 2\epsilon_{\mathbf{q}+\mathbf{p}}}|\mathcal{M}(\mathbf{q},\mathbf{p}\to\mathbf{q}+\mathbf{p})|^2 \times (f_{\mathbf{p}} - f_{\mathbf{q}+\mathbf{p}})(2\pi)\delta(\epsilon_{\mathbf{q}}+\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{q}+\mathbf{p}}). \quad (27)$$

The width of the Tkachenko mode due to this process is $g^2(TE)^{3/2}$, which means that the Tkachenko mode is still a well-defined resonance.

The estimate above assumes that the hard Tkachenko quanta participating in the scattering process has no width and is valid only when the energy of the Tkachenko mode under consideration is larger than the width of a typical thermal mode, which is, by dimensional analysis, $g^2T^3$. Thus the estimate $\Gamma(E) \sim g^2(TE)^{3/2}$ is valid in the interval $g^2T^3 \ll E \ll T$. The regime $E \ll g^2T^3$ is the hydrodynamic regime, the analysis of which we defer to future work.

We note that our formulas for the width of the Tkachenko mode, both at zero and nonzero temperature, are in conflict with a previous result obtained from a microscopic calculation [13]. For bosons on the LLL, the authors of Ref. [13] found that at zero temperature ratio of the width to the energy of the Tkachenko mode is a constant independent of the energy (which depends only on the filling factor), and at nonzero temperature the mode is overdamped. The results are untypical for a NGB, and we cannot reconcile them with the symmetries of the system. This discrepancy needs to be investigated further.

*Conclusion.* In this Letter, we have provided a new interpretation of the Tkachenko mode in a rotating superfluid: it is a noncommutative Nambu-Goldstone boson that arises from the breaking of $U(1)$ and translation symmetries. Noncommutative field theory provides a convenient way to impose the invariance of the theory with respect to $U(1)$ and magnetic translations, and the resulting theory gives us a prediction for the decay width of the Tkachenko mode at low momentum.

*Acknowledgments.* The authors thank Clay Córdova, Austin Joyce, Umang Mehta, Nathan Seiberg, Lev Spodyneiko, and Wilhelm Zwerger for discussions and comments. S.M. is supported by Vetenskapsrådet (Grant No. 2021- 322 03685) and Nordita. The work of D.X.N. is supported, in part, by Grant No. IBS-R024-D1. This work of D.T.S. and Y.-H.D. is supported, in part, by the U.S. DOE Grant No. DE-FG02-13ER41958, a Simons Investigator Grant (DTS), and by the Simons Collaboration on Ultra-Quantum Matter, which is a Grant from the Simons Foundation (Grant No. 651440, DTS).

[1] V. K. Tkachenko, On vortex lattices, Sov. Phys. JETP **22**, 1282 (1965).

[2] V. K. Tkachenko, Stability of vortex lattices, Sov. Phys. JETP **23**, 1049 (1966).

[3] V. K. Tkachenko, Elasticity of vortex lattices, Sov. Phys. JETP **29**, 945 (1969).

[4] E. B. Sonin, Tkachenko waves, JETP Lett. **98**, 758 (2014).

[5] E. Sonin, Vortex-lattice vibrations in a rotating helium II, JETP Lett. **43**, 1027 (1976).

[6] G. E. Volovik and V. S. Dotsenko, Jr., Poisson brackets and continuous dynamics of the vortex lattice in rotating He II, JETP Lett. **29**, 576 (1979).

[7] G. Baym, Tkachenko modes of vortex lattices in rapidly rotating Bose-Einstein condensates, Phys. Rev. Lett. **91**, 110402 (2003).

[8] H. Watanabe and H. Murayama, Redundancies in Nambu-Goldstone bosons, Phys. Rev. Lett. **110**, 181601 (2013).

[9] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.6.L012040 for discussion of symmetry-breaking redundancies, explicit derivation of the Lifshitz effective theory, resolution of the nonlinear constraint, and the explicit derivation of the GMP algebra.

[10] I. Coddington, P. Engels, V. Schweikhard, and E. A. Cornell, Observation of tkachenko oscillations in rapidly rotating bose-einstein condensates, Phys. Rev. Lett. **91**, 100402 (2003).

[11] M. Ruderman, Long period oscillations in rotating neutron stars, Nature (London) **225**, 619 (1970).

[12] G. Grinstein, Anisotropic sine-Gordon model and infinite-order phase transitions in three dimensions, Phys. Rev. B **23**, 4615 (1981).

[13] S. I. Matveenko and G. V. Shlyapnikov, Tkachenko modes and their damping in the vortex lattice regime of rapidly rotating bosons, Phys. Rev. A **83**, 033604 (2011).

[14] M. R. Douglas and N. A. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. **73**, 977 (2001).

[15] V. A. Rubakov, *Classical Field Theories. Theories with Fermions. Noncommutative Theories* (URSS, 2005) (in Russian).

[16] L. Susskind, The quantum Hall fluid and noncommutative Chern-Simons theory, arXiv:hep-th/0101029.

[17] A. P. Polychronakos, Quantum Hall states as matrix Chern-Simons theory, J. High Energy Phys. 04 (2001) 011.

[18] S. Hellerman and M. Van Raamsdonk, Quantum Hall physics equals noncommutative field theory, J. High Energy Phys. 10 (2001) 039.

[19] E. Fradkin, V. Jejjala, and R. G. Leigh, Noncommutative Chern-Simons for the quantum Hall system and duality, Nucl. Phys. B **642**, 483 (2002).

[20] V. Pasquier, Quantum Hall effect and non-commutative geometry, *Quantum Spaces: Poincaré Seminar 2007*, edited by B. Duplantier (Birkhäuser Basel, Basel, 2007), pp. 1–17.

[21] Z. Dong and T. Senthil, Non-commutative field theory and composite Fermi liquids in some quantum Hall systems, Phys. Rev. B **102**, 205126 (2020).

[22] Y.-H. Du, U. Mehta, and D. T. Son, Noncommutative gauge symmetry in the fractional quantum Hall effect, arXiv:2110.13875.

[23] T.-L. Ho, Bose-Einstein condensates with large number of vortices, Phys. Rev. Lett. **87**, 060403 (2001).

[24] J. Sinova, C. B. Hanna, and A. H. MacDonald, Quantum melting and absence of Bose-Einstein condensation in two-dimensional vortex matter, Phys. Rev. Lett. **89**, 030403 (2002).

[25] E. B. Sonin, Ground state and Tkachenko modes of a rapidly rotating Bose-Einstein condensate in the lowest-Landau-level state, Phys. Rev. A **72**, 021606(R) (2005).

[26] M. Pretko, Subdimensional particle structure of higher rank $U(1)$ spin liquids, Phys. Rev. B **95**, 115139 (2017).

[27] M. Pretko and L. Radzihovsky, Fracton-elasticity duality, Phys. Rev. Lett. **120**, 195301 (2018).

[28] R. M. Nandkishore and M. Hermele, Fractons, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).

[29] A. Gromov, Towards classification of fracton phases: the multipole algebra, Phys. Rev. X **9**, 031035 (2019).

[30] J. Wang, K. Xu, and S.-T. Yau, Higher-rank tensor non-Abelian field theory: Higher-moment or subdimensional polynomial global symmetry, algebraic variety, Noether's theorem, and gauging, Phys. Rev. Res. **3**, 013185 (2021).

[31] M. Pretko, X. Chen, and Y. You, Fracton phases of matter, Int. J. Mod. Phys. A **35**, 2030003 (2020).

[32] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Res. **2**, 033124 (2020).

[33] N. Seiberg and S.-H. Shao, Exotic symmetries, duality, and fractons in 2+1-dimensional quantum field theory, SciPost Phys. **10**, 027 (2021).

[34] D. X. Nguyen, A. Gromov, and S. Moroz, Fracton-elasticity duality of two-dimensional superfluid vortex crystals: defect interactions and quantum melting, SciPost Phys. **9**, 076 (2020).

[35] E. Lake, M. Hermele, and T. Senthil, Dipolar Bose-Hubbard model, Phys. Rev. B **106**, 064511 (2022).

[36] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, Global dipole symmetry, compact Lifshitz theory, tensor gauge theory, and fractons, Phys. Rev. B **106**, 045112 (2022).

[37] A. Kapustin and L. Spodyneiko, Hohenberg-Mermin-Wagner-type theorems and dipole symmetry, Phys. Rev. B **106**, 245125 (2022).

[38] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, 2+1d compact Lifshitz theory, tensor gauge theory, and fractons, arXiv:2209.10030.

[39] L. Radzihovsky, Lifshitz gauge duality, Phys. Rev. B **106**, 224510 (2022).

[40] A. Gromov and L. Radzihovsky, Fracton matter, Rev. Mod. Phys. **96**, 011001 (2024).

[41] D. X. Nguyen and S. Moroz, On quantum melting of superfluid vortex crystals: from Lifshitz scalar to dual gravity, arXiv:2310.13741.

[42] Y.-H. Du, H. T. Lam, and L. Radzihovsky, Quantum vortex lattice via Lifshitz duality, arXiv:2310.13794.

[43] N. Cooper, Rapidly rotating atomic gases, Adv. Phys. **57**, 539 (2008).

[44] H. Leutwyler, Phonons as Goldstone bosons, Helv. Phys. Acta **70**, 275 (1997).

[45] D. E. Soper, *Classical Field Theory* (Wiley, New York, 1976).

[46] M. E. Peskin, Mandelstam-'t hooft duality in abelian lattice models, Ann. Phys. **113**, 122 (1978).

[47] C. Dasgupta and B. I. Halperin, Phase transition in a lattice model of superconductivity, Phys. Rev. Lett. **47**, 1556 (1981).

[48] The hat symbol $\hat{a}$ on the two sides of Eq. (14) have different meanings: on the left-hand side, it denotes an operator of field theory, on the right-hand side, an operator in the sense of a function of noncommutative coordinates.

[49] One way to obtain the structure [Eq. (17b)] is to start from $U^{ab} = \partial_i X^a \star \partial_i X^b$. To leading order in derivatives, $U^{ab}$ is a symmetric matrix with determinant equal to 1, since the transformation from $x^i$ to $X^a$ is area preserving, and thus has two independent components. The two components of $D_{ab}\phi$ are then related to $U^{ab}$ by a rotation by 90 degrees:

$$D_{ab}\phi = -\frac{1}{4\theta}(\epsilon^{ac}U^{cb} + \epsilon^{bc}U^{ac}).$$

.

[50] M. Greiter, F. Wilczek, and E. Witten, Hydrodynamic relations in superconductivity, Mod. Phys. Lett. B **03**, 903 (1989).

[51] S. Moroz, C. Hoyos, C. Benzoni, and D. T. Son, Effective field theory of a vortex lattice in a bosonic superfluid, SciPost Phys. **5**, 039 (2018).

[52] S. Moroz and D. T. Son, Bosonic superfluid on the lowest Landau level, Phys. Rev. Lett. **122**, 235301 (2019).

[53] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Magneto-roton theory of collective excitations in the fractional quantum Hall effect, Phys. Rev. B **33**, 2481 (1986).

[54] L. Pitaevskii and S. Stringari, Uncertainty principle and off-diagonal long-range order in the fractional quantum Hall effect, Phys. Rev. B **47**, 10915 (1993).

[55] G. Baym, Vortex lattices in rapidly rotating Bose-Einstein condensates: Modes and correlation functions, Phys. Rev. A **69**, 043618 (2004).