Letter

Non-Hermitian chiral anomalies in interacting systems

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The emergence of chiral anomaly entails various fascinating phenomena such as anomalous quantum Hall effect and chiral magnetic effect in different branches of (non-)Hermitian physics. While in the single-particle picture, anomalous currents merely appear due to the coupling of massless particles with background fields, many-body interactions can also be responsible for anomalous transport in interacting systems. In this Letter, we study anomalous chiral currents in systems where interacting massless fermions with complex Fermi velocities are coupled to complex gauge fields. Our results reveal that incorporating non-Hermiticity and many-body interactions gives rise to additional terms in anomalous relations beyond their Hermitian counterparts. We further present that many-body corrections in the subsequent non-Hermitian chiral magnetic field or anomalous Hall effect are nonvanishing in nonequilibrium or inhomogeneous systems. Our findings advance efforts in understanding anomalous transport in interacting non-Hermitian systems.

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Introduction. The chiral anomaly emerges due to the violation of classical chiral symmetry by quantum fluctuations in odd spatial dimensions. This quantum anomaly has given rise to a plethora of exotic phenomena, including anomalous decay of neutral pion in high-energy physics [1–5], anomaly-induced charges in baryons in quantum chromodynamics [6], anomalous transport in condensed matter physics [7–13], and the magnetic helicity transfer in the early universe due to chiral asymmetry in cosmology [14,15]. Aside from deepening our understanding and their experimental realizations [16–20], chiral anomalies are proposed to be used in advancing quantum computing, e.g., as chiral qubits [21,22].

While chiral anomaly in condensed matter physics usually describes noninteracting massless fermions under electromagnetic fields, generalizations of this theory allow incorporating Weyl node-mixing terms [23] and treating short-range interactions between fermions [24–26]. The latter unveils novel contributions to the anomaly equations through its nonperturbative formulation [27]. This differs from the traditional convention of perturbative treatment of interactions in high-energy physics to explore the chiral anomaly [3]. These perturbative studies reported the cancellation of higher-order corrections upon respecting Lorentz and chiral symmetries [28] and concluded the universality of chiral anomaly. As Lorentz and chiral symmetries are usually broken in interacting condensed matter systems [23,25,28], violating the chiral symmetry by many-body interactions in the absence of background fields opens new directions to explore chiral anomaly in interacting lattice models [29].

Condensed matter systems with different constituent particles, e.g., electrons and phonons, can be studied as closed or open quantum systems. While in the framework of closed systems, all degrees of freedom are treated self-consistently, the formulation of open quantum systems takes advantage of tracing out some degrees of freedom with the expense of losing unitarity. It has been shown that the path-integral formulations for open or closed systems can provide an effective non-Hermitian description for these systems [30–36]. Here, non-Hermiticity originates from the dissipative nature of open quantum systems or the imaginary parts of self-energies accounting for interactions between different subsystems in closed quantum systems.

Upon constructing non-Hermitian models, the underlying physics of open/closed systems can be unraveled using methods in non-Hermitian physics and their unique properties with no counterparts in Hermitian physics [37–41]. The emergence of defective [42–47] and nondefective [48–50] degeneracies and the occurrence of exotic boundaries modes [51] exemplify the fascinating features of non-Hermitian models. While most studies focus on exploring noninteracting systems, investigating non-Hermitian many-body physics and their dynamics have gained momentum in recent years [52–66]. Despite these studies, the rich transport properties of interacting non-Hermitian models are mainly unexplored. Addressing the anomalous chiral response of interacting Dirac fermions in non-Hermitian models is the primary goal of this work.

In this Letter, we explore the chiral anomaly in $(1+1)$ and $(3+1)$ dimensions for non-Hermitian Dirac fermions with complex Fermi velocities coupled to complex background gauge fields. By introducing a unified notation, we bring the non-Hermitian model and its symmetrized version under the same umbrella, enabling us to identify purely non-Hermitian contributions in anomalous currents. We further present the

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TABLE I. Mapping f, $\tilde{A}$, and $\tilde{\mathcal{F}}$ from the unified notation into Hermitianized and non-Hermitian notation. A stands for gauge fields V and W. $\tilde{\mathcal{F}}$ is presented for two and four dimensions. The matrix B for non-Hermitian models is given in Eq. (11). The elements of the diagonal matrix M are complex-valued Fermi velocities.

$\tilde{\mathcal{S}}$	f_μ^ν	$\tilde{A}_\mu$	$\tilde{\mathcal{F}}_2$	$\tilde{\mathcal{F}}_4$
$\mathcal{S}_{\rm h}$	$\mathrm{Re}[M_\mu^\nu]$	$\mathrm{Re}[M_\mu^\nu A_\nu]$	$4\pi\,\lvert\mathrm{Re}[v_{\rm f}]\rvert$	$32\pi^2\lvert\det[\mathrm{Re}[M]]\rvert$
$\mathcal{S}_{\rm nh}$	M_μ^ν	$M_\mu^\nu A_\nu$	$4\pi\sqrt{\det[B]}$	$32\pi^2\sqrt{\det[B]}$

physical consequences of the non-Hermitian chiral anomaly in non-Hermitian systems and discuss plausible platforms to realize them.

Non-Hermitian chiral anomaly in many-body systems. We consider a non-Hermitian model of interacting massless fermions Ψ, with complex Fermi velocities, in the presence of non-Hermitian gauge fields (V, W) in even d dimensions. To facilitate later comparison of our non-Hermitian results with previous Hermitian calculations, we employ a notation, shown by a tilde, which unifies non-Hermitian (nh) and its symmetrized form, a.k.a., Hermitionized (h), models [67].

In this notation, the partition function ($\mathcal{Z}$), the Euclidean-space action ($\mathcal{S}$), and the Dirac operator $\mathscr{D}$ for our model in the units where $c = e = \hbar = 1$ read

$$\tilde{\mathcal{Z}} \propto \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}e^{\tilde{\mathcal{S}}}, \quad \text{with } \tilde{\mathcal{S}} = \tilde{\mathcal{S}}_0 + \tilde{\mathcal{S}}_{\rm int}, \tag{1}$$

$$\tilde{\mathcal{S}}_0 = \mathrm{i}\int d^dx\overline{\Psi}\gamma^\mu\tilde{\mathscr{D}}_\mu\Psi, \tag{2}$$

$$\tilde{\mathcal{S}}_{\rm int} = \int d^dx\left(-\frac{\lambda_{\mu\nu}^2}{2}j^\mu j^\nu - \frac{\lambda_{5,\mu\nu}^2}{2}j^{5,\mu}j^{5,\nu}\right), \tag{3}$$

$$\tilde{\mathscr{D}} = \gamma^\mu\tilde{\mathscr{D}}_\mu = \gamma^\mu\tilde{d}_\mu - \mathrm{i}\,\gamma^\mu\tilde{V}_\mu - \mathrm{i}\,\gamma^\mu\gamma^5\tilde{W}_\mu, \tag{4}$$

with $\tilde{d}_\mu = f_\mu^\nu\partial_\nu$. The mapping between elements of the unified notation and their counterparts in the Hermitionized and non-Hermitian models is introduced in Table I. The gamma matrices γ^μ satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with the Euclidean metric being $g^{\mu\nu} = -\delta^{\mu\nu}$ and Greek indices run from 1 to d. The Hermitian fifth gamma matrix reads $\gamma^5 = -\prod_\mu\gamma^\mu$ and $\gamma^0 = \mathrm{i}\,\gamma^d$ used in obtaining the Dirac adjoint $\overline{\Psi} = \Psi^\dagger\gamma^0$. The Fermi velocities are elements of the rank d diagonal matrix $M = \mathrm{diag}[v_1, \ldots, v_d]$ with $v_d = 1$ and $v_{i\neq d}$ are in general complex-valued Fermi velocities [68].

The interacting action $\tilde{\mathcal{S}}_{\rm int}$ consists of short-range four-fermion interactions between currents ($j^\mu = \overline{\Psi}\gamma^\mu\Psi$) and chiral currents ($j^{5,\mu} = \overline{\Psi}\gamma^\mu\gamma^5\Psi$) with real-valued interaction strengths $\lambda_{\mu\nu}^2 = \lambda_{\mu\alpha}\lambda_\nu^\alpha$ and $\lambda_{5,\mu\nu}^2 = \lambda_{5,\mu\alpha}\lambda_{5,\nu}^\alpha$, respectively [69]. The current-current interaction with interaction strengths $\lambda_{\mu\nu}^2 = \lambda^2 g_{\mu d}g_{\nu d}$ describes a density-density interaction. Similarly, the interaction between chiral currents with $\lambda_{5,\mu\nu}^2 = \lambda_5^2 g_{\mu d}g_{\nu d}$ in $d = 4$ dimensions embed the spin-spin interaction in its spatial part [26].

The action $\tilde{\mathcal{S}}$ respects $U_A(1)\times U_V(1)$ symmetry classically [25,26], where $U_{A(V)}(1)$ denotes the chiral (vector) symmetry. However, this classical symmetry does not hold in the presence of quantum fluctuations, resulting in the emergence of the chiral anomaly. In the following, we present the covariant form of this anomaly using Fujikawa's path integral approach [70–74].

We start with introducing two auxiliary fields a and s which brings the interaction action $\tilde{\mathcal{S}}$ in Eq. (1) into a free-fermion action $\tilde{\mathcal{S}}_{a,s}$ through a Hubbard-Stratonovich transformation such that

$$\int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}e^{\tilde{\mathcal{S}}} = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\mathcal{D}a\mathcal{D}s\, e^{\tilde{\mathcal{S}}_{a,s}} \equiv \tilde{\mathcal{Z}}_{a,s}, \tag{5}$$

$$\tilde{\mathcal{S}}_{a,s} = \int d^dx\left[\mathrm{i}\,\overline{\Psi}\gamma^\mu\tilde{\mathscr{D}}_{a,s,\mu}\Psi + \frac{1}{2}a_\mu a^\mu + \frac{1}{2}s_\mu s^\mu\right], \tag{6}$$

$$\tilde{\mathscr{D}}_{a,s,\mu} = \tilde{d}_\mu - \mathrm{i}\,\tilde{V}_\mu - \mathrm{i}\,\gamma^5\tilde{W}_\mu - \mathrm{i}\,\lambda_{\mu\nu}a^\nu - \mathrm{i}\,\lambda_{5\mu\nu}\gamma^5 s^\nu. \tag{7}$$

Integration over a and s fields in the above equations reproduces $\tilde{\mathcal{S}}$ in Eq. (1). Performing an infinitesimal chiral transformation with angle β on the spinor $\Psi_{\rm rot} = \exp[-\mathrm{i}\,\gamma^5\beta(x)]\Psi$, keeps the action invariant but gives rise to an anomalous term due to a change in the Jacobian of the path integral measure such that

$$\mathcal{D}\Psi_{\rm rot}\mathcal{D}\overline{\Psi}_{\rm rot} = e^{\mathrm{i}\int d^dx\beta(x)\tilde{\mathcal{A}}_{a,s}^5}\mathcal{D}\Psi\mathcal{D}\overline{\Psi}. \tag{8}$$

To evaluate this anomalous contribution, we express Ψ and $\overline{\Psi}$ in terms of eigenbasis of the Hermitian Laplacian operators $\tilde{\mathscr{D}}_{a,s}\tilde{\mathscr{D}}_{a,s}^\dagger$ and $\tilde{\mathscr{D}}_{a,s}^\dagger\tilde{\mathscr{D}}_{a,s}$ [24,25], and employ the Heat-kernel method [73–75] to regularize the divergent sum in the exponent of the Jacobian; see Ref. [67] for further details on a similar approach. Introducing $\overline{V}_\mu = \tilde{V}_\mu + \lambda_{\mu\nu}a^\nu$ and $\overline{W}_\mu = \tilde{W}_\mu + \lambda_{5\mu\nu}s^\nu$, $\tilde{\mathcal{A}}_{a,s}^5$ in $d = 2$ dimensions and up to the first order in fields, casts

$$\tilde{\mathcal{A}}_{a,s}^5 = \frac{-\varepsilon^{\mu\nu}}{\tilde{\mathcal{F}}_2}[\mathrm{i}(\tilde{F}_{\mu\nu}[\overline{V}^\dagger] - \tilde{F}_{\mu\nu}^\dagger[\overline{V}])]. \tag{9}$$

In $d = 4$ dimensions and up to the second-order in the fields, $\tilde{\mathcal{A}}_{a,s}^5$ reads

$$\tilde{\mathcal{A}}_{a,s}^5 = \frac{\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}[\tilde{F}_{\mu\nu}[\overline{V}^\dagger]\tilde{F}_{\eta\zeta}[\overline{V}^\dagger] + \tilde{F}_{\mu\nu}^\dagger[\overline{V}]\tilde{F}_{\eta\zeta}^\dagger[\overline{V}] + \tilde{F}_{\mu\nu}^\dagger[\overline{W}]\tilde{F}_{\eta\zeta}^\dagger[\overline{W}] + \tilde{F}_{\mu\nu}[\overline{W}^\dagger]\tilde{F}_{\eta\zeta}[\overline{W}^\dagger]]. \tag{10}$$

Here, $\tilde{F}_{\mu\nu}[\overline{A}] = \tilde{d}_\mu\overline{A}_\nu - \tilde{d}_\nu\overline{A}_\mu$ and $\tilde{F}_{\mu\nu}^\dagger[\overline{A}] = \tilde{d}_\mu^\dagger\overline{A}_\nu - \tilde{d}_\nu^\dagger\overline{A}_\mu$ with $\tilde{d}_\mu^\dagger = -f_\mu^{*\nu}\partial_\nu$. $\tilde{\mathcal{F}}_d$ for Hermitionized and non-Hermitian models in $d = 2, 4$ dimensions are presented in Table I, where it is written in terms of the determinant of a matrix B given in Euclidean space, with matrix elements

$$B^{\alpha\beta} = \delta^{\mu\nu}f_\mu^{*\alpha}f_\nu^\beta - \frac{1}{2}[\gamma^\mu, \gamma^\nu]\frac{f_\mu^{*\alpha}f_\nu^\beta - f_\mu^\alpha f_\nu^{*\beta}}{2}. \tag{11}$$

Carrying out the same steps for an infinitesimal vector transformation $\Psi_{\rm rot} = \exp[-\mathrm{i}\,\kappa(x)]\Psi$ with $\mathcal{D}\Psi_{\rm rot}\mathcal{D}\bar{\Psi}_{\rm rot} = \exp[\mathrm{i}\int \mathrm{d}^dx\kappa(x)\tilde{\mathcal{A}}_{a,s}]\mathcal{D}\Psi\mathcal{D}\bar{\Psi}$ in $d = 2$ dimensions and up to the first order in the fields, results in

$$\tilde{\mathcal{A}}_{a,s} = \frac{-\varepsilon^{\mu\nu}}{\tilde{\mathcal{F}}_2}[\mathrm{i}(\tilde{F}_{\mu\nu}^\dagger[\overline{W}] - \tilde{F}_{\mu\nu}[\overline{W}^\dagger])], \tag{12}$$

and in $d = 4$ dimensions and up to the second order in fields, $\tilde{\mathcal{A}}_{a,s}$ reads

$$\tilde{\mathcal{A}}_{a,s} = \frac{-\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}[\tilde{F}_{\mu\nu}[\overline{V}^\dagger]\tilde{F}_{\eta\zeta}[\overline{W}^\dagger] + \tilde{F}_{\mu\nu}^\dagger[\overline{V}]\tilde{F}_{\eta\zeta}^\dagger[\overline{W}]]. \tag{13}$$

When $\lambda_{\mu\nu}$ and $\lambda_{5\mu\nu}$ are zero, $\tilde{\mathcal{A}}_{a,s}$ and $\tilde{\mathcal{A}}^5_{a,s}$ reproduce the results of Ref. [67].

Combining all results, the rotated generalized action in Eq. (6) under the vector and chiral transformation casts

$$\tilde{\mathcal{S}}^{\mathrm{rot}}_{a,s} - \tilde{\mathcal{S}}_{a,s} = -\int \mathrm{d}^2x\big[-\beta(x)\tilde{d}_\mu j^{5,\mu}_{a,s} - \kappa(x)\tilde{d}_\mu j^{\mu}_{a,s}\big]. \quad (14)$$

Enforcing the invariance of the partition function $\tilde{\mathcal{Z}}_{a,s}$ in Eq. (5) under the vector and chiral transformations, results in satisfying $\tilde{\mathcal{A}}^5_{a,s} = -\,\mathrm{i}\,\tilde{d}_\mu j^{5,\mu}_{a,s}$ and $\tilde{\mathcal{A}}_{a,s} = -\,\mathrm{i}\,\tilde{d}_\mu j^{\mu}_{a,s}$. To obtain the anomalous relations for the interacting model, we should shift the auxiliary fields by their on-shell values as $a_\mu \to a_\mu - \lambda_{\mu\alpha} j^\alpha$ and $s_\mu \to s_\mu - \lambda_{5\mu\alpha} j^{5,\alpha}$ and integrate over Hubbard-Stratonovich fields a and s [26]. The subsequent anomalous equation in the Euclidean space in $d = 2$ dimensions is

$$\tilde{d}_\mu j^{5,\mu} = \frac{\varepsilon^{\mu\nu}}{\tilde{\mathcal{F}}_2}\big(\tilde{F}_{\mu\nu}[\tilde{V}^\dagger] - \tilde{F}^\dagger_{\mu\nu}[\tilde{V}] - 4\,\mathrm{Re}\,[f^\eta_\mu]\partial_\eta[\lambda^2_{\nu\alpha} j^\alpha]\big), \quad (15)$$

and in $d = 4$ dimensions reads

$$\begin{aligned}\tilde{d}_\mu j^{5,\mu} = \frac{\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}\big(&\tilde{F}_{\mu\nu}[\tilde{V}^\dagger]\tilde{F}_{\eta\zeta}[\tilde{V}^\dagger] + \tilde{F}^\dagger_{\mu\nu}[\tilde{V}]\tilde{F}^\dagger_{\eta\zeta}[\tilde{V}]\\ &+ \tilde{F}^\dagger_{\mu\nu}[\tilde{W}]\tilde{F}^\dagger_{\eta\zeta}[\tilde{W}] + \tilde{F}_{\mu\nu}[\tilde{W}^\dagger]\tilde{F}_{\eta\zeta}[\tilde{W}^\dagger]\\ &- 4\tilde{d}_\eta\big[\lambda^2_{\zeta\xi} j^\xi\big]\tilde{F}_{\mu\nu}[\tilde{V}^\dagger] - 4\tilde{d}^\dagger_\eta\big[\lambda^2_{\zeta\xi} j^\xi\big]\tilde{F}^\dagger_{\mu\nu}[\tilde{V}]\\ &- 4\tilde{d}_\eta\big[\lambda^2_{5,\zeta\xi} j^{5,\xi}\big]\tilde{F}_{\mu\nu}[\tilde{W}^\dagger] - 4\tilde{d}^\dagger_\eta\big[\lambda^2_{5,\zeta\xi} j^{5,\xi}\big]\tilde{F}^\dagger_{\mu\nu}[\tilde{W}]\\ &+ 8\,\mathrm{Re}[f^\kappa_\mu f^\iota_\eta]\partial_\kappa\big[\lambda^2_{\nu\alpha} j^\alpha\big]\partial_\iota\big[\lambda^2_{\zeta\alpha} j^\alpha\big]\\ &+ 8\,\mathrm{Re}\,[f^\kappa_\mu f^\iota_\eta]\partial_\kappa\big[\lambda^2_{5,\nu\alpha} j^{5,\alpha}\big]\partial_\iota\big[\lambda^2_{5,\zeta\alpha} j^{5,\alpha}\big]\big). \end{aligned} \quad (16)$$

When the gauge field V is real, W is absent and $M = \mathbb{1}_{d\times d}$, the above $\tilde{d}_\mu j^{5,\mu}$ are in agreement with the Hermitian results [25,26]. We note that in the absence of interactions, the above relations reproduce results of Ref. [67].

In the Minkowski space, the divergence of chiral currents in Eqs. (15) and (16) read

$$\tilde{d}_\mu j^{5,\mu} = \frac{2}{\tilde{\mathcal{F}}_2}(\tilde{E}^\dagger_1 + \tilde{E}_1 - 4\,\mathrm{Re}[v_\mu]\delta^{\mu\eta}\partial_\eta[\lambda^2_{\nu\alpha} j^\alpha]) \text{ in } d = 1+1, \quad (17)$$

$$\tilde{d}_\mu j^{5,\mu} = \frac{8}{\tilde{\mathcal{F}}_4}(\overline{\boldsymbol{E}}^\dagger\cdot\overline{\boldsymbol{B}}^\dagger + \overline{\boldsymbol{E}}^{5\dagger}\cdot\overline{\boldsymbol{B}}^{5\dagger} + \overline{\boldsymbol{E}}\cdot\overline{\boldsymbol{B}} + \overline{\boldsymbol{E}}^5\cdot\overline{\boldsymbol{B}}^5) \text{ in } d = 3+1. \quad (18)$$

Here, the generalized electric and magnetic fields cast

$$\overline{E}_\mu = v^*_\mu\delta^{\mu\iota}\tilde{E}_\iota - \big(\partial_0\lambda^2_{\mu\alpha} - v^*_\mu\delta^{\mu\iota}\partial_\iota\lambda^2_{0\alpha}\big)j^\alpha, \quad (19)$$

$$\overline{E}^5_\mu = v^*_\mu\delta^{\mu\iota}\tilde{E}^5_\iota - \big(\partial_0\lambda^2_{5,\mu\alpha} - v^*_\mu\delta^{\mu\iota}\partial_\iota\lambda^2_{5,0\alpha}\big)j^{5,\alpha}, \quad (20)$$

$$\overline{B}_{\xi\zeta} = v^*_\xi v^*_\zeta\delta^{\xi\iota}\delta^{\zeta\kappa}\tilde{B}_{\iota\kappa} - \tfrac{1}{2}\big(v^*_\xi\delta^{\xi\iota}\partial_\iota\lambda^2_{\zeta\alpha} - v^*_\zeta\delta^{\zeta\kappa}\partial_\kappa\lambda^2_{\xi\alpha}\big)j^\alpha, \quad (21)$$

$$\overline{B}^5_{\xi\zeta} = v^*_\xi v^*_\zeta\delta^{\xi\iota}\delta^{\zeta\kappa}\tilde{B}^5_{\iota\kappa} - \tfrac{1}{2}\big(v^*_\xi\delta^{\xi\iota}\partial_\iota\lambda^2_{5,\zeta\alpha} - v^*_\zeta\delta^{\zeta\kappa}\partial_\kappa\lambda^2_{5,\xi\alpha}\big)j^{5,\alpha}. \quad (22)$$

The complex electric fields are also given by $\tilde{E}_j = (\exp[2\,\mathrm{i}\,\phi_j]\partial_t V_j - \partial_j V_0)$, and $\tilde{E}^5_j = (\exp[2\,\mathrm{i}\,\phi_j]\partial_t W_j - \partial_j W_0)$ with $i, j, k \neq t$. Similarly, the complex magnetic fields cast $\tilde{B}^i = \varepsilon^{ijk}\tilde{B}_{jk}$ and $\tilde{B}^{5,i} = \varepsilon^{ijk}\tilde{B}^5_{jk}$ with $\tilde{B}_{jk} = \exp[2\,\mathrm{i}\,\phi_k]\partial_j V_k - \exp[2\,\mathrm{i}\,\phi_j]\partial_k V_j$ and $\tilde{B}^5_{jk} = \exp[2\,\mathrm{i}\,\phi_k]\partial_j W_k - \exp[2\,\mathrm{i}\,\phi_j]\partial_k W_j$. The phase ϕ_j with j being a spatial index satisfies $\exp[\mathrm{i}\,\phi_j] = v_j/|v_j|$.

Our main results in Eqs. (17) and (18) retain the general forms of chiral anomaly, namely $\tilde{d}_\mu j^{5,\mu} \propto E$ in $(1+1)$ dimensions and $\tilde{d}_\mu j^{5,\mu} \propto \boldsymbol{E}.\boldsymbol{B}$ in $(3+1)$ dimensions, in Hermitian systems [25,26,70,73]. We emphasize that anomalous relations carry additional terms originating from complex Fermi velocities and complex gauge fields not present in Hermitionized anomalous equations. Note also that the terms proportional to interaction strengths can be written as total derivatives of physical quantities. The prefactors of $\tilde{\mathcal{F}}_d$ in non-Hermitian systems with complex Fermi velocities also differ from their counterparts in Hermitionized models.

The complex generalized fields in Eqs. (19)–(22) can be viewed as complex background fields screened by interactions between currents and densities of spinors. These screened fields are then responsible for breaking the chiral symmetry and giving rise to the chiral anomaly. In other words, anomalous relations in Eqs. (17) and (18) account for violating the chiral symmetry by the gauge fields (V, W) and also by the induced contributions from interactions between different constituents of our systems.

Considering $(1+1)$-dimensional systems with real Fermi velocities and $\lambda^{\mu\nu} = \lambda^2\delta^{\mu\nu}$ with λ being a constant, we rewrite Eq. (17) as

$$\tilde{d}_\mu j^{5,\mu} = \frac{1}{1 + 4\lambda^2/\tilde{\mathcal{F}}_2}\frac{2}{\tilde{\mathcal{F}}_2}(\tilde{E}^\dagger_1 + \tilde{E}_1), \quad (23)$$

where we use the relation between the chiral and vector currents $j^{5,\mu} = \epsilon^{\mu\nu} j_\nu$ to obtain the above relation. In the absence of the axial W field in $(3+1)$ dimensions, keeping $M \in \mathbb{R}$, $\lambda^{\mu\nu} = \lambda^2\delta^{\mu\nu}$ and setting $\boldsymbol{E} = E_z\hat{z}$ and $\boldsymbol{B} = B_z\hat{z}$ results in

$$\tilde{d}_\mu j^{5,\mu} = \frac{1}{1 + \frac{8\lambda^2(\tilde{B}_z + \tilde{B}^\dagger_z)}{\tilde{\mathcal{F}}_4}}\frac{8}{\tilde{\mathcal{F}}_4}(\tilde{E}^\dagger_z\tilde{B}^\dagger_z + \tilde{E}_z\tilde{B}_z), \quad (24)$$

with the relation $\varepsilon^{12\mu\nu} j_\nu = j^{5,\mu}$. We note that the current-current contributions, e.g., the last two terms in Eq. (16), do not appear in the above relation. This is due to the translational and rotational symmetry on the x-y plane, with fields along the z direction. We can interpret our results in Eqs. (23) and (24) as renormalization of $(\boldsymbol{E}, \boldsymbol{B})$ fields by interactions. These equations coincide with the Hermitian results in $(1+1)$ [26,76] and $(3+1)$ [25,26] dimensions upon imposing $(\boldsymbol{E}, \boldsymbol{B})$ to be real fields.

Physical consequences in non-Hermitian Weyl semimetals. To obtain physical phenomena stemming from the non-Hermitian chiral anomaly in interacting systems, we proceed with presenting the Chern-Simons description of our $(3+1)$ dimensional model in the absence of the axial field W and without interactions between chiral currents [77]. In this case, the change of action $\tilde{S}$ under an infinitesimal rotation β reads

$$\tilde{\mathcal{S}}_{\mathrm{rot}} - \tilde{\mathcal{S}} = \int dt d^3x \beta(t, x)[\tilde{d}_\mu j^{5,\mu} - \tilde{A}_5], \quad (25)$$

where the first term in the r.h.s. of the above relation is due to the classical shift of the action, and the anomalous second term $(\tilde{S}^5)$ accounts for the change of measure and should satisfy $\tilde{A}_5 = \tilde{d}_\mu j^{5,\mu}$ as we discussed in the previous section. The anomaly induced action after performing integration by parts and neglecting a total derivative term can be rewritten as the Chern-Simons action in the Minkowski spacetime as

$$\begin{aligned}\tilde{S}^5[\beta] = &- \int dt d^3x \frac{8\varepsilon^{\mu\nu\eta\zeta}}{\tilde{F}_4}[\tilde{d}_\eta \tilde{j}_\zeta \tilde{d}_\mu \beta \tilde{V}_\nu^\dagger + \tilde{d}_\eta^\dagger \tilde{j}_\zeta \tilde{d}_\mu^\dagger \beta \tilde{V}_\nu] \\ &+ \int dt d^3x \frac{8\varepsilon^{\mu\nu\eta\zeta}}{\tilde{F}_4} \mathrm{Re}[f_\mu^\kappa f_\eta^\iota]\partial_\kappa \beta \tilde{j}_\nu \partial_\iota \tilde{j}_\zeta \\ &+ \int dt d^3x \frac{4\varepsilon^{\mu\nu\eta\zeta}}{\tilde{F}_4} \tilde{d}_\mu \beta \tilde{V}_\nu^\dagger \tilde{d}_\eta \tilde{V}_\zeta^\dagger \\ &+ \int dt d^3x \frac{4\varepsilon^{\mu\nu\eta\zeta}}{\tilde{F}_4} \tilde{d}_\mu^\dagger \beta \tilde{V}_\nu \tilde{d}_\eta^\dagger \tilde{V}_\zeta, \end{aligned} \tag{26}$$

where $\tilde{j}_\mu = \lambda_{\mu\alpha}^2 j^\alpha$. The associated currents for the above action are then evaluated by summing the functional derivatives of $\tilde{S}^5$ with respect to V and $V^\dagger$. These currents are given by

$$\begin{aligned} M_\nu^\alpha j^\nu = &\frac{8\varepsilon^{\mu\nu\eta\zeta}\partial_\delta\beta}{\tilde{F}_4} \mathrm{Re}\left[M_\mu^\delta M_\eta^\iota M_\zeta^\rho M_\nu^{*\alpha} \partial_\iota V_\rho\right] \\ &- \frac{16\varepsilon^{\mu\nu\eta\zeta}\partial_\delta\beta}{\tilde{F}_4} \mathrm{Re}\left[M_\eta^\iota M_\mu^\delta M_\nu^{*\alpha}\right]\partial_\iota \tilde{j}_\zeta. \end{aligned} \tag{27}$$

Imposing the V field to be real, and $M = \mathbb{1}_{4\times 4}$ in Eq. (27) recovers the Hermitian chiral magnetic effect when $\delta = 0$ and the Hermitian anomalous Hall effect with δ being a spatial index [7]. We note that the interaction-induced terms, second line in Eq. (27), are merely present in nonequilibrium systems ($\partial_\iota j_\nu \neq 0$ with $\nu \neq \iota = 0$) or in inhomogeneous systems with $\partial_\iota j_0 \neq 0$ with $\iota \neq 0$. Hence, in nonequilibrium or inhomogeneous systems, Eq. (27) describes the interacting non-Hermitian chiral magnetic effect when the temporal component of β ($\delta = 0$) is nonzero and when a spatial component of β with $\delta \in \{1, 2, 3\}$ is nonvanishing, Eq. (27) results in the interacting non-Hermitian anomalous Hall effect.

$\partial_0\beta$ and $\partial_\delta\beta$ are related to the complex-valued energy and spatial separation of the Weyl nodes in non-Hermitian Weyl semimetals [67]. We emphasize that these Weyl points should be nondefective degeneracies. This is because the coalescence of eigenvectors in defective degeneracies prevents introducing a well-defined basis to express the Laplacian operators for Fujikawa's path integral method. As all nondefective degeneracies are symmetry protected [49], respecting their underlying symmetry maintains nodal intersections and offers platforms to realize non-Hermitian chiral anomalies. One approach for constructing non-Hermitian Weyl semimetals is the effective non-Hermitian description of open quantum systems. Starting from a Hermitian model for Weyl semimetals and allowing the coupling of this system with external environments, e.g., by a Boltzmann factor (see Refs. [78,79] for further details), results in an effective Hamiltonian whose imaginary part of the spectrum is nonpositive [80].

Conclusion and outlook. In conclusion, we have presented how non-Hermitian chiral anomalies for massless fermions with complex Fermi velocities coupled to complex background fields are modified in the presence of four-particle interactions. We have shown that anomalous relations cast the same form as the chiral anomaly in Hermitian (non)interacting systems. Despite the similar structure, the embedded terms in the presented non-Hermitian chiral anomaly exceed those in the Hermitianized model. Our results show that nonperturbative interaction corrections to chiral anomalies are nonvanishing in nonequilibrium or inhomogeneous systems. This can be seen in the presented non-Hermitian chiral magnetic effect and non-Hermitian anomalous Hall effect.

An experimental study on light-driven anomalous Hall effect in graphene reported that the Hall conductance is unquantized, despite the theoretical expectation of quantized conductivity in Hermitian systems [81]. Theoretical efforts to explain this observation revealed the essential roles played by out-of-equilibrium and dissipative properties of the experimental system [78,82]. As these two factors are well incorporated within our theory and measuring additional terms proportional to currents in Eqs. (15) and (16) are experimentally feasible, we expect to find signatures of our findings in light-driven Weyl semimetals with a similar experimental setup as in Ref. [81]. In addition, combining circuits to explore real-time chiral dynamics [83] with algorithms to simulate open quantum systems [84] may also pave the way to realize our findings digitally.

Finally, extending these results to understand the paritylike anomaly [85,86], the axial-torsional anomaly [87], and axial-gravitational anomaly [88,89] in non-Hermitian interacting systems is also of interest which we leave for future studies.

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[1] J. S. Bell and R. Jackiw, A PCAC puzzle: $\pi_0 \to \gamma\gamma$ in the σ-model, Nuovo Cimento A **60**, 47 (1969).
[2] S. L. Adler, Axial-vector vertex in spinor electrodynamics, Phys. Rev. **177**, 2426 (1969).
[3] S. L. Adler and W. A. Bardeen, Absence of higher-order corrections in the anomalous axial-vector divergence equation, Phys. Rev. **182**, 1517 (1969).
[4] A. Bilal, Lectures on Anomalies, arXiv:0802.0634.
[5] J. Zinn-Justin, Quantum anomalies: A few physics applications, in *From Random Walks to Random Matrices* (Oxford University Press, Oxford, 2019).
[6] M. Eto, K. Hashimoto, H. Iida, T. Ishii, and Y. Maezawa, Anomaly-induced charges in baryons, Phys. Rev. D **85**, 114038 (2012).
[7] A. A. Zyuzin, S. Wu, and A. A. Burkov, Weyl semimetal with broken time reversal and inversion symmetries, Phys. Rev. B **85**, 165110 (2012).
[8] P. Hosur and X. Qi, Recent developments in transport phenomena in Weyl semimetals, C. R. Phys. **14**, 857 (2013).
[9] K. Landsteiner, Notes on anomaly induced transport, Acta Phys. Pol. B **47**, 2617 (2016).

[10] J. Fröhlich, Chiral anomaly, topological field theory, and novel states of matter, Rev. Math. Phys. **30**, 1840007 (2018).
[11] Y. Araki, Magnetic textures and dynamics in magnetic Weyl semimetals, Annalen der Physik **532**, 1900287 (2020).
[12] R. Arouca, A. Cappelli, and T. H. Hansson, Quantum field theory anomalies in condensed matter physics, SciPost Phys. Lecture Notes 62 (2022).
[13] J. Froehlich, Gauge invariance and anomalies in condensed matter physics, J. Math. Phys. **64**, 031903 (2023).
[14] A. Boyarsky, J. Frohlich, and O. Ruchayskiy, Self-consistent evolution of magnetic fields and chiral asymmetry in the early universe, Phys. Rev. Lett. **108**, 031301 (2012).
[15] A. Brandenburg, J. Schober, I. Rogachevskii, T. Kahniashvili, A. Boyarsky, J. Froehlich, O. Ruchayskiy, and N. Kleeorin, The turbulent chiral magnetic cascade in the early universe, Astrophys. J. **845**, L21 (2017).
[16] X. Huang, L. Zhao, Y. Long, P. Wang, D. Chen, Z. Yang, H. Liang, M. Xue, H. Weng, Z. Fang, X. Dai, and G. Chen, Observation of the chiral-anomaly-induced negative magnetoresistance in 3D Weyl semimetal TaAs, Phys. Rev. X **5**, 031023 (2015).
[17] M. Hirschberger, S. Kushwaha, Z. Wang, Q. Gibson, S. Liang, C. A. Belvin, B. A. Bernevig, R. J. Cava, and N. P. Ong, The chiral anomaly and thermopower of Weyl fermions in the half-Heusler GdPtBi, Nat. Mater. **15**, 1161 (2016).
[18] C.-L. Zhang, S.-Y. Xu, I. Belopolski, Z. Yuan, Z. Lin, B. Tong, G. Bian, N. Alidoust, C.-C. Lee, S.-M. Huang *et al.*, Signatures of the Adler–Bell–Jackiw chiral anomaly in a Weyl fermion semimetal, Nat. Commun. **7**, 10735 (2016).
[19] Q. Li, D. E. Kharzeev, C. Zhang, Y. Huang, I. Pletikosić, A. V. Fedorov, R. D. Zhong, J. A. Schneeloch, G. D. Gu, and T. Valla, Chiral magnetic effect in $ZrTe_5$, Nat. Phys. **12**, 550 (2016).
[20] N. P. Ong and S. Liang, Experimental signatures of the chiral anomaly in Dirac–Weyl semimetals, Nat. Rev. Phys. **3**, 394 (2021).
[21] D. E. Kharzeev and Q. Li, The Chiral Qubit: quantum computing with chiral anomaly, arXiv:1903.07133.
[22] E. Babaev, D. Kharzeev, M. Larsson, A. Molochkov, and V. Zhaunerchyk, in *Chiral Matter* (World Scientific, Singapore, 2023), p. 276.
[23] Z. M. Raines and V. M. Galitski, Enriched axial anomaly in Weyl materials, Phys. Rev. B **96**, 161115(R) (2017).
[24] H. Kikuchi, Chiral phase dependence of the massive fermionic path integral, Phys. Rev. D **46**, 4704 (1992).
[25] C. Rylands, A. Parhizkar, A. A. Burkov, and V. Galitski, Chiral anomaly in interacting condensed matter systems, Phys. Rev. Lett. **126**, 185303 (2021).
[26] A. Parhizkar, C. Rylands, and V. Galitski, On the path integral approach to quantum anomalies in interacting models, arXiv:2302.14191.
[27] We note that, in $1+1$ dimensions, the massless interacting fermions can be described by the theory of Luttinger liquids. The chiral charge conservation equation for these particles based on the Luttinger liquids, in the absence of any electromagnetic field, has a part that depends on the density-density interaction strength. This term agrees with the correction to the chiral anomaly equation in interacting systems.
[28] A. Giuliani, V. Mastropietro, and M. Porta, Anomaly non-renormalization in interacting Weyl semimetals, Commun. Math. Phys. **384**, 997 (2021).
[29] We note that all reported additional terms in Refs. [24–26] originated from many-body corrections in the anomaly equation can be written as total derivatives of physical quantities.
[30] A. Kamenev, Many-body theory of non-equilibrium systems, arXiv:cond-mat/0412296.
[31] R. van Leeuwen, N. E. Dahlen, G. Stefanucci, C.-O. Almbladh, and U. von Barth, Introduction to the keldysh formalism and applications to time-dependent density-functional theory, arXiv:cond-mat/0506130.
[32] L. M. Sieberer, M. Buchhold, and S. Diehl, Keldysh field theory for driven open quantum systems, Rep. Prog. Phys. **79**, 096001 (2016).
[33] Y. Michishita and R. Peters, Equivalence of effective non-Hermitian Hamiltonians in the context of open quantum systems and strongly correlated electron systems, Phys. Rev. Lett. **124**, 196401 (2020).
[34] H.-G. Zirnstein, G. Refael, and B. Rosenow, Bulk-boundary correspondence for non-Hermitian Hamiltonians via green functions, Phys. Rev. Lett. **126**, 216407 (2021).
[35] A. Gomez-Leon, T. Ramos, A. Gonzalez-Tudela, and D. Porras, Bridging the gap between topological non-Hermitian physics and open quantum systems, Phys. Rev. A **106**, L011501 (2022).
[36] Y. L. Gal, X. Turkeshi, and M. Schirò, Volume-to-area law entanglement transition in a non-Hermitian free fermionic chain, SciPost Phys. **14**, 138 (2023).
[37] A. A. Zyuzin and P. Simon, Disorder-induced exceptional points and nodal lines in Dirac superconductors, Phys. Rev. B **99**, 165145 (2019).
[38] Y. Ashida, Z. Gong, and M. Ueda, Non-Hermitian physics, Adv. Phys. **69**, 249 (2020).
[39] H. Wang, X. Zhang, J. Hua, D. Lei, M. Lu, and Y. Chen, Topological physics of non-Hermitian optics and photonics: A review, J. Opt. **23**, 123001 (2021).
[40] E. J. Bergholtz, J. C. Budich, and F. K. Kunst, Exceptional topology of non-Hermitian systems, Rev. Mod. Phys. **93**, 015005 (2021).
[41] N. Okuma and M. Sato, Non-Hermitian topological phenomena: A review, Annu. Rev. Condens. Matter Phys. **14**, 83 (2023).
[42] T. Yoshida, R. Peters, and N. Kawakami, Non-Hermitian perspective of the band structure in heavy-fermion systems, Phys. Rev. B **98**, 035141 (2018).
[43] T. Yoshida, R. Peters, N. Kawakami, and Y. Hatsugai, Symmetry-protected exceptional rings in two-dimensional correlated systems with chiral symmetry, Phys. Rev. B **99**, 121101(R) (2019).
[44] S. Sayyad, J. Yu, A. G. Grushin, and L. M. Sieberer, Entanglement spectrum crossings reveal non-Hermitian dynamical topology, Phys. Rev. Res. **3**, 033022 (2021).
[45] P. Delplace, T. Yoshida, and Y. Hatsugai, Symmetry-protected multifold exceptional points and their topological characterization, Phys. Rev. Lett. **127**, 186602 (2021).
[46] K. Ding, C. Fang, and G. Ma, Non-Hermitian topology and exceptional-point geometries, Nat. Rev. Phys., 1 (2022).
[47] S. Sayyad and F. K. Kunst, Realizing exceptional points of any order in the presence of symmetry, Phys. Rev. Res. **4**, 023130 (2022).
[48] X. Zhang, K. Xu, C. Liu, X. Song, B. Hou, R. Yu, H. Zhang, D. Li, and J. Li, Gauge-dependent topology in non-reciprocal hopping systems with pseudo-hermitian symmetry, Commun. Phys. **4**, 166 (2021).

[49] S. Sayyad, Protection of all nondefective twofold degeneracies by antiunitary symmetries in non-Hermitian systems, Phys. Rev. Res. **4**, 043213 (2022).
[50] S. Sayyad, M. Stalhammar, L. Rodland, and F. K. Kunst, Symmetry-protected exceptional and nodal points in non-Hermitian systems, SciPost Phys. **15**, 200 (2023).
[51] X. Zhang, T. Zhang, M.-H. Lu, and Y.-F. Chen, A review on non-Hermitian skin effect, Adv. Phys.: X **7**, 2109431 (2022).
[52] T. Fukui and N. Kawakami, Breakdown of the Mott insulator: Exact solution of an asymmetric Hubbard model, Phys. Rev. B **58**, 16051 (1998).
[53] T. Yoshida, K. Kudo, and Y. Hatsugai, Non-Hermitian fractional quantum Hall states, Sci. Rep. **9** (2019).
[54] B. Buča, C. Booker, M. Medenjak, and D. Jaksch, Bethe ansatz approach for dissipation: Exact solutions of quantum many-body dynamics under loss, New J. Phys. **22**, 123040 (2020).
[55] T. Yoshida, K. Kudo, H. Katsura, and Y. Hatsugai, Fate of fractional quantum Hall states in open quantum systems: Characterization of correlated topological states for the full Liouvillian, Phys. Rev. Res. **2**, 033428 (2020).
[56] X. Z. Zhang and Z. Song, η-pairing ground states in the non-Hermitian hubbard model, Phys. Rev. B **103**, 235153 (2021).
[57] M. Nakagawa, N. Kawakami, and M. Ueda, Exact Liouvillian spectrum of a one-dimensional dissipative Hubbard model, Phys. Rev. Lett. **126**, 110404 (2021).
[58] T. Hyart and J. L. Lado, Non-Hermitian many-body topological excitations in interacting quantum dots, Phys. Rev. Res. **4**, L012006 (2022).
[59] R. Shen and C. H. Lee, Non-Hermitian skin clusters from strong interactions, Commun. Phys. **5**, 238 (2022).
[60] H. Yoshida and H. Katsura, Liouvillian gap and single spin-flip dynamics in the dissipative Fermi-Hubbard model, Phys. Rev. A **107**, 033332 (2023).
[61] S. Sayyad and J. L. Lado, Topological phase diagrams of exactly solvable non-Hermitian interacting Kitaev chains, Phys. Rev. Res. **5**, L022046 (2023).
[62] J. Mak, M. J. Bhaseen, and A. Pal, Statics and dynamics of non-Hermitian many-body localization, arXiv:2301.01763.
[63] R. Shen, T. Chen, F. Qin, Y. Zhong, and C. H. Lee, Proposal for observing Yang-Lee criticality in Rydberg atomic arrays, Phys. Rev. Lett. **131**, 080403 (2023).
[64] S. Han, D. J. Schultz, and Y. B. Kim, Complex fixed points of the non-Hermitian Kondo model in a Luttinger liquid, Phys. Rev. B **107**, 235153 (2023).
[65] T. Yoshida and Y. Hatsugai, Fate of exceptional points under interactions: Reduction of topological classifications, Phys. Rev. B **107**, 075118 (2023).
[66] S. Sayyad and J. L. Lado, Transfer learning from Hermitian to non-Hermitian quantum many-body physics, J. Phys.: Condens. Matter **36**, 185603 (2024).
[67] S. Sayyad, J. D. Hannukainen, and A. G. Grushin, Non-Hermitian chiral anomalies, Phys. Rev. Res. **4**, L042004 (2022).
[68] We note that the presence of symmetries imposes constraints on the structure of the M matrix. To keep our findings as generic as possible, we did not enforce any restriction on M.
[69] We emphasize that merely real-valued interaction strengths can be handled in this work as the key Hubbard-Stratonovich transformation breaks down for complex coefficients.
[70] K. Fujikawa, Path-integral measure for gauge-invariant fermion theories, Phys. Rev. Lett. **42**, 1195 (1979).
[71] K. Fujikawa, Path integral for gauge theories with fermions, Phys. Rev. D **21**, 2848 (1980).
[72] K. Fujikawa, Energy-momentum tensor in quantum field theory, Phys. Rev. D **23**, 2262 (1981).
[73] R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Oxford University Press, New York, 1996).
[74] K. Fujikawa and H. Suzuki, *Path Integrals and Quantum Anomalies* (Clarendon Press, Oxford, 2004).
[75] M. Nakahara, *Geometry, Topology and Physics*, Graduate student series in physics (Hilger, Bristol, 1990).
[76] S.-S. Shei, Anomaly of the axial-vector current in one space and one time dimension, Phys. Rev. D **6**, 3469 (1972).
[77] A similar analysis for noninteracting systems is presented in Ref. [7,90] for Hermitian and in Ref. [67] for non-Hermitian models.
[78] M. Nuske, L. Broers, B. Schulte, G. Jotzu, S. A. Sato, A. Cavalleri, A. Rubio, J. W. McIver, and L. Mathey, Floquet dynamics in light-driven solids, Phys. Rev. Res. **2**, 043408 (2020).
[79] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.6.L012028 includes details on calculating an effective non-Hermitian Weyl Hamiltonian.
[80] Similar conclusion can be drawn when the non-Hermitian part of the effective Hamiltonian is obtained from the imaginary part of the zero energy self-energy, known as the quasiparticle lifetime. Here, again, the causality imposes that the imaginary part of the spectrum is nonpositive [33,42].
[81] J. W. McIver, B. Schulte, F-U Stein, T. Matsuyama, G. Jotzu, G. Meier, and A. Cavalleri, Light-induced anomalous Hall effect in graphene, Nat. Phys. **16**, 38 (2020).
[82] S. A. Sato, J. W. McIver, M. Nuske, P. Tang, G. Jotzu, B. Schulte, H. Hübener, U. De Giovannini, L. Mathey, M. A. Sentef, A. Cavalleri, and A. Rubio, Microscopic theory for the light-induced anomalous Hall effect in graphene, Phys. Rev. B **99**, 214302 (2019).
[83] D. E. Kharzeev and Y. Kikuchi, Real-time chiral dynamics from a digital quantum simulation, Phys. Rev. Res. **2**, 023342 (2020).
[84] H. Kamakari, S.-N. Sun, M. Motta, and A. J. Minnich, Digital quantum simulation of open quantum systems using quantum imaginary–time evolution, PRX Quantum **3**, 010320 (2022).
[85] A. A. Burkov, Mirror anomaly in Dirac semimetals, Phys. Rev. Lett. **120**, 016603 (2018).
[86] T. Matsushita, S. Fujimoto, and A. P. Schnyder, Topological piezoelectric effect and parity anomaly in nodal line semimetals, Phys. Rev. Res. **2**, 043311 (2020).
[87] Y. Ferreiros, Y. Kedem, E. J. Bergholtz, and J. H. Bardarson, Mixed axial-torsional anomaly in Weyl semimetals, Phys. Rev. Lett. **122**, 056601 (2019).
[88] J. Gooth, A. Niemann, T. Meng, A. Grushin, K. Landsteiner, B. Gotsmann, F. Menges, M. Schmidt, C. Shekhar, V. Süß, R. Hühne, B. Rellinghaus, C. Felser, B. Yan, and K. Nielsch, Experimental signatures of the mixed axial-gravitational anomaly in the Weyl semimetal NbP, Nature (London) **547**, 324 (2017).
[89] M. N. Chernodub, Y. Ferreiros, A. G. Grushin, K. Landsteiner, and M. A. H. Vozmediano, Thermal transport, geometry, and anomalies, Phys. Rep. **977**, 1 (2022).
[90] A. A. Zyuzin and A. A. Burkov, Topological response in Weyl semimetals and the chiral anomaly, Phys. Rev. B **86**, 115133 (2012).