Letter

Recursive quantum eigenvalue and singular-value transformation: Analytic construction of matrix sign function by Newton iteration

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Quantum eigenvalue transformation (QET) and its generalization, quantum singular value transformation (QSVT), are versatile quantum algorithms that allow us to apply broad matrix functions to quantum states, which cover many significant quantum algorithms such as Hamiltonian simulation. However, finding a parameter set which realizes preferable matrix functions in these techniques is difficult for large-scale quantum systems: there is no analytical result other than trivial cases as far as we know and we often suffer also from numerical instability. In this Letter, we propose recursive QET or QSVT (R-QET or R-QSVT) in which we can execute complicated matrix functions by recursively organizing block-encoding by low-degree QET or QSVT. Owing to the simplicity of recursive relations, it works only with a few parameters with exactly determining the parameters, while its iteration results in complicated matrix functions. In particular, by exploiting the recursive relation of Newton iteration, we construct the matrix sign function, which can be applied for eigenstate filtering for example, in a tractable way. We show that an analytically obtained parameter set composed of only eight different values is sufficient for executing QET of the matrix sign function with an arbitrarily small error ε. Our protocol will serve as an alternative protocol for constructing QET or QSVT for some useful matrix functions without numerical instability.

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Introduction. Recently, quantum eigenvalue transformation (QET) has become a versatile technique for various quantum algorithms [1]. For a hermitian matrix of interest A, QET executes parallel processing of its eigenvalues and thereby allows us to apply broad matrix polynomial functions $\sum_n c_n A^n$ to arbitrary quantum states. With its generalization to general matrices, called quantum singular value transformation (QSVT), it covers various todays important quantum algorithms, such as Hamiltonian simulation [2,3] and search algorithms [4], by properly constructing polynomial approximations. Not only does it provide unified understanding of quantum algorithms [5], but also it can serve more efficient alternative algorithms for various purposes.

QET is executed by repeating parametrized unitary gates on ancilla qubits and unitary gates embedding the target matrix, called block-encoding. While tunability of the parameters ensures realization of broad functions by quantum signal processing (QSP) [2], we must accurately determine a proper parameter set for a desired function. Although finding the parameters for degree-q polynomials within an error ε can be executed by poly$(q, \log(1/\varepsilon))$-time classical computation, its numerical instability has become one of the central problems for accurate implementation of QET/QSVT. In fact, several numerical algorithms such as optimization try to solve this instability [6–9]. By contrast, there are only a few results on analytical parameter determination. As far as we know, they are limited to trivial cases for Chebyshev polynomials useful for Grover's search [4,10,11].

In this Letter, we propose recursive QET/QSVT (R-QET/R-QSVT) that can potentially determine all the parameters in an analytical or numerically much cheaper way. We recursively organize block-encoding by low-degree QET/QSVT so that it can reproduce recursive relations of matrix functions, and then obtain complicated matrix functions by iteration. For instance, we can exploit Newton iteration as the recursive relation [12]. Then, with a sufficient number of iterations for its convergence, R-QET/R-QSVT organizes nontrivial matrix functions only with a smaller number of parameters that can be easily determined. As a prominent consequence, we obtain an analytical parameter set for realizing matrix sign functions with arbitrarily small error. Furthermore, the parameter set has constant unique values which do not depend on either an allowable error ε or any parameter of the matrix. Although our construction expenses the computational cost compared to the optimal protocol [13,14] due to the strong limitation on the parameters, it suffers from no numerical instability and even

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can overwhelm the optimal one when we perform recovery of coherent errors [15]. With various recursive constructions of matrix functions, such as Newton iteration [12] and logistic map [16,17], R-QET/R-QSVT will give a promising candidate for executing complicated operations on quantum computers in accurate and stable ways.

Quantum Eigenvalue Transformation (QET). Throughout the main text, we concentrate on QET and thus, R-QET for hermitian matrices for simplicity (See Sec. S2 of the Supplemental Material for QSVT, whose discussion is completely parallel). We begin with briefly introducing QET here. Let a hermitian matrix A have spectral decomposition $A = \sum_a a |a\rangle \langle a|$ $(a \in \mathbb{R})$ on a finite-dimensional Hilbert space $\mathcal{H}$. Block-encoding of A is defined by a unitary gate O_A, satisfying

$$\langle 0|O_A|0\rangle_b = \frac{A}{\alpha}, \quad \alpha > 0. \tag{1}$$

Here, $|0\rangle_b$ denotes a reference state in an ancillary Hilbert space $\mathcal{H}_b$. We set $\alpha = 1$ by the renormalization $A \to A/\alpha$ below. Construction of block-encoding is known for a linear combination of unitaries, a sparse-access matrix, and so on [3].

Combining parametrized unitary operations on the ancilla system,

$$R_\phi = e^{i\phi(2|0\rangle\langle 0|_b - I_b)} \otimes I, \tag{2}$$

we define a degree-q QET operator by

$$\mathrm{QET}[O_A, \vec{\phi}] = \begin{cases} R_{\phi_1} O_A \prod_{i=1}^{(q-1)/2} [R_{\phi_{2i}} O_A^\dagger R_{\phi_{2i+1}} O_A], & q\text{: odd}, \\ \prod_{i=1}^{q/2} [R_{\phi_{2i-1}} O_A^\dagger R_{\phi_{2i}} O_A], & q\text{: even}. \end{cases} \tag{3}$$

By properly tuning the parameter set $\vec{\phi} \in \mathbb{R}^q$, it realizes various polynomial functions $f(x) = \sum_n c_n x^n$ $(c_n \in \mathbb{C})$ as

$$\langle 0|\mathrm{QET}[O_A, \vec{\phi}]|0\rangle_b = f(A) = \sum_n c_n A^n. \tag{4}$$

There exists a parameter set $\vec{\phi}$ if and only if $f(x)$ satisfies all the following conditions [1]:

(1) $f(x)$ has a degree at most q and a parity $(-1)^q$;

(2) $|f(x)| \leqslant 1$ for any $x \in [-1, 1]$ and $|f(x)| \geqslant 1$ for any $x \in (-\infty, -1] \cup [1, \infty)$;

(3) (If q is even) $|f(ix)f^*(ix)| \geqslant 1$ for any $x \in \mathbb{R}$, where $f^*(x)$ is defined by $f^*(x) = \sum_n c_n^* x^n$.

By $\mathcal{O}(1)$ controlled operations $\mathrm{QET}[O_A, \vec{\phi}]$, generic renormalized matrix functions $f(x)$ with $|f(x)| \leqslant 1/4$ $({}^\forall x \in [-1, 1])$ are also realizable.

For a desired function $f(x)$ satisfying (i)–(iii), how can we find a proper parameter set $\vec{\phi}$? As far as we know, the analytical result is restricted to the Chebyshev polynomials $f(x) = T_n(x)$ with the trivial angles $\vec{\phi} = ((q-1)\pi/2, -\pi/2, -\pi/2, \dots, -\pi/2)$, which is useful for Grover's search and its family, such as amplitude amplification [4]. In general, it requires finding all the roots of $1 - f(x)f^*(x)$, which is a degree-q polynomial in x^2, and iteratively decomposing $\mathrm{QET}[O_A, \vec{\phi}]$ into lower-degree QET operators [1]. However, for useful functions such as e^{-itA} (Hamiltonian simulation [2,3]), $A^{-1}/2\kappa$ (quantum linear system problem, QLSP [18]), and $\mathrm{sign}(A)$ (eigenstate filtering [14]), the typical degree q is quite large, as $q \in \mathrm{poly}(N, \log(1/\varepsilon))$ depending on the system size N and the allowable error ε. Thus, it is difficult to accurately compute $\vec{\phi}$ generally having $\mathrm{poly}(N, \log(1/\varepsilon))$ different values in a numerically stable way, although it can be done by $\mathrm{poly}(N, \log(1/\varepsilon))$-time classical calculation. This instability has been partially resolved by refining root-finding problems and decomposition into lower degrees [6,7], or employing optimization [8,9]. They have numerically succeeded up to $q \sim 10^4$ within $10^2 \sim 10^4$ seconds.

Recursive QET (R-QET) and Newton iteration. Here, we propose the protocol named recursive QET (R-QET), and provide the formulation combined with Newton iteration for matrix functions [12]. It aims to implement complicated matrix functions by QET with keeping tractability of the parameter set $\vec{\phi}$ in a stable way. Our strategy is to employ recursive relations: while each step operation is simple, its repetition forms rather complicated functions.

Suppose that we want to execute complicated matrix functions of A with its block-encoding O_A. Based on the fact that QET generates block-encoding from block-encoding as Eq. (4), we organize R-QET by recursively defining a series of block-encodings $\{O_{X_n}\}_n$ by

$$O_{X_{n+1}} = \mathrm{QET}[O_{X_n}, \vec{\phi}^g], \quad n = 0, 1, 2, \dots. \tag{5}$$

Here, the initial input O_{X_0} depends on O_A (e.g., $O_{X_0} = O_A$), and we have options for the parameter set $\vec{\phi}_g$ including its values and dimension. The above construction forms a recursive relation of matrices $X_n = \langle 0|O_{X_n}|0\rangle_b$,

$$X_{n+1} = g(X_n), \quad X_0 = \langle 0|O_{X_0}|0\rangle_b, \tag{6}$$

with a variety of polynomial functions g. Recursive relations of matrices like Eq. (6) are available for complicated matrix functions exemplified by a matrix logistic map [16,17].

One of the most promising candidates for the recursive relation is Newton iteration, which was originally invented for solving nonlinear equations [12]. Newton iteration for matrices enables us to efficiently compute various matrix functions $f(A)$ with properly choosing the function g as a result of iteration $\lim_{N\to\infty} X_n = f(A)$. For instance, $g(X) = (3X - X^3)/2$ and $X_0 = A$ generates the matrix sign function $X_n \to \mathrm{sign}(A)$ (defined later), and the one by $g(X) = 2X - XAX$ and $X_0 = \theta A$ $(0 < \theta \ll 1)$ generates the matrix inversion $X_n \to A^{-1}$ [12]. When R-QET combined with Newton iteration, we organize the parameter set $\vec{\phi}_g$ so that the polynomial g reproduces Newton iteration. The iteration continues until it achieves an allowable error ε as $\|X_n - f(A)\| \leqslant \varepsilon$ ($\|\cdot\|$ denotes the operator norm), and thus the iteration number n depends on ε. Then, the resulting unitary gate O_{X_n} provides an accurate block-encoding for the function $f(A)$.

By iterative substitution of the recursive relation Eq. (5), the block-encoding O_{X_n} is rewritten by

$$O_{X_n} = \mathrm{QET}[\mathrm{QET}[\dots[\mathrm{QET}[O_{X_0}, \vec{\phi}^g], \vec{\phi}^g], \dots, \vec{\phi}^g], \tag{7}$$

which has the form of $O_{X_n} = \mathrm{QET}[O_{X_0}, \vec{\phi}_n]$. The $(\deg(g^n))$-dimensional parameter set $\vec{\phi}_n$ is determined solely by the $(\deg(g))$-dimensional one $\vec{\phi}^g$. Since the degree of g is a constant independent of any parameter, i.e., the input X_0, the allowable error ε, and the iteration number n, we can obtain

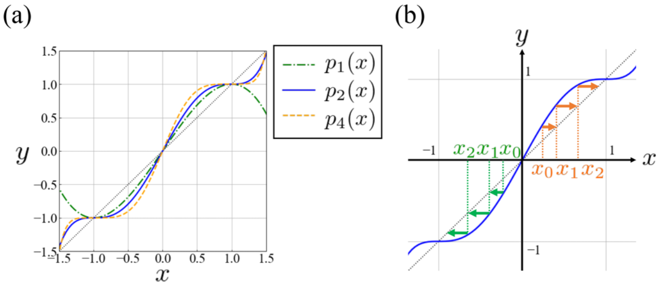

FIG. 1. (a) Some members of Padé family. (b) Intuitive picture of Newton iteration based on $p_2(x)$. Each eigenvalue of X_n, represented by x_n, approaches 1 or -1 depending on its sign.

a parameter set valid for any situation in an analytical or numerically stable way.

While we use the standard QET for implementing the function g here, we note that other types of QET/QSVT are also available. For instance, if the recursive relation $g(X)$ includes some matrices other than X (e.g., $g(X) = 2X - XAX$ for matrix inversion), we need at-least multi-variate QET/QSVT [19,20].

Analytical R-QET for Matrix Sign Function. In R-QET, we should carefully choose the recursive relation so that the tractable function g can be realized by QET, and the resulting function X_n is meaningful. Here, we show the power of R-QET combined with Newton iteration by analytically constructing the parameter set of QET for matrix sign functions.

Suppose that a hermitian matrix $A = \sum_a a\,|a\rangle\langle a|$ has a spectral gap 2Δ (>0) around zero as $a \in [-1, -\Delta] \cup [\Delta, 1]$. The matrix sign function is defined by

$$\mathrm{sign}(A) = \sum_a \mathrm{sign}(a)\,|a\rangle\langle a|, \quad \mathrm{sign}(a) = \frac{a}{|a|}, \tag{8}$$

which is useful for various tasks such as eigenstate filtering [21]. The matrix sign function sign(A) can be generated by Newton-Shultz iteration using a series of rational functions called Padé family $\{p_l(x)\}_{l\in\mathbb{N}}$ with the initial input $X_0 = A$ [12,22]. As the recursive relation $g(X)$, we adopt the second simplest case,

$$p_2(X) = \tfrac{1}{8}(15X - 10X^3 + 3X^5), \quad \deg(p_2) = 5 \tag{9}$$

since the simplest one $p_1(X) = (3X - X^3)/2$ violates condition (ii) for QET [See Fig. 1(a)]. The convergence to sign(A) is confirmed by the fact that every eigenvalue of X_n, denoted by x_n, moves to $+1$ or -1 based on the recursive relation $x_{n+1} = p_2(x_n)$ [See Fig. 1(b)].

We determine the parameter set $\vec{\phi}^{p_2} \in \mathbb{R}^5$, which constructs the relation

$$O_{X_{n+1}} = \mathrm{QET}[O_{X_n}, \vec{\phi}^{p_2}], \quad O_{X_0} = O_A. \tag{10}$$

To find $\vec{\phi}^q$, we compute all the roots of the quintic polynomial $1 - p_2(x)p_2^*(x)$. Although there is no formula for quintic equations, we find them by the factorization,

$$1 - p_2(x)p_2^*(x) = -\tfrac{9}{64}(1 - x^2)^3(x^2 - s)(x^2 - s^*), \tag{11}$$

with $s = (11 + 3\sqrt{15}i)/6$. This leads to the following analytical parameter set $\vec{\phi}^{p_2}$ (See Sec. S1 of the Supplemental Material for its detailed calculation [23]):

$$\phi_1^{p_2} = 0, \quad \phi_2^{p_2} = \pi + \frac{1}{2}\arctan\frac{\sqrt{15}}{7},$$
$$\phi_3^{p_2} = \pi + \frac{1}{2}\arctan\sqrt{15}, \quad \phi_4^{p_2} = -\frac{1}{2}\arctan\sqrt{15},$$
$$\phi_5^{p_2} = -\frac{1}{2}\arctan\frac{\sqrt{15}}{7}. \tag{12}$$

Therefore, by repeated substitution, Eq. (7), we obtain an analytical QET operator QET$[O_A, \vec{\phi}_n]$ for the matrix sign function sign(A). The parameter set $\vec{\phi}_n$ has only eight different angles $\pm\phi_i^{p_2}$ for $i = 2, 3, 4, 5$, and they always appear in the fixed orders of $\phi_2^{p_2} \to \phi_3^{p_2} \to \phi_4^{p_2} \to \phi_5^{p_2}$ or $-\phi_5^{p_2} \to -\phi_4^{p_2} \to -\phi_3^{p_2} \to -\phi_2^{p_2}$.

Cost and comparison with standard QET. Let us evaluate the cost for the matrix sign function. We repeat the recursive relation until the desirable error $\varepsilon \in [0, 1]$ is achieved as $\|X_n - \mathrm{sign}(A)\| \leqslant \varepsilon$. Convergence to sign($A$) based on the Newton iteration, Eq. (9), is dominated by the gap Δ as follows [22]:

$$\|X_n - \mathrm{sign}(A)\| \leqslant (1 - \Delta^2)^{3^n}. \tag{13}$$

By using the relation $\log(1 - \Delta^2)^{-1} \geqslant \Delta^2$ for $\Delta \in [0, 1)$, it is sufficient to choose the iteration number by

$$n = \left\lceil \log_3\left(\frac{1}{\Delta^2}\log(1/\varepsilon)\right)\right\rceil. \tag{14}$$

The cost is measured by the query complexity, i.e., the number of O_A in the unitary gate O_{X_n} giving $X_n = \mathrm{sign}(A) + \mathcal{O}(\varepsilon)$. The query complexity q_n is immediately obtained by Eq. (10), which results in $q_{n+1} = 5q_n$ and $q_0 = 1$. Under the proper iteration number n by Eq. (14), the query complexity yields

$$q_n = 5^n \leqslant 5 \times \left(\frac{1}{\Delta^2}\log(1/\varepsilon)\right)^{\log_3 5} \tag{15}$$
$$\in \Theta\left(\frac{1}{\Delta^{2\log_3 5}}\log^{\log_3 5}(1/\varepsilon)\right). \tag{16}$$

Let us compare with the standard QET approach [1,5,13], which uses the optimal approximation by the shifted Chebyshev polynomials. This yields the query complexity $\Theta(\Delta^{-1}\log(1/\varepsilon))$, which is optimal both in Δ and ε. Considering $\log_3 5 \simeq 1.465$, the query complexity of R-QET, Eq. (16), is polynomially larger than the optimal one. This difference comes from flexibility of the parameter set $\vec{\phi}$. The standard QET approach uses $\mathcal{O}(\Delta^{-1}\log(1/\varepsilon))$ different parameters $\vec{\phi}$ obtained by numerically solving an $\mathcal{O}(\Delta^{-1}\log(1/\varepsilon))$-degree equation. In contrast, our approach employs only eight different values in a fixed order independent of Δ and ε. At some expense of cost, R-QET can serve sign(A) up to arbitrarily small error ε, with completely avoiding numerical instability and working only with a few kinds of gates.

While R-QET fails to achieve the optimal scaling, it can overwhelm the optimal protocol when correcting coherent error is considered. In QET, a multiplicative coherent error on the parameter set $\phi_i \to \phi_i(1 + \delta)$ $(i = 1, 2, \ldots, q)$ is a possible obstacle to accurate implementation. Recently, Tan *et al.* [15] have shown that the coherent error δ on degree-q QET can be suppressed up to $\mathcal{O}(\delta^{k+1})$ $(k \in \mathbb{N})$ by additional query complexity,

$$q_{\text{correct}} \in \mathcal{O}(2^k (c_{\vec{\phi}})^{k^2} q). \tag{17}$$

Here, $c_{\vec{\phi}}$ denotes the number of different values in $\vec{\phi} \in \mathbb{R}^q$. The standard QET approach achieving optimality is expected to have $c_{\vec{\phi}} \in \mathcal{O}(q)$ different values in $\vec{\phi}$, and its total query complexity amounts to $\mathcal{O}(\Delta^{-k^2-1} \log^{k^2+1}(1/\varepsilon))$. By contrast, R-QET employs constant values with $c_{\vec{\phi}} = 8$ regardless of any parameters. The total query complexity including the recovering remains, Eq. (16), as long as $k \in \mathcal{O}(1)$, which becomes smaller than that of the originally optimal protocol for $k \geqslant 2$.

Errors in each recursive relation caused by deviation of parameters may bring instability to convergence of Newton iterations. However, in contrast to the standard QET suffering from numerical errors in parameter determination and large cost for recovering implementation errors, R-QET can analytically provide parameters and efficiently deals with coherent errors. Such errors are much suppressed, and hence R-QET has a strong advantage in stability.

Generalization of R-QET for matrix sign functions. Our result for matrix sign functions can be generalized to other members of the Padé family or implementation of polar decomposition by R-QSVT.

In the first case, the recursive relation $X_{n+1} = p_l(X_n)$ with $X_0 = A$ $(l \in \mathbb{N})$ by the Padé family,

$$p_l(x) = x \sum_{k=0}^{l} \frac{(2k-1)!!}{2^k k!} (1-x^2)^k, \tag{18}$$

casts matrix sign functions as $\lim_{n\to\infty} X_n = \text{sign}(A)$. Conditions (i)–(iii) for QET can be satisfied only when l is even [24]. It has only $2l$ different values in the parameter set for the degree-$(2l+1)$ polynomial $p_l(X)$, where one of them can be zero. As well as $l = 2$, we can analytically determine the parameter set for $l = 4$ by the quartic formula. Even for larger $l \geqslant 6$, the numerical instability for parameter determination is much more suppressed than the standard QET. The advantage of this generalization is the computational cost. Convergence to sign(A) becomes faster as $\|X_n - \text{sign}(A)\| \leqslant (1-\Delta^2)^{(l+1)^n}$ [22], and hence the query complexity becomes smaller as

$$q_n \in \Theta\left(\frac{1}{\Delta^{2(1+\nu_l)}} \log^{1+\nu_l}(1/\varepsilon)\right), \tag{19}$$

$$\nu_l = \frac{\log(2l+1)}{\log(l+1)} - 1. \tag{20}$$

The scaling becomes $\Theta(\Delta^{-(2+o(1))} \log^{1+o(1)}(1/\varepsilon))$ for large l [24]. While it is still not optimal in the gap Δ, it can reach the optimal one in the desirable error ε.

Next, we consider generalization to R-QSVT. QSVT produces block-encoding executing polynomial transformation of every singular value from block-encoding of a generic matrix A, and hence R-QSVT can be composed by recursive iteration of QSVT (See Sec. S2 of the Supplemental Material for its detail [23]). When we use the same parameter set $\vec{\phi}^{p_2}$ for the matrix sign function, R-QSVT organizes block-encoding $\{O_{X_n}\}_n$, reproducing

$$X_{n+1} = \frac{X_n}{8}\{15 - 10X_n^\dagger X_n + 3(X_n^\dagger X_n)^2\}, \quad X_0 = A. \tag{21}$$

When A is nonsingular, X_n converges to $A(\sqrt{A^\dagger A})^{-1}$, which is the unitary part of polar decomposition. With the same iteration number, Eq. (14), R-QSVT achieves an arbitrarily small error ε and analyticity (or numerical stability) of the parameter set for the polar decomposition.

Discussion and Conclusion. In this Letter, we propose recursive QET/QSVT that executes recursive relations by QET/QSVT. All the parameters can be determined by low-degree polynomials for recursive relations, which enables analytical or numerically stable calculation. Particularly, the construction of matrix sign functions when combined with Newton iteration is the first analytical result on parameters for useful functions other than the trivial Chebyshev polynomials. Indeed, with the analytically obtained parameters Eq. (12), we can execute eigenstate filtering, and thereby solve quantum linear system problems [21] (See Sec. S3 of the Supplemental Material for its detailed discussion [23]).

Although R-QET/R-QSVT have polynomially large query complexity compared to the optimal standard QET/QSVT, they are expected to be useful in the following scenarios. In the early fault-tolerant quantum computation (early-FTQC) era, where the ability to correct errors is restricted, suppressing errors in the algorithmic level will be of importance. Then the availability of the coherent error protocol will make R-QET/R-QSVT advantageous compared to the standard QET/QSVT, as discussed in Eq. (17). In the FTQC era, our interest will move to huge quantum systems, which requires high-degree QET/QSVT. Inaccurate parameters make the standard QET/QSVT meaningless even if we can apply arbitrarily accurate gates, but R-QET/R-QSVT avoids such a problem due to its numerical stability. Therefore, R-QET/R-QSVT will be a significant option which ensures stability.

We conclude this Letter with a future direction. While we focus on the matrix sign function with using the standard QET for the recursive relation, these will be generalized. For instance, Newton iteration covers various matrix functions like matrix inversion with ensuring quadratic or faster convergence. Another interesting example is the matrix logistic map [16,17], which generates either chaotic or nonchaotic behaviors of each eigenvalue. As subroutines for reproducing recursive relations, we can also exploit a series of QET/QSVT protocols. For instance, we can realize QET/QSVT of generic renormalized functions with fixed parity with an extra qubit [1]. While we avoid the odd order Padé family, it can also be exploited for R-QET to realize matrix sign functions with some additional cost (See Sec. S4 of the Supplemental Material for its detailed discussion [23]). A series of recently proposed QET/QSVT for Fourier series [25–27] and multivariate polynomials [19,20] will also be promising candidates. These broad options of R-QET/R-QSVT will provide a variety of matrix functions for various quantum tasks while efficiently and accurately providing the required parameters.

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[23] See Supplemental Materials at http://link.aps.org/supplemental/10.1103/PhysRevResearch.6.L012007. It includes the detailed way of computing the parameters in QET/QSVT (S1) and the detailed construction of the R-QSVT (S2). We also discuss some potential applications of the marix sign function by R-QET (S3) and some other recursive relations available in R-QET/R-QSVT (S4).
[24] Exactly speaking, we were not able to find proof of the latter part of the condition (ii) for generic even l, $|p_l(x)| \geqslant 1$ for any $x \notin [-1, 1]$, although it seems to be correct from their graphs. By computing the derivatives of $p_l(x)$ at $x = \pm 1$, we have proven that $p_l(x)$ is compatible with the standard QET at least up to $l = 10^3$. While we believe that the exponent ν_l [See Eq. (19)] can be arbitrarily small under $l \to \infty$, it is ensured to reach $\nu_{10^3} \simeq 0.100$ at the present stage.
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