Universality classes in out-of-equilibrium systems: An encompassing theorem for a one-dimensional fusing particles model

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This paper supports the idea that some out-of-equilibrium systems can be described by universality classes. A specific out-of-equilibrium fusing particles model is studied in detail, resulting in a method for determining the number and mass of the final particles based solely on the initial conditions, eliminating the need to evolve the particle system. This method reveals the basis for a universality class encompassing theorem, which is developed to define other models within the same universality class. This result establishes an infinite number of models with the same behavior and scaling, two of which are described in detail.

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I. INTRODUCTION

The significance of nonequilibrium phenomena in physics is profound, as they capture the intrinsic dynamic nature of complex systems beyond their states of thermodynamic equilibrium. While it can be argued that nearly every observable macroscopic event occurs under nonequilibrium conditions, a comprehensive framework for understanding such systems remains elusive. This challenge arises from the diverse array of nonequilibrium phenomena observed in nature. Examples include biological processes [1], chemical systems [2], turbulent flows [3], quantum transport in novel materials [4], vehicular movement on road networks [5,6], competitive dynamics for resources among populations [7], and plasma instabilities [8], among others. Most notably, these phenomena manifest across scales, ranging from the microscopic [9] to the cosmological [10]. In seeking a unifying framework, it was proposed that universality classes for out-of-equilibrium systems might exist [11]. Furthermore, experimental evidence has been presented [12,13] to support this idea.

The concept of universality classes was initially proposed for equilibrium critical phenomena. In equilibrium systems, second-order phase transitions exhibit universality. Diverse setups and models exhibit the same critical exponents and asymptotic behavior near phase transitions, indicating they belong to the same class. This implies that, although systems may be very different in terms of their microscopic interactions, geometry, dimensions, etc., they will exhibit universal properties and behavior near critical points or phase transitions. For thermodynamic systems of an infinite number of particles, the theory of renormalization groups has proven to be a valuable tool for characterizing different mathematical models that produce phase transitions. More importantly, it has been found to be a key factor in determining if any two models belong to the same universality class. However, for finite systems such as the one studied in this paper, no definitive results are yet known. Moreover, there is no method for either finite or infinite systems that sheds light on how to define a new model that belongs to the same universality class of a given model. This paper intends to makes progress in this direction.

As a starting point, a fairly simple out-of-equilibrium finite particle system is studied. This system corresponds to a gas of N identical point particles in an open one-dimensional space undergoing perfectly plastic collisions [14]. This elegant model has been the subject of rigorous studies, and a thorough characterization has been achieved. For this model, an encompassing theorem is presented to define other models in the same universality class. Based on this theorem's result, an infinite collection of mathematical models belonging to the same universality class is introduced, and two of these models are described.

II. A 1D PHYSICAL FUSING PARTICLE MODEL

At time zero, there are N identical point particles of mass m in a one-dimensional space at different arbitrary positions. Let the farthest left particle be considered particle 1, the second be particle 2, and so on, with particle N being the rightmost one, i.e., their initial positions verify $Y_1 < Y_2 < \cdots < Y_N$ respectively. In this model, velocity will take a protagonist role. The initial velocities of each particle $V_1, V_2, \ldots, V_N$ are considered as a sequence of iid random variables with an absolute continuous distribution function $F(x) := \mathbb{P}(V_1 \leqslant x)$. Each particle evolves at constant velocity, $Y_i(t) = V_i t + Y_i$, until it eventually collides with another particle. At this point, a perfectly plastic collision is generated, resulting in a single particle with a mass that is equal to the sum of the individual particles' masses. The velocity in which this particle moves is determined by the conservation of momentum, which dictates

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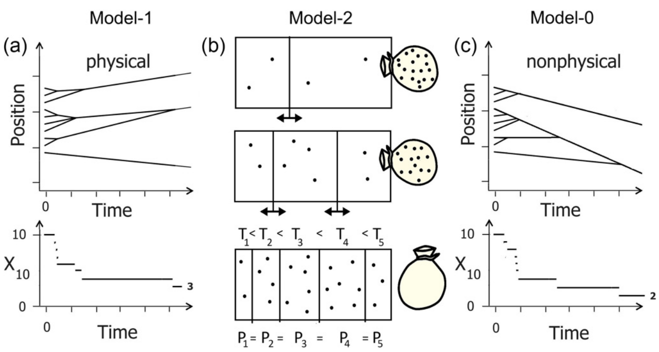

FIG. 1. Example of the evolution of a system of $N = 10$ particles suffering: (a) perfectly plastic collisions and (c) artificial collisions. (b) Example of a model of $N = 22$ noninteracting particles escaping into a larger container. Model 2 illustrates the use and positioning of movable walls to create a temperature gradient. Recipients are at the same pressure (P).

that the velocity of the fused particle will be the weighted average particles' velocities prior to collision. The merged particles evolve equally to the nonmerged ones, i.e., they move at constant velocity until a new plastic collision eventually happens.

In this paper, the asymptotic properties of the stochastic process $X_N(t) =$ number of particles at time t, which starts with $X_N(0) = N$ particles, are studied. $X_N(t)$ converges to a random variable $\tilde{X}_N$ that naturally depends on N,

$$X_N(t) \underset{t\to\infty}{\rightarrow} \tilde{X}_N.$$

On the left panel of Fig. 1, a realization of a system of $N = 10$ identical particles undergoing perfectly plastic collisions is shown, identified as Model 1. The evolution of the number of particles, $X_{10}(t)$, is shown below. For this random realization, at the time when the last collision occurs, henceforth referred to as τ, only three particles remain in the system. In other words, for this realization, $\tilde{X}_{10} = 3$ [or equivalent $X_{10}(t) = 3$ for $t \geqslant \tau$]. When looking at the mass of these three resulting particles and comparing them to the mass of the initial ones, one of them will be three times the original value; another, six times the original value; and the last one, the one which has not suffered any collision, will have the same mass of the original particles. Specifically, the final masses are represented as $\mathbb{M} = (1, 6, 3)$. This notation assigns the first coordinate of the vector to the particle in the farthest left position, the second to the next position, and so forth.

A. Literature review

The model was first studied numerically in [14], and later, in [15], interesting analytical results were obtained. Three significant results, listed below, have been proven. The first result shows that changing the initial positions of each particle, while maintaining their order, does not alter the number of final particles or their masses.

Theorem 1 (Sibuya et al. *1990 [15]).* $\tilde{X}_N$ and $\mathbb{M}$ do not depend on the initial positions.

These position changes can only affect the sequence of collisions and, trivially, the timing of the last collision (τ). However, they do not affect the final configuration ($\tilde{X}_N$ and $\mathbb{M}$). The second result presents the distribution of $\tilde{X}_N$.

Theorem 2 (Sibuya et al. *1990 [15]).* The probability mass function of $\tilde{X}_N$ is given by

$$\mathbb{P}(\tilde{X}_N = k) = \frac{|c(N, k)|}{N!}, \tag{1}$$

where $c(n, k)$ is the Stirling number of the first kind given by the equality $a(a-1)\cdots(a-N+1) = \sum_{k=0}^{N} c(N, k)a^k$.

In addition, the mean and variance of $\tilde{X}_N$ is given by

$$\langle \tilde{X}_N \rangle = \sum_{k=1}^{N} \frac{1}{k} \overset{N\gg 1}{\approx} \ln(N) + \gamma, \tag{2}$$

$$\langle \tilde{X}_N^2 \rangle - \langle \tilde{X}_N \rangle^2 = \sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{N} \frac{1}{k^2} \overset{N\gg 1}{\approx} \ln(N) + \gamma - \frac{\pi^2}{6}, \tag{3}$$

where γ is the Euler-Mascheroni constant. As Theorem 2 states, $\tilde{X}_N$ is universal, also known as distribution-free; i.e., Eq. (1) [and the ones derived from it, Eqs. (2) and (3)] is valid for any continuous initial velocity distribution (F). In essence, the specific distribution of initial particle positions and velocities does not matter. This behavior is expected in systems approaching a critical phase transition [16,17], or in self-organized criticality systems [18–23]. In criticality, scale-free behavior is also expected for some variables. As explained in Theorem 3, the model exhibits power-law behavior, and a characterization of the final mass distribution is presented. In particular, the expected number of final particles with mass s, denoted as n_s, is studied, where

$$n_s := \sum_{i=1}^{\tilde{X}_N} \delta_{\mathbb{M}[i],s}, \tag{4}$$

$\delta_{i,j}$ is the Kronecker delta, and $\mathbb{M}[i]$ is the mass of final particle i.

Theorem 3 (Sibuya et al. *1990 [15]).* For $s = \{1, 2, \ldots, N\}$, the expected number of final particles with mass s is given by

$$\langle n_s \rangle = \frac{1}{s}. \tag{5}$$

The result is particularly interesting, showing the first indication of criticality with a power-law decay of exponent 1. Equally interestingly, this expectation does not depend on N.

As demonstrated in the previous paragraph, both the number of final particles [Eq. (1)] and the expected number of final particles with mass s [Eq. (5)] are found to be universal. However, some nonuniversal properties of this model have also been studied. For instance, both the size and velocity of the rightmost final particle were investigated in [24]. Another example of nonuniversal behavior was examined in [25], where the velocity distribution of the final particles was analyzed.

B. New results

Further evidence for the critical behavior of this out-of-equilibrium system, as well as scaling behaviors, is shown in this section. Additionally, a formula for the calculation of $\tilde{X}_N$ and $\mathbb{M}$ is provided. This formula will be crucial for Sec. III.

1. The last collision time distribution

The last collision time (τ) distribution depends on the initial conditions of the particles. To study this distribution, two factors must be considered. First, their initial positions: in this section, a grid of length N is used, where particles are placed at $Y_k = k$. Second, their initial velocities: as mentioned before, these are considered i.i.d. continuous random variables with cumulative distribution F. In this section different F distributions are studied: Normal(0,1), Normal(0,10), Normal(10,1), Uniform(-1,1), Uniform(0,1), Uniform(-10,10), and Beta(0.5,0.5).

As the last collision time is a random variable that depends on F and N, it will be referred to as $\tau_{F,N}$ from this point forward. In Fig. 2(a) the $\tau_{F,N}$ distribution is examined by plotting $\mathbb{P}(\tau_{F,N} > t) =: h_{F,N}(t)$ as a function of t for each of the different F distributions, and each of the four values of $N = \{10, 30, 100, 300\}$. Note that all the curves exhibit a similar behavior. In scale-invariant systems, such as those observed in criticality, it is well known that the different behaviors of $h_{F,N}(t)$ can be collapsed into a single curve, $h(t)$, with the appropriate scaling. Let σ_F be the variance of F distribution, and let

$$W := \tau_{F,N}\sigma_F N^{-\alpha}$$

be a rescaled version of $\tau_{F,N}$. The new variable has length units. Figure 2(b) shows the distribution of W for the curves shown in Fig. 2(a), where the different F distributions and N are studied. The value of α is constant and will be equal to 1.65. When the scaling given by W is applied, as seen in Fig. 2(b), all the curves will collapse into a single curve. The distribution of W remains the same for any F with finite variance. For example, for a power-law distribution with a power exponent $\beta > 2$ in the probability density function, expressed as $\sim 1/(x^{1+\beta})$. Moreover, this scaling goes as $1/t$,[1] $\mathbb{P}(W > t) \sim 1/t$. This phenomenon provides additional evidence to support the hypothesis that the system is in a critical regime.

Interestingly, if the initial position configuration is changed, the curves will also be collapsed under the appropriate scaling, as far as the study shows. An example of this behavior is having N particles be placed in a confined region, such as the one given by random positions from a Uniform(-1,1) distribution. The same will also be true even with random positions from a Normal(0,1) distribution. Regardless of the specifics, as long as the initial position configuration differs from that of the grid, the only change reflected in the scaling is in α, now adopting the value 0.65, exactly one less than that of the grid. Unfortunately, an explanation as to why the exponent changes has not yet been determined. These results are shown in Appendix A.

2. Expected number of particles evolution

This section introduces the time evolution of a specific quantity of the fusing particles model: the expected number of particles, $\langle X_N(t)\rangle$. It begins at time zero with N particles,

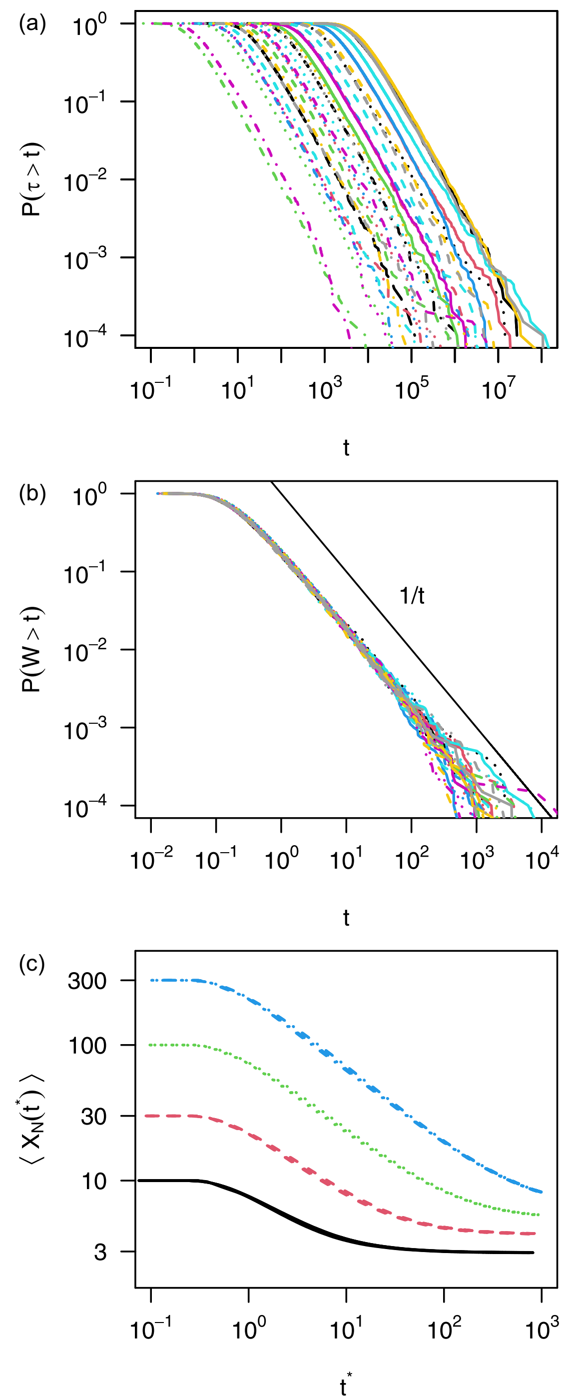

FIG. 2. (a) Last collision time distribution, $\mathbb{P}(\tau_{F,N} > t)$ and (b) $\tau_{F,N}$ rescaled version $\mathbb{P}(\tau_{F,N}\sigma_F N^{-\alpha} > t)$ as a function of t. (c) Evolution of the mean number of particles $\langle X_N(t^\star)\rangle$ as a function of a rescaled time $t^\star$. For F: Normal(0,1) ($\sigma_F = 1$), Normal(0,10) ($\sigma_F = 10$), Normal(10,1) ($\sigma_F = 1$), Uniform(-1,1) ($\sigma_F = \sqrt{1/3}$), Uniform(-10,10) ($\sigma_F = 10\sqrt{1/3}$), Uniform(0,1) ($\sigma_F = 1/\sqrt{12}$), and Beta(0.5,0.5) ($\sigma_F = \sqrt{1/8}$), and considering $N = \{10, 30, 100, 300\}$.

and it ends at $\sum_{k=1}^{N} \frac{1}{k}$ [see Eq. (2)]. Although the specific evolution of this quantity has not yet been explored, as with any nonuniversal variable, it is safe to assume it will depend on N and F. With this in mind, this section will attempt to

[1] A fit yields a power exponent $\delta = 0.98 \pm 0.04$ for $\mathbb{P}(W > t) \sim 1/t^{\delta}$.

explore the evolution of this value in detail. In order to do so, the same initial velocity distributions (F) and values of N previously studied will be considered. Additionally, the initial position of the particles will be given by the grid introduced in the previous section. Different curves are obtained for each F and N condition studied (data not shown), similarly to the observations for $\tau_{F,N}$ in Fig. 2(a). However, when examining the expected value of the rescaled variable, $X_N(t^\star) := X_N(t\sigma_F)$, the curves collapse into a single unified curve. Figure 2(c) shows the results of $\langle X_N(t^\star)\rangle$ as a function of the rescaled time $t^\star$ for each F and N combination. Note that for each N, $\langle X_N(t^\star)\rangle$ exhibits consistent behavior across all F distributions studied.

3. A calculus for $\tilde{X}_N$ and $\mathbb{M}$ that does not require time evolution

For any given initial condition, the way to compute the number of final particles ($\tilde{X}_N$) and their masses has been to evolve the particles system numerically and study the results. However, is there a way to calculate these values without evolving the system? The main purpose of this section is to show that it is indeed possible, and to present a proposal for calculating both $\tilde{X}_N$ and the mass of each final particle, with no need for time evolution.

Before presenting the main result, two points must be understood. First, note that in a system characterized by perfectly plastic collisions, any fused particle (regardless of the order in which the collisions occurred) will have a velocity equal to the average of the velocity of the particles that formed it. Specifically, when two particles with masses m_1 and m_2 and velocities V_1 and V_2 respectively collide, the resulting fused particle of mass $m_1 + m_2$ has a velocity equal to $V_1 p_1 + V_2(1 - p_1)$ with $p_1 = m_1/(m_1 + m_2)$. Calculating the velocity of a system of particles of equal masses, such as the one in this case, is surprising simple: the final velocity of any fused particle, $\tilde{V}_f$, formed by N_f identical particles, such as particles $k, k+1, \ldots, k+N_f-1$, ends up being the average velocity of the fused particles,

$$\tilde{V}_f = \overline{V}_k^{k+N_f-1}, \tag{6}$$

where $\overline{V}_i^j := \frac{1}{j-i}\sum_{k\in\mathbb{N}_i^j} V_k$ with $\mathbb{N}_i^j := \{k \in \mathbb{N} : i \leqslant k \leqslant j\}$.

Before formally presenting the main result of this section (Theorem 4), let's first look at a simple explanation. In order to do this, the time evolution of the system shown in Fig. 1(a) will be taken as reference. As seen in this example, after a time τ, the number of final particles remains the same. These final particles, from left to right, have increasing[2] velocities. This increasing behavior is always found in any configuration, as additional collisions would occur otherwise. Therefore, if the system culminates with more than one final particle ($\tilde{X}_N > 1$), then the condition

$$\overline{V}_1^{j_1} < \overline{V}_{j_1+1}^{j_2} < \overline{V}_{j_2+1}^{j_3} < \cdots < \overline{V}_{j_{\tilde{X}_N-1}+1}^{N}$$

must be satisfied. A key step in obtaining the final configuration is calculating the values $j_1, j_2, \ldots, j_{\tilde{X}_N}$ and $\tilde{X}_N$ which (as seen below) are coupled.

The procedure for determining the final number of particles ($\tilde{X}_N$) and the masses of each final ($j_k - j_{k-1} - 1$) particle is as follows:

(1) Identify a particle that meets a certain condition.

(2) Combine the identified particle with lower index particles.

(3) From the remaining particles, find another one that satisfies the given condition.

(4) Combine this new particle with the remaining lower index ones.

(5) Repeat this iterative process until particle N satisfies the condition.

The specific condition that the target particle, denoted by particle k, must satisfy is for the resulting merged particle to have a lower velocity than any "merged" particle containing particle $k+1$.

At this point, some definitions, as well as the main result, are introduced. Consider the vector v_i representing the initial velocities of N particles, expressed as $v_i = (V_1, V_2, \ldots, V_N)$, and let $w_i = (V_1, V_2, \ldots, V_N, 2\max\{V_1, V_2, \ldots, V_N\})$ be the velocity vector augmented by an additional coordinate. This coordinate has a value greater than the maximum velocity of the N particles. The purpose of introducing w is only to simplify the notation. Let v and w be a general vector obtained by potentially excluding some of the first consecutive coordinates, such as $v = (V_4, V_5, \ldots, V_N)$. The set of all real vectors v with lengths ranging from 1 to N will be referred to as $\Lambda := \cup_{k\in\mathbb{N}_1^N}\mathbb{R}^k$. Let s be an arbitrary vector, $s[j]$ correspond to element j, and $s[j:k]$ correspond to the vector starting at coordinate j and ending at element k of the sequence.

Definition 1. Let $\tilde{m} : \Lambda \to \mathbb{N}_1^N$ be the function

$$\tilde{m}(w) = \min\left\{j \in \mathbb{N}_1^{L(w)-1} : \overline{w}_1^j < \overline{w}_{j+1}^i \quad \forall i \in \mathbb{N}_{j+1}^{L(w)}\right\},$$

where $\overline{w}_i^j = \frac{1}{j-i}\sum_{k=i}^j w[k]$.

Definition 2. Let G be a function that verifies

$$G(w) = \begin{cases} w[(\tilde{m}(w)+1) : L(w)] & \text{if } \tilde{m}(w) < L(w) - 1 \\ \emptyset & \text{if } \tilde{m}(w) = L(w) - 1 \end{cases}$$

and $G(\emptyset) = \emptyset$.

Definition 3. Let G^k be the k-times composition of the function G, with G^0 the identity function. For example, $G^3(w) = G(G(G(w)))$.

Theorem 4. It is possible to calculate the number of final particles ($\tilde{X}_N$) and their individual masses ($\mathbb{M}$) without evolving the particle system (in time). Moreover, for a given augmented initial velocity vector w_i,

$$\tilde{X}_N(w_i) = \min\left\{k \in \mathbb{N}_1^N : G^k(w_i) = \emptyset\right\}, \tag{7}$$

$$\mathbb{M}(w_i) = (\tilde{m}(G^0(w_i)), \tilde{m}(G^1(w_i)), \ldots, \tilde{m}(G^{\tilde{X}_N(w_i)-1}(w_i))). \tag{8}$$

See Appendix B for a proof. Based on the previous theorem, a simple algorithm for computing $\tilde{X}_N$ and $\mathbb{M}$ can be developed. It is worth noting that this is not the first paper to present a result for $\tilde{X}_N$ and $\mathbb{M}$ without the need to evolve the system. In [15], the authors show that these values can

[2] Although the condition for particles not merging requires that the final velocities form a nondecreasing sequence, we will specifically consider an increasing sequence and disregard the case of equal velocities, as the probability of this occurring is zero with continuous random variables.

be obtained geometrically. The convex hull of the function representing the cumulative momentum as a function of the cumulative mass contains the information of both $\tilde{X}_N$ and $\mathbb{M}$. Theorem 4 is the analytical procedure of this geometric approach, which, as seen in the next section, provides the basis for defining other models within the same universality class.

Finally, an example is presented.

Example 1. Let $v_i = (5.4, 2.1, 9.5, 1.3, 4.5, 3.7, 8.1)$, and therefore $w_i = (v_i, 19)$ then

$$\begin{aligned} &\tilde{m}(G^0(w_i)) = 2, && G^1(w_i) = (9.5, 1.3, 4.5, 3.7, 8.1, 19),\\ &\tilde{m}(G^1(w_i)) = 4, && G^2(w_i) = (8.1, 19),\\ &\tilde{m}(G^2(w_i)) = 1, && G^3(w_i) = \emptyset, \end{aligned}$$

$\tilde{X}_7(w_i) = 3$, and $\mathbb{M}(w_i) = (2, 4, 1)$. For completeness, the velocities of the final particles are computed. The velocity of the leftmost merged particle is $(5.4 + 2.1)/2 = 3.75$, and the velocities of the remaining final particles (from left to right) are 4.75, and 8.1. Note the increasing velocity behavior of the final particles.

III. UNIVERSALITY CLASS OF THE FUSING PARTICLE MODEL AND ITS POSSIBLE ENCOMPASSING THEOREM

It has been shown that the out-of-equilibrium fusing particle model behaves as a system in the critical regime. This naturally leads to question its universality class and, in particular, which other models can be constructed within this class. As far as it is known, the latter question has not been answered for either equilibrium nor nonequilibrium systems. A proposal for the fusing particles model is presented here.

The universality class of the model under consideration will be referred to as the universality class of partition models. In this fusing particles model, the final number of particles and their masses are determined by a partition of their initial velocity sequence. Given the initial velocities, there is only one partition that reproduces the physics of the final state. Theorem 4 shows how to construct this partition. What makes this model special is that the partition verifies two conditions: the final number of particles, $\tilde{X}_N$, has a probability mass function given by Eq. (1); and the expected number of clusters of size s is given by Eq. (5). If the function $\tilde{m}$ is modified, a new partitioning or clustering model will be generated using the same theorem. However, not all functions $\tilde{m}$ will ensure the previously mentioned both conditions. Consequently, the formal question becomes: given a sequence of i.i.d. random variables, which $\tilde{m}$ functions satisfy these two conditions when Eqs. (7) and (8) are applied?

Enhancing the definition of the function $\tilde{m}$ is the first step in answering this question, in order to the limit the alternatives. A new function g, explained below, will be used for this purpose.

Definition 4. Let $\tilde{m} : \Lambda \to \mathbb{N}_1^N$ be the function

$$\begin{aligned} \tilde{m}(w) = \min \big\{ j \in \mathbb{N}_1^{L(w)-1} : g(w[1:j]) \\ < g(w[j+1:i]) \forall i \in \mathbb{N}_{j+1}^{L(w)} \big\}, \end{aligned} \quad (9)$$

with $g : \bigcup_{k \in \mathbb{N}_1^N} \mathbb{R}^k \to \mathbb{R}$ a function.

Now that $\tilde{m}$ has been specified, the next step is finding which g functions generate models within the same universality class. This is answered in Theorem 5, which is the result of combining two theorems found in two older papers ([26] and [15]).

Theorem 5. A function g that verifies that, for $1 \leqslant a < b \leqslant N$,

$$\begin{aligned} \min\{g(w[1:a]), g(w[a+1:b])\} \leqslant g(w[1:b]) \\ \leqslant \max\{g(w[1:a]), g(w[a+1:b])\} \quad \text{and} \\ \times\, g(w[1:a]) \neq g(w[1:b]) \end{aligned} \quad (10)$$

also satisfies Eqs. (1) and (5) when $\tilde{X}_N$ and $\mathbb{M}$ are given by Eqs. (7) and (8), considering $\tilde{m}$ given by (9) and $w_i = (V_1, V_2, \ldots, V_N, 2\max\{V_1, V_2, \ldots, V_N\})$ with $V_1, V_2, \ldots, V_N$ being i.i.d. continuous random variables with an arbitrary distribution F.

See Appendix C for a proof.

Examples of g functions that verify Eq. (10) include the following:

$$\text{Model 0:} \quad g(w[i:j]) = w[j], \quad (11)$$

$$\text{Model 1:} \quad g(w[i:j]) = \frac{1}{j-i}\sum_{h=i}^{j} w[h], \quad (12)$$

$$\text{Model 2:} \quad g(w[i:j]) = \frac{1}{j-i}\sum_{h=i}^{j} (w[h])^2. \quad (13)$$

However, these examples are not exhaustive; an infinite collection of models can be generated. Specifically, Model k can be defined as one that has $g(w[i:j]) = \frac{1}{j-i}\sum_{h=i}^{j}(w[h])^k$ for $k \geqslant 1$. More generally, $g(w[i:j]) = \frac{1}{j-i}\sum_{h=i}^{j} f(w[h])$ with f being any real function (with domain in Λ) that is not constant on any interval.

Model 1 corresponds to the 1D physical fusing particles model studied in the previous section. Model 2 clusters particles according to the empirical second moment of the distribution. To understand this, consider the following example: suppose there are N noninteracting point particles in a recipient at a given temperature, and a small hole is made so that the particles escape one by one into a larger container. At this point, a theoretical question arises: assuming that the velocities of each of the particles at the moment they enter the larger container are known at time zero, is it possible to use walls to create a temperature gradient? In other words, if movable walls (or partitions) can be used to "trap" the particles that have already entered, when and where should they be positioned for a gradient of increasing temperatures to be created? This model is portrayed in Fig. 1(b). The number of particle clusters (number of walls plus one) generated using the criteria given by Eq. (12) (which will also be called $\tilde{X}_N$ in this section) has a probability law given by Eq. (1), proven by Theorem 5. In addition, the mass vector $\mathbb{M}$, which contains the information of the number of particles trapped by the walls in this particular model, verifies that the expected number of clusters of trapped particles of size s follows Eq. (5). An example can be seen in Fig. 1(b), where $\mathbb{M} = (2, 4, 6, 7, 3)$. The observable difference between Models 1 and 2 lies in their mass vectors; each criterion yields a different probability law for $\mathbb{M}$, always satisfying Eq. (5).

There is another interpretation for Model 2, and a general interpretation for Model k ($k \geqslant 1$) that relates to the fusing particles model. Assuming that $w[h] \geqslant 0$ (positive velocities), the segmentation produced by $g(w[i:j]) = \frac{1}{j-i}\sum_{h=i}^{j}(w[h])^k$ is the same as the one produced by $f(g(w[i:j]))$, with f being an increasing monotonic function. If f is considered the kth root, $f(g(w[i:j])) = \sqrt[k]{\frac{1}{j-i}\sum_{h=i}^{j}(w[h])^k}$, then the interpretation of the model that f defines becomes straightforward when working with fusing particles model. Assuming that all particles now have positive initial velocities $V_1, V_2, \ldots, V_N$, then, Model k will represent a fusing particles model with conservation of the quantity defined here as

$$Q_k = \sum_{j=1}^{N} m_j V_j^k. \tag{14}$$

For example, when $k = 1$, this quantity is momentum, while, when $k = 2$, it is energy. In terms of energy conservation, the model for $k = 1$ exhibits energy loss, and for $k > 2$, the model is nonphysical due to energy being created as particles fuse. In any case, when two particles with masses m_1 and m_2 and velocities v_1 and v_2, respectively, collide, the resulting fused particle, which mass is the sum of m_1 and m_2, will have a velocity v_f given by

$$v_f = \sqrt[k]{\frac{m_1 v_1^k + m_2 v_2^k}{m_1 + m_2}}.$$

More generally, the final velocity of any fused particle, $\tilde{V}_f$, formed by N_f identical particles (particles $j, j+1, \ldots, j+N_f-1$), is the kth power average velocity of the fused particles,

$$\tilde{V}_f = \sqrt[k]{\frac{1}{N_f}\sum_{h=j}^{j+N_f-1} V_h^k}.$$

Model 0: A 1D nonphysical fusing particle model

Here Model 0 is studied. Its $\tilde{m}$ function can be rewritten,

$$\tilde{m}(w) = \min\left\{j \in \mathbb{N}_1^{L(w)-1} : w[j] < w[i] \quad \forall i \in \mathbb{N}_{j+1}^{L(v)+1}\right\}$$
$$= \left\{k \in \mathbb{N}_1^{L(w)-1} : w[k] = \min(w)\right\},$$

where it is made explicit that the model partitions the sequence of velocities based on the minimum velocities. This new model can be formulated as follows: taking the fusing particles model,as reference, a new nonphysical collision process for the same initial conditions is developed. In this artificial process, once two particles collide, the resulting merged particle continues at a velocity equal to the minimum velocity of both particles involved in the collision. The only difference between this new process and the original one is that there is no conservation of momentum: the velocity of the merged particle, which is the average velocity of the original particles prior to the merger, is replaced by the minimum velocity of the colliding particles.

A realization of this nonphysical case is shown in Fig. 1(c). The initial conditions are the same as the ones in the physical system shown in the left panel. Note that in this particular realization, the number of final particles is equal to 2, while in the physical model it is equal to 3. The evolution of the total number of particles is shown in the lower panel of Fig. 1(c). The pattern for calculating the number of final particles can be easily found by looking at this figure. First, the particle with the minimum velocity must be found, let us say it is particle j. Then, particles $1, 2, \ldots, j-1$ merge with particle j, producing the leftmost final particle with mass j. Then the minimum velocity particle among the remaining particles $j+1, j+2, \ldots, N$ is identified; let us say it is particle k. Then all the previous particles in this group $(j+1, j+2, \ldots, k)$ are merged, producing a second final particle of mass $k-j$. This process is repeated until the minimum velocity particle is particle N. For example, in Fig. 1(c) the minimum velocity is seen to correspond to particle number 7, which merges particles from 1 to 7, and the second minimum velocity of the remaining particles to correspond to particle 10, which merges particles from 8 to 10. As a result, the two final particles ($\tilde{X}_{10} = 2$) have masses of 7 and 3, respectively $\mathbb{M} = (7, 3)$. It is important to note that this final configuration remains the same [$\tilde{X}_{10} = 2$ and $\mathbb{K} = (7, 3)$] for any initial position satisfying $Y_1 < Y_2 < \cdots < Y_{10}$. This will always be the case; the positions will be irrelevant, the order of the particle(s) velocities being the only key element, rather than the specific values. This is why the behavior of $\tilde{Z}_{10}$ and their masses is, once again, universal. Theorem 5 ensures that $\mathbb{P}(\tilde{X}_N = k) = \frac{|c(N,k)|}{N!}$ and and that the scale-free behavior given by Eq. (5) is satisfied for the expected number of final particles of size s.

The last collision time distribution

In this section, the distribution of the last collision time (τ) is studied. Unlike the physical model, this one allows for an explicit expression for τ. The derived expression for τ is presented below.

Let $\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_{\tilde{X}_N}$ be the velocities of the final particles. In this nonphysical model, these final velocities correspond to the velocities of certain specific particles. As expected,

$$\tilde{V}_k = V_{M_k} \quad \text{for} \;\; k = \{1, 2, \ldots, \tilde{X}_N\},$$

where $M_k = \sum_{h=1}^{k} m(G^{h-1}(w_i))$. A final particle, let's say final particle k, is considered to be born when all its constituent particles have merged. Let this time be denoted by τ_k, and this variable can be defined by the following equation:

$$\tau_k = \begin{cases} \max\limits_{M_{k-1}<i<M_k} \frac{Y_{M_k}-Y_i}{V_i - V_{M_k}} & \text{if } m(G^{k-1}(w_i)) > 1 \\ 0 & \text{if } m(G^{k-1}(w_i)) = 1 \end{cases} \tag{15}$$

with $M_0 = 0$ and Y_i the initial position of particle i. With this definition in mind, the last collision time is given by

$$\tau = \max\left\{\tau_1, \tau_2, \ldots, \tau_{\tilde{X}_N}\right\}. \tag{16}$$

Based on Theorem 4, as well as Eqs. (15) and (16), it is straightforward to define an algorithm that quickly computes τ without the need to evolve the particle system.

Obtaining an analytical result for $\mathbb{P}(\tau > t)$ is challenging, so numerical simulations are presented in this case instead. As

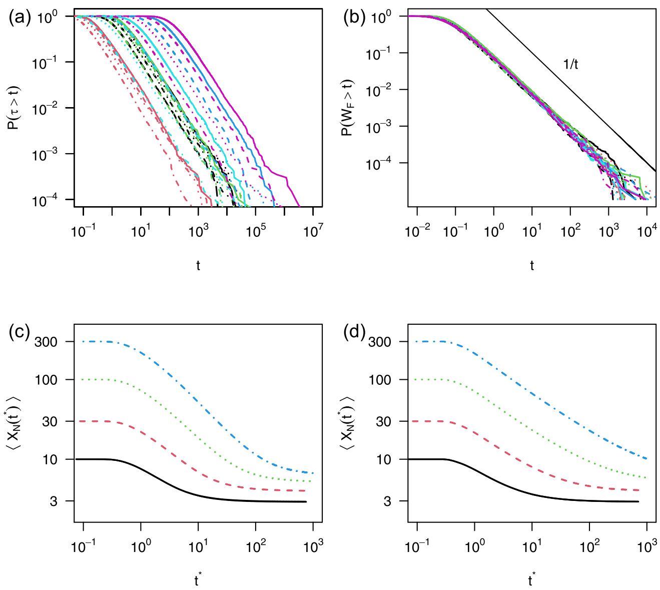

FIG. 3. (a) Last collision time distribution, $\mathbb{P}(\tau_{F,N} > t)$, (b) $\tau_{F,N}$ rescaled version $\mathbb{P}(\tau_{F,N}\sigma_F N^{-\alpha_F} > t)$ as a function of t. For F: Normal(0,1) ($\sigma_F = 1$), Normal(0,10) ($\sigma_F = 10$), Normal(10,1) ($\sigma_F = 1$), Uniform(−1,1) ($\sigma_F = \sqrt{1/3}$), Uniform(-10,10) ($\sigma_F = 10\sqrt{1/3}$), Uniform(0,1) ($\sigma_F = 1/\sqrt{12}$), and Beta(0.5,0.5) ($\sigma_F = \sqrt{1/8}$), and considering $N = \{10, 30, 100, 300, 1000\}$. The evolution of the mean number of particles $\langle X_N(t^\star)\rangle$ as a function of a rescaled time $t^\star$, considering F the three previous (c) normal and (d) uniform distributions and $N = \{10, 30, 100, 300, 1000\}$.

seen in Eqs. (15) and (16), τ depends on N, F, and the initial positions of the particles. Figure 3(a) shows an estimation of $\mathbb{P}(\tau > t)$ as a function of t, considering initial positions on the grid of length equal N previously used, as well as Uniform and Normal initial velocity distributions. It is noteworthy that, across all studied values of $N = 10, 30, 100, 300$, the curves exhibit similar behavior. In order for these curves to collapse, the parameter α_F must vary, resulting in the rescaled τ:

$$W_F := \tau_{F,N}\sigma_F N^{-\alpha_F}.$$

The results for W_F are shown in Fig. 3(b). For the three Uniform distributions [Uniform(0,1), Uniform(−1,1), Uniform(−10,10)], the parameter α_F equals 2. Conversely, for the three Normal distribution cases [Normal(0,1), Normal(0,10), Normal(10,1)] $\alpha_F = 1.15$.

Unlike the physical model discussed earlier, the α_F parameter in this model varies depending on the initial velocity distribution. Furthermore, not all distributions collapse into a single curve. Further analysis is necessary to fully understand this specific behavior.

Finally, in Figs. 3(c) and 3(d), the evolution of the number of particles in the rescaled time $t^\star = \sigma_F t$ is presented. Figure 3(c) displays the results for the three Normal distributions studied, while Fig. 3(d) shows those for the three Uniform distributions. The evolution collapses within the same distribution type, but different distributions exhibit distinct behaviors.

IV. CONCLUSIONS

A 1D particle system of N identical point particles undergoing perfectly plastic collisions has been studied in detail. One notable property of this model is its analytical tractability for finite N. The distribution of the number of final particles, along with its mean and variance, has already been achieved [15]. Additionally, the expected number of final particles of mass s follows a power-law behavior, indicating that this nonequilibrium system behaves as a system in criticality. Moreover, these results are universal, meaning they do not depend on the initial conditions of the particle system.

Further evidence supporting the criticality hypothesis in this out-of-equilibrium system has been presented. Specifically, the last collision time (τ) and the evolution of the number of particles [$\langle X_N(t)\rangle$] have been studied. The different τ curves collapse under the appropriate scaling, yielding a probability density that decays as $1/t^2$. The behavior of $\langle X_N(t)\rangle$ is more complex, but for each N, all different curves collapse into a single curve.

Additionally, a procedure for calculating the number of final particles and each of their masses has been developed (Theorem 4). This result allows for the calculation of these quantities without the need for the system to evolve. This procedure is significant and particularly useful for studying very large systems, primarily due to its computational efficiency. Moreover, this finding also raises an important question: what other conditions might lead to the same fundamental statistical properties observed in the fusing particles model? In other words, which other models belong to this same universality class in the context of criticality? Whether Theorem 5 fully comprehends every model within this universality class or not is a question that extends beyond the scope of this work. Nevertheless, this theorem represents a comprehensive attempt to provide an infinite set of models within this class, offering valuable tools for understanding and addressing other nonequilibrium processes.

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APPENDIX A: τ DISTRIBUTION FOR UNIFORM AND NORMAL INITIAL POSITIONS

This Appendix presents results for $\tau_{F,N}$ when the initial positions of the particles are randomly distributed according to a Uniform(−1,1) or Normal(0,1) distribution. The rescaled versions of $\tau_{F,N}$ are denoted as W_U and W_G for Uniform and Gaussian initial positions, respectively, where W_U (W_G) is defined as $W_U := \tau_{F,N}\sigma_F N^{-\alpha_U}$. Figure 4(a) shows the distribution of W_U, while Fig. 4(b) shows the results for W_G. Note that for the different velocity distributions studied, the results collapse to a single curve in both cases. The only difference lies in the α parameter: $\alpha_U = 0.6$ and $\alpha_G = 0.65$. Further analysis is needed to understand the impact of other initial

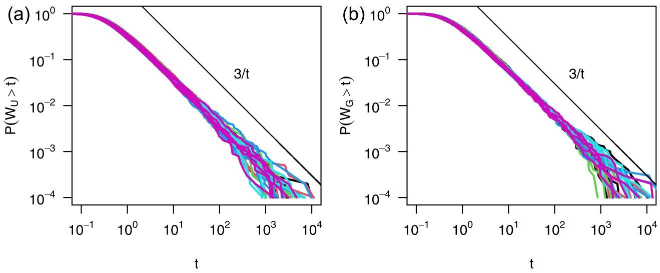

FIG. 4. $\mathbb{P}(W_U > t)$, (b) $\mathbb{P}(W_G > t)$ as a function of t. For F: Normal(0,1) ($\sigma_F = 1$), Normal(0,10) ($\sigma_F = 10$), Normal(10,1) ($\sigma_F = 1$), Uniform(−1,1) ($\sigma_F = \sqrt{1/3}$), Uniform(−10,10) ($\sigma_F = 10\sqrt{1/3}$), and Uniform(0,1) ($\sigma_F = 1/\sqrt{12}$), and considering $N = \{10, 30, 100, 300, 1000\}$.

velocity distributions, as well as different distributions for the initial positions.

APPENDIX B: PROOF OF THEOREM 4

The process begins with a configuration of velocities $v_1 := v_i = (V_1, V_2, \ldots, V_N)$ and a set of positions $Y_1 < Y_2 < \cdots < Y_N$. The specific positions determine the sequence of collisions, dictating which ones occur initially and which ones follow. Importantly, this sequence doesn't impact the total number of final particles or the mass of each individual particle (see Theorem 1). Consequently, in the following, we will assume a particular order in which the collisions take place.

For particles 1 and 2 to merge, one of two alternatives must occur: (1) $V_1 > V_2$, or (2) $V_1 > \frac{1}{s-1}\sum_{i=2}^{s} V_i$ for some $s > 2$ (i.e., particle 1 collide with a fused particle that contains particle 2). Alternatively, we can express that particle 1 merges with other particles if $s_1(v_i) \neq \emptyset$, where

$$s_1(v_1) = \min\left\{s \in \mathbb{N}_2^N : V_1 > \overline{V}_2^s\right\},$$

with $\overline{V}_j^k = \frac{1}{k-j}\sum_{i=j}^{k} V_i$. In this case, particle 1 will collide with a merged particle comprising particles $2, 3, \ldots, s_1(v_1)$. In essence, a merged particle will form, incorporating particles $1, 2, \ldots, s_1(v_1)$, and it will have a velocity $\overline{V}_1^{s_1(v_1)}$.

Now, with this new merged particle, we work as we did before, i.e., as if it were the original particle 1, which can be merged when $s_2(v_i) \neq \emptyset$, where

$$s_2(v_1) = \min\left\{s \in \mathbb{N}_{s_1(v_1)+1}^N : \overline{V}_1^{s_1(v_1)} > \overline{V}_{s_1(v_1)+1}^s\right\}.$$

If $s_2(v_1) \neq \emptyset$, then the merged particle containing particles $1, 2, \ldots, s_1(v_1)$ will inevitably collide with another merged particle containing particles $s_1(v_1) + 1, 3, \ldots, s_2(v_1)$. Consequently, a new merged particle will form, incorporating particles $1, 2, \ldots, s_2(v_1)$, and it will possess a velocity of $\overline{V}_1^{s_2(v_1)}$. This process repeats until, for the first time,

$$\tilde{k}(v_1) = \min\left\{k \in \mathbb{N}_1^N : s_k(v_1) = \emptyset \text{ or } s_k(v_1) = N\right\}$$

with

$$s_k(v_1) = \min\left\{s \in \mathbb{N}_{s_{k-1}(v_1)+1}^N : \overline{V}_{s_0(v_1)}^{s_{k-1}(v_1)} > \overline{V}_{s_{k-1}(v_1)+1}^s\right\},$$

and $s_0(v_1) = 1$.

Finally, the resulting final particle 1, the leftmost particle, will be a fusion of particles $1, 2, \ldots, s_{\tilde{k}(v_1)}(v_1)$. That is, the mass will be $s_{\tilde{k}(v_1)}(v_1)$, and if its mass is smaller than N, the following condition will be satisfied:

$$\overline{V}_1^{s_{\tilde{k}(v_1)}(v_1)} < \overline{V}_{s_{\tilde{k}(v_1)}(v_1)+1}^s \quad \forall s \in \mathbb{N}_{s_{\tilde{k}(v_1)}+1}^N. \tag{B1}$$

The general case of $s_{\tilde{k}(v_1)}(v_1)$, which includes the value N, can be written as

$$s_{\tilde{k}(v_1)}(v_1) = \min\left\{j \in \mathbb{N}_1^N : \overline{V}_1^j < \overline{V}_{j+1}^i \quad \forall i \in \mathbb{N}_{j+1}^{N+1}\right\} \tag{B2}$$

$$=: \tilde{\tilde{m}}(v_1), \tag{B3}$$

where, to improve the notation, an additional "phantom" particle is added: particle $N + 1$, positioned to the right of particle N, with a velocity equal to $2V_{\max} := 2\max\{V_1, V_2, \ldots, V_N\}$. The upper row in Fig. 5 presents an illustration of a system involving $N = 9$ particles, solved through the temporal evolution of the process. In contrast, the analogous scenario without temporal evolution, as outlined by Eq. (B2), is portrayed in the upper row of Fig. 6.

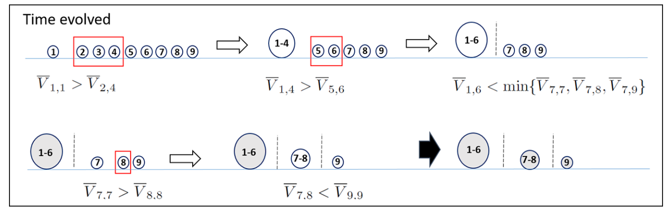

FIG. 5. Solution based on system evolution.

Now that the composition of the final particle 1 is understood, consisting of the particles $1, 2, \ldots, \tilde{\tilde{m}}(v_1)$, a similar analysis can be applied to the remaining particles on the right-hand side. Assuming $\tilde{m}(v_1) < N$, we initiate the analysis starting with particle $\tilde{\tilde{m}}(v_1) + 1$ to investigate potential mergers with the remaining particles. Let $v_2 = v_1[\tilde{\tilde{m}}(v_1) + 1 : N] = (V_{\tilde{\tilde{m}}(v_1)+1}, V_{\tilde{\tilde{m}}(v_1)+2}, \ldots, V_N)$ and define

$$s_1(v_2) = \min\left\{s \in \mathbb{N}_{\tilde{\tilde{m}}(v_1)+2}^N : V_{\tilde{\tilde{m}}(v_1)+1} > \overline{V}_{\tilde{\tilde{m}}(v_1)+2}^s\right\}.$$

If $s_1(v_2) \neq \emptyset$, particle $\tilde{\tilde{m}}(v_1) + 1$ will collide with a merged particle comprising particles $\tilde{\tilde{m}}(v_1) + 2, \tilde{\tilde{m}}(v_1) + 3, \ldots, s_1(v_2)$. A merged particle will form, incorporating particles $\tilde{\tilde{m}}(v_1) + 1, \tilde{\tilde{m}}(v_1) + 2, \ldots, s_1(v_2)$, and it will have a velocity $\overline{V}_{\tilde{\tilde{m}}(v_1)+1}^{s_1(v_2)}$. This fused particle can fuse with other particles if $s_2(v_2) \neq \emptyset$, with

$$s_2(v_2) = \min\left\{s \in \mathbb{N}_{s_1(v_2)+2}^N : \overline{V}_{s_1(v_2)+1}^s > \overline{V}_{s_1(v_2)+2}^s\right\}.$$

If $s_2(v_2) \neq \emptyset$, the fused particle will collide. The new fused particle will be formed by particles $s_1(v_1) + 1, s_1(v_1) +$

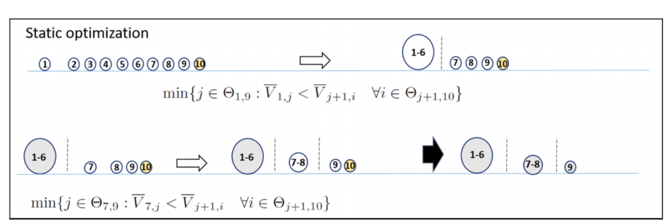

FIG. 6. Solution without relying on system evolution.

$2, \dots, s_2(v_2)$, and this process continues until the first time that

$$\tilde{k}(v_2) = \min\left\{k \in \mathbb{N}_1^N : s_k(v_2) = \emptyset \text{ or } s_k(v_2) = N\right\}$$

with

$$s_k(v_2) = \min\left\{s \in \mathbb{N}_{s_{k-1}(v_2)+1}^N : \overline{V}_{s_0(v_2)}^{s_{k-1}(v_2)} > \overline{V}_{s_{k-1}(v_2)+1}^{s}\right\},$$

and $s_0(v_2) = s_{\tilde{k}(v_1)}(v_1) + 1$. It is crucial to reemphasize that, within the context of this proof,

$$s_{\tilde{k}(v_2)}(v_2) = \tilde{\tilde{m}}(v_2).$$

This continues until we have $\tilde{j}$ final particles, with

$$\tilde{j} = \min\left\{j \in \mathbb{N}_1^N : \sum_{h=1}^{j} \tilde{k}(v_h) = N\right\}.$$

Finally, evolving the system, we obtain the following final masses:

$$\begin{aligned}\mathbb{M} &= (s_{\tilde{k}(v_1)}(v_1), s_{\tilde{k}(v_2)}(v_2), \dots, s_{\tilde{k}(v_{\tilde{j}})}(v_{\tilde{j}})),\\ &= (\tilde{\tilde{m}}(v_1), \tilde{\tilde{m}}(v_2), \dots, \tilde{\tilde{m}}(v_{\tilde{j}})),\\ &= (\tilde{\tilde{m}}(G^0(v_i)), \tilde{\tilde{m}}(G^1(v_i)), \dots, \tilde{\tilde{m}}(G^{\tilde{X}_N(v_i)-1}(v_i))).\end{aligned}$$

To conclude the proof, note that $\tilde{\tilde{m}}(v) = \tilde{m}(w)$ and the last equality is valid, since $v_j = G^{j-1}(v_i)$ and

$$\begin{aligned}\tilde{X}_N(v_i) &= \min\left\{k \in \mathbb{N}_1^N : G^k(v_i) = \emptyset\right\},\\ &= \min\left\{k \in \mathbb{N}_1^N : \sum_{j=0}^{k-1} \tilde{\tilde{m}}(G^j(v_i)) = N\right\}.\end{aligned}$$

APPENDIX C: PROOF OF THEOREM 5

The combination of two older results led to this theorem. One is an old mathematical statistics paper [26] that addresses the statistical test problem of comparing different means ($H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$) under the ordered alternative ($H_A : \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k$) [27]. In the context of our study, Brunk [26] demonstrates that functions g which satisfy Eq. (10) also verify Eq. (1). The other result comes from Sibuya *et al.* [15], which establishes that the only condition required to obtain an expected number of clusters verifying Eq. (5) is that the number of final particles satisfies Eq. (1) and the sequence of velocities is i.i.d. with a common continuous distribution. The combination of these two results forms the basis of the proof.

The proof of Theorem 3 presented in [15] does not use any detail about the particles system model, it just uses that the number of final particles has a probability mass function given by 1 and then combinatorial arguments are used taking into account that initial velocities are iid. Therefore, Theorem 3 can be reformulated in the following way.

Theorem 3′. If a sequence of N iid random variables is partitioned in such a way that the final number of clusters, $\tilde{X}_N$, has a probability mass function given by

$$\mathbb{P}(\tilde{X}_N = k) = \frac{|c(N,k)|}{N!}, \tag{C1}$$

where $c(n,k)$ is the Stirling number of the first kind, then the number of final particles with mass s, $n_s := \sum_{i=1}^{\tilde{X}_N} \delta_{\mathbb{M}(i),s}$, verify

$$\langle n_s \rangle = \frac{1}{s}.$$

The other important result is given in [27]. The author proposed an statistics for the statistical testing problem of determining if the means $\mu_1, \mu_2, \dots, \mu_k$ of k independent random variables $V_1, V_2, \dots, V_k$ with $V_i \sim \text{Normal}(\mu_i, \sigma_i)$ are equal, $H_0 : \mu_1 = \mu_2 = .. = \mu_k$, or the ordered alternative hypothesis is true, $H_A : \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k$ with at least one greater inequality. In the following, the simplest case $\sigma_i = 1$ is examined, which is the case of interest in our context. The proposed statistics was

$$T = \sum_{i=1}^{k} \left(\hat{\mu}_i - \overline{V}_1^N\right)^2,$$

where $\overline{V}_1^N$ is the overall average and $\hat{\mu}_i$ in our notation would be

$$\hat{\mu}_i = \sum_{h=1}^{\tilde{X}_N(w_i)} \overline{V}_{M_{h-1}+1}^{M_h} \mathbb{1}_{\mathbb{N}_{M_{h-1}+1}^{M_h}}(i),$$

with $M_0 = 0$, $M_h = \sum_{j=0}^{h-1} \tilde{m}(G^j(w_i))$ for $h \geqslant 1$, $w_i = (V_1, V_2, \dots, V_N, 2\max\{V_1, V_2, \dots, V_N\})$. and $\mathbb{1}_A(x)$ the indicator function. In other words, $\hat{\mu}_i$ is the velocity of fused final particle that contains particle i. Moreover, the statistics T for $\tilde{X}_N(w_i) > 1$ can be written as

$$T = \sum_{h=1}^{\tilde{X}_N(w_i)} \mathbb{M}[h]\left(\overline{V}_{M_{h-1}+1}^{M_h} - \overline{V}_1^N\right)^2.$$

The distribution of the statistics under the null hypothesis verifies

$$\begin{aligned}\mathbb{P}(T > t) &= \sum_{k=1}^{N} \mathbb{P}(T > t | \tilde{X}_N(w_i) = k)\mathbb{P}(\tilde{X}_N(w_i) = k)\\ &= \mathbb{P}\left(\chi_{k-1}^2 > t\right)\mathbb{P}(\tilde{X}_N = k),\end{aligned}$$

where χ_{k-1}^2 is a random variable with a χ^2 distribution with $k-1$ degrees of freedom. Bartholomew [27] also calculates $\mathbb{P}(\tilde{X}_N = k)$ for the particular cases of $N = \{2, 3, 4\}$. In a subsequent paper, Miles [28] proved that $\mathbb{P}(\tilde{X}_N = k)$ is given by Eq. (1). Finally, Brunk [26] answered what other partitions of the particles give the same $\mathbb{P}(\tilde{X}_N = k)$. In our context, this is equivalent to asking which g functions verify $\mathbb{P}(\tilde{X}_N = k)$ considering Eqs. (7)–(9). The answer is given by Eq. (10).

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