Fluctuation-dominated quantum oscillations in excitonic insulators

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The realization of excitonic insulators in transition metal dichalcogenide systems has opened the door to explorations of the exotic properties that such a state exhibits. We study theoretically the potential for excitonic insulators to show an anomalous form of quantum oscillations: the de Haas–van Alphen effect in an insulating system. We focus on the role of the interactions that generate the energy gap and show that it is crucial to consider quantum fluctuations that go beyond the mean-field treatment. Remarkably, quantum fluctuations can be dominant and lead to quantum oscillations that are significantly larger than those predicted using mean-field theory. Indeed, in experimentally accessible parameter regimes these fluctuation-generated quantum oscillations can even be larger than what would be found for the corresponding gapless system.

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I. INTRODUCTION

Materials that become insulating as the result of interactions have attracted significant interest in recent years. Excitonic insulators, formed by the spontaneous condensation of electron-hole bound states, were theoretically proposed more than 50 years ago [1–4]. They have now been realized in single- [5] and double-layer [6,7] transition metal dichalcogenide (TMD) systems, a class of materials which itself has garnered wide and significant attention [8]. Kondo insulators [9,10], in particular topological Kondo insulators [11,12], have also been intensely studied because of their nontrivial topological properties and significant bulk gaps at low temperature.

The measurement of oscillations of the magnetization with magnetic field—i.e., the de Haas–van Alphen effect [13]—in the Kondo insulators SmB_6 [14–17] and YbB_{12} [18–20] was particularly unexpected, since the presence of a Fermi surface was long believed to be a necessary condition to realize this effect [21,22]. Consequently, a large amount of theoretical work has gone into understanding how quantum oscillations (QOs) can manifest in insulators, some focused specifically on these Kondo systems [23–32], and others considering more general insulating systems [33–45], including excitonic insulators, which will be our focus here.

Direct calculations show that even simple models of noninteracting band insulators can exhibit QOs [33–39,42–45]: provided the minimum band gap traces out some closed area in reciprocal space, the free energy contains an oscillatory component and therefore so do thermodynamic quantities like the magnetization. The resulting QOs have several properties that do not depend on specific details of the model. First, the oscillation frequency is determined by the area noted above, just as the area of the Fermi surface determines this frequency in metals. Second, at small magnetic field B these oscillations are suppressed by a factor of the form $\exp(-B_0/B)$, where B_0 is proportional to the size of the band gap. While this has the same form as the Dingle suppression of QOs in metals due to impurity scattering [22], we emphasize that here B_0 is an intrinsic property of the system. We have previously shown that—at least for the lowest frequency oscillatory response—these results extend beyond the case of noninteracting particles, and arise also for interaction-driven excitonic and Kondo insulators when the interactions are treated within mean-field theory [43]. Our mean-field results have recently been confirmed in Ref. [46].

In this paper we go beyond the mean-field approximation for a model of an excitonic insulator and calculate QOs arising from quantum fluctuations of the gap. This mechanism for realizing QOs is fundamentally different from the typical semiclassical picture of electrons executing closed orbits with quantized area, or indeed any single-particle picture, and we find the surprising result that it is the principal mechanism for generating QOs in the system—the mean-field result is orders of magnitude smaller. Indeed, as shown in Fig. 1, in experimentally accessible low-electron-density parameter regimes, for instance, in TMD double-layer systems [6,7], the oscillations from this quantum fluctuation effect can even dominate those for the corresponding gapless system obtained by "turning off" the interaction. Counterintuitively, for low-density systems QOs can be significantly amplified by introducing interactions that destroy the Fermi surface. We study the system in a regime where quantum fluctuations give a correction to the free energy Ω_{fluct} that is much smaller than the total

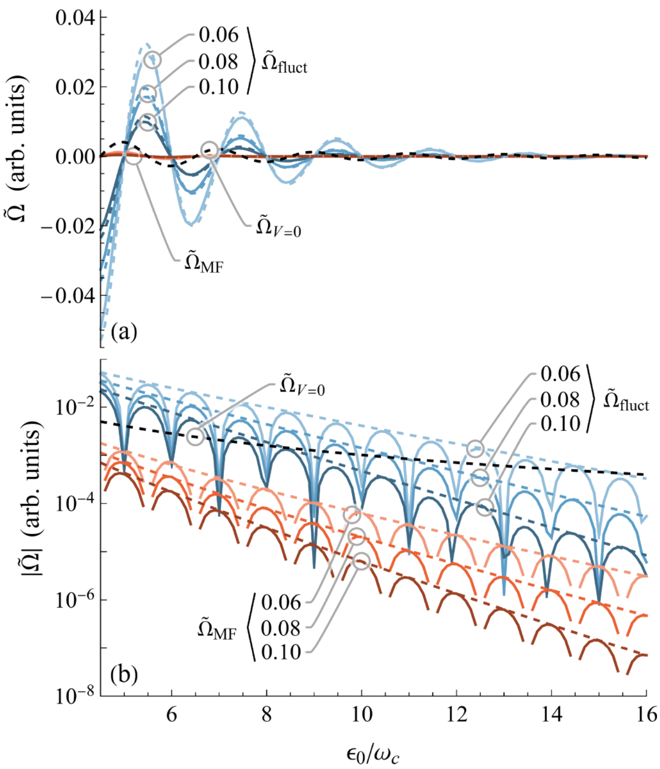

FIG. 1. The oscillatory free energies $\tilde{\Omega}_{\rm fluct}$ (blue), $\tilde{\Omega}_{\rm MF}$ (red), and $\tilde{\Omega}_{V=0}$ (black) are plotted on (a) linear and (b) logarithmic scales as functions of inverse cyclotron frequency ($\propto 1/B$), for the case of $T = 0$. The numbers labeling different curves correspond to three choices of the size of the gap Δ_0, given in units of ϵ_0. Solid lines are numerical results and dashed lines of the same color are corresponding analytic results, Eqs. (12), (20), and (21). In (a) the three mean-field results are too small to distinguish. In (b) only the amplitude of analytic results are plotted. Beyond the clear difference in amplitudes, also note the $\pi/2$ phase shift of the fluctuation result.

free energy Ω, as is necessary for a stable mean-field theory. Since the part of the free energy that oscillates with magnetic field, $\tilde{\Omega}$, is a very small fraction of the total free energy, there is no contradiction that the quantum fluctuations can be the dominant source of $\tilde{\Omega}$ while still being small compared to Ω.

As we explain below we expect that this effect is very generic. However, in order to establish the importance of the effect most clearly, we study a concrete model for which we can provide an exact calculation of the leading order fluctuation effects. This paper is therefore outlined as follows. In Sec. II we introduce the two-band model with an interband interaction, then write the excitonic mean-field theory and the action for fluctuations above the mean field. We then show the effects of coupling an external magnetic field. In Sec. III we obtain an expression for the fluctuation contribution to the free energy, and in Sec. IV we find and analyze its oscillatory part. Finally, in Sec. V we discuss the results and conclude.

II. MODEL

We consider a two-dimensional, two-band system of spinless electrons with an interband interaction. For vanishing magnetic field it is described by the Hamiltonian

$$\hat{H} = \sum_{\mathbf{k}} \sum_{i=c,v} \xi_{i,\mathbf{k}} \hat{\psi}^\dagger_{i,\mathbf{k}} \hat{\psi}_{i,\mathbf{k}} + \frac{V}{A} \sum_{\mathbf{q},\mathbf{k},\mathbf{k}'} \hat{\psi}^\dagger_{c,\mathbf{k}+\mathbf{q}} \hat{\psi}^\dagger_{v,\mathbf{k}'-\mathbf{q}} \hat{\psi}_{v,\mathbf{k}'} \hat{\psi}_{c,\mathbf{k}}, \tag{1}$$

where c, v label the conduction and valence bands and A is the area of the system. The constant $V > 0$ parameterizes the strength of the attractive Coulomb interaction between particles and holes, here approximated as a contact interaction. We consider a particle-hole symmetric model, with chemical potential $\mu = 0$ and with dispersions $\xi_{c,\mathbf{k}} = -\xi_{v,\mathbf{k}} = |\mathbf{k}|^2/2m^* - \epsilon_0/2 \equiv \xi_k$. This assumption simplifies our analysis but is not essential for arriving at our main result: broken particle-hole symmetry should not have a significant qualitative effect. (We note, however, that TMD double-layer systems that realize the sort of system we are interested in are well approximated as particle-hole symmetric). The energy $\epsilon_0 > 0$ is the offset of the two band edges. We also define a UV cutoff for $\xi_{\mathbf{k}}$, which we set at an energy Λ from the band minimum so that $\max(\xi_k) = \Lambda - \epsilon_0/2$. (In terms of real material parameters, this Λ is related to the bandwidth.) Thus the total electron density in the system is $n_e = \rho_F \Lambda$, with densities $n_c = \rho_F \epsilon_0/2$ and $n_v = \rho_F(\Lambda - \epsilon_0/2)$ in the conduction and valence bands, respectively, where $\rho_F = m^*/2\pi$ is the density of states for a spinless 2D electron gas. Here and throughout we set $\hbar = c = 1$.

A systematic analysis of the thermodynamics of this model is performed using standard finite-temperature field theoretical methods. We thus introduce Grassmann fields for the conduction and valence electrons, $\psi_{c,k}$ and $\psi_{v,k}$, with the subscript k representing both momentum $\mathbf{k}$ and fermionic Matsubara frequency $\epsilon_n = (2n+1)\pi/\beta$ at inverse temperature β. The interaction decouples with a Hubbard-Stratonovich transformation in terms of a bosonic field Δ_q related to the pairing of electrons and holes between the two bands. Here q represents both momentum $\mathbf{q}$ and bosonic Matsubara frequency $\omega_m = 2m\pi/\beta$. We separate $\Delta_q = \delta_{q,0}\Delta + \eta_q$ into a static, spatially uniform mean field Δ and a dynamic, spatially nonuniform fluctuation field η_q. Choosing Δ to be real, we identify the real and imaginary parts of η_q as the Higgs (or amplitude) mode and phase mode, respectively. The resulting action is $S = S_{\rm MF} + S_{\rm fluct}$ with

$$S_{\rm MF} = \frac{\beta A}{V}\Delta^2 + \sum_k \bar{\Psi}_k \begin{pmatrix} -i\epsilon_n + \xi_{\mathbf{k}} & -\Delta \\ -\Delta & -i\epsilon_n - \xi_{\mathbf{k}} \end{pmatrix} \Psi_k, \tag{2}$$

$$S_{\rm fluct} = \frac{\beta A}{V} \sum_q \bar{\eta}_q \eta_q - \sum_{k,q} \bar{\Psi}_{k+\frac{q}{2}} \begin{pmatrix} 0 & \eta_q \\ \bar{\eta}_{-q} & 0 \end{pmatrix} \Psi_{k-\frac{q}{2}}, \tag{3}$$

where $\Psi_k = (\psi_{c,k} \quad \psi_{v,k})^T$. $S_{\rm MF}$ is the mean-field action for the electrons, in which Δ is determined self-consistently to minimize the free energy. Diagonalizing the mean-field single-particle Hamiltonian yields the new set of gapped bands, $\pm E_k = \pm\sqrt{\xi_k^2 + \Delta^2}$. These and the relevant energies of the system are shown in Fig. 2.

The action $S_{\rm fluct}$ describes the effects of quantum fluctuations beyond mean-field theory. The consequences of these fluctuations for the thermodynamics of related BCS superconductors have been worked out in detail previously [47–49]. Much can be gleaned from these studies for our $B = 0$ model

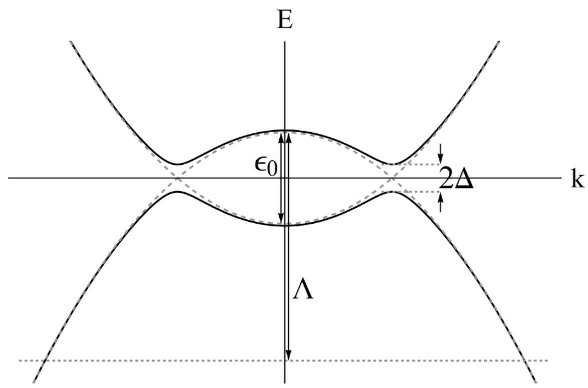

FIG. 2. The particle-hole symmetric band structure we consider. The solid lines are $\pm E_k$ and the dashed lines are $\pm\xi_k$. The band gap 2Δ, band edge offset ϵ_0, and UV cutoff Λ in the valence band are indicated.

of the excitonic insulator, which is equivalent to a superconductor under a particle-hole transformation. In particular, because of the mathematical similarity of the present model to BCS theory we can expect that the mean-field state is stable—the correction to the free energy from fluctuations is small, and they have only a small effect on the value of the order parameter, i.e., determining Δ from the mean-field sector of the theory alone, neglecting fluctuations, is a very good approximation. Our analytic results are consistent with these statements, and our numerical analysis verifies this stability explicitly, as discussed in Appendix D.

Coupling to external magnetic field

To include an external magnetic field B perpendicular to the system we consider minimally coupling the fermionic theory to a static vector potential in the Landau gauge. The modified kinetic energy term in Eq. (1) is then diagonal in the basis of Landau level states, with energies $\xi_l = \omega_c(l + 1/2) - \epsilon_0/2$ that are evenly spaced by the cyclotron energy $\omega_c = eB/m^*$, and corresponding wave functions

$$\Phi_{l,k_y}(x, y) = e^{ik_y y}\phi_l\big(x - \ell_B^2 k_y\big), \tag{4}$$

using

$$\phi_l(x) = \frac{1}{\sqrt{2^l l!}}\left(\frac{1}{\pi\ell_B^2}\right)^{1/4} e^{-x^2/(2\ell_B^2)}H_l(x/\ell_B), \tag{5}$$

where $H_l(x)$ are the Hermite polynomials and $\ell_B = 1/\sqrt{eB}$ is the magnetic length. The action Eqs. (2) and (3) for the system at $B = 0$ translate to their nonzero field equivalents with simple substitutions: $\xi_k \to \xi_l$, so the mean-field bands are $E_k \to E_l = \sqrt{\xi_l^2 + \Delta^2}$, and with our choice of gauge $\sum_{\mathbf{k}} \to \sum_l \sum_{k_y}$. Dependence on k_x is thus replaced by l and the momentum k_y now labels the degenerate states in each Landau level so that $\sum_{k_y} = AB/\Phi_0 \equiv N_\Phi$ is the degeneracy of each Landau level, where A is the system's area and $\Phi_0 = h/e$ is the magnetic flux quantum. In this basis we now use subscript k on fermions for Matsubara frequency and the remaining momentum index, with the Landau level index written separately.

The fluctuation field η_q, being a neutral bosonic degree of freedom, does not directly couple to an electromagnetic field at the level of minimal coupling, and the primary effect of this change to the basis of Landau levels is to introduce a nontrivial coupling between fluctuations and fermions:

$$S_\eta = \frac{\beta A}{V}\sum_q \bar{\eta}_q\eta_q - \sum_{l,l'}\sum_{q,k}\bar{\Psi}_{l,k+\frac{q}{2}}\, g^{ll'}_{\mathbf{q},k_y}\begin{pmatrix} 0 & \eta_q \\ \bar{\eta}_{-q} & 0\end{pmatrix}\Psi_{l',k-\frac{q}{2}}, \tag{6}$$

with the coupling

$$g^{ll'}_{\mathbf{q},k_y} \equiv e^{i\ell_B^2 q_x k_y}\int \mathrm{d}x\, \phi_l\left(x - \ell_B^2\frac{q_y}{2}\right)\phi_{l'}\left(x + \ell_B^2\frac{q_y}{2}\right)e^{iq_x x}. \tag{7}$$

III. FREE ENERGY

The free energy is obtained by integrating out all fields in the action, both electrons and fluctuations. Because the action is represented as a mean-field term and a fluctuation term, this leads to a free energy that is likewise the sum of mean-field and fluctuation contributions. The behavior of the mean-field energy Ω_{MF} obtained from S_{MF} has been previously analyzed in detail [43] and is the most relevant point of comparison for the contribution from the fluctuations. We therefore reproduce relevant aspects of that analysis in Appendix A. One important result of this analysis is that the mean-field gap must itself become a function of the magnetic field strength, $\Delta = \Delta(B)$.

Turning to S_η, after integrating out the fermions we expand up to second order in fluctuations and obtain a Gaussian action,

$$S_{\mathrm{fluct}} = \beta A\sum_q (h_{-q}\ \varphi_{-q})\left(\frac{\hat{\mathbf{1}}}{V} + \hat{\Pi}_q\right)\begin{pmatrix} h_q \\ \varphi_q\end{pmatrix}, \tag{8}$$

where we have put $\eta_q = h_q + i\varphi_q$, with h_q and φ_q real bosonic fields representing the Higgs and phase modes respectively. The polarization $\hat{\Pi}_q$ now contains all information about coupling to the underlying electrons, and has the form

$$\begin{aligned}\hat{\Pi}_q = \frac{N_\Phi}{2\beta A}\sum_{\epsilon_n}\sum_{l,l'}&\frac{|\langle l|e^{iq\hat{x}}|l'\rangle|^2}{\big[(i\epsilon_n)^2 - E_l^2\big]\big[(i\epsilon_n^+)^2 - E_{l'}^2\big]}\\ &\times\{[(i\epsilon_n + \xi_l)(i\epsilon_n^+ - \xi_{l'}) + (i\epsilon_n - \xi_l)(i\epsilon_n^+ + \xi_{l'})]\hat{\sigma}_0\\ &- [(i\epsilon_n + \xi_l)(i\epsilon_n^+ - \xi_{l'}) - (i\epsilon_n - \xi_l)(i\epsilon_n^+ + \xi_{l'})]\hat{\sigma}_2\\ &+ 2\Delta^2\hat{\sigma}_3\},\end{aligned} \tag{9}$$

where $\epsilon_n^+ = \epsilon_n + \omega_m$, $\hat{\sigma}_i$ are the Pauli matrices indexing the 2D fluctuation space, and the squared matrix element is from two factors of the electron-fluctuation coupling g,

$$\begin{aligned}\frac{1}{2\pi}\sum_{k_y} g^{ll'}_{\mathbf{q},k_y}g^{l'l}_{-\mathbf{q},k_y} &= N_\Phi\left|\int \mathrm{d}x\phi_l(x)\phi_{l'}(x)e^{iqx}\right|^2\\ &\equiv N_\Phi|\langle l|e^{iq\hat{x}}|l'\rangle|^2.\end{aligned} \tag{10}$$

This polarization has a similar form to what has been obtained previously when analyzing the fluctuations in BCS theory [47–49], but there are some notable differences. In particular, particle-hole symmetry ensures that the Higgs and phase modes decouple in systems at $B = 0$, but here there are nontrivial off-diagonal elements in $\hat{\Pi}_q$ due to the breaking of time-reversal symmetry by the magnetic field so these modes become mixed. To obtain the fluctuation contribution to the free energy we finally integrate out the bosonic fields and find

$$\Omega_{\text{fluct}}(B) = \frac{1}{2\beta}\sum_q \text{tr}\ \ln(\hat{\mathbf{1}} + V\hat{\Pi}_q). \tag{11}$$

IV. RESULTS

We are interested in the oscillatory components of these thermodynamic quantities as functions of the (inverse) magnetic field. We shall consider only regimes of weak magnetic fields where all of $\Delta(B)$, $\Omega_{\text{MF}}(B)$, and $\Omega_{\text{fluct}}(B)$ have oscillation amplitudes that are small compared to their zero-field values. We thus separate these into their nonoscillatory values taken for $B = 0$, denoted Δ_0, $\Omega_{\text{MF},0}$ and $\Omega_{\text{fluct},0}$, and their oscillatory parts yielding QOs, denoted $\tilde{\Delta}(B)$, $\tilde{\Omega}_{\text{MF}}(B)$, and $\tilde{\Omega}_{\text{fluct}}(B)$, which are our primary interest. We shall focus on the behavior of oscillations at the fundamental frequency, which is related to the area in reciprocal space in which the unhybridized bands overlap, set by the condition that $\delta[\epsilon_0/(2\omega_c)] = 1$, i.e., a frequency in $1/B$ of $m^*\epsilon_0/(2e)$. For a more detailed discussion of how we separate oscillatory and nonoscillatory contributions, and where contributions to the fundamental frequency oscillation arise, see Appendix B.

At the mean-field level the fundamental frequency oscillation of $\tilde{\Omega}_{\text{MF}}(B)$ is

$$\tilde{\Omega}_{\text{MF}}(B) \approx -\frac{2N_\Phi}{\pi}\cos\left(2\pi\frac{\epsilon_0}{2\omega_c}\right)R_1(T,\omega_c), \tag{12}$$

where temperature dependence is encapsulated in the function

$$R_p(T,\omega_c) = 2\pi T\sum_{n=0}^{\infty}\exp\left[-\frac{2\pi p}{\omega_c}\sqrt{\epsilon_n^2+\Delta_0^2}\right], \tag{13}$$

and $N_\Phi = eBA/h$ is the number of electrons in each filled Landau level. Note that $\Delta_0 = \Delta_0(T)$, with the temperature dependence of the gap determined through the gap equation. In the $T \to 0$ limit we have

$$R_p(T\to 0,\omega_c) \to \Delta_0 K_1\left(2\pi p\frac{\Delta_0}{\omega_c}\right), \tag{14}$$

where K_1 is the modified Bessel function of the second kind, which recovers the result reported in Ref. [43]. For the weak field regime $\omega_c \ll 2\pi\Delta_0$ the asymptotic form $K_1(x \gg 1) \sim \sqrt{\pi/2x}e^{-x}$ shows exponential suppression in $x = 2\pi\Delta_0/\omega_c$.

The oscillatory contribution of quantum fluctuations $\tilde{\Omega}_{\text{fluct}}(B)$ can be obtained by exact numerical evaluation of the full fluctuation free energy Eq. (11). We can also derive analytic results based on physically motivated approximations, giving better physical insight, and compare the two to verify both approaches. To begin this analytic approach we first note that fluctuations provide a small contribution to the total free energy, $|\Omega_{\text{fluct}}| \ll |\Omega_{\text{MF}}|$, meaning that the mean-field state is stable. Furthermore, examining the dependence of the polarization on frequency and momentum we see that the trace log in Eq. (11) is always the same sign, so the smallness of Ω_{fluct} cannot be attributed to cancellations from contributions at different q—Ω_{fluct} is small because $V\hat{\Pi}_q$ is small. Therefore, we can expand the logarithm in Eq. (11) and keep just the first term, proportional to the trace of the polarization. The remaining sums can be done exactly (see Appendix C), and we obtain

$$\Omega_{\text{fluct}} \approx \frac{N_\Phi^2 V}{4A}\sum_{l,l'}\left[1 - \frac{\xi_l\xi_{l'}}{E_lE_{l'}}\tanh\left(\frac{E_l}{2T}\right)\tanh\left(\frac{E_{l'}}{2T}\right)\right]\Bigg|_{\Delta_0}. \tag{15}$$

The first term here is a constant proportional to the total number of electrons squared, which can be ignored since we are concerned only with oscillatory quantities. We are then left with a constant multiplying two factors of the same sum, which can be evaluated with the Poisson summation formula,

$$\sum_{l=0}^{\infty}\frac{\xi_l}{E_l}\tanh\left(\frac{E_l}{2T}\right) = \int_0^\infty dx\frac{\xi(x)}{E(x)}\tanh\left(\frac{E(x)}{2T}\right) \times\left[1 + 2\sum_{p=1}^{\infty}(-1)^p\cos(2\pi px)\right], \tag{16}$$

where $\xi(x) = \omega_c x - \epsilon_0/2$ so that $\xi_l = \xi(l+1/2)$, and $E(x) = \sqrt{\xi(x)^2 + \Delta_0^2}$. Imposing the UV cutoff on these integrals and using $\Lambda \gg \epsilon_0 \gg \Delta_0$ to approximate them, the nonoscillatory term can be simply integrated, and the oscillatory term can be evaluated in terms of an infinite sum via contour integration, giving

$$\sum_{l=0}^{\infty}\frac{\xi_l}{E_l}\tanh\left(\frac{E_l}{2T}\right) \approx \frac{\delta n_e}{\rho_F\omega_c} - \frac{4}{\omega_c}\sum_{p=1}^{\infty}(-1)^p\sin\left(2\pi p\frac{\epsilon_0}{2\omega_c}\right)R_p(T,\omega_c), \tag{17}$$

written in terms of the constant

$$\begin{aligned}\delta n_e &= 2T\rho_F\log\left[\cosh\left(\frac{\sqrt{(\epsilon)^2+\Delta_0^2}}{2T}\right)\right]\Bigg|_{\epsilon=-\epsilon_0/2}^{\Lambda-\epsilon_0/2}\\ &\approx \rho_F\left(\sqrt{\left(\Lambda-\frac{\epsilon_0}{2}\right)^2+\Delta_0^2} - \sqrt{\left(\frac{\epsilon_0}{2}\right)^2+\Delta_0^2}\right)\\ &\approx \rho_F(\Lambda-\epsilon_0) = n_v - n_c,\end{aligned} \tag{18}$$

which is well approximated by the difference of the original valence and conduction band electron densities for $T \lesssim \Delta_0 \ll \epsilon_0 \ll \Lambda$, and the function

$$R_p(T,\omega_c) \equiv 2\pi T\sum_{n=0}^{\infty}\exp\left(-\frac{2\pi p}{\omega_c}\sqrt{\epsilon_n^2+\Delta_0^2}\right), \tag{19}$$

which after these approximations contains all remaining temperature dependence. The free energy is proportional to the square of Eq. (17), so the fundamental frequency term of $\tilde{\Omega}_{\text{fluct}}$ is found to be

$$\tilde{\Omega}_{\text{fluct}}(B,T) \approx -2\rho_F V\,\delta n_e A\sin\left(2\pi\frac{\epsilon_0}{2\omega_c}\right)R_1(T,\omega_c). \tag{20}$$

The behavior of this oscillatory energy is compared to the mean-field result for the case of $T = 0$ in Fig. 1, and the

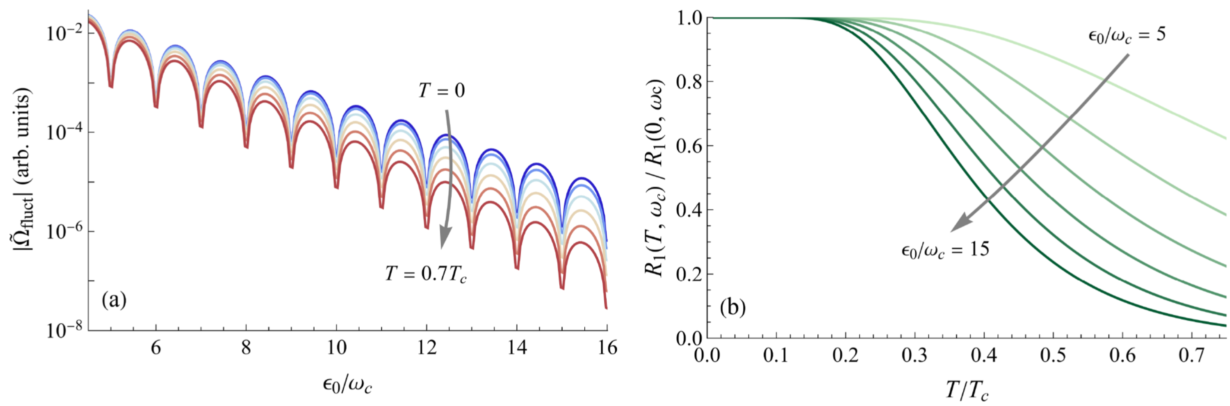

FIG. 3. Temperature dependence: (a) The oscillatory fluctuation free energy $\tilde{\Omega}_{\rm fluct}$ in Eq. (20) for $\Delta_0(T = 0) = 0.1\epsilon_0$ as a function of the inverse cyclotron frequency for several choices of temperature, from $T = 0$ (blue) up to $T = 0.7T_c$ (red). (b) The normalized function $R_1(T, \omega_c)$ giving the temperature dependence of the oscillation amplitudes for a range of values of the inverse cyclotron frequency as indicated. In both plots we see that oscillations are suppressed at higher temperatures, more strongly for larger values of the inverse cyclotron frequency (weaker magnetic fields).

effect of nonzero temperature on this result, including the self-consistent temperature dependence of the $B = 0$ gap $\Delta(T)$ determined numerically from the zero-field gap equation (see Appendix A), is shown in Fig. 3. We see that it is most sensitive to changes in temperature for large ϵ_0/ω_c, meaning weaker magnetic field strengths, and that for low temperatures ($T \lesssim 0.2T_c$) the oscillation amplitude is approximately constant for all considered values of ω_c. We consider only temperatures up to $T \approx 0.7T_c$, where T_c is the critical temperature of the excitonic insulating state, since closer to T_c the gap quickly becomes small and the picture of a large mean-field gap with small fluctuations, which is the fundamental assumption of this calculation, may start to break down. Though we do not provide the calculation here, the same temperature factor $R_p(T, \omega_c)$ is obtained for the oscillatory mean-field free energy as well, so a complementary picture as given in Fig. 3(a) for $\tilde{\Omega}_{\rm fluct}$ applies for $\tilde{\Omega}_{\rm MF}$. Because of this, temperature dependence of the oscillations are not a viable means to determine whether the mean-field or fluctuation contributions dominate the effect.

We verify the above approximate calculation by comparison with direct numerical evaluation of the entire fluctuation free energy Eq. (11), from which we extract the oscillatory part [50]. For this comparison we restrict to the low-temperature regime where the effect is largest and easiest to compute numerically. In our numerical analysis we use ϵ_0 as our unit of energy and set $\Lambda = 10$ and $\Delta_0(T = 0) = 0.06, 0.08$, or 0.10 (with $\rho_F V$ then fixed by the zero-field gap equation). The details of our numerical procedure are outlined in Appendix E. The numerical and analytic results for the oscillatory parts of the free energy using these parameters are plotted in Fig. 1. The close agreement we find between them validates the approximations used to derive Eq. (20) within this parameter regime.

V. DISCUSSION

Comparing $\tilde{\Omega}_{\rm MF}$ and $\tilde{\Omega}_{\rm fluct}$ in Eqs. (12) and (20) and Fig. 1, we find that the oscillatory part of the total free energy is easily dominated by contributions from quantum fluctuations. Both are exponentially suppressed for weak fields by the same factor R_1, but while the prefactor of $\tilde{\Omega}_{\rm MF}$ has only a linear dependence on the (small) magnetic field strength, $\tilde{\Omega}_{\rm fluct}$ depends on the interaction strength and the imbalance of electron densities between the two bands δn_e. This δn_e may be very large depending on ϵ_0, setting the carrier density, and Λ, parametrizing the valence bandwidth or the electron density in the valence band. The dependence on V shows that the size of these quantum oscillations may be used to probe interaction strengths in these insulating materials. Here the signature is in the fundamental oscillation frequency, so it differs from theories of interaction-induced harmonics for metallic systems [51,52]. We also find that, for both the mean field and fluctuations, the temperature dependence is rather different than the normal Lifshitz-Kosevich (LK) result, featuring a broad low-temperature regime almost completely insensitive to changes in temperature.

Perhaps most surprisingly we find there exists a parameter regime where the amplitude of $\tilde{\Omega}_{\rm fluct}$ can be even larger than oscillations found for the corresponding noninteracting gapless system obtained as the $\Delta \to 0$ limit of Eq. (12), equivalent to putting $V = 0$ from the start,

$$\tilde{\Omega}_{V=0}(B) \approx -\frac{N_\Phi \omega_c}{\pi^2} \cos\left(2\pi \frac{\epsilon_0}{2\omega_c}\right), \tag{21}$$

also shown in Fig. 1. Though oscillations for the insulator are exponentially suppressed as $B \to 0$, for low-electron density materials the regime where the oscillations remain large is in readily accessible ranges of magnetic fields. Our comparison of free energies here directly translates to the size of the de Haas–van Alphen effect, since magnetization is related to free energy by a derivative with respect to B; in all cases, the largest contribution is from this derivative acting on the oscillatory function, yielding a common factor of $\pi\epsilon_0/(B\omega_c)$.

Dominance of this fluctuation effect would have several hallmarks in experiment. First, we find a $\pi/2$ phase difference from LK and mean-field results; note the cosine in Eqs. (12) and (21) and the sine in Eq. (20). Second, the amplitude of Eq. (20) depends explicitly on V, beyond the implicit dependence through Δ_0 common to both Eq. (12) and

Eq. (20). Third, the amplitude goes as a specific power law in B; after accounting for the shared exponential suppression at low temperatures and small B, the leading contribution to the amplitude of de Haas–van Alphen oscillations is $\propto B^{-1/2}$ for the mean-field term, and $\propto B^{-3/2}$ for the fluctuation term.

To connect to excitonic insulators that can be realized experimentally we consider the $\mathrm{MoSe_2/WSe_2}$ devices examined in Ref. [6]. Carrier densities of $\sim 10^{12}\,\mathrm{cm}^{-2}$ can be achieved in these TMD double layers through gating, and using the effective masses $m_{c,v} \approx m_e/2$ with m_e the bare electron mass, this corresponds to $\epsilon_0 \sim 20\,\mathrm{meV}$. The range of ϵ_0/ω_c shown in Fig. 1 thus corresponds to $B \sim 5$–$20\,\mathrm{T}$. The bandwidth of the $\mathrm{WSe_2}$ valence band, related to Λ, is $\sim 1\,\mathrm{eV}$ [53], and the exciton binding energy, corresponding to Δ_0, is $\sim 100\,\mathrm{meV}$. This is further into the strong coupling regime than the theory we consider (indeed, we want $\Delta_0 \ll \epsilon_0$), but weaker interactions can be achieved in principle with larger spacing between TMD layers. Taking the spacing to be ~ 50 times larger than in [6] (~ 30–$50\,\mathrm{nm}$), so the binding energy is ~ 50 times smaller ($\Delta_0 \sim 0.2\,\mathrm{meV} \sim 0.1\epsilon_0$), we estimate that for $\omega_c \sim \Delta_0$, giving $B \sim 8.5\,\mathrm{T}$, the amplitude of magnetization oscillations from the fluctuation effect is $|\tilde{M}_{\mathrm{fluct}}| \sim 10\,\mathrm{A/m}$, well within the capabilities of standard torque measurements to detect.

We expect that the effect is very generic. Its origin can be alternatively understood with a complementary calculation beginning with the mean-field, Landau quantized fermionic action plus bosonic fluctuations. Instead of integrating out electrons to give a theory of the fluctuations alone, we can consider the fluctuations as mediating a residual interaction between electrons in the mean-field state—the mean field does not fully account for all effects of the interaction it decouples. We can then include the effect of the fluctuating modes by evaluating the Hartree and Fock self-energies that this interaction bestows on the fermions in the gapped mean-field bands. Demanding that the Hartree self-energy is equal to the gap Δ yields the gap equation, i.e., this condition enforces the constraints of mean-field theory. The Fock self-energy, on the other hand, is a purely quantum mechanical effect, and the contribution it provides to the free energy at first order in V with $\Delta = \Delta_0$, which can now be understood as an exchange energy, recovers Eq. (15) and therefore the oscillatory behavior generated by the fluctuations. This calculation in terms of a self-energy and the calculation presented in Sec. III are the two complementary ways of evaluating the diagrams in Fig. 4. As the field is swept at fixed total electron number, there is an oscillation in the density difference between the two bands, and this exchange energy contribution oscillates. Because the oscillation amplitude is determined by the relative densities of conduction and valence electrons it is much larger than the effect of the mean field, which instead has amplitude determined by the filling of individual Landau levels.

With this generic understanding of the effect, it seems natural that it will manifest in any system that is driven to an insulating state by interactions and in particular will surely be important for understanding all possible contributions to the observation of the de Haas–van Alphen effect in Kondo insulators [14–20]. Some theoretical works [23,26,28,29,32] have proposed that the observed large effect in these experiments results from a Fermi surface of exotic neutral degrees of freedom, guided by intuition from well-established theories of quantum oscillations which demand the presence of a Fermi surface [21,22]. Our results, however, provide a different starting point for understanding these experiments. Specifically, we find a surprisingly large effect in an interacting system with only gapped excitations—a Fermi surface is not a fundamental requirement for large quantum oscillations in some systems.

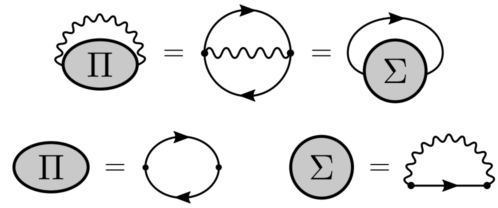

FIG. 4. The first-order loop diagram giving the main result for the fluctuation free energy, Eq. (15), expressed in terms of both the polarization Π of the bosonic fluctuations and the Fock self-energy Σ of the electrons. The straight solid lines are fermion Green's functions, the wavy lines are the fluctuation-mediated interaction parameterized by V, and the dots are the electron-fluctuation coupling as in Eqs. (6) and (7).

We have shown how the nature of quantum oscillations in excitonic insulators is principally determined by quantum fluctuations of the gap. Not only are these QOs significantly larger than what is obtained from just a mean-field treatment of these systems, for low-carrier-density semiconductors they can be even larger than the oscillations obtained from the noninteracting gapless state from which the excitonic insulator state arises. We suggest that the sort of TMD systems already shown to host excitonic insulating states are prime candidates to see this effect realized. Furthermore, though we have specifically focused on excitonic insulators here, we expect that an effect of this sort will generically manifest in interaction-driven insulators and will be important to consider in more general circumstances.

Data supporting this publication are available in the Apollo repository [54].

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APPENDIX A: MEAN-FIELD FREE ENERGY AND GAP

Integrating out the fermionic fields for S_{MF} yields the mean-field free energy,

$$\Omega_{\mathrm{MF}}(B) = \frac{\Delta^2 A}{V} - N_\Phi T \sum_{l,\alpha=\pm} \log(1 + e^{-\alpha E_l/T}). \tag{A1}$$

For $T \to 0$ this reduces to the energy of the mean field itself (the Δ^2 term) plus the sum over the occupied electronic states, i.e., the entire lower band. The stationary condition on this free energy determines the mean-field gap,

$$\frac{1}{V} = \frac{N_\Phi}{A}\sum_l \frac{\tanh\left(\frac{\sqrt{\xi_l^2+\Delta^2}}{2T}\right)}{2\sqrt{\xi_l^2+\Delta^2}}. \tag{A2}$$

For this equation to have a solution the gap itself must depend on B, so that $\Delta = \Delta(B)$. The same procedure for $B = 0$ would instead produce

$$\frac{1}{V} = \frac{1}{A}\sum_{\mathbf{k}} \frac{\tanh\left(\frac{\sqrt{\xi_k^2+\Delta_0^2}}{2T}\right)}{2\sqrt{\xi_k^2+\Delta_0^2}} \xrightarrow[T\to 0]{} \rho_F\left[\operatorname{arsinh}\left(\frac{\Lambda-\frac{\epsilon_0}{2}}{\Delta_0}\right)+\operatorname{arsinh}\left(\frac{\epsilon_0}{2\Delta_0}\right)\right], \tag{A3}$$

where we define Δ_0 as the value of the gap at $B = 0$, and also give the explicit solution for the zero-temperature limit, invoking a UV cutoff scale Λ.

We now introduce some general notation and assumptions we will use throughout the rest of our analysis: with the zero-field gap satisfying Eq. (A3), we can define $\delta\Delta(B) \equiv \Delta(B) - \Delta_0$ which contains all of the gap's magnetic field dependence. We assume $\delta\Delta(B)$ vanishes continuously as $B \to 0$ and restrict our focus to the regime of magnetic field for which $|\delta\Delta(B)| \ll \Delta_0$. We denote the specifically oscillatory part of $\delta\Delta(B)$ as $\tilde{\Delta}(B)$. For a generic $f(B)$ the two quantities $\delta f(B)$ and $\tilde{f}(B)$ defined in this way need not be the same, but with the approximations we make in our model we find that the two are equivalent for all quantities we will consider. In Ref. [43] this field-dependent component of the gap and its effect on QO in the mean-field approximation was explored in detail for model excitonic and Kondo insulators.

APPENDIX B: OSCILLATORY FREE ENERGY CONTRIBUTIONS

We are specifically interested in the oscillatory part of the free energy, $\tilde{\Omega}(B, \Delta(B))$, which is responsible for quantum oscillations of thermodynamic quantities like the magnetization via $\tilde{M}(B) = -\partial\tilde{\Omega}(B)/\partial B$. We can isolate this part as in Ref. [43] by first using $\Delta(B) = \Delta_0 + \tilde{\Delta}(B)$ with $|\tilde{\Delta}(B)| \ll \Delta_0$ to expand the free energy around $\Delta = \Delta_0$ in powers of $\tilde{\Delta}(B)$, then separating $\Omega(B, \Delta_0)$ into its $B = 0$ part $\Omega_0(\Delta_0)$ and its B-dependent oscillatory part $\tilde{\Omega}(B, \Delta_0)$. As done for Δ_0 and $\tilde{\Delta}$, we assume that $|\tilde{\Omega}| \ll \Omega_0$, which we verify *post hoc* by numerically evaluating the free energy without approximation. Writing the total free energy as the sum of mean field and fluctuation terms, altogether we obtain

$$\Omega(B, \Delta(B)) \approx \Omega_{\mathrm{MF},0} + \Omega_{\mathrm{fluct},0} + \tilde{\Omega}_{\mathrm{MF}}(B) + \tilde{\Omega}_{\mathrm{fluct}}(B) + \tilde{\Delta}(B)\frac{\partial}{\partial\Delta_0}(\Omega_{\mathrm{MF},0} + \Omega_{\mathrm{fluct},0}), \tag{B1}$$

where every Ω is evaluated at $\Delta = \Delta_0$. Any further terms in this expansion are necessarily at least second order in small oscillatory quantities, which contribute only to second and higher harmonic oscillations. Since our interest is in oscillations at the fundamental frequency, we drop these terms. The final term this expression, proportional to $\tilde{\Delta}$, also does not contribute further; Δ_0 is the value for which the $B = 0$ part of the free energy is stationary, so the derivative vanishes by definition. For the sort of model we have here it can be shown that the correction to the gap from fluctuations above the purely mean field value is negligible [48,49], so to a very good approximation Δ_0 is determined from just the mean field term, recovering exactly Eq. (A2). Thus, the largest oscillatory part of the free energy is the sum of two terms, $\tilde{\Omega}_{\mathrm{MF}}(B)$ and $\tilde{\Omega}_{\mathrm{fluct}}(B)$, which are just the oscillatory parts of the mean field and fluctuation free energies evaluated with the zero-field gap Δ_0.

APPENDIX C: EXPANSION OF THE TRACE LOG

As discussed in the main text, the fact that Eq. (11) is small can be attributed to $V\hat{\Pi}_q$ being small (it has eigenvalues with absolute value $\ll 1$). Using this and the above approximations we can expand the log to first order and write

$$\begin{aligned}
\Omega_{\mathrm{fluct}}(B) &\approx \frac{V}{2\beta}\sum_q \operatorname{tr}\hat{\Pi}_q\Big|_{\Delta=\Delta_0} \\
&= \frac{N_\Phi V}{2\beta^2 A}\sum_{q,\epsilon_n}\sum_{l,l'}\frac{|\langle l|e^{iq\hat{x}}|l'\rangle|^2}{[(i\epsilon_n)^2 - E_l^2][(i\epsilon_n^+)^2 - E_{l'}^2]} \\
&\quad\times[(i\epsilon_n + \xi_l)(i\epsilon_n^+ - \xi_{l'}) + (i\epsilon_n - \xi_l)(i\epsilon_n^+ + \xi_{l'})]|_{\Delta_0} \\
&= \frac{N_\Phi^2 V}{\beta^2 A}\sum_{\epsilon_n,\epsilon_{n'}}\sum_{l,l'}\frac{i\epsilon_n + \xi_l}{(i\epsilon_n)^2 - E_l^2}\frac{i\epsilon_{n'} - \xi_{l'}}{(i\epsilon_{n'})^2 - E_{l'}^2}\Big|_{\Delta_0} \\
&= \frac{N_\Phi^2 V}{4A}\sum_{l,l'}\left[1 - \frac{\xi_l}{E_l}\tanh\left(\frac{E_l}{2T}\right)\right]\left[1 + \frac{\xi_{l'}}{E_{l'}}\tanh\left(\frac{E_{l'}}{2T}\right)\right]\Big|_{\Delta_0} \\
&= \frac{N_\Phi^2 V}{4A}\sum_{l,l'}\left[1 - \frac{\xi_l\xi_{l'}}{E_l E_{l'}}\tanh\left(\frac{E_l}{2T}\right)\tanh\left(\frac{E_{l'}}{2T}\right)\right]\Big|_{\Delta_0}.
\end{aligned}$$

Because of these approximations, the off-diagonal elements of $\hat{\Pi}_q$ do not contribute, so the coupling between Higgs and phase modes is found not to be relevant for the dominant contribution to thermodynamic QO. Going the third line we perform the sum over $\mathbf{q}$, which can be done exactly, and seeing that ω_m only appears in the combination $\epsilon_n^+ = \epsilon_n + \omega_m$ we exchange the sum over ω_m with a sum over this fermionic frequency redefined as ϵ_n'. The Matsubara sums can then be performed exactly.

APPENDIX D: STABILITY OF THE MEAN FIELD

To obtain Eq. (15) we assumed that the total fluctuation free energy is small, allowing expansion of the trace log, and we now show that the results we obtain are consistent with the mean field being stable, $|\Omega_{\mathrm{fluct}}| \ll |\Omega_{\mathrm{MF}}|$. The primary contribution to Ω_{fluct} in this approximation is the nonoscillatory part, which has two contributions. The first is provided by the 1 inside the brackets in Eq. (15), which gives the square of the total electron density times $V/4A$. The second is the full nonoscillatory part of the remaining term, which is simply the product of the nonoscillatory terms of the Landau level sums, already obtained from the Poisson summation formula in Eq. (17). Overall, at low temperatures the nonoscillatory terms are thus

$$\begin{aligned}\Omega_{\text{fluct},0} &= \frac{VA}{4}\left(n_e^2 - \delta n_e^2\right) \approx VA\, n_v n_c \\ &= VA\,\rho_F\left(\Lambda - \frac{\epsilon_0}{2}\right)\rho_F\frac{\epsilon_0}{2} \sim \rho_F^2 VA\Lambda\,\epsilon_0. \end{aligned} \tag{D1}$$

This is to be compared to the nonoscillatory part of Ω_{MF}, which is straightforward to calculate for small T. For $T \to 0$ the mean-field energy is

$$\Omega_{\text{MF}} = \frac{\Delta^2 A}{V} - N_\Phi \sum_l E_l, \tag{D2}$$

and applying the Poisson summation formula the nonoscillatory part of this quantity is given by

$$\begin{aligned}\Omega_{\text{MF},0} &= \frac{\Delta_0^2 A}{V} - \rho_F A \int_0^\Lambda \mathrm{d}\xi \sqrt{\left(\xi - \frac{\epsilon_0}{2}\right)^2 + \Delta_0^2} \\ &\sim -\rho_F A\Lambda^2, \end{aligned} \tag{D3}$$

keeping just the largest term which sets the overall scale. The ratio of these energies is

$$\left|\frac{\Omega_{\text{fluct},0}}{\Omega_{\text{MF},0}}\right| \sim \frac{\rho_F^2 VA\Lambda\,\epsilon_0}{\rho_F A\Lambda^2} = \rho_F V\frac{\epsilon_0}{\Lambda} \ll 1, \tag{D4}$$

which is small since $\epsilon_0 \ll \Lambda$ and $\rho_F V < 1$ since we are considering the weak-coupling regime. Our approximate analysis is therefore consistent with the necessary condition $|\Omega_{\text{fluct}}| \ll |\Omega_{\text{MF}}|$. Note that the oscillatory components of these energies are irrelevant for this comparison; the overall coefficients of Eqs. (12) and (20) are both smaller than these nonoscillatory energies, and they are further suppressed by $K_1(2\pi\,\Delta_0/\omega_c)$, which is small even for $\omega_c \sim \Delta_0$ since $K_1(2\pi) \approx 0.001$.

While this demonstrates the consistency of our analysis with the assumption of stability of the mean field, to demonstrate that the mean field is actually stable we turn to numerical evaluation of this energy, which can be performed without these assumptions. This numerical analysis, the details of which are discussed in Section E, confirms that the fluctuation energy is indeed small compared to the mean-field energy; we find that $|\Omega_{\text{fluct}}/\Omega_{\text{MF}}| \sim 0.02$, with the exact value depending on the values of Δ_0 and ω_c. Furthermore, viewing the total free energy $\Omega(\Delta)$ as a potential that is minimized by the true value of Δ, we can compute Ω_{fluct} as a function of Δ for some fixed magnetic field strength to see how it shifts Δ from the value computed from the mean-field sector alone. The result of this analysis is shown in Fig. 5 for $\omega_c = \epsilon_0/5.5$, a value which should give the maximum dependence of Ω_{fluct} on Δ based on our analytic results Eq. (20). We see that at this point Δ_{tot}, the value minimizing $\Omega = \Omega_{\text{MF}} + \Omega_{\text{fluct}}$, is shifted by only a small fraction from Δ_{MF}, the value minimizing Ω_{MF} alone, confirming that the system is well approximated by mean-field theory.

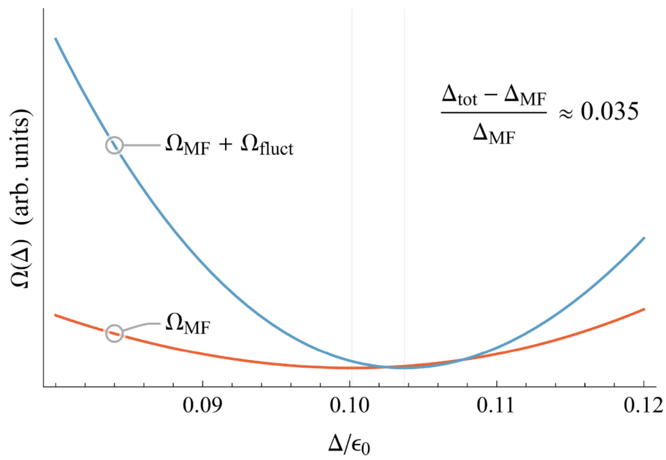

FIG. 5. The mean-field free energy Ω_{MF} and total free energy $\Omega_{\text{MF}} + \Omega_{\text{fluct}}$ as functions of the parameter Δ for $T = 0$ and $\omega_c = \epsilon_0/5.5$, giving a maximum in $|\tilde{\Omega}_{\text{fluct}}|$ and $\tilde{\Omega}_{\text{MF}} = 0$. The dimensionless interaction strength $\rho_F V$ used here is chosen so that Ω_{MF} is minimized for $\Delta_0 = 0.1$. We see that the inclusion of the fluctuation free energy shifts the value of Δ minimizing the total free energy by only ≈3.5%, confirming that the system is well approximated by mean-field theory. Since this value of ω_c maximizes $|\tilde{\Omega}_{\text{fluct}}|$ this is representative of the largest this shift in Δ can be.

APPENDIX E: NUMERICAL EVALUATION OF THE FLUCTUATION ENERGY

Computing the magnetic field dependence of the free energy of the bosonic fluctuations Eq. (11) for low temperatures first requires determining an appropriate set of values of ω_c to evaluate the function on. These values are chosen to satisfy several conditions. First, we restrict the range of values to be $\lesssim \Delta_0$ in order to compare with our analytic results, which are valid in this regime. Next, we note that by imposing a cutoff Λ we will naively find "quantum oscillations" at a large frequency associated with the "Fermi surface" of this cutoff, so we choose our values of ω_c so that these oscillations vanish, i.e., we use only ω_c such that $\sin(2\pi\,\Lambda/\omega_c) = 0$.

Numerical evaluation of the energy for each of these values is then simply a matter of computing a number of nested sums, which can be sped up by first analytically computing as many of the sums as possible. Starting with the outermost, we first note that only the magnitude of the momentum $\mathbf{q}$ appears, so the integral over its angle is trivial. Furthermore, this momentum appears only inside the matrix element

$$|\langle l|e^{iq\hat{x}}|l+r\rangle|^2 = \frac{l!}{(l+r)!}\left(\frac{\ell_B q}{\sqrt{2}}\right)^{2r} e^{-\frac{\ell_B^2 q^2}{2}} L_l^r\left(\frac{\ell_B^2 q^2}{2}\right)^2, \tag{E1}$$

where $L_l^r(x)$ are the associated Laguerre polynomials and ℓ_B is the magnetic length. The form of this function motivates a change of variables from q to the dimensionless $Q = \ell_B q/\sqrt{2}$. Analyzing this function, we find that it is exponentially small for Q much larger than $\sim\sqrt{l+r}$. Since the system is cut off at high energy by Λ, there is a largest value l_Λ that $l + r$ can take, defined as the largest l such that $\xi_l \leqslant \Lambda$, so there is a largest value of Q that we need to consider, which we take to be $Q_{\text{max}} = 2\sqrt{l_\Lambda}$.

We now consider the Matsubara sum over the frequency ω_m. In the $T \to 0$ limit, this sum becomes an integral, which we then approximate with another sum; this final sum is not the same as the original Matsubara sum because the integrand is evaluated at $T = 0$,

$$T \sum_{\omega_m} f(\omega_m, T) \to \int \mathrm{d}\omega f(\omega, 0) \approx \sum_{\omega} \delta\omega\, f(\omega, 0). \quad \text{(E2)}$$

For us the integrand is the trace log, so we need the polarization at $T = 0$. The form of $\hat{\Pi}_q$ in Eq. (9) is useful when anticipating an expansion of the trace log because it allows the two Matsubara sums to be treated on equal footing (and for nonzero temperature), but it is not in a convenient form for performing the sum over ϵ_n and then taking the $T \to 0$ limit. Therefore, we instead use

$$\begin{aligned}
\hat{\Pi}_q &= \frac{N_\Phi}{4\beta A} \sum_{\epsilon_n} \sum_{l,l'} \sum_{\alpha,\alpha'=\pm} \frac{|\langle l|e^{\mathrm{i}q\hat{x}}|l'\rangle|^2}{[\mathrm{i}\epsilon_n^+ e^{-\mathrm{i}\epsilon_n^+ 0^+} - \alpha E_l][\mathrm{i}\epsilon_n e^{-\mathrm{i}\epsilon_n 0^+} - \alpha' E_{l'}]} \\
&\times \begin{pmatrix} \left(1 - \frac{\xi_l \xi_{l'} - \Delta^2}{E_l E_{l'}}\right)\delta_{\alpha,\alpha'} + \left(1 + \frac{\xi_l \xi_{l'} - \Delta^2}{E_l E_{l'}}\right)\delta_{\alpha,-\alpha'} & -\mathrm{i}\left(\frac{\xi_l}{E_l} - \frac{\xi_{l'}}{E_{l'}}\right)\delta_{\alpha,\alpha'} - \mathrm{i}\left(\frac{\xi_l}{E_l} + \frac{\xi_{l'}}{E_{l'}}\right)\alpha\, \delta_{\alpha,-\alpha'} \\ \mathrm{i}\left(\frac{\xi_l}{E_l} - \frac{\xi_{l'}}{E_{l'}}\right)\delta_{\alpha,\alpha'} + \mathrm{i}\left(\frac{\xi_l}{E_l} + \frac{\xi_{l'}}{E_{l'}}\right)\alpha\, \delta_{\alpha,-\alpha'} & \left(1 - \frac{\xi_l \xi_{l'} + \Delta^2}{E_l E_{l'}}\right)\delta_{\alpha,\alpha'} + \left(1 + \frac{\xi_l \xi_{l'} + \Delta^2}{E_l E_{l'}}\right)\delta_{\alpha,-\alpha'} \end{pmatrix} \\
&\approx -\frac{N_\Phi}{4A} \sum_{l,l'} |\langle l|e^{\mathrm{i}q\hat{x}}|l'\rangle|^2 \Bigg[\left(\frac{1}{\mathrm{i}\omega_m e^{\mathrm{i}\omega_m 0^+} + E_l - E_{l'}} - \frac{1}{\mathrm{i}\omega_m e^{-\mathrm{i}\omega_m 0^+} + E_l - E_{l'}}\right) \begin{pmatrix} 1 - \frac{\xi_l \xi_{l'} - \Delta^2}{E_l E_{l'}} & -\mathrm{i}\left(\frac{\xi_l}{E_l} - \frac{\xi_{l'}}{E_{l'}}\right) \\ \mathrm{i}\left(\frac{\xi_l}{E_l} - \frac{\xi_{l'}}{E_{l'}}\right) & 1 - \frac{\xi_l \xi_{l'} + \Delta^2}{E_l E_{l'}} \end{pmatrix} \\
&+ \left(\frac{1}{i\omega_m e^{\mathrm{i}\omega_m 0^+} + E_l + E_{l'}} - \frac{1}{i\omega_m e^{-\mathrm{i}\omega_m 0^+} - E_l - E_{l'}}\right) \begin{pmatrix} 1 + \frac{\xi_l \xi_{l'} - \Delta^2}{E_l E_{l'}} & 0 \\ 0 & 1 + \frac{\xi_l \xi_{l'} + \Delta^2}{E_l E_{l'}} \end{pmatrix} \\
&+ \left(\frac{1}{i\omega_m e^{\mathrm{i}\omega_m 0^+} + E_l + E_{l'}} + \frac{1}{\mathrm{i}\omega_m e^{-i\omega_m 0^+} - E_l - E_{l'}}\right) \begin{pmatrix} 0 & -\mathrm{i}\left(\frac{\xi_l}{E_l} + \frac{\xi_{l'}}{E_{l'}}\right) \\ \mathrm{i}\left(\frac{\xi_l}{E_l} + \frac{\xi_{l'}}{E_{l'}}\right) & 0 \end{pmatrix} \Bigg],
\end{aligned} \quad \text{(E3)}$$

where $\epsilon_n^+ = \epsilon_n + \omega_m$ as before. This form is obtained by writing the fermionic Green's functions underlying this polarization in their diagonal basis, and the second equality is the result of performing the Matsubara sum over ϵ_n and discarding terms containing $n_F(E_l)$, which is exponentially small as $T \to 0$.

Notice that we have explicitly written the exponential regulators that ensure convergence of conditionally convergent Matsubara sums to the physically correct values. Here the sum over ϵ_n is absolutely convergent so these factors are not needed, but notice that after the sum the regulators do not all have the form we would naively assume—in half the terms, the exponents in the regulators have changed sign from the usual form. If we were not careful about tracking this sign change, then the bosonic sum would not give the correct result, and if we were to expand the trace log, we would not obtain Eq. (15)—we would obtain a -1 inside the bracket instead of a $+1$.

We take the infinitesimal 0^+ to be a small but finite number, the size of which determines the accuracy of the sum and the rate at which it converges; smaller values give more accurate answers at the expense of slower convergence. We take $0^+ \to 0.001$, which balances these two competing goals. For fixed Q and ω, $\hat{\Pi}_q$ can therefore be computed as the double-sum over Landau level indices given in Eq. (E3), using the full B-dependent gap Δ determined by numerically solving Eq. (A2) for $T = 0$ and the values of B corresponding to the chosen values of ω_c. Denoting the frequency sum over the trace log for a given Q as $W(Q)$ we have

$$\Omega_{\text{fluct}} = 2\rho_F \omega_c \int_0^{Q_{\max}} \mathrm{d}Q\, Q\, W(Q), \quad \text{(E4)}$$

which can be approximated by evaluating $W(Q)$, which is a well-behaved function, on a dense set of values of Q.

The result of this analysis is a value of Ω_{fluct} for each value of ω_c, which we can then straightforwardly analyze to extract the oscillatory component. A nonoscillatory background can be identified by using the expected sinusoidal behavior to identify where the oscillations should vanish, then this background can be subtracted to give the oscillatory behavior alone. Alternatively, we can Fourier transform the data as a function of $1/\omega_c$ and discard the low-frequency components, corresponding to the slowly varying nonoscillatory background.

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