Quantum state over time is unique

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(Received 5 December 2023; revised 6 May 2024; accepted 20 June 2024; published 6 August 2024)

The conventional framework of quantum theory treats space and time in vastly different ways by representing temporal correlations via quantum channels and spatial correlations via multipartite quantum states—an imbalance absent in classical probability theory. Since Leifer and Spekkens [Phys. Rev. A **88**, 052130 (2013)] called for a causally neutral formulation of quantum theory in their seminal work, numerous attempts have been made to rectify this asymmetry by proposing a dynamical description of a quantum system encapsulated by a static *quantum state over time*, without a definite consensus on which one is most appropriate. In this paper, we propose sets of operationally motivated axioms for quantum states over time alternative to the ones proposed by Fullwood and Parzygnat [Proc. R. Soc. A **478**, 20220104 (2022)], which we show is unable to induce a unique quantum state over time. Our proposed axioms are better suited to describe quantum states over any spacetime regions beyond two points. Through this reformulation, we prove that the Fullwood-Parzygnat state over time uniquely satisfies all these operational axioms, unifying the bipartite spacetime correlations of quantum systems.

DOI: 10.1103/PhysRevResearch.6.033144

I. INTRODUCTION

Quantum theory has been considered a generalization of the classical probability theory in the sense it is essentially for calculating probabilities of measurement outcomes [1]. However, there is a fundamental asymmetry between space and time in the conventional formalism of quantum theory. In the classical theory, both timelike and spacelike correlations can be described with joint distributions, but in quantum theory, the spacelike correlation can be expressed as a multipartite quantum state, while the time evolution is described with quantum channels. Leifer and Spekkens encapsulated this problem into the following question in their seminal work [2]: Is it really impossible to formalize a causally neutral quantum theory? There have been various attempts to solve this problem [3–16] and a particular effort lies in constructing *quantum state over time*. It enables us to map dynamical quantum processes to static quantum states over time, the quantum counterpart of the joint probability distributions, so every correlation can be expressed as a quantum state regardless of their causal structure, as in classical probability theory.

Multiple candidates have been presented as states over time (or quantum Bayes maps that could be induced by states over time through the result of Ref. [17]) including those by Ohya [18], Leifer-Spekkens [2], Wilde [19], Fitzsimons-Jones-Vedral (FJV) [20] and Sutter-Tomamichel-Harrow [21]. (Reference [17] gives a comprehensive introduction to states over time and their applications to the quantum Bayes' rule.) At one point, a no-go result by Horsman *et al.* [22] seemed to forbid the existence of a state over time that satisfies several mathematical axioms. Nevertheless, Fullwood and Parzygnat circumvented this [23] by appropriately adjusting the axioms to physically relevant forms and introducing a state over time based on the Jordan product that is equivalent to the FJV function for qubits [22]. However, whether the Fullwood-Parzygant (FP) function is the only one among its class was an open problem [23,24].

In this paper, we answer this in the affirmative. We achieved this by identifying a minimal set of axioms, such that a unique description for a quantum state over spacetime emerges. Our strategy is to elucidate the logical relation between different axioms that have been imposed on QSOT functions. We first propose a set of axioms whose operational meaning is clearer than the previous one. In particular, we show that the axioms employed previously are not strong enough to uniquely characterize a QSOT function. We also show that our uniqueness result is robust against a slight change of the axioms by identifying four equivalent yet different sets of axioms and studying the logical relations between them.

II. QUANTUM STATE OVER TIME

First, we introduce some basic notation: quantum systems and their associated Hilbert spaces are denoted by A, B, etc. Let $\mathfrak{C}(A,B)$ denote the set of quantum channels from A to B, and $\mathfrak{B}(A,B)$ as the set of operators acting on A and output to B. If $A=B$, we denote them by $\mathfrak{C}(A)$ and $\mathfrak{B}(A)$. Finally, let $\mathfrak{S}(A)$ and $\mathfrak{H}(A)$ be the respective set of quantum states and Hermitian operators on A.

Let A and B be quantum systems at different time steps connected by a quantum channel $\mathcal{E}_{B|A}$. When the initial

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quantum state is prepared as ρ_A, the bottom-line requirement for the quantum state over time would be as follows.

Definition 1. A *quantum state over time* (QSOT) *function*, or a *star product*, $\star : \mathfrak{C}(A, B) \times \mathfrak{S}(A) \to \mathfrak{B}(AB)$, is a function that satisfies

$$\mathrm{Tr}_A \mathcal{E}_{B|A} \star \rho_A = \mathcal{E}_{B|A}(\rho_A), \quad (1)$$

$$\mathrm{Tr}_B \mathcal{E}_{B|A} \star \rho_A = \rho_A. \quad (2)$$

The resultant bipartite operator $\rho_{AB} = \mathcal{E}_{B|A} \star \rho_A$ is called a quantum state over time.

One can immediately notice the lack of positivity in ρ_{AB} in contrast to a conventional quantum state. However, it has been pointed out in previous works on QSOT that positivity poses severe difficulty [22] that can only be circumvented by compromising operationally more meaningful properties such as linearity [2]. In fact, one should refrain from blindly imposing positivity to QSOTs, considering positivity is equivalent to nonnegative measurement probabilities, while it is still unclear what the characterization of measurements implementable upon QSOTs is.

Moreover, the framework of the nonpositive quantum state is rapidly gaining relevance in the recent development of quantum information theory. For example, *virtual* quantum processes utilize simulatability of nonpositive quantum states at the cost of computational overhead [25], with various applications such as quantum resource distillation [26], error mitigation [27–30], and quantum state broadcasting [31]. This implies that quantum state over time allows for spatial simulation of temporal quantum correlation through such virtual processes. See Supplemental Material Sec. IV [32] for further discussion.

III. ON LINEARITY

If there is uncertainty in the input state and process so they are probabilistic mixtures of other states or processes, then we would naturally expect the resultant QSOT to be the mixture of the corresponding QSOTs. Therefore, it has been commonly argued [22,23] that a QSOT function should satisfy two linearities: *process linear*, namely, linear in the first argument, and *state linear*, i.e., linear in the second argument. If a QSOT function sarisfies both properties, then it is *bilinear* [33]. Our first result is the identification of two physical axioms, completeness and compositionality, which jointly underpin bilinearity.

Axiom (E): Completeness

For any quantum state over spacetime ρ_{AE} with two arbitrary regions A and E in spacetime, and any quantum channel $\mathcal{E}_{B|A}$, the action of QSOT function on a subsystem $\mathcal{E}_{B|A} \star \rho_{AE}$ can be defined and has the following properties: For any completely positive trace nonincreasing operation $\mathcal{I}_E$ on system E,

$$\mathcal{I}_E[\mathcal{E}_{B|A} \star \rho_{AE}] = \mathcal{E}_{B|A} \star \mathcal{I}_E(\rho_{AE}). \quad (3)$$

The completeness axiom, i.e., **(E),** conveys the requirement that the state over time provides a consistent description when seeing A as a subsystem of a larger system AE. This is analogous to how complete positivity is defined as positivity of a linear map acting on any subsystem of a joint system. Axiom **(E)** is therefore very intuitive, yet powerful enough to induce many useful properties. For instance, one can easily observe that **(E)** implies $\mathrm{Tr}_E[\mathcal{E}_{B|A} \star \rho_{AE}] = \mathcal{E}_{B|A} \star \rho_A$ and $\mathcal{E}_{B|A} \star (\rho_A \otimes \sigma_E) = (\mathcal{E}_{B|A} \star \rho_A) \otimes \sigma_E$.

While completeness is motivated by purely physical considerations, it turns out that Axiom **(E)** implies state linearity [34] and can be extended to the converse. In other words, for any QSOT function $\star$ satisfying **(E)**, the following holds for any state ρ_{AE} over spacetime:

$$\mathcal{E}_{B|A} \star \rho_{AE} = (\mathcal{E}^\star \otimes \mathrm{id}_E)(\rho_{AE}), \quad (4)$$

where $\mathcal{E}^\star$ is the mapping $\rho_A \mapsto \mathcal{E}_{B|A} \star \rho_A$ and id_E is the identity channel on E. Conversely, every state-linear QSOT function satisfies **(E)** through Eq. (4). The proofs of these claims are found in Supplemental Material Sec. I [32].

Although the completeness axiom is a natural property to expect, surprisingly not every known QSOT satisfies it. In particular, the Leifer-Spekkens QSOT function violates it [2]. Given the established role of the Leifer-Spekkens function in inducing the Petz recovery map via Bayesian retrodiction, and its extensive use in thermodynamics [35], its incompleteness poses a conceptual challenge.

Observing in Eq. (1) that the map $\rho_A \mapsto \mathrm{Tr}_A \circ \mathcal{E}_{B|A} \star \rho_A$ is linear, for any bipartite state ρ_{AE} over spacetime, one can define $\mathrm{Tr}_A[\mathcal{E}_{B|A} \star \rho_{AE}] = (\mathcal{E}_{B|A} \otimes \mathrm{id}_E)(\rho_{AE})$ consistently. This leads us to the introduction of the compositionality axiom, i.e., **(P)**, independent of **(E)**.

Axiom (P): Compositionality [24]

A QSOT function should be compatible with composition of quantum channels. In other words, for any two quantum channels $\mathcal{E}_{B|A}$ and $\mathcal{F}_{C|B}$, we have

$$\mathrm{Tr}_B[\mathcal{F}_{C|B} \star (\mathcal{E}_{B|A} \star \rho_A)] = (\mathcal{F} \circ \mathcal{E})_{C|A} \star \rho_A. \quad (5)$$

Fundamentally speaking, **(P)** says that the QSOT function is essentially determined by the corresponding time expansion, $(\mathrm{id}_{A'|A} \star \rho_A)$, the function that expands a single-time state ρ_A via trivial dynamics. More concretely, we denote the identity channel between isomorphic systems A and B as $\mathrm{id}_{B|A}$ [36].

Proposition 1. For any QSOT function $\star$, axiom **(P)** is equivalent to that for any $\mathcal{E}_{B|A} \in \mathfrak{C}(A, B)$ and $\rho_A \in \mathfrak{S}(A)$:

$$\mathcal{E}_{B|A} \star \rho_A = (\mathrm{id}_A \otimes \mathcal{E}_{B|A'})(\mathrm{id}_{A'|A} \star \rho_A). \quad (6)$$

Moreover, it implies that $\star$ is process linear, and the definition of $\star$ can be linearly extended to arbitrary linear maps $\mathcal{E}_{B|A}$ that may not be a quantum channel through Eq. (6).

We prove Proposition 1 in Supplemental Material Sec. II A [32]. In summary, axioms **(E)** and **(P)** force a QSOT function to be bilinear. In other words, the failure of fulfilling bilinearity leads to the malfunction of QSOT in multipartite settings. This partially answers, in the negative, one of the open problems on the possibility of operationally meaningful states over time that are process nonlinear [17]. However, we remark that not every bilinear QSOT function satisfies **(P)**—highlighting that one should base guiding intuition on the physical considerations while understanding bilinearity as a necessary implication instead. (Remark 2, Supplemental Material Sec. IV [32]).

The compositionality axiom is arguably simpler and more operational compared to the associativity [37] given in Ref. [23]. Perhaps a simpler form of associativity could be [23]

$$(\mathcal{F} \star \mathcal{E}) \star \rho = \mathcal{F} \star (\mathcal{E} \star \rho), \tag{7}$$

where the star product of $\mathcal{E}$ and $\mathcal{F}$ is understood as $(\mathcal{F} \star \mathcal{E})(\sigma) := \mathcal{F} \star \mathcal{E}(\sigma)$ for all $\sigma \in \mathfrak{S}(A)$. However, it is unclear if Eq. (7) is applicable to QSOT functions that may not satisfy Eq. (6) because it is not immediately clear if the definition of $\star$ can be extended to arbitrary linear maps. On the other hand, Corollary 1 and Proposition 1 show that axioms **(E)** and **(P)** circumvent this issue, and thus we have

$$\begin{aligned}(\mathcal{F}_{C|B} \star \mathcal{E}_{B|A}) \star \rho_A &= [(\mathcal{F}_{C|B} \star \cdot_B) \circ \ \mathcal{E}_{B|A}] \star \rho_A \\ &= [(\mathcal{F}_{C|B} \star \cdot_B) \circ \ \mathcal{E}_{B|A'}](\mathrm{id}_{A'|A} \star \rho_A) \\ &= \mathcal{F}_{C|B} \star (\mathcal{E}_{B|A} \star \rho_A),\end{aligned} \tag{8}$$

i.e., Eq. (7) follows. This means **(P)** can safely replace associativity in Eq. (7), whenever we assume state-linearity. Furthermore, we also note that compositionality for any two quantum channels ensures compositionality for arbitrarily many channels (Supplemental Material Sec. II B [32]).

IV. ON TIME-REVERSAL SYMMETRY

Fundamentally speaking, the state over time should exhibit a causally neutral structure between inputs and outputs of a process. In particular, suppose we have a classical state P_{XY} describing the input and output of a channel E. When E is the identity channel, $P_{XY} = P_{YX}$, i.e., the classical state over time is invariant under a swap of inputs and outputs. This property yields the following axiom for quantum states over time:

> **Axiom (T): Time-reversal symmetry**
> A state over time corresponding to the trivial evolution should be symmetric under the time reversal transformation, i.e., $F_{AB}(\mathrm{id}_{B|A} \star \rho_A)F_{AB} = \mathrm{id}_{B|A} \star \rho_A$ for all $\rho_A \in \mathfrak{S}(A)$, where F_{AB} denotes the swap gate between systems A and B.

In previous literature, Hermiticity is more commonly discussed compared to **(T)**. They are equivalent if one associates time-reversal operation with complex conjugation, as in Ref. [23]. However, although it is plausible when applied to the Kraus operator of quantum channels [38], there is no *a priori* reason for complex conjugation to represent the time reversal of quantum states. Meanwhile, **(T)** is already effective in operationally capturing the essence of causally neutral structures. We discuss in detail the difference in technical implications for both axioms in a later section.

A. On classical limit

Naturally, a given state over time should reduce to a classically correlated bipartite state between input and output systems when the process is effectively a classical channel. To this end, the *classical limit axiom* has been widely employed [17,22,23]. In particular, given channel $\mathcal{E}$, let $\mathscr{D}[\mathcal{E}]$ denote the (Jamiołkowski) channel state [39],

$$\mathscr{D}[\mathcal{E}] := (\mathrm{id}_A \otimes \mathcal{E}_{B|A'})(F_{AA'}), \tag{9}$$

recalling that $F_{AA'}$ denotes the swap gate between systems A and A', which is a copy of A. Furthermore, denote the identity operator on B as $\mathbb{1}_B$. The classical limit axiom states that whenever the commutator $[\mathscr{D}[\mathcal{E}], \rho_A \otimes \mathbb{1}_B] = 0$, then

$$\mathcal{E}_{B|A} \star \rho_A = \mathscr{D}[\mathcal{E}](\rho_A \otimes \mathbb{1}_B). \tag{10}$$

However, it appears to be stronger than what it aims to achieve. One direct consequence of Eq. (10) is that the state over time for the maximally mixed input state $\pi_A := |A|^{-1}\mathbb{1}_A$ is the channel state $\mathscr{D}[\mathcal{E}]$ up to the normalization factor. We first explicitly spell it out as an independent axiom.

> **Axiom (J): Jamiołkowski**
> For a system A with dimension $|A|$, the state over time associated with the maximally mixed state π_A is
> $$\mathcal{E}_{B|A} \star \pi_A = \frac{1}{|A|}(\mathrm{id}_A \otimes \mathcal{E}_{B|A'})(F_{AA'}). \tag{11}$$

Favoring the Jamiołkowski state over other alternatives such as the Choi matrix is a quantum-exclusive feature that does not appear in classical systems. Hence, let us reformulate the classical limit axiom to eliminate its reliance on **(J)**. This refinement yields a conceptually clearer criterion, which we term classical conditionability, whose formal definition is elaborated on in Appendix A.

> **Axiom (CC): Classical conditionability** (informal)
> When the input state and the channel are prepared in an ensemble $\{\lambda_i, \pi_{A_i}, \mathcal{E}_{B|A_i}\}$, then the corresponding QSOT is given as $\mathcal{E}_{B|A} \star (\sum_i \lambda_i \pi_{A_i}) = \sum_i \lambda_i \mathcal{E}_{B|A_i} \star \pi_{A_i}$, where $\mathcal{E}_{B|A} = \sum_i \mathcal{E}_{B|A_i}$.

As its name suggests, **(CC)** requires the QSOT to be consistent with conditioning on classical information. A noteworthy insight is that the classical limit axiom is effectively a composite result derived from the interplay of axioms **(J)** and **(CC)**. In other words, Axioms **(CC)** and **(J)** are jointly equivalent to the classical limit axiom as represented by Eq. (10). We present the proof in Supplemental Material Sec. III A [32]. Moreover, Axioms **(CC)** and **(J)** have logical dependence under other assumptions. See Appendix A for further discussion.

B. Uniqueness of the Fullwood-Parzygnat state

Having carefully reasoned out a set of physically motivated, fundamental axioms, we now introduce the Fullwood-Parzygnat QSOT function $\star_{\mathrm{FP}}$ (or the *FP function* for short) given as $\mathcal{E}_{B|A} \star_{\rm FP} \rho_A := \frac{1}{2}\{\rho_A \otimes \mathbb{1}_B, \mathscr{D}[\mathcal{E}]\}$, where $\{X, Y\} = XY + YX$ is the Jordan product (or anticommutator).

The FP function has several useful mathematical properties, i.e., *Hermiticity, bilinearity, preservation of classical limit* and *associativity*. These properties were proposed earlier by Fullwood and Prazygnat as axioms that should characterize any reasonable candidate for a state over time [17,22–24]. Although the trace of a quantum state over time should be normalized to one, it has been observed that such a state need *not* be positive. This does not necessarily mean that its definition is pathological, similarly to the negative sign associated with time in the spacetime interval in relativity [24]. We carefully discuss the subtleties of linearity and nonpositive properties in Supplemental Material Sec. IV [32]. It was an open problem if the FP function is unique in satisfying such conditions [23]. Our main result is to solve this problem in the affirmative.

Theorem 1. The FP function $\star_{\rm FP}$ is the only QSOT function that satisfies **(E), (P), (T),** and **(CC)**.

The proof is contained in Supplemental Material Sec. V [32]. In other words, the FP function is indeed the unique QSOT function satisfying the operationally motivated set of axioms formulated in consideration of multipartite settings, considering that a satisfactory state over time should be consistent with mixed causal structures [2].

V. ALTERNATIVE CHARACTERIZATIONS OF THE FP STATE OVER TIME

A natural following question is whether the set of axioms in Theorem 1 is unique and minimal for singling out the FP function as the QSOT function. We show that they are not by exploring alternative sets of axioms for QSOT functions.

We start by discussing an axiom that has often been used to impose a reduction to the classical limit. Let us first observe in classical probability theory a simple way to update probability distributions given a process map. Suppose we start with some input distribution $P_X(x)$, which we may encode in a diagonal matrix $[D_p]_{xx} := P_X(x)$. Furthermore, any classical channel can be fully specified by the conditional distribution matrix $C_{yx} := P_{Y|X}(y|x)$, and the joint distribution of input and output distributions of this channel, $J_{yx} := P_{XY}(x, y)$, can be obtained from a simple matrix multiplication:

$$J = CD_p. \tag{12}$$

Observe also that the matrix D_p is a natural way to *render* the input distribution into a self-adjoint, positive-semidefinite operator, and furthermore $D_p|x\rangle = P_X(x)|x\rangle$. One may then ask themselves: What would be the quantum analog of this?

Axiom (QC): Quantum conditionability

For every state $\rho \in \mathfrak{S}(A)$, there exists a state-rendering function Θ_ρ [17,40,41] on $\mathfrak{B}(A)$ such that

$$\mathcal{E}_{B|A} \star \rho_A = (\Theta_\rho \otimes \mathrm{id}_B)(\mathcal{E}_{B|A} \star \mathbb{1}_A) \tag{13}$$

for all $\mathcal{E} \in \mathfrak{C}(A, B)$, where Θ_ρ is linear, and for any $M \in \mathfrak{B}(A)$, whenever $[\rho, M] = 0$, we have $\Theta_\rho(M) = \rho M$.

This axiom is a generalization of Eq. (12): First, Θ_ρ renders the input state analogously to D_p. Meanwhile, the conditional state over time $\mathcal{E}_{B|A} \star \mathbb{1}_A$ acts as a propagator encoding the full process, and therefore is analogous to C, while $\mathcal{E}_{B|A} \star \rho_A$ is analogous to the joint distribution J.

Axiom **(QC)** is often employed to enable conditioning on arbitrary input states ρ through Θ_ρ and the conditional quantum state $\mathcal{E}_{B|A} \star \mathbb{1}_A$. In particular, **(QC)** tries to start with a causally neutral formulation of quantum theory resembling classical probability theory, where conditioning is *always* possible.

Proposition 2. A self-adjoint state-rendering function that is linear in ρ must be of the form $\Theta^\mu_\rho(M) = \mu\rho M + (1-\mu)M\rho$ for real number μ. If Θ_ρ is also positive-semidefinite, then μ is between 0 and 1.

The proof of Proposition 2 can be found in Supplemental Material Sec. VI B [32]. Two extreme cases of such functions are known as the left bloom $\Theta^L_\rho(M) := M\rho$ and the right bloom $\Theta^R_\rho(M) := \rho M$. However, all the convex sums other than the symmetric bloom, $\Theta^S_\rho(M) := \frac{1}{2}(\rho M + M\rho)$ yield a state over time that is not Hermitian, and would yield a time-expansion function $\mathrm{id}_{B|A} \star \rho_A$ that is asymmetric under time reversal. This observation motivates us to investigate the axiom of Hermiticity and discuss its interplay with **(T)**.

Axiom (H) Hermiticity

For any quantum channel and state $\mathcal{E}_{B|A}$, ρ_A, the state over time $\mathcal{E}_{B|A} \star \rho_A$ must be Hermitian.

Recall from Theorem 1 that **(E)**, **(P)**, **(T)**, and **(CC)** give us a unique QSOT function. What happens if we substitute **(T)** with **(H)**? To answer this, note that our previous discussions, together with our discussions on compositionality versus associativity, show that **(E)**, **(P)**, **(H)**, and **(CC)** imply the original set of axioms considered in Ref. [23]: Hermiticity, bilinearity, preservation of classical limit and *associativity* (except for positivity, see Supplemental Material Sec. IV [32]). However, we show that **(H)** is weaker than **(T)** because it cannot induce a unique QSOT function (Supplemental Material Sec. VII [32]). Alternatively, Proposition 2 tells us that **(H)** requires a stronger version of **(QC)**, namely, **(QC+SA)**, which requires Θ_ρ to be self-adjoint to arrive at a unique characterization of the FP function. See Appendix B for further discussion. We summarize the equivalence relation between sets of axioms that singles out the FP function in the following theorem.

Theorem 2. The following sets of axioms are equivalent, and satisfied only by the FP function:

$$\textbf{(E)} + \textbf{(P)} + \textbf{(CC)} + \textbf{(T)} \quad \textbf{or} \quad \textbf{(E)} + \textbf{(P)} + \textbf{(J)} + \textbf{(T)}$$
$$\textbf{or}\ \textbf{(E)} + \textbf{(QC)} + \textbf{(T)} \quad \textbf{or} \quad \textbf{(E)} + \textbf{(QC + SA)} + \textbf{(H)}.$$

VI. CONCLUSIONS

We demonstrate the restoration of symmetry between spatial and temporal correlations in quantum theory by establishing the uniqueness of quantum states over spacetime: (1) We first showed that the axioms introduced in literature do not yield a unique QSOT function and (2) introduced a set of operationally motivated axioms for QSOT functions with a focus on its application in multipartite settings that is stronger than the one given in Ref. [23]. (3) We also analyzed the (in)equivalence relations between alternative sets of axioms and (4) characterized the Fullwood-Parzygnat function as the unique QSOT function. Interestingly, the mathematical technique [42] originally employed in demonstrating the no-go result [22] turns out to also be the key in establishing uniqueness. This achievement marks a significant milestone in developing a causally neutral quantum theory framework. With this groundwork being laid, further attempts can now be directed towards discovering the various applications of this formalism [24,43].

Recently, we became aware of an independent work of Parzygnat *et al.* that characterizes the FP function from a different set of axioms [31].

ACKNOWLEDGMENTS

We thank A. J. Parzygnat and J. Fullwood for helpful discussions. This work was supported by the startup grant of the Nanyang Assistant Professorship awarded to N.H.Y.N. of Nanyang Technological University, Singapore. S.H.L. was also supported by the startup fund of the Ulsan National Institute of Science and Technology, Korea.

APPENDIX A: ON CLASSICALITY AXIOMS

First, we introduce the technical definition of axiom **(CC)** that was informally introduced in the main text.

Axiom (CC): Classical conditionability

Consider a quantum channel $\mathcal{E}_{B|A}$ such that $\mathcal{E}_{B|A}\circ(\sum_i \mathrm{Ad}_{\mathbb{1}_{A_i}}) = \mathcal{E}_{B|A}$, where $A = \bigoplus_i A_i$ and $\mathrm{Ad}_X(\cdot) := X(\cdot)X^\dagger$. Then, for any probability distribution $\{\lambda_i\}$, we have

$$\mathcal{E}_{B|A} \star \left(\sum_i \lambda_i \pi_{A_i}\right) = \sum_i \lambda_i \mathcal{E}_{B|A_i} \star \pi_{A_i}, \qquad \text{(A1)}$$

where $\mathcal{E}_{B|A_i}$ is the limitation of $\mathcal{E}_{B|A}$, i.e., $\mathcal{E}_{B|A_i} = \mathcal{E}_{B|A}\circ \mathrm{Ad}_{\mathbb{1}_{A_i}}$.

Can we weaken the constraint of the classical limit axiom, say, to either axiom **(CC)** or **(J)**? We answer this affirmatively, and comment on the proof technique with the full proof in Supplemental Material Sec. III B [32].

Proposition 3. Assuming axioms **(E)** and **(P)**, axioms **(CC)** and **(J)** are equivalent, and furthermore they imply quantum conditionability **(QC)**.

We remark that **(J)** is applicable when A is a subspace of a larger Hilbert space. This is physically plausible; for example, a system with an energy cutoff could be considered a system on its own. If we reject this and consider a weaker version of **(J)** called **($\hat{\mathbf{J}}$)** applicable only to full systems, then the equivalence with **(CC)** under **(E)** and **(P)** breaks down. See Supplemental Material Sec. III C [32] for more information.

APPENDIX B: ON QUANTUM CONDITIONABILITY

Let us scrutinize the condition $\Theta_\rho(M) = \rho M$ whenever $[\rho, M] = 0$. This condition is motivated by a reduction to the classical limit with commuting algebras. However, the following result shows that, without **(T)**, this often-employed condition is actually not strong enough to induce either of **(CC)** or **(J),** assuming axioms **(E)** and **(P)**. This highlights the importance of a proper quantum conditional state. See Supplemental Material Sec. VI A [32] for the proof.

Proposition 4. Axiom **(QC)** with **(E)** implies **(P)**, however, they cannot imply either **(CC)** or **(J)**. Nevertheless, when supplemented with axiom **(T)**, axiom **(QC)** implies both **(CC)** and **(J)**.

Propositions 3 and 4 show that **(QC)** is logically weaker than **(CC)** or **(J)**. What if we strengthen it by requiring the state-rendering function Θ_ρ to be self-adjoint or, more strongly, positive-semidefinite with respect to the Hilbert-Schmidt inner product like its classical counterpart? It is shown in Proposition 2 that such modifications of **(QC)** can narrow down the set of QSOT functions, but not to a unique one.

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[32] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.6.033144 for further discussion on physical significance of the axioms in the main text and mathematical proofs.
[33] Note that we can extend the domain to make $\star$ homogeneous in each argument by letting $(\lambda\mathcal{E}_{B|A})\star\rho_A=\mathcal{E}_{B|A}\star(\lambda\rho_A)=\lambda[\mathcal{E}_{B|A}\star(\rho_A)]$ for all $\lambda\in\mathbb{C}$.
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[36] Technically, there cannot be "the" identity channel between different spaces A and B, but we follow the convention in which isomorphic quantum systems are identified so one can talk about the same states in different spaces, e.g., $|1\rangle_A$ and $|1\rangle_B$ are the same states but in different spaces.
[37] This axiom states that

$$\mathscr{D}^{-1}\left[|A|\mathcal{F}_{C|B}\star\left(\frac{\mathscr{D}[\mathcal{E}_{B|A}]}{|A|}\right)\right]\star\rho_A=\mathcal{F}_{C|B}\star(\mathcal{E}_{B|A}\star\rho_A). \quad \text{(B1)}$$

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