Thermometry with a dissipative heavy impurity

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Improving the measurement precision of low temperature is significant in fundamental science and advanced quantum technology application. However, the measurement precision of temperature T usually diverges as T tends to zero. Here, by utilizing a heavy impurity to measure the temperature of a Bose gas, we obtain the Landau bound to precision $\delta^2 T \propto T^2$ to avoid the divergence. When the initial momentum of the heavy impurity is close to be fixed and nonzero, the measurement precision can be $\delta^2 T \propto T^3$ to break the Landau bound. We derive the momentum distribution of the heavy impurity at any moment and obtain the optimal measurement precision of the temperature by calculating the Fisher information. As a result, we find that increasing the expectation value P_0 and reducing the variance $\Delta/2$ of the initial momentum can help to improve the measurement precision. Moreover, under certain conditions, $\sqrt{\Delta}/P_0$ is a relevant parameter, the smallness of which helps improve the thermometric precision of the probe. In addition, the momentum measurement is the optimal measurement of the temperature in the case that the initial momentum is close to be fixed and not equal to zero. The kinetic energy measurement is the optimal measurement in the case that the expectation value of the initial momentum is close to zero. Finally, we obtain that the temperatures of two Bose gases can be measured simultaneously. The simultaneous measurement precision is proportional to T^2 when two temperatures are close to T.

I. INTRODUCTION

With the rapid development of quantum technology, precise measurement of low temperature is becoming an important and significant subject in the field of quantum metrology [1–7] and quantum thermodynamics [8–10]. The measurement of low temperature has always been a challenging task due to the fact that the uncertainty of temperature diverges as the temperature tends to zero [11,12]. Strong couplings [13,14], periodic driving [15], and correlations among multiple probes induced by the common bath [16] have been used to slow down the divergence. In order to truly avoid the divergence, quantum system with a vanishing gap [17], a non-Markovian reservoir [18], and invariant subspaces due to the polariton thermalization [19] have been utilized to obtain the Landau bound to precision $\delta^2 T \propto T^2$. In these cases, the measurement uncertainty is not only not divergent but also getting smaller and smaller as the temperature decreases to zero.

With the progress of experimental techniques in quantum gases, the high-resolution imaging of distinguishable impurities has been realized [20–22]. The strength of the impurity-environment coupling can be tuned to obtain many significant physical results. The motion of mobile impurities through one-dimensional quantum liquid or gas has been extensively studied [23–27]. A heavy impurity moving through a Luttinger liquid [28] was explored to find that the friction force experienced by the impurity behaves as the fourth power of temperature (T^4). By controlling the system parameter, the friction force can dramatically change its temperature dependence from T^4 to T^8 in Bose liquid [29,30]. In recent work [31], the low-temperature T^2 dependence of the friction force has been obtained for a strong coupling between a heavy impurity and a Bose gas, which is contrasted with the expected T^4 scaling for a weak coupling. In order to obtain the friction force from the Bose gas, the temperature of the Bose gas should be measured accurately in advance.

In this article, we use a heavy impurity (probe) to measure the temperature of a Bose gas. Based on the relation between the friction force and the momentum being linear, we analytically derive the momentum distribution of the heavy impurity at any time given by the initial Gaussian distribution. The optimal temperature measurement precision can be analytically derived by achieving the Fisher information from the momentum distribution of the heavy impurity. As a result, we show that the Landau bound to precision $\delta^2 T \propto T^2$ can be obtained in the general case. More importantly, when the initial momentum of the heavy impurity is close to be fixed and nonzero, the measurement precision can be $\delta^2 T \propto T^3$ to break the Landau bound. And we find that enhancing the expectation value and reducing the variance of the initial momentum can help to improve the measurement precision. Under certain conditions, the smaller the relevant parameter $\sqrt{\Delta}/P_0$ is, the higher the measurement precision of the temperature. In the case that the initial momentum is close to being fixed and not equal to zero, we show that the momentum measurement is

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the optimal measurement of the low temperature. When the initial momentum is not fixed and the expectation value is not equal to zero, the measurement precision proportional to T^2 can still be obtained by using the momentum measurement. The kinetic energy measurement just happens to be the optimal measurement in the case that the expectation value of the initial momentum is close to zero. Finally, we show that the simultaneous measurement precision is proportional to T^2 when the temperatures of two gases are close to T.

This article is organized as follows. In Sec. II, we introduce the model of the heavy impurity in the Bose gas. In Sec. III, the momentum distribution of the heavy impurity is derived given by the initial Gaussian distribution. In Sec. IV, the optimal estimation precision is obtained by the Fisher information. In Sec. V, two practical measurements are used to obtain the temperature measurement precision. The simultaneous estimation of two temperatures is discussed in Sec. VI. We make a conclusion and a discussion on the feasibility of the experiment in Sec. VII.

II. HEAVY IMPURITY IN A BOSE GAS

We consider the system of a mobile heavy impurity in a Bose gas composed of one-dimensional interacting bosons. The total Hamiltonian is described by [32]

$$H_T = H_B - \int dx \hat{\Psi}^\dagger \frac{\hbar^2}{2M} \partial_x^2 \hat{\Psi} + G \int dx \hat{\Psi}^\dagger \hat{\Psi} \hat{\psi}^\dagger \hat{\psi}, \quad (1)$$

where H_B denotes the Lieb-Liniger Hamiltonian of the Bose gas

$$H_B = - \int dx \hat{\psi}^\dagger \frac{\hbar^2}{2m} \partial_x^2 \hat{\psi} + g \int dx \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi}, \quad (2)$$

$\hat{\Phi}(x) = \Psi(x) [\psi(x)]$ denotes the bosonic field operator for the heavy impurity with the mass M (the single boson with the mass m), which satisfies the commutation relations $[\hat{\Phi}(x), \hat{\Phi}^\dagger(x')] = \delta(x - x')$ and $[\hat{\Phi}(x), \hat{\Phi}(x')] = 0$ with $\delta(x - x')$ being the Dirac delta function. In the Bose gas, the contact interaction between bosons are repulsive and weak, i.e., $\hbar^2 n_0/m \gg g > 0$ with n_0 being the mean density of the bosons. In the case of the weak interaction between bosons, the quasiparticles of the Hamiltonian H_B have the Bogoliubov dispersion relation [33] $\epsilon_P = \sqrt{v^2 p^2 + p^4/4m^2}$ with the sound velocity $v = \sqrt{gn_0/m}$, and G denotes the density-density interaction strength between the heavy impurity and the Bose gas.

III. DISTRIBUTION FUNCTION OF THE IMPURITY

Due to the fact that the heavy impurity collides with thermally excited bosons in the Bose gas, the heavy impurity motion is stochastic. The momentum distribution function $f(t, P)$ of the heavy impurity ($M \gg m$) can be characterized by the Fokker-Planck form [34,35]

$$\frac{\partial f(t, P)}{\partial t} = \frac{\partial}{\partial P} \left[-F f(t, P) + \frac{1}{2} \frac{\partial D f(t, P)}{\partial P} \right], \quad (3)$$

where P represents the momentum of the heavy impurity. F denotes the friction force due to scattering f thermally excited

quasiparticles in the Bose gas, which is given by

$$F = -\frac{m^2 v^2 P}{2\pi \hbar M \tilde{T}} \int_0^\infty dk \frac{k^2 |r(k, \tilde{G})|^2 (2 + k^2)}{\sinh^2 \left(k \frac{\sqrt{4+k^2}}{4\tilde{T}} \right) \sqrt{4 + k^2}}, \quad (4)$$

where the dimensionless parameters $\tilde{T} = \frac{T}{mv^2}$ and $\tilde{G} = \frac{G}{\hbar v}$, and $r(k, \tilde{G})$ denotes the reflection amplitude of the Bogoliubov quasiparticles scattering off a heavy impurity, which has been studied by the Bogoliubov-de Gennes theory [36]. At low temperatures, $\tilde{T} \ll 1$ (only the low-energy quasiparticles are excited), in the cases of weak coupling ($\tilde{G} \ll 1$) or strong coupling ($\tilde{G} \gg 1$) the friction force F can be simplified as [31]

$$F = -\Gamma(G) P T^n, \quad (5)$$

where the exponent n of the temperature depends on the relationship between $\tilde{T}$ and $\tilde{G}$. By referring to Ref. [31], we can get the following specific classification. For $1/\tilde{G} \ll \tilde{T} \ll 1$, $n = 2$ and $\Gamma(G) = \frac{2\pi}{3\hbar M v^2}$. For $\tilde{T} \ll 1/\tilde{G} \ll 1$, $n = 4$ and $\Gamma(G) = \frac{8\pi^3 G^2}{15 \hbar^3 m^2 M v^8}$. For $\tilde{G} \ll 1$ and $\tilde{T} \ll 1$, $n = 4$ and $\Gamma(G) = \frac{2\pi^3 G^2}{15 \hbar^3 m^2 M v^8}$. D is the impurity diffusion coefficient, which is given by

$$D = -2FMT/P. \quad (6)$$

When the friction force F is proportional to the momentum P as shown in Eq. (5), the Fokker-Planck form in Eq. (3) can be solved analytically (see Appendix A). In general, the initial momentum distribution $f(0, P)$ is Gaussian, which is described by

$$f(0, P) = \frac{1}{\sqrt{\pi \Delta}} \exp \left[-\frac{(P - P_0)^2}{\Delta} \right], \quad (7)$$

where $\Delta/2$, P_0 represent the variance and the expectation value of the initial momentum, respectively.

By analytically solving the Fokker-Planck equation in Eq. (3), we can obtain the momentum distribution at time t with the initial Gaussian momentum distribution as shown in Eq. (7),

$$f(t, P) = \frac{1}{\sqrt{2\pi M T (1 - e^{-2t/\tau}(1 - \Delta'))}}$$

$$\times \exp \left[-\frac{(P - P_0 e^{-t/\tau})^2}{2M T (1 - e^{-2t/\tau}(1 - \Delta'))} \right], \quad (8)$$

where $\Delta' = \Delta/(2MT)$ and $\tau = 1/(\Gamma T^n)$ with the abbreviation $\Gamma \equiv \Gamma(G)$ throughout the rest of this article.

IV. FISHER INFORMATION FROM GAUSSIAN MOMENTUM DISTRIBUTION

According to the Cramér-Rao bound [37], the measurement precision of the temperature can be given by

$$(\delta T)^2 \geqslant \frac{1}{N \mathcal{F}[T]}, \quad (9)$$

where N represents the total number of repeated experiments. Since the content of our next study has nothing to do with the number of measurements, we simply set $N = 1$. $\mathcal{F}[T]$

denotes the Fisher information of the temperature T, which is described by

$$\mathcal{F}(T) = \int_{-\infty}^{\infty} dP \frac{[\partial_T f(t, P)]^2}{f(t, P)}, \tag{10}$$

where the partial derivative $\partial_T f(t, P) = \frac{\partial f(t, P)}{\partial T}$.

The general form of the Fisher information is shown in Appendix B. When $\Delta \gg MT$ and the measurement time t is not infinite, we can obtain the Fisher information in the low temperature limit $T^n t \Gamma \ll 1$:

$$\mathcal{F}(T, \Delta \gg MT) = \frac{2(n t \Gamma)^2 T^{2n-2}(P_0^2 + \Delta)}{\Delta}, \tag{11}$$

where we need to emphasize that Γ and n are dependent on the values of $\tilde{G}$. From the above equation, we can see that the Fisher information can be increased by increasing the initial expectation values of momentum. When $\tilde{T} \ll 1/\tilde{G}$, the coefficient Γ increases by four times with the coupling strength, i.e., $\Gamma(\tilde{G} \gg 1) = 4\Gamma(\tilde{G} \ll 1)$. Namely, the Fisher information increase 4 times with the coupling strength. When the coupling strength continues to increase until $\tilde{G} \gg 1/\tilde{T}$, the power of temperature goes from $n = 4$ to $n = 2$. Hence, in the case of $\tilde{G} \gg 1/\tilde{T}$, the Fisher information is proportional to the square of the temperature $[\mathcal{F}(T) \propto T^2]$, and it is independent of the coupling strength, which shows that the measurement precision of temperature will not be improved when the coupling strength increases to a certain extent. As the temperature T approaches zero, the Fisher information $\mathcal{F}(T)$ will approach zero. It means that the uncertainty of the temperature T is divergent at $T = 0$. More importantly, we find that Fisher information $\mathcal{F}(T, \Delta \gg MT)$ increases linearly as the relevant parameter P_0^2/Δ increases. In other words, the smaller $\sqrt{\Delta}/P_0$ is, the higher the measurement precision of the temperature.

When $\Delta \ll MRT$ with $R = e^{2T^n \Gamma t} - 1$ and t is a nonzero finite value ($t > 0$), the Fisher information in the low temperature limit $T^n t \Gamma \ll 1$ is given by

$$\mathcal{F}(T, \Delta \ll MRT) = \left[\frac{(1+n)^2}{2T^2} + \frac{n^2 P_0^2 T^{n-3} \Gamma t}{2M}\right](1 - \xi/2)$$
$$- n(1+n)\Gamma t T^{n-2} \xi + \mathcal{O}(\xi^2), \tag{12}$$

where the small quantity $\xi = \frac{\Delta}{MRT}$ and $\mathcal{O}(\xi^2)$ represents infinitesimally small quantities of the second order. We emphasize that one can not take the $t \to 0$ limit, as for small values of t, the above formula becomes invalid (the condition $\Delta \ll MRT$ becomes invalid).

When T is close to zero ($n = 4$), we obtain the Fisher information $\mathcal{F}(T) \approx \frac{25}{2T^2} \ll \frac{25}{2}\left(\frac{2M\Gamma t}{\Delta}\right)^{2/5}$. The achievable Fisher information is bounded from above, and the bound depends on how small Δ is. At close to a fixed initial momentum, i.e., $\Delta \to 0$, the uncertainty of T will be close to zero as T is close to zero, which avoids the divergence of measurement uncertainties at extremely low temperatures. A simple summary is that when the uncertainty of the initial momentum distribution is close to zero, the measurement precision of the temperature changes from infinity to zero at the temperature $T \to 0$. This result tells us that reducing the uncertainty of the initial momentum distribution can effectively further improve the measurement precision of the temperature when $\Delta \ll MRT$.
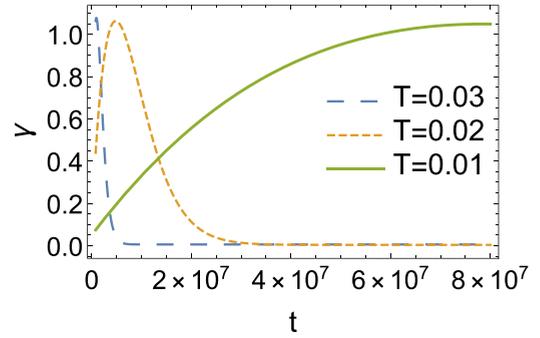
FIG. 1. Evolution diagram of the ratio of the Fisher information at different low temperatures. Here, γ denotes the ratio of the Fisher information $\mathcal{F}(T)$ at time t and the Fisher information $\mathcal{F}_\tau(\Delta \to 0)$ at the characteristic time τ. The dimensionless parameters chosen are given by $P_0 = 1$, $M = 1$, $n = 4$, and $\Gamma = 1$.

It is consistent with the previous trend in Eq. (11), except that P_0^2/Δ is not a relevant parameter in Eq. (12).

Next, we consider that the measurement time t can be arbitrarily large. By optimizing the measurement time to obtain a better measurement precision, we consider that the measurement time is the characteristic time of the distribution function $f(t, P)$, i.e., $t = \tau = 1/(\Gamma T^n)$. When T is close to zero and $\Delta \gg MT$, the Fisher information is given by

$$\mathcal{F}_\tau(\Delta \gg MT) = \frac{2n^2(P_0^2 + \Delta)}{T^2 \Delta}. \tag{13}$$

The above equation recovers the scaling in Ref. [13]. Contrary to the result in Eq. (11), we find that $\mathcal{F}_\tau(\Delta \gg MT)$ becomes larger and larger as the temperature T decreases. The essential reason is that the measurement precision of temperature close to zero is improved by using the time resource tending to be infinite. The Fisher information $\mathcal{F}_\tau(\Delta \gg MT)$ also increases linearly with the relevant parameter P_0^2/Δ, which is the same as the previous result.

When T is close to zero, $\Delta \ll MT$ and $P_0 \neq 0$, the Fisher information is given by

$$\mathcal{F}_\tau(\Delta \ll MT) = \frac{n^2 P_0^2}{MT^3(e^2 - 1)}\left(1 - \frac{\xi'}{2e^2 - 2}\right) + \mathcal{O}(\xi'^2), \tag{14}$$

where $\xi' = \frac{\Delta}{MT}$.

As shown in Fig. 1, we can see that the maximum ratio $\gamma = \mathcal{F}(T)/\mathcal{F}_\tau(\Delta \to 0)$ approaches one. It means that $\mathcal{F}_\tau(\Delta \to 0)$ is close to the optimal Fisher information. When $\Delta \to 0$, the characteristic time τ is close to the optimal measurement time at low temperature. Comparing Eqs. (12) and (14), the Fisher information changes from being proportional to $1/T^2$ to $1/T^3$ in the case of $P_0 \neq 0$. This is also due to the use of time resources that tend to be infinite.

When the initial momentum P_0 is zero, the Fisher information at the characteristic time τ is given by

$$\mathcal{F}_\tau(\Delta \to 0, P_0 = 0) = \frac{(2n + e^2 - 1)^2}{2T^2(e^2 - 1)^2}. \tag{15}$$

This result shows that the initial nonzero momentum (P_0) can help to effectively use the time resources, and thus improve the measurement precision of the temperature.

V. TEMPERATURE MEASUREMENT WITH MOMENTUM

In the previous section, we give the optimal measurement precision for a given probability distribution by the Fisher information. Then we will use the error propagation formula to show whether the specific measurement can approach the optimal measurement result.

First, we obtain the information of the temperature T by the measurement of the momentum P. Given by the momentum distribution in Eq. (2), we achieve the expectation value $\bar{P}$ and the variance $\delta^2 P = \bar{P^2} - \bar{P}^2$, which are described by

$$\bar{P} = P_0 e^{-t/\tau}, \tag{16}$$

$$\delta^2 P = MT[1 - e^{-2t/\tau}(1 - \Delta')]. \tag{17}$$

The uncertainty of the temperature T can be derived by the error propagation formula [38]

$$\delta^2 T = \frac{\delta^2 P}{|\partial_T \bar{P}|^2} = \frac{MT(1 - e^{-2t/\tau}(1 - \Delta'))}{(P_0 nt\, \Gamma)^2 T^{2n-2} e^{-2t/\tau}}. \tag{18}$$

From the above equation, we can derive that the optimal measurement time can be close to the characteristic time τ. At time τ, the corresponding measurement precision of the temperature T obtained by the measurement of momentum can be given by

$$\delta^2 T|_{t=\tau} = \frac{MT(1 - e^{-2}(1 - \Delta'))}{(P_0 n)^2 T^{-2} e^{-2}} \tag{19}$$

$$\approx \frac{(e^2 - 1)MT^3 + T^2\Delta}{(P_0 n)^2}. \tag{20}$$

When $P_0 \neq 0$ and $\Delta \rightarrow 0$, we can find that $\delta^2 T|_{t=\tau} = 1/\mathcal{F}_\tau(\Delta \rightarrow 0)$. It shows that the momentum measurement is the optimal measurement of the low temperature in the case that the initial momentum is close to being fixed ($\Delta \rightarrow 0$) and not equal to zero ($P_0 \neq 0$). When the initial momentum is not fixed ($\Delta > 0$) and the expectation value is not equal to zero ($P_0 \neq 0$), the measurement precision proportional to T^2 can still be obtained by using the momentum measurement. This means that the momentum measurement is close to the optimal measurement as T tends to zero in the case of $P_0 \neq 0$ and $\Delta > 0$.

However, when $P_0 = 0$, one can obtain that the uncertainty $\delta^2 T|_{t=\tau} \rightarrow \infty$. It shows that the measurement of momentum is the worst measurement, which can not obtain any information of the temperature T in the case of $P_0 = 0$.

A. Temperature measurement with kinetic energy

In order to deal with the case of $P_0 = 0$, we try to use the kinetic energy measurement $P^2/(2M)$. The expectation value and the variance of P^2 are given by

$$\bar{P^2} = MT[1 - e^{-2t/\tau}(1 - \Delta')], \tag{21}$$

$$\delta^2 P^2 = 2M^2T^2[1 - e^{-2t/\tau}(1 - \Delta')]^2. \tag{22}$$

By substituting the above equations into the error propagation formula, the measurement precision of the temperature is derived

$$\delta^2 T = \frac{T^2[2(e^{2t/\tau} - 1)MT + \Delta]^2}{2[MT(e^{2t/\tau} - 1) + nt(2MT - \Delta)/\tau]^2}. \tag{23}$$

For the infinite measurement time $t \rightarrow \infty$, the measurement precision of the low temperature obtained by the kinetic energy measurement is $\delta^2 T = 2T^2$, which is consistent with the results obtained by the Fisher information in the previous section. For $t = \tau$ and $\Delta \rightarrow 0$, the measurement precision of the low temperature is $\delta^2 T = \frac{2(e^2-1)^2 T^2}{(e^2 - 1 + 2n)^2}$, which is consistent with the result as shown in Eq. (15). For $\Delta > 0$, the measurement precision of the low temperature is $\delta^2 T = \frac{T^2}{2n^2}$, which is also consistent with the result by the Fisher information in Eq. (13). Therefore, in the case of $P_0 = 0$, the temperature measurement with the kinetic energy is the optimal measurement.

VI. SIMULTANEOUS MEASUREMENT OF TWO TEMPERATURES

We consider that the heavy impurity passes through the first Bose gas with temperature T_1 and the second Bose gas with temperature T_2. Given the initial Gaussian momentum distribution of the Heavy impurity as shown in Eq. (7), the momentum distribution at time $t_1 + t_2$ is described by

$$f(t_1 + t_2, P) = \frac{1}{\sqrt{2\pi MT_2\left[1 - e^{-2t_2/\tau_2}\left(1 - \frac{\Delta_1}{2MT_2}\right)\right]}}$$
$$\times \exp\left[-\frac{(P - P_0 e^{-t_1/\tau_1 - t_2/\tau_2})^2}{2MT_2\left[1 - e^{-2t_2/\tau_2}\left(1 - \frac{\Delta_1}{2MT_2}\right)\right]}\right], \tag{24}$$

where t_1 (t_2) is the interaction time between the heavy impurity and the first (second) Bose gas, $\Delta_1 = 2MT_1[1 - e^{-2t_1/\tau_1}(1 - \frac{\Delta}{2MT_1})]$, and the characteristic time of the ith Bose gas is $\tau_i = 1/(\Gamma_i T_i^n)$ for $i = \{1, 2\}$.

At time $t = t_1 + t_2$, the two temperatures can be simultaneously measured by the momentum distribution. The estimation precision of (T_1, T_2) is governed by its covariance matrix $\text{Cov}(T_1, T_2)$, which is lower bounded via the multiparameter Cramér-Rao bound [39]

$$\text{Cov}(T_1, T_2) \geqslant \frac{1}{\chi}, \tag{25}$$

where χ is the Fisher information matrix, which is derived by

$$\chi_{jk} = \int_{-\infty}^{\infty} dP \frac{\partial_{T_j} f(t, P) \partial_{T_k} f(t, P)}{f(t, P)}, \tag{26}$$

where $j, k = \{1, 2\}$. The total simultaneous uncertainty of the two temperatures is achieved by

$$(\delta^2 T_1 + \delta^2 T_2)|_{\text{sim}} = \text{tr}[\text{Cov}(T_1, T_2)] \tag{27}$$

$$\geqslant \frac{\chi_{11} + \chi_{22}}{\chi_{11}\chi_{22} - |\chi_{12}|^2}. \tag{28}$$

The simultaneous uncertainty of the two temperatures can be analytically derived by the above equation. Especially,

when $\Delta \to 0$, $P_0 \neq 0$, $T_1 = T_2 = T$, and $t_j = \tau_j$, the simultaneous estimation precision of the two low temperatures is given by

$$(\delta^2 T_1 + \delta^2 T_2)|_{\text{sim}} \geqslant 6.89625 T^2. \tag{29}$$

This result shows that the simultaneous estimation of the two temperatures can also obtain very high measurement precision, which is proportional to T^2 when both temperatures are close to T. However, the scaling T^3 can not be obtained like the measurement precision of a single temperature. This is mainly due to the fact that the variance of the momentum distribution, $\Delta_1/2$, after passing through the first Bose gas is not zero. Therefore, by controlling the initial momentum distribution, the measurement precision obtained by two separate temperature measurements is higher than that obtained by the simultaneous two-temperature measurement.

VII. CONCLUSION

In this article, we have proposed that the temperature of the Bose gas can be measured by the mobile heavy impurity. Due to the scattering off thermally excited quasiparticles in the Bose gas, the momentum distribution of the heavy impurity will carry the information about the temperature of the Bose gas. Based on the Fisher information from the momentum distribution of the heavy impurity, the optimal measurement precision can be analytically derived. Due to the fact that the momentum is continuous, the Landau bound to precision $\delta^2 T \propto T^2$ can be obtained in the general case. When the initial momentum of the heavy impurity is close to be fixed and nonzero, we obtain the measurement precision beyond the Landau bound to be $\delta^2 T \propto T^3$. Enhancing the expectation value of the initial momentum can help to improve the measurement precision. Reducing the uncertainty of the initial momentum distribution can further improve the measurement precision of the temperature. More importantly, when $\Delta \gg MT$, $\sqrt{\Delta}/P_0$ is the relevant parameter, the smallness of which helps improve the thermometric precision of the probe. In addition, we show that the momentum measurement is the optimal measurement when the initial momentum is close to be fixed and not equal to zero. The kinetic energy measurement just happens to be the optimal measurement when the expectation value of the initial momentum is equal to zero. Finally, we find that the simultaneous measurement precision of the temperatures of the two Bose gases can also reach the Landau bound. However, the scaling T^3 can not be obtained like the measurement precision of a single temperature.

Our scheme can be performed in the setup with cold atoms, such as a one-dimensional quantum liquid of ^{4}He atoms confined within a porous material [40], and an ultracold Rb gas with single neutral Cs impurity atoms [20]. The interaction strength between the impurity and host atoms is tunable by the Feshbach resonances, and powerful measurement techniques have been developed [23,41].

APPENDIX A: THE SOLUTION OF THE FOKKER-PLANCK FORM

The Fokker-Planck form can be rewritten as

$$\frac{\partial f(t, P)}{\partial t} = -\frac{1}{\tau} \mathcal{H} f(t, P), \tag{A1}$$

where $\tau = |P/F| = (\Gamma T^n)^{-1}$ and the operator $\mathcal{H}$ is described by

$$\mathcal{H} = -MT \frac{\partial^2}{\partial P^2} - \frac{\partial}{\partial P} P. \tag{A2}$$

The general form of the solution can de described as

$$f(t, P) = e^{-\mathcal{H}t/\tau} f(0, P). \tag{A3}$$

Defining $\tilde{y} = P/\sqrt{2MT}$ and $\tilde{p} = -i\frac{\partial}{\partial P}$, the operator $\mathcal{H}$ is rewritten as

$$\mathcal{H} = \tfrac{1}{2}(\tilde{p} - i\tilde{y})^2 + \tfrac{1}{2}\tilde{y}^2 - \tfrac{1}{2}. \tag{A4}$$

Then, let us make a transformation to get an operator $\tilde{\mathcal{H}}$, which is similar with the Hamiltonian of a harmonic oscillator

$$\tilde{\mathcal{H}} = \exp(\tilde{y}^2/2)\mathcal{H}\exp(-\tilde{y}^2/2) = \tfrac{1}{2}\tilde{p}^2 + \tfrac{1}{2}\tilde{y}^2 - \tfrac{1}{2}. \tag{A5}$$

The eigenvalues (E_n) and eigenvectors $[\phi_n(\tilde{y})]$ of the operator $\tilde{\mathcal{H}}$ can be given by

$$E_n = n, \phi_n(\tilde{y}) = (-1)^n \left(\frac{1}{\sqrt{\pi}2^n n!}\right)^{1/2} e^{y^2/2} \frac{d^n}{dy^n} e^{-y^2}. \tag{A6}$$

Setting $f(t, P) = e^{-\tilde{y}^2/2}\tilde{f}(t, P)$, we can obtain that

$$\frac{\partial \tilde{f}(t, \tilde{y})}{\partial t} = -\frac{1}{\tau} \tilde{\mathcal{H}} \tilde{f}(t, \tilde{y}). \tag{A7}$$

The solution of theabove equation can be achieved by

$$\tilde{f}(t, \tilde{y}) = e^{-\tilde{\mathcal{H}}t/\tau} \tilde{f}(0, \tilde{y}) \tag{A8}$$

$$= \sum_n e^{-\tilde{\mathcal{H}}t/\tau} A_n \phi(\tilde{y}) \tag{A9}$$

$$= \sum_n e^{-nt/\tau} A_n \phi(\tilde{y}), \tag{A10}$$

where the coefficient A_n is derived by

$$A_n = \int_{-\infty}^{\infty} \tilde{f}(0, P) \phi^*(\tilde{y}) dy. \tag{A11}$$

We consider that the initial momentum distribution is Gaussian, which is described by

$$f(0, P) = \frac{1}{\sqrt{\pi \Delta}} \exp\left[-\frac{(P - P_0)^2}{\Delta}\right]. \tag{A12}$$

The corresponding momentum distribution $f(0, \tilde{y})$ is given by

$$f(0, \tilde{y}) = \frac{1}{\sqrt{\pi \Delta'}} \exp\left[-\frac{(\tilde{y} - P_0')^2}{\Delta'}\right], \tag{A13}$$

where $\Delta' = \frac{\Delta}{2MT}$ and $P_0' = P_0/\sqrt{2MT}$. Next, the initial momentum distribution after the transformation is given by

$$\tilde{f}(0, \tilde{y}) = \frac{1}{\sqrt{\pi \Delta'}} \exp\left[-\frac{(\tilde{y} - P_0')^2}{\Delta'} + \frac{\tilde{y}^2}{2}\right]. \quad (A14)$$

Expanding it with the eigenvectors of the harmonic oscillator operator $\tilde{\mathcal{H}}$, the expansion coefficient can be obtained

$$A_n = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi \Delta'}} \exp\left[-\frac{(\tilde{y} - P_0')^2}{\Delta'} + \frac{\tilde{y}^2}{2}\right] \phi^*(\tilde{y}) dy \quad (A15)$$

$$= \int_{-\infty}^{\infty} d\omega \frac{(i\omega)^n \exp\left[-\frac{\omega^2}{4} - \frac{\Delta'\omega^2 + 4iP_0'\omega - 4P_0'^2}{4(1-\Delta')}\right]}{\sqrt{4\pi}\sqrt{\pi}2^n n!(1-\Delta')}, \quad (A16)$$

where we have utilized the following formulas:

$$e^{-\tilde{y}^2} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} d\omega e^{i\omega\tilde{y} - \omega^2/4}. \quad (A17)$$

$$\frac{d^n}{d\tilde{y}^n}(e^{-\tilde{y}^2}) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} d\omega (i\omega)^n e^{i\omega\tilde{y} - \omega^2/4}. \quad (A18)$$

Substituting the above equations into Eq. (A10) and making an inverse transformation, we can obtain the solution of the Fokker-Planck formula in Eq. (A1):

$$f(t, P) = \frac{1}{\sqrt{2\pi MT(1 - e^{-2t/\tau}(1 - \Delta'))}}$$

$$\times \exp\left[-\frac{(P - P_0 e^{-t/\tau})^2}{2MT(1 - e^{-2t/\tau}(1 - \Delta'))}\right]. \quad (A19)$$

APPENDIX B: GENERAL FORM OF FISHER INFORMATION

For the general initial Gaussian momentum distribution, the general form of the Fisher information can be analytically achieved by substituting Eq. (8) into Eq. (10):

$$\mathcal{F}(T) = 2\Gamma^2[T(2RMT + \Delta)]^{-2}\{n^2 t^2 T^{2n}[(P_0^2 - 4MT)\Delta$$
$$+ 2MT(RP_0^2 + 2MT) + \Delta^2] + (RMT/\Gamma)^2$$
$$+ 2nMRtT^{n+1}(2MT - \Delta)/\Gamma\}, \quad (B1)$$

where $R = e^{2t\Gamma T^n} - 1$.

[1] M. Mehboudi, A. Sanpera, and L. A. Correa, Thermometry in the quantum regime: recent theoretical progress, J. Phys. A: Math. Theor. **52**, 303001 (2019).

[2] F. Giazotto, T. T. Heikkilä, A. Luukanen, A. M. Savin, and J. P. Pekola, Opportunities for mesoscopics in thermometry and refrigeration: Physics and applications, Rev. Mod. Phys. **78**, 217 (2006).

[3] S. Jevtic, D. Newman, T. Rudolph, and T. M. Stace, Single-qubit thermometry, Phys. Rev. A **91**, 012331 (2015).

[4] P. P. Hofer, J. B. Brask, M. Perarnau-Llobet, and N. Brunner, Quantum thermal machine as a thermometer, Phys. Rev. Lett. **119**, 090603 (2017).

[5] S. Campbell, M. Mehboudi, G. D. Chiara, and M. Paternostro, Global and local thermometry schemes in coupled quantum systems, New J. Phys. **19**, 103003 (2017).

[6] A. H. Kiilerich, A. De Pasquale, and V. Giovannetti, Dynamical approach to ancilla-assisted quantum thermometry, Phys. Rev. A **98**, 042124 (2018).

[7] D. Xie, F. X. Sun, and C. Xu, Quantum thermometry based on a cavity-QED setup, Phys. Rev. A **101**, 063844 (2020).

[8] M. Campisi, P. Hänggi, and P. Talkner, Colloquium: Quantum fluctuation relations: Foundations and applications, Rev. Mod. Phys. **83**, 771 (2011).

[9] F. Brandão, M. Horodecki, N. Ng, J. Oppenheim, and S. Wehner, The second laws of quantum thermodynamics, Proc. Natl. Acad. Sci. USA **112**, 3275 (2015).

[10] S. Deffner and S. Campbell, *Quantum Thermodynamics* (Morgan & Claypool Publishers, 2019).

[11] L. A. Correa, M. Mehboudi, G. Adesso, and A. Sanpera, Individual quantum probes for optimal thermometry, Phys. Rev. Lett. **114**, 220405 (2015).

[12] G. De Palma, A. De Pasquale, and V. Giovannetti, Universal locality of quantum thermal susceptibility, Phys. Rev. A **95**, 052115 (2017).

[13] K. V. Hovhannisyan and L. A. Correa, Measuring the temperature of cold many-body quantum systems, Phys. Rev. B **98**, 045101 (2018).

[14] M. Mehboudi, A. Lampo, C. Charalambous, L. A. Correa, M. A. García-March, and M. Lewenstein, Using polarons for sub-nK quantum nondemolition thermometry in a Bose-Einstein condensate, Phys. Rev. Lett. **122**, 030403 (2019).

[15] V. Mukherjee, A. Zwick, A. Ghosh, X. Chen, and G. Kurizki, Enhanced precision bound of low-temperature quantum thermometry via dynamical control, Commun. Phys. **2**, 162 (2019).

[16] G. Planella, M. F. B. Cenni, A. Acín, and M. Mehboudi, Bath-induced correlations enhance thermometry precision at low temperatures, Phys. Rev. Lett. **128**, 040502 (2022).

[17] M. G. A. Paris, Achieving the Landau bound to precision of quantum thermometry in systems with vanishing gap, J. Phys. A: Math. Theor. **49**, 03LT02 (2016).

[18] N. Zhang, C. Chen, S. Bai, W. Wu, and J. An, Non-Markovian quantum thermometry, Phys. Rev. Appl. **17**, 034073 (2022).

[19] D. Xie and C. Xu, Optimal temperature estimation in polariton Bose-Einstein condensate, Adv. Quantum Technol. **5**, 2200085 (2022).

[20] N. Spethmann, F. Kindermann, S. John, C. Weber, D. Meschede, and A. Widera, Dynamics of single neutral impurity atoms immersed in an ultracold gas, Phys. Rev. Lett. **109**, 235301 (2012).

[21] M.-G. Hu, M. J. Van de Graaff, D. Kedar, J. P. Corson, E. A. Cornell, and D. S. Jin, Bose polarons in the strongly interacting regime, Phys. Rev. Lett. **117**, 055301 (2016).

[22] M. Cetina, M. Jag, R. S. Lous, I. Fritsche, J. T. M. Walraven, R. Grimm, J. Levinsen, M. M. Parish, R. Schmidt, M. Knap, and E. Demler, Ultrafast many-body interferometry of impurities coupled to a Fermi sea, Science **354**, 96 (2016).

[23] F. Meinert, M. Knap, E. Kirilov, K. Jag-Lauber, M. B. Zvonarev, E. Demler, and H.-C. Nägerl, Bloch oscillations in the absence of a lattice, Science **356**, 945 (2017).

[24] M. Knap, C. J. M. Mathy, M. Ganahl, M. B. Zvonarev, and E. Demler, Quantum flutter: Signatures and robustness, Phys. Rev. Lett. **112**, 015302 (2014).

[25] A. G. Volosniev, H.-W. Hammer, and N. T. Zinner, Real-time dynamics of an impurity in an ideal Bose gas in a trap, Phys. Rev. A **92**, 023623 (2015).

[26] N. J. Robinson, J.-S. Caux, and R. M. Konik, Motion of a distinguishable impurity in the Bose gas: Arrested expansion without a lattice and impurity snaking, Phys. Rev. Lett. **116**, 145302 (2016).

[27] S. I. Mistakidis, F. Grusdt, G. M. Koutentakis, and P. Schmelcher, Dissipative correlated dynamics of a moving impurity immersed in a Bose-Einstein condensate, New J. Phys. **21**, 103026 (2019).

[28] A. H. Castro Neto and M. P. A. Fisher, Dynamics of a heavy particle in a Luttinger liquid, Phys. Rev. B **53**, 9713 (1996).

[29] K. A. Matveev and A. V. Andreev, Scattering of hole excitations in a one-dimensional spinless quantum liquid, Phys. Rev. B **86**, 045136 (2012).

[30] A. Petković and Z. Ristivojevic, Dynamics of a mobile impurity in a one-dimensional Bose liquid, Phys. Rev. Lett. **117**, 105301 (2016).

[31] A. Petković and Z. Ristivojevic, Dissipative dynamics of a heavy impurity in a Bose gas in the strong coupling regime, Phys. Rev. Lett. **131**, 186001 (2023).

[32] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, Phys. Rev. **130**, 1605 (1963).

[33] P. P. Kulish, S. V. Manakov, and L. D. Faddeev, Comparison of the exact quantum and quasiclassical results for a nonlinear Schrödinger equation, Theor. Math. Phys. **28**, 615 (1976).

[34] E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics*, Course of Theoretical Physics (Oxford University Press, Oxford, 1983), Vol. 10.

[35] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier, New York, 2007).

[36] A. Petković, Local spectral density of an interacting one-dimensional Bose gas with an impurity, Phys. Rev. A **105**, 043305 (2022).

[37] S. L. Braunstein, and C. M. Caves, Statistical distance and the geometry of quantum states, Phys. Rev. Lett. **72**, 3439 (1994).

[38] G. Tóth and I. Apellaniz, Quantum metrology from a quantum information science perspective, J. Phys. A: Math. Theor. **47**, 424006 (2014).

[39] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).

[40] A. Del Maestro, N. Nichols, T. Prisk, G. Warren, and P. E. Sokol, Experimental realization of one dimensional helium, Nat. Commun. **13**, 3168 (2022).

[41] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Feshbach resonances in ultracold gases, Rev. Mod. Phys. **82**, 1225 (2010).