# Saturation of exponents and the asymptotic fourth state of turbulence

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(Received 11 August 2022; accepted 24 June 2024; published 19 July 2024)

A recent discovery about the inertial range of homogeneous and isotropic turbulence is the saturation of the scaling exponents $\zeta_n$ for large $n$, defined via structure functions of order $n$ as $S_n(r) = \langle(\delta_r u)^n\rangle = A(n)r^{\zeta_n}$. We focus on longitudinal structure functions for $\delta_r u$ between two positions that are $r$ apart in the same direction as $u$. In a previous work [Phys. Rev. Fluids **6**, 104604 (2021)], two of the present authors developed a theory for $\zeta_n$, which agrees with measurements for all $n$ for which reliable data are available, and shows saturation for large $n$. Here, we derive expressions for the probability density functions of $\delta_r u$ for four different states of turbulence, including the asymptotic fourth state defined by the saturation of exponents for large $n$. This saturation means that the scale separation is violated in favor of strongly coupled quasiordered flow structures, which likely take the form of long and thin (worm-like) structures of length $L$ and thickness $l = O(L/Re)$.

DOI: 10.1103/PhysRevResearch.6.033087

## I. PROBLEM DEFINITION

We are interested in fluid flows described by the Navier-Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f} \tag{1}$$

subject also to the incompressibility condition $\nabla \cdot \mathbf{u} = 0$; $\mathbf{f}$ is a body force. Of particular interest to this study is the case when $\mathbf{f}$ is a random stirring force generating turbulent cascade. Depending on the Reynolds number, Eq. (1) describes both laminar and turbulent flows. A goal of proper theory is to deduce observed properties of turbulence from (1). We focus here on the small-scale properties of the turbulent state. To make progress, it is customary [1] to introduce the velocity increments $u_r \equiv \delta_r u \equiv u(x+r) - u(x)$ and define structure functions as their moments. In particular, there is a natural expectation following Kolmogorov [1] that power laws of the form

$$S_n(r) = \langle u_r^n \rangle = A(n)r^{\zeta_n}, \tag{2}$$

where the angular brackets indicate a suitable average, hold for the intermediate separation distances $L \gg r \gg \eta$, known as the inertial range; here, $L$ is the large scale of turbulence and $\eta = (\nu^3/\varepsilon)^{1/4}$ is the dissipation scale, $\nu$ being the kinematic viscosity of the fluid and $\varepsilon$ the rate of energy dissipation. Kolmogorov's theory [1], based on the assumption of localness in the space of interacting scales and isotropy, led subsequently to the famous linear relation, $\zeta_n = n/3$. This elegant result invariably fails for high-Reynolds-number velocity fields in three dimensions (see, e.g., [2–4]), because of the occurrence of rare and extreme events corresponding to the tails of the probability density function (PDF). Increasingly intense fluctuations can be probed by considering $\langle(u_r)^n\rangle^{1/n}$ for increasing moment order $n$ (i.e., by knowing $\zeta_n$ for increasing $n$). Thus, a major problem of the turbulence theory, similar to those of high-energy and condensed matter physics, is the evaluation of the exponents $\zeta_n$ in Eq. (2). We show here that four different states of turbulence can be identified depending on the qualitative behavior of $\zeta_n$ versus $n$. Our specific goal here is to define these four states on the basis of the results of our previous work [5].

## II. BACKGROUND

As a statistical mechanical result, it would be instructive to create a basic argument beyond the Kolmogorov-type reasoning. Renormalization group and related techniques have been applied to turbulence (e.g., [6–9]), with focus on low-order moments. For high-order moments, the following two ideas were combined in our recent work [5] to produce the semblance of a theory. The first idea is that an initially Gaussian flow state stirred by a random force acquires anomalous scaling at and beyond a presumably universal Taylor microscale Reynolds number of $R_\lambda \approx 9$ in direct numerical simulations (DNS) [10]. The second idea is that the inertial range of turbulence also has an $R_\lambda \approx 9$ when based on the scale dependent viscosity. The intuitive hypothesis of our previous work [5] linking (a) and (b) is that the anomalous scaling of the inertial range manifests as a marginally unstable state given by the stability boundary of the transition to anomalous scaling of

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a stochastically forced random Gaussian field. Note that our previous work [5] draws heavily on conclusions from successively eliminating high wavenumber bands and rescaling until a fixed point is reached, based on a "dynamic renormalization group technique" supplemented by a certain degree of phenomenology.

The theory from our previous work [5] combines Hopf equations with dynamic renormalization mentioned above, for the pressure gradient equations for white-in-time Gaussian forcing at large scales, valid for all moments $n > 0$. In particular, the following expression was derived for the scaling exponents of the longitudinal structure functions:

$$\zeta_{2n} = \frac{0.366n}{0.05n + 0.475} = 7.32 - \frac{69.54}{n + 9.5}. \tag{3}$$

This expression was found to be in excellent agreement with the available simulations data [5] up to the highest-order $n$ for which data were available ($\sim$10).

Despite this success, we should point out that this key conclusion of our previous work [5] applies to the randomly stirred fluid. Thus, *a priori*, that work [5] is not an exact theory applicable to fluid turbulence driven by actual physical shear—especially because it assumes that the delicate properties of large moments defined by a functional equation are stable with respect to the change in dynamics from a flow driven by a white-in-time stirrer to the Navier-Stokes turbulence in a physical system. Basic intuition combined with experiment and numerics suggests that these conclusions can perhaps be generalized, but it implies a considerable jump in physical reasoning.

## III. SPECIFIC CONTRIBUTIONS OF THIS ARTICLE AND THEIR SIGNIFICANCE

As already mentioned, we distinguish four distinct states of turbulence on the basis of the theory from our previous work [5], and derive expressions for the PDFs of velocity increments in all these states, including what we regard as the asymptotic (or the fourth) state, which is defined by the saturation of $\zeta_n$ with respect to $n$, as given by (3) for the longitudinal case. This saturation shows that $\langle (u_r)^n \rangle^{1/n}$, which is characteristic of intense small-scale fluctuations when $n$ is large, approaches an $r$-independent constant quite rapidly in $n$, suggesting that fluctuations on the smallest scales could often be as large as $u_0$, the fluctuation at the largest scale $L$ itself—thus calling to question the concept of scale separation and confirming the violation of local Galilean invariance.

To strengthen this last point, we recall from the work of Monin and Yaglom [11] that the Kolmogorov theory assumes turbulence in the inertial range to be universal and independent of both $L$ and $\eta$. However, at very high Reynolds numbers (i.e., $\nu \to 0$), the length scale $L$ remains $O(1)$ but $\eta \to 0$. Since the amplitude of fluctuation on scales of $O(\eta)$ or smaller (as we shall see) can occasionally be of the order of large-scale fluctuations $u_0$, both $L$ and $\eta$ are simultaneously important. This nudges us in the direction of coherent, "worm-like" or "pancake-like" structures, with $L$ and $\eta$ (or a smaller scale) as their linear dimensions. This provides a physical picture for the notion that the turbulence problem is one of strong interactions. Indeed, if we take the only characteristic length scale in the inertial range to be the space increment $r$ and its characteristic velocity $u_r$, the effective local viscosity $\nu(r) \approx r u_r$ and the characteristic Reynolds number is $R_\lambda = r u_r/\nu(r) =$ constant, as readily seen in perturbation expansions such as those of Wyld [12]. A fixed point has been evaluated to be $\approx 9$ in a prior work [13], implying, as is well known, that the perturbation theory is divergent, and that turbulence belongs to the class of strong interactions. Making progress on this strongly coupled problem was a major contribution of our previous work [5].

## IV. THE FIRST STATE OF TURBULENCE

According to other works [10,14,15], in a low-Reynolds number regime defined by $\mathrm{Re} = u_0 L/\nu \leqslant 9$, the weak fluctuations generated by forcing on a large length scale $L$ reside in the scale range $r > L$ and obey Gaussian statistics. (If the forcing is different, one expects the PDF in this state of turbulence to be accordingly different.) For this condition, there is no distinction to be made between $L$ and the Taylor microscale $\lambda$, so we might as well state that $\mathrm{Re} = R_\lambda$, where the microscale Reynolds number is based on $\lambda$ or $L$. This is the first stage of "turbulence." Its character is dependent entirely on details of forcing.

## V. THE SECOND STATE OF TURBULENCE

In statistically isotropic turbulence, if the moments of velocity increments $u_r$ are given by power laws (2), their PDF can be found from the Mellin transform

$$P(u_r, r) = \frac{1}{u_r} \int_{C-i\infty}^{C+i\infty} A(n) r^{\zeta(n)} u_r^{-n} dn, \tag{4}$$

where we have set the integral scale $L$ and the dissipation rate $\varepsilon$ to unity. Multiplying (4) by $u_r^k$ and evaluating the integral yields $S_k = A(k) r^{\zeta_k}$. At the transition or instability point at $R_\lambda \approx 9$ (for a discussion that is similar in spirit, see Ref. [16]), this Gaussian state becomes unstable due to nonlinearity, giving rise to the smaller scale fluctuations in the interval $L > r > \eta$, making the problem hard [10,13]. The dynamics of spreading energy in the range between $L$ and $\eta$ has often been thought of in terms of a cascade with constant energy flux, leading to the formation of successively smaller scales, $L_1 = L/2$, $L_2 = L/4$, $L_3 = L/8$, and so on, proceeding all the way to $\eta$. It had been assumed in early models that each step of the cascade was space filling, but past work [2–4] has shown that such models are, in general, not adequate.

To make use of (4), we need dynamic information on both the amplitudes $A(n)$ and the exponents $\zeta_n$ in (2). We obtain the expression for $A(n)$ from the large-scale boundary condition for the PDF. To do this, we first have to define the scale $L$ more precisely. Based on experimental data and theoretical considerations discussed in the following, $L$ is the scale at which the energy flux toward small scales changes sign or tends to zero. This suggests [11] that at small scales $r < L$, the structure function $S_3(r) < 0$, whereas for the larger scales, $r > L$, $S_3 \geqslant 0$. Typically, at this scale $L$, which depends upon the geometric details of the flow, the odd moments $S_{2n+1}(L) = 0$ and the even moments saturate [i.e., $\partial_r S_{2n}(L) = 0$]. From other works [6,13], $L \approx 5.88/k_f$ when $S_3(L) = 0$. This scale also appears naturally in Navier-Stokes equations defined on an infinite domain driven by the white-in-time forcing function $f(k)$, $k$ being the wave number, with the variance

$$\langle f^2(k)\rangle = \frac{\mathcal{P}}{2(2\pi)^4}\delta(k-k_f)/k^2,$$

where $\mathcal{P}$ is the forcing power and $k_f$ is the forcing wave number. The exact calculation of the relation for the third-order structure function $S_3(r)$ gives an oscillating expression [17]

$$S_3 = -\mathcal{P}\frac{-36r\cos r + 12\sin r - 12(-2+r^2)\sin r}{r^4}.$$

In the limit $r \to 0$, we have the Kolmogorov-like relation

$$S_3 = -(4/5)\mathcal{P}r.$$

It is interesting that no viscosity, and therefore no dissipation scale $\eta$, appears in the preceding relation that resembles Kolmogorov's 4/5 law, though not identical to it. We would like to stress that the integral scale defined this way typically corresponds to the top of the inertial range marking constant energy flux toward small scales. The Gaussian boundary condition at $r = L$ is the result of a fluctuation-dissipation theorem and leads to the expression $A(2n) = (2n-1)!!$ (well tested both experimentally and numerically [5]).

For $n$ between 0 and 4, say, the behavior of $\zeta_n$ is close to normal scaling; in other words, $\zeta_n \approx (1/3)n$, where the proportionality coefficient of 1/3 is a consequence of the Kolmogorov-like relation $S_3 \propto r$ with $S_3(r=L)=0$. Writing $(2n-1)!! = \frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2}x^{2n}dx$ and rotating the integration axis by 90°, we have

$$\begin{aligned} P(u_r, r) &= \frac{1}{\sqrt{\pi}u_r}\int_{-\infty}^{\infty} e^{-x^2}dx\int_{-\infty}^{\infty} e^{in\left(ln\frac{r^a\sqrt{2}}{u_r}+ln\, x\right)}dn \\ &= \frac{1}{\sqrt{\pi}u_r}\int_{-\infty}^{\infty} e^{-x^2}\delta\left(ln\frac{r^a\sqrt{2}}{u_r}+ln\, x\right)dx, \end{aligned} \quad (5)$$

where $\delta$ is the standard delta function. This integral is evaluated readily to yield the result

$$P(u_r) = \frac{1}{\sqrt{2\pi}r^a}e^{-\left(\frac{u_r^2}{2r^{2a}}\right)}. \quad (6)$$

As we see, an unbounded flow governed by (1) is characterized by a single length scale in this state.

## VI. THE THIRD STATE OF TURBULENCE

We now consider a somewhat larger range of $n$ for which the anomalous behavior has set in, but they are not so large that saturation has become visible. We account for the emergence of anomalous scaling by introducing small deviations from the linear relation for $\zeta_n$, as

$$\zeta_n = an - bn^2. \quad (7)$$

For a moment order $n$ that is not too large, Eq. (7) can be perceived as containing the first two terms of the Taylor expansion of $\zeta_n$ near $n = 0$, independent of the detailed nature of the problem. Using the Kolmogorov constraint $\zeta_3 = 1$, we get $b = (3a-1)/9$. The PDF of $u_r$ is then given by

$$P(u_r, r) = \frac{2}{\pi u_r\sqrt{4\ln r^b}}\int_{-\infty}^{\infty} e^{-x^2}\exp[-\frac{\left(\ln\frac{u_r}{r^a\sqrt{2x}}\right)^2}{4b\ln r}]dx. \quad (8)$$

It is clear that the expansion (7) cannot be correct for all $n$. Indeed, in accordance with the Hölder inequality, $\zeta_n$ is a concave and nondecreasing function giving $\zeta_n/n \geqslant 1/n$ as $n \to \infty$. So, it is a pleasant surprise that for $n \leqslant 10$, the experimental data on strong turbulence are consistent with $a \approx 0.383$ and $b \approx 0.0166$, and that the expression (8) is accurate up to $n = O(10)$ (see Ref. [5]).

## VII. THE FOURTH AND FINAL STATE OF TURBULENCE

This state is defined by the effects of saturated $\zeta_n$. We first explore some qualitative consequences of the saturation of exponents, which do not depend on the precise saturation value. As indicated a few lines after Eq. (2), the largest fluctuations of scale $r$ have amplitudes given by $\langle u_r^n\rangle^{1/n}$ for large $n$, which, as a result of saturation of $\zeta_n$ in Eq. (2), will have amplitudes as large as $u_0$ itself. The Reynolds number of the finest of these large fluctuations should be unity, which specifies the scale $l$ via the requirement that $u_0 l/\nu = 1$, to be $l = \eta Re^{-1/4}$. In other words, there are very large excursions on scales that are smaller than the Kolmogorov scale $\eta$ by the factor $\mathrm{Re}^{1/4}$, with their amplitudes of the order of $u_0$ itself. The corresponding finest time scale will also be smaller than the conventional estimate by the factor $\mathrm{Re}^{-1/4}$. Technically, then, computationally resolving the smallest scales of motion requires grid size that is finer than $\eta$ by the factor $\mathrm{Re}^{1/4}$, and an improved time resolution by the same factor of $\mathrm{Re}^{1/4}$, than is adopted in standard DNS of Eq. (1). This requirement becomes quite demanding as the Reynolds number becomes large.

This observation blunts the great progress made by DNS in the past 50 years. The development of powerful computers has resulted in larger computational domains from $N = 32^3$ in the early 1970s [18] to $N = 32,768^3$ today [19]—a billion-fold increase in $N$, the number of grid points in each direction of a periodic box within which forced turbulence is studied. Despite this accomplishment, it is becoming clearer that DNS is unable to keep up, in resolving extreme events, because the scale ranges in both time and space widen well beyond the Kolmogorov estimates, as demonstrated above. (We do not imply that existing data resolved on Kolmogorov scales are inadequate for studying low-order moments.) As a result, theoretical models based on various averaging methods and dynamic renormalization have important roles to play, especially in engineering. In particular, they supplement the one-loop renormalized perturbation expansions which have led to various models for turbulent viscosity, which have been successful in simulations of large flow features.

## VIII. THE PDF OF $u_r$ IN THE FOURTH STATE

To obtain the PDF, one needs the limiting value of $\zeta_n$ as $n \to \infty$; according to our previous work [5], $\zeta_\infty = 7.3$. It should be pointed out that similar saturation properties are shared, for less extreme values of $n$, by the random Burgers equation [20–22], the passive scalar [23], the transverse structure functions [4], and the Lagrangian scaling exponents [24]. The PDF

$$P(u_r, r) = \frac{2}{\sqrt{\pi} u_r} \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-i\infty}^{i\infty} x^{2n} r^{\zeta_n} u_r^{-n} dn \quad (9)$$

can be evaluated for constant $\zeta_n$ by the use of the steepest descent approximation, yielding

$$P(u_r, r) \propto \frac{1}{u_r} \left( \frac{r^a}{u_r} \right)^{\frac{1}{b}} e^{-\sqrt{|\ln u_r||\ln r^a|}}. \quad (10)$$

It describes the PDF with algebraically decaying tails familiar in the literature on three-dimensional turbulence.

Now let us extend the use of Eq. (4) beyond the original range $\eta \ll r \ll L$ into the range $r \gg L$. Then the main contribution in the PDF is due to $n \to \infty$ so that we have the PDF that is qualitatively different from (8). We then have

$$P(u_r, r) = \frac{2}{\sqrt{\pi} u_r} \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-i\infty}^{i\infty} x^{2n} r^{\zeta_n} u_r^{-n} dn$$
$$= \left( \frac{r}{L} \right)^{7.3} g(U), \quad (11)$$

where $U = 2u_0$ and the single-point PDF is equal to

$$g(U) \propto \exp(-U^2/2), \quad (12)$$

for $r \geqslant L$, so that the single-point PDF is given by $P(U) = \exp(-U^2/2)/\sqrt{2\pi}$. Needless to say, this holds for the saturation state with the asymptotic exponent of 7.3.

## IX. THE BREAKDOWN OF LOCAL GALILEAN INVARIANCE IN THE FINAL STATE

The nonlinearity in (1) is a consequence of Galilean invariance. Indeed, the transformation to a frame moving with the velocity $\mathbf{u} + \mathbf{V}$, where $\mathbf{V} =$ constant, keeps the Navier-Stokes equations for fluctuations unchanged. However, it is clear from (12) that Galilean invariance is broken locally. The corresponding dynamics was experimentally studied [25] and shown to possess a single-point, non-Galilean-invariant contribution to $P(u_r, r; U)$. This breakdown of Galilean invariance was previously obtained [26] in the context of the Burgers equation (which, as already pointed out, also possesses saturated scaling exponents). We stress that the saturation emerged in our previous work [5] as a solution to the Hopf equation, and that one cannot expect universality in this regime.

## X. GEOMETRIC STRUCTURE

The analysis so far demonstrates the relevance of both large and small scales when fluctuations are intense. This points to the existence of powerful structures with one dimension that is very small, of the order $l = \eta Re^{-1/4}$, and the other of the order $L$. Between vortex sheets and tubes, it would appear that the inherent instability of the sheets and their tendency to roll up suggests that the final structures are likely to be in the form of tubes. If tubes are the most likely objects, it is clear that they are like the "worms" described in many simulations (see especially Ref. [27]), with lengths that are much larger than their diameters. Qualitative observations from simulations suggest that they could be sometimes of the order of $L$. These vortical motions, which are thin and long at the same time, are a feature of all high-Reynolds-number flows such as homogeneous turbulence, thermal convection, and meteorological flows.

Given this picture of high-Reynolds-number turbulence, it is clear that no local filtering procedure can be applied successfully. Neither Kolmogorov-like arguments nor other qualitative or approximate approaches can account for structures that are very small and very large simultaneously. It might have been reasonable to do so if they were extremely rare, but the saturation of exponents makes them not so rare. In particular, this property of the asymptotic state does not augur well for large-eddy-simulation methods.

## XI. SUMMARY

In our previous work [5], we considered the dynamics of spatially infinite fluid driven by white-in-time Gaussian random force acting on a length scale $L$. Our goal was to probe the velocity field in terms of structure functions defined in (2). Of particular interest is $(\langle u_r^n \rangle)^{1/n}$ for large $n$, the magnitude of local mean velocity corresponding to the far tails of the PDFs, dominated by extreme events.

We note that many years of experimental and numerical work [2–4] has revealed power laws with anomalous exponents departing from $\zeta_n = n/3$ (except for $n = 3$). The theory of turbulence giving the result (3) was produced only recently (see Ref. [5] and references cited therein). Here, based on that expression for the exponents, we have analyzed the PDFs of velocity increments to investigate the structure of turbulence. To estimate the effect of intermittency in the limit of large $n$, we approximate (3) as $\zeta_n^{41}/\zeta_n = 0.0455n + 0.864$. When $n \leqslant 4$, this ratio is within a few percent of unity, showing that intermittency is practically not important in this range. However, as $n \to \infty$ (say, for $n \geqslant 20$), the intermittent behavior dominates the velocity field and all models based on arguments of Kolmogorov [1], which do not recognize intermittency, are quite problematic. For other quantities such as transverse exponents and Lagrangian exponents, the state of saturation sets in at much lower $n$.

We can now state the basic ingredients of the present theory, valid in the entire range of Reynolds numbers, as follows:

1. *Linear regime and the onset of turbulence:* The velocity $\mathbf{u}$ is proportional to the forcing $\mathbf{f}$. This corresponds to the lowest Reynolds number, $R_\lambda \to 0$ [5]. In the Reynolds number range $R_\lambda < 9$, the quasilaminar Gaussian random flow consists of random patches of the typical scale $L$ [10,14,15]. In this range, $L$ is the only relevant scale, and, in particular, it is of the same order as $\lambda$.

2. *Approximately K41 range:* At $R_\lambda \approx 9$, local transition to anomalous scaling ensues, seen by the broadening of the PDF tails of velocity increments. For small orders ($n < 4$), this broadening does not matter significantly and the dynamics can be regarded as being close to K41.

3. *Significant deviations from K41:* For higher orders in $n$, deviations from K41 become stronger. This is the range most often studied in the literature.

4. *Well-defined structures and the asymptotic fourth state:* With further increase of $R_\lambda$, the tails of the PDFs broaden and the middle part becomes sharper [28]. This is accompanied by the appearance of elongated, thin structures responsible for limiting intermittency. One might regard the entire flow as a Gaussian *gas* of long spaghetti-like structures with length of the order $L$ and thickness $l = O(L/Re)$. Note that the immediate appearance of non-Gaussian statistics at the transitional $R_\lambda \approx 9$ is indicative of a strong dynamic coupling between large and small scales. The corresponding geometric concept of stretched "worms" is already implicit in the suggestion that turbulence dynamics is organized around such a critical state. At larger Reynolds numbers, such a dynamic process is further excited so that the flow is locked to this marginally stable state that is effectively transitional in $R_\lambda$: a self-organized criticality due to the direct balance between the large-scale stretching and small-scale excitation and dissipation of worms. Quantitative verification and developments of this concept allow simulation and modeling of a variety of flows (see, e.g., Ref. [29]).

## XII. WE CONCLUDE ON THIS OPTIMISTIC NOTE

The theory developed in our previous work [5] and in the present article is for a specific type of turbulence. While there are gaps between the conditions of the theory and the case of Navier-Stokes turbulence under actual physical conditions, one expects that the physics can be generalized. Likewise, we believe that the theory can be extended to compressible hydrodynamic turbulence and gravitational collapse of interstellar dust which are of interest for star formation and related applications. They will be topics of future studies.

## ACKNOWLEDGMENTS

Over the years, we have interacted with many colleagues too numerous to list here but acknowledge the early influence of the late S. Orszag, as well as discussions with D. A. Donzis, G. L. Eyink, K. P. Iyer, A. A. Migdal, A. M. Polyakov, J. Schumacher, and L. Smith.

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