Hawking-Page transition on a spin chain

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The accessibility of the Hawking-Page transition in AdS_5 through a one-dimensional (1D) Heisenberg spin chain is demonstrated. We use the random matrix formulation of the Loschmidt echo for a set of spin chains, and randomize the ferromagnetic spin interaction. It is shown that the thermal Loschmidt echo, when averaged, detects the predicted increase in entropy across the Hawking-Page transition. This suggests that a 1D spin chain exhibits characteristics of black hole physics in $4+1$ dimensions. We show that this approach is equally applicable to free fermion systems with a general dispersion relation.

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I. INTRODUCTION

A fundamental discovery in the search for a quantum theory of gravity was the realization that quantum information is the foundation, while spacetime and the particles within it are emergent. This "it from qubit" approach, originally proposed by Wheeler [1], has resulted in significant progress in comprehending the quantum properties of black holes. The utilization of quantum information techniques in exploring the emergence of spacetime resulted in various developments, including the ER = EPR conjecture [2], holographic quantum error-correcting codes [3–5], and proposals aimed at resolving the information paradox [6,7], among others.

Alongside this new method for studying quantum gravity is the advancement of quantum simulations. In brief, quantum simulators are devices designed to replicate physical phenomena impossible to study through experimentation.

An essential query that arises from the amalgamation of these two areas is the degree to which gravity's manifestations can emerge from a set of qubits. An example in this direction was the quantum simulation of a traversable wormhole [8].

In this paper, we examine another important aspect of gravity: the formation of a black hole (BH). Unlike wormholes, which harbor innate quantum properties, BHs are prominently thermal entities. Hence, they can be analyzed through conventional means such as finite-temperature lattices. A BH forms in anti-de Sitter (AdS) spacetime with the Hawking-Page transition [9]. Our main point asserts that this phenomenon can be simulated on a simple spin chain, the Heisenberg XX chain.

To elaborate, we will begin with thermal AdS_5. As the temperature T rises, a BH solution emerges, which becomes dominant above a threshold T_{HP} [9]. It can be inferred that, at $T > T_{HP}$, the entropy would be significantly larger than thermal AdS, due to the presence of the BH horizon. This follows from the Bekenstein-Hawking formula [10,11]

$$S_{BH} = \frac{A_{hor}}{4G_N}, \tag{1}$$

with A_{hor} the area of the BH horizon and G_N the Newton constant in AdS_5 (in units $c = \hbar = 1$).

The celebrated AdS/CFT correspondence [12–14] posits that gravitational asymptotically AdS solutions are encoded into a conformal field theory (CFT) on the boundary. The most widely studied example, and the one we will focus on, is the duality between AdS_5 and 4D $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) [12].

We will show that the entropy S_{BH} can be computed from a coupling average of thermal, or imaginary time, Loschmidt echoes in the XX chain. Preparing the spins in the initial state

$$|\psi_0\rangle = |\underbrace{\downarrow\downarrow\ldots\downarrow}_{N}\uparrow\uparrow\ldots\uparrow\rangle, \tag{2}$$

we claim, schematically,

$$\boxed{S_{BH} \sim \ln\left(\int_0^\infty dJ\, e^{-\frac{J^2}{4a}}\,\langle\psi_0|e^{-JH_{XX}}|\psi_0\rangle\right),} \tag{3}$$

with average over the coupling J of the XX chain [the accurate statement is (26)]. The parameter a, controlling how J is sampled, is a monotonic function of the BH temperature T, cf. (7). In particular, we show that the averaged Loschmidt echo detects the jump in entropy predicted by the Hawking-Page transition. Our result implies that a one-dimensional (1D) spin chain captures features of BH physics in $4+1$ dimensions.

In the AdS_5/CFT_4 dictionary, the entry

$$G_N \sim 1/N^2 \tag{4}$$

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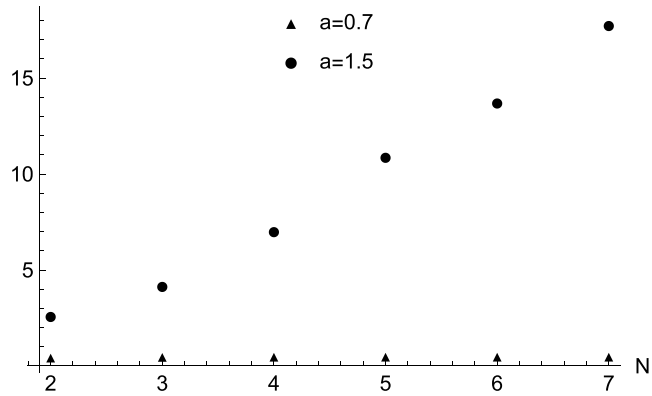

FIG. 1. $\ln\langle\sqrt{\hat{\mathcal{L}}_N}\rangle_{2a}$ as a function of N at $T < T_{\rm HP}$ (▲) and $T > T_{\rm HP}$ (●).

relates the Newton constant and the rank of SYM. The regime $G_{\rm N} \to 0$, in which gravity becomes tractable, corresponds via (4) to the large N limit of $SU(N)\,\mathcal{N} = 4$ SYM [12]. Combining the Bekenstein-Hawking formula (1) with the AdS/CFT relation (4), a Hawking-Page transition is characterized by

$$S \sim \begin{cases} s_{\rm low} & T < T_{\rm HP} \\ N^2 s_{\rm high} & T > T_{\rm HP} \end{cases}, \tag{5}$$

where S is the entropy and $s_{\rm low}$, $s_{\rm high}$ are $\mathcal{O}(1)$ constants. Enormous effort has been put to test and sharpen this prediction [15–80].

We demonstrate how the distinctive behavior (5) can be replicated on the spin chain (Fig. 1), primarily focusing on weak gauge coupling. Remarkably, finite coupling corrections—a key factor in achieving more credible results—incur no further complexity in the setup of the spin chain (see Appendix B 4), although it requires repeating the measurement multiple times.

II. HAWKING-PAGE TRANSITION

In this paper we focus on the BH formation in maximally supersymmetric AdS_5, whose holographic dual is $\mathcal{N} = 4$ SYM. On the one hand, maximal supersymmetry plus conformal symmetry tightly constrain the theory, making it the perfect playground to explore 4D physics. On the other hand, it bears universal properties akin to pure Yang-Mills and QCD [81–83], befitting phenomenological applications.

The AdS/CFT correspondence dictates that the Hawking-Page transition [9] in AdS_5 is dual to a first-order transition in $\mathcal{N} = 4$ SYM on Euclidean $\mathbb{S}^3 \times \mathbb{S}^1$ [84] (details in Appendix A), with thermal $\mathbb{S}^1$ of radius $1/T$. An effective description for this theory, in the weak coupling limit, was devised in Refs. [15,16,18]. The result, written in terms of the AdS_5 entropy S, is

$$S = \ln\left[\oint dU\ \exp\left(\sum_{n\geqslant 1}\frac{a_n}{n}\mathrm{Tr}(U^n)\mathrm{Tr}(U^{-n})\right)\right]. \tag{6}$$

U is the holonomy of the gauge field around $\mathbb{S}^1$, dU is the normalized Haar measure and a_n are effective couplings produced integrating out the massive modes. They depend on T, for instance,

$$a \equiv a_1 = \frac{2(3e^{1/(2T)} - 1)}{(e^{1/(2T)} - 1)^3}. \tag{7}$$

At large N, (6) is approximated by the simpler matrix model [18]

$$\hat{S} = \ln\left[\oint dU\ \exp[a\mathrm{Tr}(U)\mathrm{Tr}(U^{-1})]\right] \tag{8}$$

with higher couplings $a_{n>1}$ being irrelevant perturbations, negligible in the analysis of the phase transition (refinements by higher couplings are dealt with in Sec. IV). The notation $\hat{S}$ stresses the difference with the exact quantity S.

The large N limit of (8) was solved in Ref. [19]. One rewrites

$$e^{\hat{S}} = \int_0^\infty \sigma d\sigma\ \exp\left[-N^2\left(\frac{\sigma^2}{4a} - \mathcal{F}(\sigma)\right)\right], \tag{9}$$

with $\mathcal{F}(\sigma) = \frac{1}{N^2}\log \mathcal{Z}_N^{\rm GWW}(N\sigma)$ the free energy of the Gross-Witten-Wadia (GWW) matrix model [85–87]

$$\mathcal{Z}_N^{\rm GWW}(N\sigma) = \oint dU\ \exp\left\{\frac{N\sigma}{2}\mathrm{Tr}(U + U^{-1})\right\}. \tag{10}$$

Then (9) is solved in two steps: (i) apply the planar limit to the GWW model, and (ii) minimize the resulting effective action by a standard saddle point analysis. The upshot is [19]:

(1) If $a < 1$, the saddle is $\sigma = 0$, thus $\hat{S}(a < 1) \sim \mathcal{O}(1)$;

(2) If $a > 1$, there is a nontrivial saddle, and $\hat{S}(a > 1) = N^2 s_{\rm high}$.

Therefore (9) undergoes a first-order transition at

$$a(T_{\rm HP}) = 1 \Longrightarrow T_{\rm HP} \approx 0.38 \tag{11}$$

separating the two phases (5).

III. MAPPING TO THE SPIN CHAIN

Our novel claim is that the Hawking-Page transition in supersymmetric AdS_5 can be encoded in a 1D quantum spin chain. The simulation of such thermodynamic process requires a chain at finite temperature. The quantity detecting the sudden change in entropy is the Loschmidt echo [88], an observable tailored to quantify how chaotic a system is. It has already found broad application in BH physics (e.g., Refs. [89–92]). Our insight is to interpret (9) as a coupling average of the Loschmidt echo in the XX chain.

Consider a spin-$\frac{1}{2}$ chain of L sites, and prepare the initial state $|\psi_0\rangle$ in (2). We are interested in the regime $1 \ll N \ll L$ and assume periodic boundary conditions, but the results hold more generally [93]. The system is at temperature $\tilde{T} > 0$ and evolves with the XX Hamiltonian [94]

$$H_{\rm XX} = -\frac{\tilde{J}}{2}\sum_{j=0}^{L-1}(\sigma_j^-\sigma_{j+1}^+ + \sigma_j^+\sigma_{j+1}^-), \tag{12}$$

where $\tilde{J} > 0$ is the ferromagnetic coupling and $\sigma_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$ are the spin-flip operators on the site j, satisfying

$$\sigma^+|\downarrow\rangle = |\uparrow\rangle, \quad \sigma^-|\uparrow\rangle = |\downarrow\rangle, \quad \sigma^+|\uparrow\rangle = 0 = \sigma^-|\downarrow\rangle \tag{13}$$

$$[\sigma_j^+, \sigma_k^-] = \sigma_j^z\delta_{jk}, \quad [\sigma_j^z, \sigma_k^\pm] = \pm 2\sigma_j^\pm\delta_{jk}. \tag{14}$$

We consider the thermal Loschmidt amplitude

$$\mathcal{G}_N(J) = \langle\psi_0|e^{-H_{\rm XX}/\tilde{T}}|\psi_0\rangle \tag{15}$$

and the corresponding echo:

$$\mathcal{L}_N(J) = |\mathcal{G}_N(J)|^2. \tag{16}$$

Note that $\tilde{T}$ is the temperature of the chain, unrelated to the BH temperature T. Furthermore, (16) only depends on $J \equiv \tilde{J}/\tilde{T}$. In (15) we used the standard replacement $it \mapsto 1/\tilde{T}$. In this way, $\mathcal{L}_N$ is not a probability, because it is not canonically normalized. It is thus convenient to work with the ratios

$$\hat{\mathcal{L}}_N(J) = \mathcal{L}_N(J)/\mathcal{L}_1(J). \tag{17}$$

One advantage of this prescription, is that $\hat{\mathcal{L}}_N$ remains finite when $L \to \infty$.

Next, we map (15) to the GWW model (10) [95,96]. We introduce $|\Uparrow\rangle \equiv |\uparrow,\uparrow,\ldots,\uparrow\rangle$ and

$$g_{j,k}(J) = \langle\Uparrow|\sigma_j^+ e^{-H_{\rm XX}/\tilde{T}}\sigma_k^-|\Uparrow\rangle. \tag{18}$$

Observe that, as $L \to \infty$ [95]

$$\langle\Uparrow|\left(\bigotimes_{j=0}^{N-1}\sigma_j^+\right)e^{-H_{\rm XX}/\tilde{T}}\left(\bigotimes_{k=0}^{N-1}\sigma_k^-\right)|\Uparrow\rangle = \det_{0\leqslant j,k\leqslant N-1}[g_{j,k}]. \tag{19}$$

Besides, $|\psi_0\rangle = \bigotimes_{k=0}^{N-1}\sigma_k^-|\Uparrow\rangle$. Using (14) to pass $H_{\rm XX}$ across σ_j^+, and $H_{\rm XX}|\Uparrow\rangle = 0$, $g_{j,k}$ satisfies

$$\frac{dg_{j,k}}{dJ} = \frac{1}{2}\langle\Uparrow|(\sigma_{j-1}^+ + \sigma_{j+1}^+)e^{-H_{\rm XX}/\tilde{T}}\sigma_k^-|\Uparrow\rangle. \tag{20}$$

Therefore $g_{j,k}$ solves

$$\frac{dg_{j,k}}{dJ} = \frac{1}{2}(g_{j-1,k} + g_{j+1,k}) \tag{21}$$

with initial condition $g_{j,k}(0) \propto \delta_{j,k}$. This is the recurrence relation of the Bessel function $I_{j-k}(J)$, thus

$$\frac{\mathcal{G}_N(J)}{\mathcal{G}_1(J)} = \frac{1}{I_0(J)}\det_{0\leqslant j,k\leqslant N-1}[I_{j-k}(J)]. \tag{22}$$

Using the Heine-Szegő identity

$$\det_{0\leqslant j,k\leqslant N-1}[I_{j-k}(J)] = \mathcal{Z}_N^{\rm GWW}(N\sigma = J) \tag{23}$$

we conclude that

$$\hat{\mathcal{L}}_N(J) = \frac{1}{I_0(\tilde{J}/\tilde{T})^2}\left|\mathcal{Z}_N^{\rm GWW}(N\sigma = \tilde{J}/\tilde{T})\right|^2. \tag{24}$$

The denominator is independent of N. The relation (24) has been previously exploited to analyze the echo of the XX chain [93,96–106]. We revert the argument and consider the Loschmidt echo as the building block for the realization of the Hawking-Page transition (5) on a quantum apparatus.

Comparing the exact identity (24) with (9), identifying $N\sigma = J$, the integral over σ in $\hat{S}$ is translated into a Gaussian average over J. This is enforced by randomizing the coupling $\tilde{J}$ at fixed $\tilde{T}$.

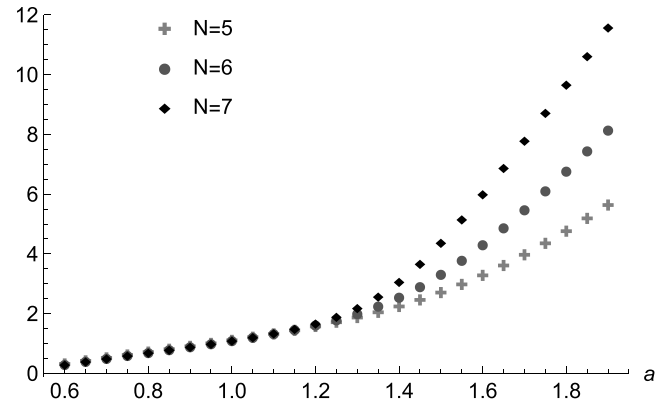

FIG. 2. $\ln\langle\sqrt{\hat{\mathcal{L}}_N}\rangle_{2a}$ as a function of $a(T)$ for various N at $L = 18$.

We thus propose the following setup. $\hat{\mathcal{L}}_N(J)$ is measured in the regime $1 \ll N \ll L$, with J as above. The experiment is run several times, sampling J from a suitably chosen random distribution. The average echo is computed:

$$\langle\sqrt{\hat{\mathcal{L}}_N}\rangle_{2a} = \int_0^\infty \frac{JdJ}{2a}\, e^{-\frac{J^2}{4a}}\sqrt{\hat{\mathcal{L}}_N(J)}. \tag{25}$$

The dependence on $\tilde{T}$ is reabsorbed in the integration variable, thus the only free parameter is $a > 0$. We refer to $2a$ as the standard deviation. The measurement can be performed at arbitrary nonzero temperature $\tilde{T}$.

Altogether we arrive at the relation between the BH entropy and the averaged Loschmidt echo:

$$\boxed{e^{\hat{S}} \sim \langle\sqrt{\hat{\mathcal{L}}_N}\rangle_{2a}.} \tag{26}$$

The symbol $\sim$ means equality up to details negligible when $N \gg 1$. The temperature $\tilde{T}$ is unimportant and distinct from the BH temperature T, which enters the left-hand side of (26) via a. Moreover, (26) is expressed using $\sqrt{\hat{\mathcal{L}}_N}$, rather than $\hat{\mathcal{G}}_N$, because only the former is a measurable quantity. We conclude that, varying a (sampling J with different law), the averaged Loschmidt echo (25) with $N \gg 1$ experiences a sharp change in behavior, signaling the Hawking-Page transition (Fig. 2).

Intriguingly, to produce a jump in the echo that simulates the BH entropy, an average is introduced in the spin chain:

(1) $0 < a \ll 1$: the average over J is sharply peaked, introducing a small amount of disorder;

(2) $a \gg 1$: the average introduces more entropy in the system because no value of J is preferred, resulting in more disorder.

A. Interpretation of the result

The AdS_5/CFT_4 correspondence led us to an equality relating the large N BH entropy and the coupling averaged Loschmidt echo. On the BH side of (26), the only parameter is the temperature T, which appears through a. The dependence (7) is monotonic, thus it can be inverted and the BH temperature can be thought of as a function of a.

On the spin chain side of (26), the free parameter a determines how the coupling is sampled. It can be varied freely,

and does not require tuning. The experiment is repeated many times, varying J as sampled randomly according to (25), and computing the average of the measured Loschmidt echo. The right-hand side of (26) is a controlled setup that should experience a sharp change in behavior, when the procedure is performed sampling the coupling at $a < 1$ versus $a > 1$ (Fig. 1). In conclusion, our formula indicates how to concretely test and validate the Hawking-Page prediction, in the spirit of quantum simulation.

B. Fermionic systems

If the chain is fermionized using the Jordan-Wigner transformation [107], (24) remains valid [101], resulting in a completely equivalent outcome. A matrix model description for the Loschmidt echo of fermionic systems corresponding to more complex spin chains is shown to hold in Appendix B 3. The fermionic approach introduces a new set of models with implications for improving the estimation of the BH entropy, as well as for potential experimental applications.

IV. REFINED PROBES

We explain four ways to refine (26) for more accurate tests of the Hawking-Page transition. We discuss:

(1) The analysis of the order parameters for the transition;
(2) The inclusion of corrections by higher operators;
(3) The inclusion of corrections in the 't Hooft coupling;
(4) The case of complex fugacities for the black hole charges.

The technical details are relegated to Appendix B.

A. Order parameters

A fine probe of the transition, from the CFT side of the holographic duality, is the Polyakov loop. This is defined as a Wilson loop winding around the Euclidean time direction. On general grounds, its expectation value $\mathcal{P}$ vanishes if $T < T_{\rm HP}$ and is nonvanishing in a BH phase $T > T_{\rm HP}$. Therefore, the study of $\mathcal{P}$ tests the properties of the transition beyond the jump in entropy. We compute $\mathcal{P}$ in the matrix model and find that indeed it is an order parameter for the phase transition (see Appendix B 1 for the derivation).

In the spin chain, the Polyakov loop is realized as an impurity in the preparation of the initial state. That is, the ket in (15) is replaced with the state

$$|\psi_\times\rangle = |\underbrace{\downarrow\downarrow\ldots\downarrow}_{N-1}\uparrow,\downarrow,\uparrow\ldots\uparrow\rangle. \tag{27}$$

Following Ref. [93], we obtain that the resulting coupling-averaged probability, divided by the coupling-averaged Loschmidt echo, equals $\mathcal{P}$. We elaborate further on this aspect in Appendix B 1. In summary, the expectation value of a Polyakov loop can be analyzed on the spin chain by testing how the results vary when a simple impurity is introduced into the initial state.

Even without any reference to the gauge theory, we predict the ratio of the Loschmidt echoes with and without impurities to be an order parameter discerning between a phase with low disorder, $0 < a \ll 1$, and a phase of large disorder, thus large entropy, $a \gg 1$. This is a novel claim for the coupling averaged Heisenberg XX chain and may be of independent interest. To quantitatively test this prediction with the current state of the art technologies seems an interesting avenue to pursue.

B. Improved estimates of the entropy

The relation (26) prevents a direct evaluation of the BH entropy from the averaged Loschmidt echo, because $\hat{S}$ is only approximately equal to the entropy S. Nonetheless, more accurate results are achieved extending the argument to Hamiltonians with interactions beyond nearest neighbor (Appendix B 2).

The spin chain can reproduce the perturbative corrections in the 't Hooft coupling to the BH entropy (Appendix B 4). Surprisingly, this major conceptual step requires no further sophistication of the experimental setup. These corrections are incorporated without complicating the Hamiltonian nor the initial state. The implementation simply requires that the random coupling is sampled with random standard deviation, see Appendix B 4 for details. The BH entropy is computed by nested averages of the Loschmidt echo.

The matrix model (6) corresponds to a convenient choice of fugacities for the BH charges. More general fugacities lead to complex couplings $a_n \in \mathbb{C}$. The model (9) at $a \in \mathbb{C}$ presents more realistic features of a Hawking-Page transition compared to its counterpart with $a \in \mathbb{R}$ [45]. In Appendix B 5 we explain how this richer behavior is encoded in the spin chain.

V. DISCUSSION

Assuming the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, we showed that the Hawking-Page transition with maximal supersymmetry is reproducible on an elementary 1D spin chain. The result is based on the mathematical identity (26), relating the entropy to the Loschmidt echo of the XX chain with an average over the randomized coupling. We also studied order operators for this transition. Additionally, we enriched the chain in various ways to simulate more realistic features of the BH entropy.

Unlike the planar integrability of $\mathcal{N} = 4$ SYM (reviewed in Appendix C), the XX chain used here is an auxiliary device manifesting the phase transition, rather than being a subsector of SYM. We map one specific observable to the averaged thermal Loschmidt echo in the spin chain. The outcome highlights a universality property: introducing entropy in the 1D chain through the average, it emulates the behavior of (superficially unrelated) BH systems in $(4 + 1)$ dimensions.

It is natural to inquire about the extent to which the spin chain captures structural properties of BH physics, besides the jump in entropy at the Hawking-Page transition. In Appendix D, we elaborate on two concrete avenues for future research, aimed at characterizing the von Neumann algebra and the microstates of the BH via spin chain techniques.

This work opens the possibility for the experimental realizations of Hawking-Page transitions with simple quantum systems. A remaining challenge is to see whether the required system sizes ($N \geqslant 4, L \geqslant 11$) and associated error estimates are feasible with the current technologies.

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APPENDIX A: HAWKING-PAGE, DECONFINEMENT, AND THE SUPERCONFORMAL INDEX

Our study of the AdS_5 entropy across the Hawking-Page transition is done in two steps: first assume the validity of the AdS/CFT correspondence, which maps the supergravity problem to a four-dimensional gauge theory problem, and then take the large N limit in the gauge theory. We now present the pertinent quantities directly in $\mathcal{N}=4$ SYM.

1. Superconformal index

Among the observables that the combination of supersymmetry and conformal symmetry allows us to evaluate, a basic one is the enumeration of supersymmetric operators in the theory. The theory being conformal, the state-operator map dictates that one can equivalently count states.

The superconformal index (SCI) [108–110] (for a review see Ref. [111]) is a twisted trace over the Hilbert space of states in radial quantization. Thus, it precisely counts states in $\mathcal{N}=4$ SYM, with sign (+ for bosons, – for fermions). It is explicitly constructed as follows (our conventions are as in Ref. [32]). Put $\mathcal{N}=4$ SYM on $\mathbb{S}^3$, with isometry $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ whose Cartan generators we denote $\mathsf{J}_\pm$. Let $\mathsf{r}_1, \mathsf{r}_2, \mathsf{r}_3$ be the generators of the Cartan $\mathfrak{u}(1)_1 \oplus \mathfrak{u}(1)_2 \oplus \mathfrak{u}(1)_3$ of the $\mathfrak{so}(6)_R$ R symmetry of $\mathcal{N}=4$ SYM, and define $\mathsf{r} = (\mathsf{r}_1+\mathsf{r}_2+\mathsf{r}_3)/3$. Besides, let Q denote the supercharge preserved by the operators of interest. The SCI is defined as

$$\mathcal{I}^{\mathcal{N}=4} = \mathrm{Tr}\,(-1)^{\mathsf{F}}\, e^{-\beta\{\mathsf{Q},\mathsf{Q}^\dagger\}}\, p^{\mathsf{J}_+ + \mathsf{J}_- + \frac{\mathsf{r}}{2}} q^{\mathsf{J}_+ - \mathsf{J}_- + \frac{\mathsf{r}}{2}}, \tag{A1}$$

where F is the fermion number, β the inverse temperature and p, q are fugacities. The trace is taken over the Hilbert space on $\mathbb{S}^3$ in radial quantization. Finally, the term $\{\mathsf{Q},\mathsf{Q}^\dagger\}$ is the generator of translations along the Euclidean time direction, as usual. We have worked in the preferred slice $p=q$ in the main text, so that

$$\mathcal{I}^{\mathcal{N}=4} = \mathrm{Tr}\,(-1)^{\mathsf{F}}\, e^{-\beta\{\mathsf{Q},\mathsf{Q}^\dagger\}}\, q^{2\mathsf{J}_+ + \mathsf{r}}. \tag{A2}$$

The SCI is protected by supersymmetry and the sign $(-1)^{\mathsf{F}}$ effectively restricts the count to BPS states.

2. Thermal partition function

We are interested in a different observable than the SCI, namely the thermal partition function on $\mathbb{S}^3 \times \mathbb{S}^1$, which we denote $\mathcal{Z}_{\mathbb{S}^3\times\mathbb{S}^1}$. It is defined as the path integral on the curved manifold $\mathbb{S}^3 \times \mathbb{S}^1$, but it is equivalently described as a trace over the Hilbert space in radial quantization on $\mathbb{S}^3$,

$$\mathcal{Z}^{\mathcal{N}=4}_{\mathbb{S}^3\times\mathbb{S}^1} = \mathrm{Tr}\, e^{-\beta\{\mathsf{Q},\mathsf{Q}^\dagger\}}, \tag{A3}$$

or refinements thereof by the fugacities p, q. It is similar to the SCI, but it does not involve the weight $(-1)^{\mathsf{F}}$. In practice, the boundary conditions for the fermions along $\mathbb{S}^1$ break supersymmetry; see, e.g., Ref. [84] for more details on placing the theory on compact Euclidean spacetime.

The SCI has the advantage of being independent of the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ [112], therefore it can be computed in the free limit and the result is exactly equal to the strong coupling limit. By contrast, the thermal partition function does depend on λ, preventing an exact extrapolation from weak to strong coupling. However, the thermal partition function has the advantage that, due to the lack of cancellations between bosons and fermions, it is more sensitive to the deconfinement transition and hence, by AdS/CFT, to the Hawking-Page transition [84].

The entropy in AdS_5 is predicted to be dual to the thermal partition function, and we have computed

$$S \sim \ln \mathcal{Z}^{\mathcal{N}=4}_{\mathbb{S}^3\times\mathbb{S}^1}, \tag{A4}$$

up to normalization terms that are not relevant for our analysis. This identification follows from noting that (A3) is counting states, whence its logarithm is directly interpreted holographically in terms of microstate counting, by using the thermodynamic relation

$$S = \log(\#\ \text{microstates}). \tag{A5}$$

An alternative take on the relation (A4) uses that, in the BH phase $T > T_{\mathrm{HP}}$, the near-horizon geometry is well known to be that of an AdS_2, which implies the dual theory to reduce to an effective quantum mechanics along a thermal circle. This effective field theory approach lies at the heart of the derivation of (6), as we now review.

3. Effective matrix model

The insight of Ref. [15], further extended and generalized in Ref. [18], was to consider the limit of extremely weak 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ of $\mathcal{Z}_{\mathbb{S}^3\times\mathbb{S}^1}$. That is, only gauge invariance is required in Ref. [15], and the computations are performed without accounting for $\mathcal{O}(\lambda)$ corrections. Protected by the extended supersymmetry, one is then able to interpolate to the opposite corner of the conformal manifold, in which the theory is strongly coupled and dual to a BH. No phase transition takes place in this interpolation and the process is

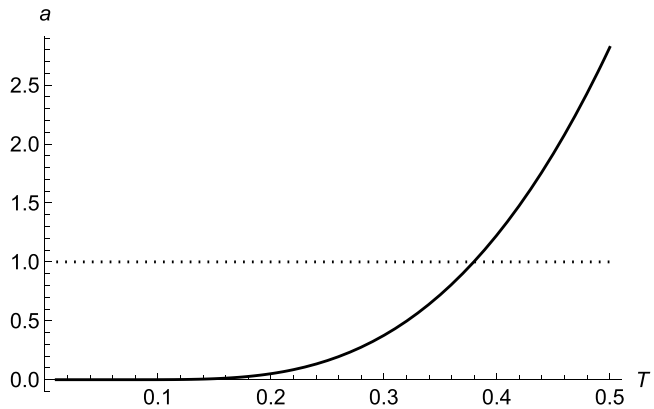

FIG. 3. Coupling $a(T)$ in (7) as a function of the temperature T. The dotted line is $a = 1$, which defines T_H.

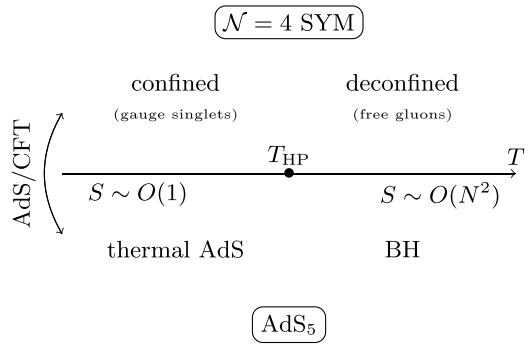

FIG. 4. Hawking-Page and deconfinement transitions.

smooth. We comment further on the inclusion of perturbative corrections in λ in Appendix A 4.

Then, one considers SYM on $\mathbb{S}^3 \times \mathbb{S}^1$, for which all the fields are massive. Integrating them out, the effective description obtained is only in terms of the holonomy U of the gauge field around the thermal circle $\mathbb{S}^1$. A careful account of the integration procedure leads to the effective matrix model description (6) for the partition function $\mathcal{Z}_{\mathbb{S}^3\times\mathbb{S}^1}$ [15,18]. Note that maximal supersymmetry requires that all the fields are in the adjoint representation of the gauge group. Therefore only effective multitrace interactions of the form $\mathrm{Tr}(U^n)\mathrm{Tr}(U^{-n})$ appear in the effective action, which express the trace taken in the adjoint in terms of the standard trace Tr in the fundamental representation.

The effective couplings a_n in the matrix model (6) depend on the masses of the heavy modes integrated out, all of which depend on the radius $1/T$ of the thermal $\mathbb{S}^1$ as well as on the radius of the spatial $\mathbb{S}^3$. The lowest coefficient $a \equiv a_1$ was given in (7), and it is shown in Fig. 3 as a function of T.

4. Weak coupling corrections and deconfinement

The basic ingredients on which our analysis is built are the matrix model (6) and the AdS/CFT correspondence. The holographic principle is used to identify the Hawking-Page transition in the bulk with a deconfinement transition on the dual $\mathcal{N} = 4$ SYM [84]. We have already explained how the matrix model is derived at zeroth order in the 't Hooft coupling λ, whereas the identification between deconfinement and Hawking-Page transition is a strong coupling effect. However, the phase transition we have studied still captures the transition from a thermal AdS to a BH phase. We refer to Ref. [18] for a more exhaustive discussion on this matter.

Notice that, from the point of view of SYM, the jump (5) has the neat interpretation of the thermal partition function being dominated by gauge-singlet bound states at low temperature, where the theory confines, and being dominated by a gas of free gluons at high temperature, after deconfinement (Fig. 4).

Let us briefly elaborate on the physical picture, following Refs. [18,21]. At finite λ, starting from very low temperature and increasing it, there is a phase transition on the gravity side at a critical temperature $T_{\rm HP}$ at which the BH solution becomes dominant compared to the thermal AdS solution. This is the Hawking-Page transition, holographically dual to the deconfinement point in $\mathcal{N} = 4$ SYM [84]. Then, at some higher temperature $T > T_{\rm HP}$, there will be a second transition in which the thermal AdS solution becomes unstable, perturbation theory around that solution becomes ill defined, and only the large BH phase remains. It was shown in Refs. [18,21] that the effect of taking the limit $\lambda \to 0$ while retaining gauge invariance, is that the two transitions coalesce. It follows that, for the sake of observing the signatures of the Hawking-Page transition, the picture considered in the main text suffices.

Including finite λ effects, the backbone of the derivation of the effective matrix model still holds [18], but one has to improve (6) and include additional terms. In particular, as we are interested in the phase transition, we can still focus on the reduced matrix model (8) and include the corrections in λ [18,21]:

$$e^{\hat{S}_\lambda} = \oint dU \, \exp\left[a\mathrm{Tr}(U)\mathrm{Tr}(U^{-1}) + \sum_{n\geqslant 1} \frac{b_n}{N^{2n}} [\mathrm{Tr}(U)\mathrm{Tr}(U^{-1})]^{n+1} \right], \tag{A6}$$

where the notation $\hat{S}_\lambda$ indicates that it includes the corrections in the 't Hooft coupling on the gauge theory side of the duality. The coefficients $b_n = b_n(T, \lambda)$ depend on T as well as on the 't Hooft coupling λ and vanish at $\lambda \to 0$. In other words, as argued in Refs. [18,21], the terms $[\mathrm{Tr}(U)\mathrm{Tr}(U^{-1})]^n$ dominate against the corrections $\mathrm{Tr}(U^n)\mathrm{Tr}(U^{-n})$ near the phase transition. Furthermore, to establish a finite-λ phase diagram, it is sufficient to include the lowest-order correction, i.e., $b_1 > 0$ and $b_{n>1} = 0$ [21].

In summary, the simplified partition function (8) captures all the distinctive properties of the Hawking-Page transition, and a more realistic approximation, which includes a coexistence region for thermal AdS and BH, is described by the improved formula (A6).

APPENDIX B: REFINED PROBES OF THE HAWKING-PAGE TRANSITION

1. Polyakov loop

From the point of view of the gauge theory, the Hawking-Page transition is holographically dual to a deconfinement transition at $T = T_{\rm HP}$ [84]. Famously, a probe of deconfinement [113] is provided by the so-called Polyakov loop [114], placed along the circle $\mathbb{S}^1$ of radius $1/T$ in this context. Let us denote $\mathcal{P}$ the Polyakov loop expectation value, and let A be the $SU(N)$ or $U(N)$ gauge field. Then,

$$\mathcal{P} = \left\langle \mathrm{Tr}\left[\exp\left(i\oint_{\mathbb{S}^1} A\right)\right]\right\rangle_{\mathcal{N}=4}, \tag{B1}$$

where $\langle\cdot\rangle_{\mathcal{N}=4}$ stands for the correlation function in the full-fledged $\mathcal{N}=4$ CFT. Wilson loops in general, and $\mathcal{P}$ in particular, are charged under the center symmetry of $\mathcal{N}=4$ SYM theory. This is directly seen by the fact that (B1) is rotated by a phase when the gauge field A is shifted by a flat one-form in the center of the gauge group. This makes $\mathcal{P}$ an order parameter for the deconfinement transition in $\mathcal{N}=4$ SYM: it must vanish in the low-temperature phase, in which the center symmetry is preserved, and acquires a vacuum expectation value in the high-temperature phase, in which deconfinement spontaneously breaks the center symmetry.

While the definition (B1) is valid at every point of the conformal manifold of $\mathcal{N}=4$ SYM, here we only consider the simplifying weak coupling limit, and consider the insertion of a Polyakov loop in the matrix model (6). The computation of $\mathcal{P}$ dramatically simplifies into

$$\mathcal{P} = \langle \mathrm{Tr}U\rangle, \tag{B2}$$

where now the symbol $\langle\cdot\rangle$ means the one-point function computed in the matrix model. If we are interested in signatures of the phase transition, the study of $\mathcal{P}$ simplifies further by taking the expectation value (B2) in the reduced matrix model (8), exactly as we did in passing from the entropy S to $\hat{S}$.

A convenient way to evaluate $\mathcal{P}$ at large N is to introduce a source term

$$\exp\left(N\varepsilon \mathrm{Tr}U\right) \tag{B3}$$

in the matrix model (9), and compute $\mathcal{P}$ by differentiating the modified entropy with respect to ε. Thanks to the charge conjugation symmetry of $\mathcal{N}=4$ SYM, manifest in the $U \leftrightarrow U^{-1}$ symmetry of (6), and thus in particular of (9), introducing the source term is equivalent to shift

$$\frac{\sigma}{2}\mathrm{Tr}(U+U^{-1}) \mapsto \frac{1}{2}(\sigma+\varepsilon)\mathrm{Tr}(U+U^{-1}), \tag{B4}$$

and, denoting $\hat{S}(\varepsilon)$ the entropy modified in this way,

$$\mathcal{P} = \frac{1}{N}\frac{\partial \hat{S}(\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}. \tag{B5}$$

At this stage, it is straightforward to evaluate $\mathcal{P}$ at large N, borrowing the solution of the matrix model in Ref. [19] and using it to evaluate (B5) at leading order, and indeed we find $\mathcal{P}=0$ if $T < T_{\rm HP}$ and $\mathcal{P}\neq 0$ if $T > T_{\rm HP}$, in agreement with the prediction of the duality between Hawking-Page transition and deconfinement.

A technical caveat on the latter statement is in order. Physically, the Polyakov loop describes the insertion of a heavy quark. When considering a theory without fields in the fundamental representation, placed on a compact space without boundaries such as $\mathbb{S}^3$, the impossibility to give the loop a nonvanishing expectation value follows from the Gauss law applied to the quark flux, thus our claim may look counterintuitive. As explained in Ref. [18], this problem is circumvented by adding an infinitesimal perturbation that breaks the center symmetry explicitly, and setting it to zero at the end. This procedure is pretty much analogous to the study of spontaneous magnetization, in which one turns on a small external field of modulus $\varepsilon > 0$ and sends $\varepsilon \to 0$ at the end.

In practice, the shift (B4), which for us is a convenient strategy at the level of matrix models, matches with the regularization prescribed in Ref. [18]. Eventually one finds $\mathcal{P}\neq 0$ if $T > T_{\rm HP}$ [18].

On the spin chain side, it was shown in Ref. [93] that matrix model averages such as (B2) correspond to impurities in the preparation of the initial state $|\psi_0\rangle$. Then, $\mathcal{P}$ corresponds to the ratio of amplitudes with and without impurity. More precisely, let $|\psi_\times\rangle$ be as in (27), and denote $\mathcal{G}_N^\times(J)$ [$\hat{\mathcal{G}}_N^\times(J)$] the analog of $\mathcal{G}_N(J)$ [$\hat{\mathcal{G}}_N(J)$] constructed using $|\psi_\times\rangle$ as initial state, instead of $|\psi_0\rangle$. Explicitly,

$$\mathcal{G}_N^\times(J) = \langle\psi_0|e^{-H_{\rm XX}/\tilde{T}}|\psi_\times\rangle, \tag{B6}$$

and $\hat{\mathcal{G}}_N^\times = \mathcal{G}_N^\times/\mathcal{G}_1$. Then, the spin chain definition of the observable (B2) is

$$\mathcal{P} = \frac{\langle\hat{\mathcal{G}}_N^\times(J)\rangle_{2a}}{\langle\hat{\mathcal{G}}_N(J)\rangle_{2a}}. \tag{B7}$$

The computation of $\langle\mathrm{Tr}U\rangle$ at fixed J was done in Ref. [93].[1] Taking that result and performing the Gaussian average over J, we get the expected result for $\mathcal{P}$.

Notice that the subtlety mentioned in the Polyakov loop, and the necessity to introduce a symmetry-breaking deformation and the $\varepsilon \to 0$ limit at the end, have a natural solution in the spin chain. Recall that the amplitudes $\hat{\mathcal{G}}_N$ are not measurable quantities, but the echoes $\hat{\mathcal{L}}_N$ are. Therefore, in a hypothetical experimental setup, $\hat{\mathcal{G}}_N^\times$ and $\hat{\mathcal{G}}_N$ in (B7) should be replaced by the square root of the averaged Loschmidt echoes,

$$\mathcal{P} = \frac{\langle\sqrt{\hat{\mathcal{L}}_N^\times(J)}\rangle_{2a}}{\langle\sqrt{\hat{\mathcal{L}}_N(J)}\rangle_{2a}} \tag{B8}$$

exactly as in (26). The center symmetry of $\mathcal{N}=4$ SYM does act on the spin chain with the defect, rotating the amplitude $\hat{\mathcal{G}}_N^\times$ by a phase. However, any physical implementation would automatically select a real saddle point for $\mathcal{P}$, since the hypothetical experiment would only be measuring the real probability (left invariant by the action of the center symmetry) and not the amplitude. Stated differently, any concrete implementation of (B7) would have built in the analog of the regulating procedure described in Ref. [18] to obtain $\mathcal{P}\neq 0$.

[1]In turn, the details of the computation in Ref. [93] mimic the Wilson loop analysis of Refs. [115,116].

It is also noteworthy that the definition of $\mathcal{P}$ can be modified in both $\mathcal{N}=4$ SYM and the spin chain. On the field theory side, it is possible to examine loops in higher-dimensional representations of the gauge group, such as the rank-p antisymmetric representation. On the spin chain side, a feasible approach is to design an initial state $|\psi_\times\rangle$ where the final spin $\downarrow$ is shifted $p \geqslant 1$ locations away from the other $N-1$ spins $\downarrow$. Notwithstanding this alteration, Identity (B7) remains valid.

An open problem would be to study the spin chain avatars of other types of defects, analogous to the one considered in Ref. [73]. The initial state in that case would be a (complicated) superposition of states with impurities.

2. Improving the estimate of the BH entropy

Our main result (26) hinges upon the mapping of $e^{\hat{S}}$ to a coupling average of the GWW matrix model. However, as explained in Ref. [18] (also Refs. [19,45]) and reviewed in the main text, this is far from being the end of the story for the BH entropy. $\hat{S}$ is the dominating contribution near the critical point, which therefore contains enough information to establish the phase transition, but does not accurately reproduce the entropy S without accounting for the operators that have been discarded in passing from (6) to (8). Nevertheless, as long as we restrict to the regime in which the matrix model description holds, we can obtain more accurate results via mapping to more complicated spin chain Hamiltonians, with more interactions.

On the one hand, by repeating the Gaussian integral trick that led to (9), we can approximate the full BH entropy in (6) up to some cutoff order K with K Gaussian integrals over auxiliary variables σ_n, with standard deviation $2a_n$:

$$e^S \sim \prod_{n=1}^{K} \int_0^\infty \sigma_n d\sigma_n \, e^{-N^2 \frac{\sigma_n^2}{4a_n}} \times \oint dU \, \exp\left\{ \sum_{n=1}^{K} \frac{N\sigma_n}{2n} \mathrm{Tr}(U^n + U^{-n}) \right\}, \tag{B9}$$

omitting uninteresting normalization factors. In fact, a (different) average point of view on (6) has been already advocated in Ref. [57].

On the other hand, the mapping to the spin chain for these more general Hamiltonians has been formalized in Refs. [99,103]. The procedure, which generalizes Ref. [95], is similar to the one reviewed in the main text and we skip the details, for which we refer to Ref. [99]. The streamlined idea is to generalize the Hamiltonian H_{XX} beyond nearest-neighbor interactions:

$$H_{\mathrm{gen}} = -\frac{1}{2} \sum_{j=0}^{L-1} \sum_{n=1}^{K} \frac{\tilde{J}_n}{n} (\sigma_j^- \sigma_{j+n}^+ + \sigma_j^+ \sigma_{j+n}^-), \tag{B10}$$

for a collection of couplings $\tilde{J}_n$ and assuming periodic boundary conditions. For ease of exposition, we rename $\tilde{J}_n = \tilde{J}_1 \gamma_n$ and denote $J \equiv \tilde{J}_1/\tilde{T}$ as in the main text. Then, running an argument analogous to Ref. [95] and Sec. III, ones finds that the Loschmidt amplitude

$$\mathcal{G}_N^{\mathrm{gen}}(J) = \langle \psi_0 | e^{-H_{\mathrm{gen}}/\tilde{T}} | \psi_0 \rangle \tag{B11}$$

is computed by a Toeplitz determinant

$$\mathcal{G}_N^{\mathrm{gen}}(J) = \det_{1 \leqslant j,k \leqslant N-1} \left[g_{j,k}^{\mathrm{gen}}(J) \right], \tag{B12}$$

where the generalization of (18) is

$$g_{j,k}^{\mathrm{gen}}(J) = \langle \Uparrow | \sigma_j^+ e^{-H_{\mathrm{gen}}/\tilde{T}} \sigma_k^- | \Uparrow \rangle. \tag{B13}$$

Reasoning as in Sec. III, we differentiate $g_{j,k}^{\mathrm{gen}}$ and, using the commutation relations (14), we find

$$\frac{d g_{j,k}^{\mathrm{gen}}}{dJ} = \frac{1}{2} \sum_{n=1}^{K} \frac{\gamma_n}{n} (g_{j-n,k} + g_{j+n,k}). \tag{B14}$$

This is the recurrence relation of the generalized Bessel function. Alternatively, it can be directly checked that the integral

$$\oint \frac{du}{2\pi i u} u^{j-k} \, \exp\left\{ \frac{J}{2} \sum_{n=1}^{K} \frac{\gamma_n}{n} (u^n + u^{-n}) \right\} \tag{B15}$$

is a solution. Plugging the result into (B12) and applying the Heine-Szegő identity, we have that the Loschmidt amplitude for this more general spin chain equals the matrix model

$$\mathcal{Z}_N^{\mathrm{gen}} = \oint dU \, \exp\left\{ \sum_{n=1}^{K} \frac{\tilde{J}_n}{2n\tilde{T}} \mathrm{Tr}(U^n + U^{-n}) \right\} \tag{B16}$$

in the limit $L \gg N$. Again, the temperature $\tilde{T}$ of the laboratory does not matter as long as it is nonzero, because the control parameters of the system are the ratios

$$J_n = \tilde{J}_n / \tilde{T}. \tag{B17}$$

Thus, with the normalization as in the main text, we find the generalized thermal Loschmidt echo

$$\hat{\mathcal{L}}_N^{\mathrm{gen}}(J_1, \ldots, J_K) = \left| \mathcal{Z}_N^{\mathrm{gen}} / \mathcal{Z}_1^{\mathrm{gen}} \right|^2. \tag{B18}$$

Comparing (B9) with the generalized spin chain matrix model (B16), we observe that the entropy S, with irrelevant interactions[2] cut off at some order K, and the generalized spin chain with up to K-neighbor interactions, satisfy the analogous relation as for the simplified entropy $\hat{S}$ and the XX chain. Once again, the spin chain coupling constants $\tilde{J}_n$ are related to the Hubbard-Stratonovich fields σ_n through $N\sigma_n = \tilde{J}_n/\tilde{T} \equiv J_n$, and in the generalized setting one is instructed to take a Gaussian average over all the spin-spin couplings J_n.

In principle, one is able to approximate the entropy S (and hence the black hole entropy S_{BH} in the phase $T > T_{\mathrm{HP}}$) to arbitrarily high-order K of irrelevant couplings, by measuring the Loschmidt echo in the generalized chain and then taking a Gaussian average over all the couplings.

We underline that the average perspective on the entropy of a BH in the holographic dual to $\mathcal{N}=4$ SYM was already undertaken in Ref. [57], but here we are pointing out a very concrete and explicit way to implement the average in a simple 1D auxiliary device.

[2]In this paper, the word "irrelevant" is always used in its precise RG flow sense.

$$\begin{array}{cccccccccc} \cdots & \circ & \circ & \bullet & \bullet & \cdots & \bullet & \circ & \circ & \cdots \\ n_j = \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots \end{array}$$

FIG. 5. The initial state (2) is equivalent, via a Jordan-Wigner transformation, to place N adjacent fermions on a lattice of L sites. $\circ$ indicates an empty site, $\bullet$ indicates a site occupied by a fermion.

3. Fermionic point of view

The XX model, also known as isotropic XY model, is well known to be equivalent to a system of free fermions, via the Jordan-Wigner (JW) map [107]. We can identify the spin at site j of the chain with a fermionic occupation number, according to

$$|\uparrow\rangle \mapsto |0\rangle, \quad |\downarrow\rangle \mapsto |1\rangle, \tag{B19}$$

where $|n_j\rangle$ indicates $n_j \in \{0, 1\}$ fermions on the site j. The initial state (2) thus corresponds to place N adjacent fermions on a periodic lattice of L sites (Fig. 5),

$$|\psi_0\rangle = |\underbrace{1, 1 \ldots, 1}_{N}, 0, 0 \ldots, 0\rangle. \tag{B20}$$

This fermionic model leads to the same Loschmidt echo used in the main text [101].

It is possible to extend this result and to show that for a 1D fermionic model

$$H_{\text{ferm}} = \frac{1}{L}\sum_k \epsilon_k \tilde{c}_k^\dagger \tilde{c}_k \tag{B21}$$

with a general dispersion relation ϵ_k, the transition amplitude is still a unitary matrix model (for fermions with periodic boundary conditions) with the dispersion relation as a confining potential.

The Hamiltonian (B21), is expressed in momentum space, with k running over the dual lattice,

$$k = \frac{2\pi}{L}q, \quad q = 0, \ldots, L-1, \tag{B22}$$

and $\tilde{c}_k^\dagger$, $\tilde{c}_k$ the fermionic creation and annihilation operators in momentum space, which satisfy the canonical anticommutation relations

$$\{\tilde{c}_k^\dagger, \tilde{c}_{k'}\} = 2\pi\delta_{q,q'}. \tag{B23}$$

They are related to the creation and annihilation operators $c_j^\dagger$, c_j for fermions on the lattice site j via the discrete Fourier transform

$$\tilde{c}_k = \sum_{j=0}^{L-1} e^{\mathrm{i}jk} c_j = \sum_{j=0}^{L-1} e^{-\mathrm{i}2\pi qj/L} c_j. \tag{B24}$$

Fermionic models of the form (B21) have been analyzed in Ref. [117], to which we refer for a careful treatment of the periodic boundary conditions. As a side remark, we note that the fermionic setup allows nonisotropic XY interactions [117], but for concreteness we do not consider such scenario.

The result equating the Loschmidt echo of the fermionic model to the Loschmidt echo of the spin chain holds true even for cases where ϵ_k is not equal to $\cos(k)$, which corresponds to the image of the XX model by the JW transformation and results in the GWW matrix model for the Loschmidt amplitude.

To derive the matrix model description, we extend the result of Ref. [101]. With the initial state $|\psi_0\rangle$ as in (B20), we consider the amplitude

$$\mathcal{G}_N^{\text{ferm}} = \langle\psi_0|e^{-H_{\text{ferm}}/\tilde{T}}|\psi_0\rangle. \tag{B25}$$

Inserting N times the resolution of the identity on the left and on the right and approximating for $L \to \infty$ (for the finite-L expression, simply replace the integrals by Riemann sums), we get

$$\mathcal{G}_N^{\text{ferm}} = \int_0^{2\pi} \frac{dk_1}{2\pi} e^{-\epsilon_{k_1}/\tilde{T}} \cdots \int_0^{2\pi} \frac{dk_N}{2\pi} e^{-\epsilon_{k_N}/\tilde{T}} \times |\langle k_1, \ldots, k_N|\psi_0\rangle|^2. \tag{B26}$$

This expression heavily relies on the fact that (B21) describes noninteracting fermions. The amplitude $|\langle k_1, \ldots, k_N|\psi_0\rangle|$ is computed by a Slater determinant. We also make a change of variables $k_j = \theta_j + \pi$, which is not necessary in principle but is required for consistency with our conventions in the spin chain, as we have (implicitly) worked with variables $\theta_j \in [-\pi, \pi]$ throughout. Then, one finally arrives at

$$\mathcal{G}_N^{\text{ferm}} = \oint dU \, \exp\left\{\frac{1}{\tilde{T}}\mathrm{Tr}\, V_\epsilon(U)\right\}, \tag{B27}$$

where U is a unitary matrix with eigenvalues $e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_N}$, and V_ϵ is the matrix model potential defined by the dispersion relation,

$$\mathrm{Tr}\, V_\epsilon(U) = \sum_{j=1}^{N} \epsilon_{k_j=\theta_j+\pi}. \tag{B28}$$

This equals (B16) upon identification of the dispersion relation

$$\epsilon_k = \sum_{n=1}^{K} (-1)^n \frac{\tilde{J}_n}{2n}(e^{\mathrm{i}k} + e^{-\mathrm{i}k}). \tag{B29}$$

Therefore, with this choice, $\mathcal{G}_N^{\text{ferm}} = \mathcal{G}_N^{\text{gen}}$.

We emphasize that, despite the equality of this specific observable, the free fermion system with generalized dispersion relation and the spin chain with interaction beyond nearest neighbor are not equivalent. It is a known fact that the JW transformation cannot map these two models for any $K > 1$.

We have obtained that the spin chains in Appendix B 2 and the fermionic models (B21) have equal return amplitude in the suitably chosen initial state (2). We now proceed to show this fact explicitly, thus yielding a consistency check of the previous derivation and demonstrating as a proof of concept that two models that are not equivalent can nevertheless have equal Loschmidt echo.

It is well known that, for an open chain, the inverse JW map [in the conventions (B19)] transforms the fermionic bilinear terms according to [107,117]

$$\frac{(-1)^n}{4} c_j^\dagger c_{j+n} \mapsto -\sigma_j^- \left(\prod_{\nu=1}^{n-1} \sigma_{j+\nu}^z\right) \sigma_{j+n}^+, \tag{B30}$$

where the sign on the left-hand side comes from the $(-1)^n$ in (B29), and the overall minus sign on the right-hand side agrees with the spin chain conventions in (B10). The spin

chain Hamiltonian resulting from the inverse JW transformation differs from the Hamiltonian (B10) by the presence of the intermediate $\sigma^z_{j+\nu}$, enforcing fermionic statistic. We refer to Refs. [117,118] for more details as well as for the subtleties arising from the periodic boundary conditions. Since (B21) only contains fermion bilinears, the fermion parity is a conserved quantity. For the observable of interest, where H_{ferm} acts only on (B20), this equals $(-1)^N$.

Consider first a Hamiltonian with only nth-nearest-neighbor interaction, and map it to spin variables via (B30). To compute the return amplitude for this term, we expand the exponential and compute, for every $m \geqslant 0$,

$$\langle\psi_0|\left(\sum_{j=0}^{L-1}\sigma_j^-\left(\prod_{\nu=1}^{n-1}\sigma^z_{j+\nu}\right)\sigma^+_{j+n}+\text{ c.c.}\right)^m|\psi_0\rangle. \tag{B31}$$

Letting the operator act on the ket gives a superposition of states with definite fermion occupation configuration. However, only even powers of m give states with nontrivial overlap with $\langle\psi_0|$. Therefore, since even powers of σ^z are trivial, the Loschmidt amplitude is the same as in the spin chain with nth-nearest-neighbor interaction.

When including all the interaction terms $1 \leqslant n \leqslant K$, the outcome is unchanged. This is guaranteed by the fact that the potential mixing of contributions [for instance, $(c^\dagger_{j+1}c_j)^{nm}$ with $(c^\dagger_{j+n}c_j)^m$], cannot produce cancellations, because they come with different coefficients. The claim can also be checked by direct computation, e.g., in the $K = 2$ case.

We highlight that the equality of the Loschmidt amplitudes for the two inequivalent models stems from two properties:

(1) The initial and final states are identical;

(2) The initial state is an eigenstate of the fermion parity operator.

4. Perturbative corrections to the BH entropy

In Appendix A 4 we have enriched the BH entropy allowing for a small but finite 't Hooft coupling $0 < \lambda \ll 1$. The matrix model (A6) was studied in Ref. [21], where it was shown that $b_{n>1} = 0$ already retains all the desired qualitative features. The authors of Ref. [21] rewrite [$b \equiv b_1(T,\lambda)$]

$$e^{\hat{S}_\lambda} = \frac{N^3}{4\sqrt{\pi b}}\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\,e^{-\frac{N^2}{4b}(\mu-a)^2}\int_0^\infty \sigma\, d\sigma \times \oint dU\,\exp\left[-N^2\left(\frac{\sigma^2}{4\mu}-\mathcal{F}(\sigma)\right)\right], \tag{B32}$$

which makes it clear that the map to the spin chain goes through. In this case, though, a nested average over the coupling J is required. First, observe that

$$\hat{S}_\lambda \sim \ln\left[\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\,e^{-\frac{N^2}{4b}(\mu-a)^2}e^{\hat{S}}|_{a\mapsto\mu}\right], \tag{B33}$$

where $e^{\hat{S}}|_{a\mapsto\mu}$ stands for (9) with a replaced by μ on the right-hand side, and the symbol $\sim$ again means that we drop terms that do not affect our discussion. Then, we run the argument of the main text and use (26) to obtain

$$\hat{S}_\lambda \sim \ln\left[\int_{-\infty}^{\infty}\frac{d\mu}{\mu}\,e^{-\frac{N^2}{4b}(\mu-a)^2}\,\langle\sqrt{\hat{\mathcal{L}}_N}\rangle_{2\mu}\right]. \tag{B34}$$

The outcome of Ref. [21] is that (B34) has a phase structure that agrees with (9) near $b \to 0$ (which corresponds to $\lambda \to 0$ in SYM), but in addition shows the properties predicted in Appendix A 4.

Our result is to argue that this outcome can be tested on the Heisenberg XX chain, by measuring the thermal Loschmidt echo and then take a nested average of the coupling. Namely, first take a Gaussian average over the spin-spin coupling J centered at $J = 0$ with standard deviation 2μ. The parameter μ is itself sampled from a Gaussian distribution, centered around $\mu = a$ with standard deviation $2b/N^2$. Interestingly, the progress in passing from $\hat{S}$ to $\hat{S}_\lambda$, which is a nontrivial step from the SYM side, reduces to averaging the averaged echo on the spin chain. That is, in practice it boils down to repeat the measurement several times, sampling both J and the standard deviation with which J is sampled.

5. Complex temperature

It is worthwhile to stress that the matrix model (6) was derived in Refs. [15,16,18] under certain simplifying assumptions. It was pointed out in Ref. [45] that, adopting most of the same simplifications but allowing for complex couplings $a_n \in \mathbb{C}$ in the matrix model (6) leads to physically more realistic predictions about the BH behavior. This is because it would allow for arbitrary fugacities in the SCI, which in turn give arbitrary fugacities for the BH charges, thus retaining more information in the computation.

Conceptually, it is straightforward to introduce complex couplings in our model, even though any hypothetical experimental realization would become more challenging. We focus again on the reduced model (9) to compute $\hat{S}$. Writing $a = |a|e^{i\varphi}$, we use (26) and (25), and find that the model of Ref. [45] satisfies

$$e^{\hat{S}} \sim \langle\hat{\mathcal{L}}_N(e^{-i\varphi/2}J)\rangle_{2|a|}. \tag{B35}$$

Namely, we still compute the average of the Loschmidt echo, with random coupling sampled with a Gaussian distribution, except that now the system is at complex temperature $e^{i\varphi/2}\tilde{T}$, i.e., finite temperature and time evolved. The complex temperature Loschmidt amplitude reduces to the GWW matrix model with complex coupling, in which the integration contour is deformed along a suitable thimble [119,120].

In a laboratory setup, one first fixes the temperature of the system at a value $\tilde{T}_{\mathbb{R}}$ and then lets the system evolve until time t, which depends on the phase φ of a in such a way that

$$\arg\left(it + \tilde{T}_{\mathbb{R}}^{-1}\right) = -\varphi/2. \tag{B36}$$

The procedure of averaging over J remains unchanged in this scenario.

6. Determinant identities from spin chains

Throughout this Appendix we have studied various ways of enriching the XX spin chain, with the aim of refining the correspondence with the Hawking-Page transition. We now comment on a byproduct of our construction, which has interesting implications for the spin chain itself, regardless of the applications to BH physics.

Consider a spin chain with interaction at next-to-nearest neighbor, but without any nearest-neighbor term. That is, we are in the setup of Appendix B 2 with $K = 2$ and

$$\tilde{J}_1 = 0, \quad \tilde{J}_2 > 0. \tag{B37}$$

In this case, the spin chain decouples into two noninteracting XX spin chains, both with $\tilde{J} = \tilde{J}_2$. One copy of the XX chain involves only the spins at even lattice sites, and the other copy involves only the spins at odd lattice sites. Therefore, as a corollary of our analysis in Appendix B 2 we derive the identity

$$\mathcal{G}_{2N}^{2\mathrm{nn}}(J) = [\mathcal{G}_N(J)]^2, \tag{B38}$$

where, on the left-hand side, $\mathcal{G}_{2N}^{2\mathrm{nn}}$ is the Loschmidt amplitude for the spin chain with only next-to-nearest-neighbor ferromagnetic interaction. This holds already at finite L, but in the following we consider $L \to \infty$ to simplify the expressions.

Utilizing (B12), the equality of the amplitudes translates into a mathematical identity:

$$\det_{1\leqslant j,k\leqslant 2N}\left[I_{j-k}^{(1,2)}(0,J)\right] = (\det_{1\leqslant j,k\leqslant N}[I_{j-k}(J)])^2. \tag{B39}$$

On the right-hand side, $I_\nu(J)$ is the modified Bessel function of first kind, while on the left-hand side $I_\nu^{(1,2)}(J_1, J_2)$ is the generalized modified Bessel function, with integral representation for $\nu \in \mathbb{Z}$

$$I_\nu^{(1,2)}(J_1, J_2) = \oint \frac{du}{2\pi i u} u^\nu \, e^{\sum_{n=1}^{2} \frac{J_n}{2n}\mathrm{Tr}(u^n + u^{-n})}. \tag{B40}$$

Therefore, as an immediate byproduct of Ref. [99] together with Appendix B 2 we derive (B39) [and its generalization (B46)]. We stress the following aspects:

(1) The argument is this Appendix entails a change in philosophy, where we use an elementary spin chain argument and the map of Ref. [99] to write an identity for Toeplitz determinants, which is seemingly less obvious from the point of view of random matrix theory.

(2) This identity involves the ubiquitous GWW model.

To strengthen our argument, we now rederive (B39) [and its generalization (B46)] in an independent way, thus providing a solid consistency check of the construction in Appendix B 2.

To begin with, we note that, in the large N limit, (B39) is a direct consequence of Szegő's strong limit theorem [121]. To study the determinants at finite N, we use that the generalized Bessel functions satisfy [122]

$$I_\nu^{(1,2)}(0,J) = \begin{cases} 0, & \forall \nu \in 1 + 2\mathbb{Z}, \\ I_{\nu/2}(J), & \forall \nu \in 2\mathbb{Z}. \end{cases} \tag{B41}$$

The case $N = 1$ of (B39) reads

$$I_0^{(1,2)}(0,J)^2 - I_{-1}^{(1,2)}(0,J) I_1^{(1,2)}(0,J) = I_0(J)^2, \tag{B42}$$

and it is proved as an immediate consequence of (B41). It is also easy to check explicitly that (B39) holds for low value of N, thanks to the relations (B41).

To prove (B39) in full generality—without resorting to the spin chain—one expands the determinant on the left-hand side,

$$\det_{1\leqslant j,k\leqslant 2N}\left[I_{j-k}^{(1,2)}(0,J)\right] = \sum_{\varpi\in S_{2N}} (-1)^\varpi \prod_{j=1}^{2N} I_{j-\varpi(j)}^{(1,2)}(0,J) \tag{B43}$$

and separates the product over j into two products, over $j \in 2\mathbb{Z} \cap [1, N]$ and $j \in (1 + 2\mathbb{Z}) \cap [1, N]$. From (B41), $I_{j-\varpi(j)}^{(1,2)}(0,J) \neq 0$ only if both j and $\varpi(j)$ are even or both are odd. Therefore the nonvanishing contributions to the Toeplitz determinant are those from permutations that factorize into $\varpi = \varpi_\mathrm{e} \circ \varpi_\mathrm{o}$, where ϖ_e (ϖ_o) permutes only the even (odd) indices. This yields

$$\begin{aligned} &\sum_{\varpi\in S_{2N}} (-1)^\varpi \prod_{j=1}^{2N} I_{j-\varpi(j)}^{(1,2)}(0,J) \\ &= \left[\sum_{\varpi_\mathrm{e}\in S_N} (-1)^{\varpi_\mathrm{e}} \prod_{\substack{j=1\\ j\ \mathrm{even}}}^{2N} I_{j-\varpi_\mathrm{e}(j)}^{(1,2)}(0,J)\right] \\ &\quad \times \left[\sum_{\varpi_\mathrm{o}\in S_N} (-1)^{\varpi_\mathrm{o}} \prod_{\substack{j=1\\ j\ \mathrm{odd}}}^{2N} I_{j-\varpi_\mathrm{o}(j)}^{(1,2)}(0,J)\right], \end{aligned} \tag{B44}$$

where we remark that this factorization is a consequence of the nontrivial identity (B41). Recognizing the two terms on the right-hand side as Toeplitz determinants, we arrive at (B39).

The above argument is easily generalized to chain with Kth-nearest-neighbor interaction, and no other interaction. Namely, one considers a spin chain with hopping Hamiltonian[3]

$$H_{K\mathrm{nn}} = -\frac{J}{2}\sum_{j=0}^{L-1}(\sigma_j^- \sigma_{j+K}^+ + \sigma_j^+ \sigma_{j+K}^-). \tag{B45}$$

The same argument as for $K = 2$ implies the factorization into K noninteracting XX chains, hence

$$\mathcal{G}_{KN}^{K\mathrm{nn}}(J) = [\mathcal{G}_N(J)]^K. \tag{B46}$$

This translates again into an identity of Toeplitz determinants, involving higher generalized Bessel functions. The generalization of property (B41) is [122]

$$I_\nu^{(1,K)}(0,J) = \begin{cases} 0, & \forall \nu \notin K\mathbb{Z}, \\ I_{\nu/K}(J), & \forall \nu \in K\mathbb{Z}. \end{cases} \tag{B47}$$

Using this property, it is easy to rederive (B46) independently of the spin chain, exactly as in the $K = 2$ case.

APPENDIX C: COMMENTS ON PLANAR INTEGRABILITY

There exists a connection between spin chains and $\mathcal{N} = 4$ SYM, known since the work of Minahan-Zarembo [123], which goes under the name of planar integrability. This property is the statement that, in the planar limit, correlation

[3]To reduce clutter, we slightly change the notation with respect to Appendix B 2, $(\tilde{J}_K/K, \tilde{T}) \mapsto (J, 1)$.

functions of certain local operators in $\mathcal{N} = 4$ SYM in flat spacetime equal those of (different) operators in a spin chain [123,124]. This relation appears in the planar limit of large N with fixed 't Hooft coupling λ.

While the setup is vaguely similar to what we consider here, we ought to emphasize the built-in differences.

(1) Planar integrability, as the name suggests, is a feature that emerges only in the planar limit. On the contrary, given any truncation of the tower of irrelevant operators appearing in (6) (see Appendix B 2 for details), we can define a spin chain Hamiltonian such that the averaged Loschmidt echo reproduces the thermal partition function of $\mathcal{N} = 4$ SYM. This holds before taking any large N limit. Moreover, the relation (26) is exact, even though the large N limit serves to render $\hat{S}$ a good approximation of the SCI.

(2) In turn, the bridge between $\mathcal{N} = 4$ SYM and the XX spin chain we find is limited to a map between specific observables, namely the partition function on thermal $\mathbb{S}^3 \times \mathbb{S}^1$ on the SYM side and the averaged Loschmidt echo on the chain side. We do not claim a full identification of the two systems, in any regime. This correspondence can be extended to include selected nonlocal operators in $\mathcal{N} = 4$ SYM, mapped to defects in the XX chain (see Appendix B 1), but falls short of yielding a complete dictionary between operators on both sides of the correspondence.

(3) Furthermore, for the correspondence to work, we have to introduce disorder in the spin chain by randomizing the coupling and taking the Gaussian average. This situation is radically different from the planar integrability.

APPENDIX D: OUTLOOK: BEYOND THE HAWKING-PAGE TRANSITION

We conclude this Appendix on a more speculative note. So far, our discussion has focused on a 1D quantum simulation to reproduce the entropy of a holographic BH in AdS_5. We now suggest and lay down two open problems, which aim at looking for deeper signatures of BH physics on the spin chain.

1. Operator algebras from the spin chain

The upshot of our analysis is that the Heisenberg XX spin chain can be used to simulate the Hawking-Page transition. This correspondence has been refined and extended in several directions in Appendix B.

It is interesting to ask whether other characteristic properties of the Hawking-Page transition are automatically encoded in the spin chain, if we look beyond the Loschmidt echo. For instance, phase transitions as the one studied presently are accompanied by a drastic change in the large-N algebra of operators [125]. Due to its relevance for questions concerning the BH horizon and its interior [126,127], it would be extremely interesting to enlarge our correspondence to include this aspect.

To address questions of this kind, the fermionic picture comes in handy. Indeed, it has been recently pointed out that these algebra transitions are detected by fermionic lattices [128]. If this result could be extended to the fermionic systems obtained in Appendix B 3, one would end up with a very concrete quantum mechanical gadget that realizes the criterion of Ref. [125] for the behavior of the algebra of operators.

A related problem is to understand the relation between the averaging procedure on the Loschmidt echo and chaos. The presence of a BH should be accompanied by maximal chaos, and it would be interesting to see how this effect appears in the spin chain. Leveraging the connection between quantum chaos and algebras of operators [129], establishing that the averaged Loschmidt echo possesses maximal chaos would shed light on how spacetime emerges from the spin chain through the Gaussian average over couplings of the measurements.

2. Identifying states

So far, our conclusions have been about the large-N behavior of the models, be it the gravitational system, its holographic dual $\mathcal{N} = 4$ SYM, or the spin chain. Nevertheless, the mapping of the effective partition function onto the spin chain is exact and thus holds for finite N. One problem, however, is that the contributions discarded in going from (6) to (8) give finite contributions at finite N, so their corrections should be taken into account. This can be done by constructing a more involved spin chain, with interactions beyond nearest neighbors as explained in Appendix B 2, which approximates the effect of accounting for higher operators in (6). The situation will remain tractable for N finite but large, but will become untamed for small N.

In principle, it is possible to encode the exact SCI of $\mathcal{N} = 4$ on a spin chain, because it admits a presentation as a unitary matrix model. Therefore, the argument in Appendix B 2 shows that with sufficiently complicated interactions one should be able to simulate the SCI and extract the microstate count at large N from the spin chain device. The SCI is protected by supersymmetry, so this procedure does not require weak 't Hooft coupling and theoretically allows nonperturbative counting. Unfortunately, the required Hamiltonian seems too complicated to be effectively implemented in practice. Moreover, the microstate counting in this proposal would suffer from the finite-size effects of the lattice, so the empirical result will inevitably differ from the setup it is supposed to simulate.

The work [66] investigated the finite N spectrum of protected operators in $\mathcal{N} = 4$ SYM, building upon [26]. One significant discovery was the identification of potential black hole microstates even at $N = 2$. It is possible to verify the findings of Ref. [66] using a quantum simulator. At present, it is unclear how to refine our setup to achieve this, as it would likely require considering more complex interactions among qubits to reproduce the significant $1/N$ corrections. Additionally, it would involve probing a more precise quantity than the Loschmidt echo. This paper lays the groundwork for exploring important quantum simulation questions in the future. The exploration will shed light on the nature of black holes.

[1] J. A. Wheeler, Information, physics, quantum: The search for links, in *Proceedings of the 3rd International Symposium on Foundations of Quantum Mechanics in Light* (JPS Conf. Proc., Tokyo, Princeton, NJ, 1989), pp. 354–358.

[2] J. Maldacena and L. Susskind, Cool horizons for entangled black holes, Fortschr. Phys. **61**, 781 (2013).

[3] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, Black holes: Complementarity or firewalls? J. High Energy Phys. 02 (2013) 062.

[4] E. Verlinde and H. Verlinde, Black hole entanglement and quantum error correction, J. High Energy Phys. 10 (2013) 107.

[5] A. Almheiri, X. Dong, and D. Harlow, Bulk locality and quantum error correction in AdS/CFT, J. High Energy Phys. 04 (2015) 163.

[6] G. Penington, Entanglement wedge reconstruction and the information paradox, J. High Energy Phys. 09 (2020) 002.

[7] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole, J. High Energy Phys. 12 (2019) 063.

[8] D. Jafferis, A. Zlokapa, J. D. Lykken, D. K. Kolchmeyer, S. I. Davis, N. Lauk, H. Neven, and M. Spiropulu, Traversable wormhole dynamics on a quantum processor, Nature (London) **612**, 51 (2022).

[9] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-de Sitter space, Commun. Math. Phys. **87**, 577 (1983).

[10] J. D. Bekenstein, Black holes and entropy, Phys. Rev. D **7**, 2333 (1973).

[11] S. W. Hawking, Particle creation by black holes, Commun. Math. Phys. **43**, 199 (1975) [Erratum: Commun. Math. Phys. 46, 206 (1976)].

[12] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **38**, 1113 (1998).

[13] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).

[14] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).

[15] B. Sundborg, The Hagedorn transition, deconfinement and $\mathcal{N} = 4$ SYM theory, Nucl. Phys. B **573**, 349 (2000).

[16] A. M. Polyakov, Gauge fields and space-time, Int. J. Mod. Phys. A **17**, 119 (2002).

[17] V. Balasubramanian, M.-X. Huang, T. S. Levi, and A. Naqvi, Open strings from $\mathcal{N} = 4$ super Yang-Mills, J. High Energy Phys. 08 (2002) 037.

[18] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk, The Hagedorn - deconfinement phase transition in weakly coupled large $\mathcal{N}$ Gauge theories, Adv. Theor. Math. Phys. **8**, 603 (2004).

[19] H. Liu, Fine structure of Hagedorn transitions, arXiv:hep-th/0408001.

[20] D. Berenstein, A toy model for the AdS/CFT correspondence, J. High Energy Phys. 07 (2004) 018.

[21] L. Alvarez-Gaume, C. Gomez, H. Liu, and S. R. Wadia, Finite temperature effective action, AdS_5 black holes, and $1/N$ expansion, Phys. Rev. D **71**, 124023 (2005).

[22] G. Festuccia and H. Liu, Excursions beyond the horizon: Black hole singularities in Yang-Mills theories. I. J. High Energy Phys. 04 (2006) 044.

[23] P. Basu and S. R. Wadia, R-charged AdS_5 black holes and large $\mathcal{N}$ unitary matrix models, Phys. Rev. D **73**, 045022 (2006).

[24] L. Alvarez-Gaume, P. Basu, M. Marino, and S. R. Wadia, Blackhole/String transition for the small Schwarzschild blackhole of $AdS_5 \times S^5$ and critical unitary matrix models, Eur. Phys. J. C **48**, 647 (2006).

[25] I. Biswas, D. Gaiotto, S. Lahiri, and S. Minwalla, Supersymmetric states of $\mathcal{N} = 4$ Yang-Mills from giant gravitons, J. High Energy Phys. 12 (2007) 006.

[26] C.-M. Chang and X. Yin, 1/16 BPS states in $\mathcal{N} = 4$ super-Yang-Mills theory, Phys. Rev. D **88**, 106005 (2013).

[27] S. M. Hosseini, K. Hristov, and A. Zaffaroni, An extremization principle for the entropy of rotating BPS black holes in AdS_5, J. High Energy Phys. 07 (2017) 106.

[28] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes, J. High Energy Phys. 10 (2019) 062.

[29] S. Choi, J. Kim, S. Kim, and J. Nahmgoong, Large AdS black holes from QFT, arXiv:1810.12067.

[30] S. Choi, J. Kim, S. Kim, and J. Nahmgoong, Comments on deconfinement in AdS/CFT, arXiv:1811.08646.

[31] F. Benini and P. Milan, A Bethe Ansatz type formula for the superconformal index, Commun. Math. Phys. **376**, 1413 (2020).

[32] F. Benini and P. Milan, Black Holes in 4D $\mathcal{N} = 4$ Super-Yang-Mills field theory, Phys. Rev. X **10**, 021037 (2020).

[33] M. Honda, Quantum black hole entropy from 4d supersymmetric Cardy formula, Phys. Rev. D **100**, 026008 (2019).

[34] A. Arabi Ardehali, Cardy-like asymptotics of the 4d $\mathcal{N} = 4$ index and AdS_5 blackholes, J. High Energy Phys. 06 (2019) 134.

[35] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, The asymptotic growth of states of the 4d $\mathcal{N} = 1$ superconformal index, J. High Energy Phys. 08 (2019) 120.

[36] A. González Lezcano and L. A. Pando Zayas, Microstate counting via Bethe Ansätze in the 4d $\mathcal{N} = 1$ superconformal index, J. High Energy Phys. 03 (2020) 088.

[37] F. Larsen, J. Nian, and Y. Zeng, AdS_5 black hole entropy near the BPS limit, J. High Energy Phys. 06 (2020) 001.

[38] A. Cabo-Bizet and S. Murthy, Supersymmetric phases of 4d $\mathcal{N} = 4$ SYM at large N, J. High Energy Phys. 09 (2020) 184.

[39] A. A. Ardehali, J. Hong, and J. T. Liu, Asymptotic growth of the 4d $\mathcal{N} = 4$ index and partially deconfined phases, J. High Energy Phys. 07 (2020) 073.

[40] F. Benini, E. Colombo, S. Soltani, A. Zaffaroni, and Z. Zhang, Superconformal indices at large N and the entropy of $AdS_5 \times SE_5$ black holes, Class. Quantum Grav. **37**, 215021 (2020).

[41] M. David, J. Nian, and L. A. Pando Zayas, Gravitational Cardy limit and AdS black hole entropy, J. High Energy Phys. 11 (2020) 041.

[42] A. Cabo-Bizet, D. Cassani, D. Martelli, and S. Murthy, The large-N limit of the 4d $\mathcal{N} = 1$ superconformal index, J. High Energy Phys. 11 (2020) 150.

[43] P. Agarwal, S. Choi, J. Kim, S. Kim, and J. Nahmgoong, AdS black holes and finite $\mathcal{N}$ indices, Phys. Rev. D **103**, 126006 (2021).

[44] A. G. Lezcano, J. Hong, J. T. Liu, and L. A. Pando Zayas, Sub-leading structures in superconformal indices: Subdominant saddles and logarithmic contributions, J. High Energy Phys. 01 (2021) 001.

[45] C. Copetti, A. Grassi, Z. Komargodski, and L. Tizzano, Delayed deconfinement and the Hawking-Page transition, J. High Energy Phys. 04 (2022) 132.

[46] K. Goldstein, V. Jejjala, Y. Lei, S. van Leuven, and W. Li, Residues, modularity, and the Cardy limit of the 4d $\mathcal{N}=4$ superconformal index, J. High Energy Phys. 04 (2021) 216.

[47] A. Amariti, M. Fazzi, and A. Segati, The SCI of $\mathcal{N}=4$ USp($2N_c$) and SO(N_c) SYM as a matrix integral, J. High Energy Phys. 06 (2021) 132.

[48] A. Amariti, M. Fazzi, and A. Segati, Expanding on the Cardy-like limit of the SCI of 4d $\mathcal{N}=1$ ABCD SCFTs, J. High Energy Phys. 07 (2021) 141.

[49] D. Cassani and Z. Komargodski, EFT and the SUSY index on the 2nd sheet, SciPost Phys. **11**, 004 (2021).

[50] O. Aharony, F. Benini, O. Mamroud, and P. Milan, A gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N}=4$ SYM index, Phys. Rev. D **104**, 086026 (2021).

[51] C. F. Uhlemann, Islands and page curves in 4d from type IIB, J. High Energy Phys. 08 (2021) 104.

[52] J. Hong, The topologically twisted index of $\mathcal{N}=4$ SU(N) Super-Yang-Mills theory and a black hole Farey tail, J. High Energy Phys. 10 (2021) 145.

[53] N. Ezroura, F. Larsen, Z. Liu, and Y. Zeng, The phase diagram of BPS black holes in AdS_5, J. High Energy Phys. 09 (2022) 033.

[54] Y. Imamura, Finite-N superconformal index via the AdS/CFT correspondence, PTEP **2021**, 123B05 (2021).

[55] D. Gaiotto and J. H. Lee, The giant graviton expansion, arXiv:2109.02545 [hep-th].

[56] A. Cabo-Bizet, Quantum phases of $4d$ SU(N) $\mathcal{N}=4$ SYM, J. High Energy Phys. 10 (2022) 052.

[57] S. Murthy, Unitary matrix models, free fermions, and the giant graviton expansion, Pure Appl. Math. Quart. **19**, 299 (2023).

[58] J. Boruch, M. T. Heydeman, L. V. Iliesiu, and G. J. Turiaci, BPS and near-BPS black holes in AdS5 and their spectrum in $\mathcal{N}=4$ SYM, arXiv:2203.01331.

[59] M. Honda and T. Yoda, String theory, $\mathcal{N}=4$ SYM and Riemann hypothesis, arXiv:2203.17091.

[60] J. H. Lee, Exact stringy microstates from gauge theories, J. High Energy Phys. 11 (2022) 137.

[61] M.-X. Huang, Modular anomaly equation for Schur index of $\mathcal{N}=4$ super-Yang-Mills, J. High Energy Phys. 08 (2022) 049.

[62] Y. Imamura, Analytic continuation for giant gravitons, PTEP **2022**, 103B02 (2022).

[63] A. Holguin and S. Wang, Giant gravitons, Harish-Chandra integrals, and BPS states in symplectic and orthogonal $\mathcal{N}=4$ SYM, J. High Energy Phys. 10 (2022) 078.

[64] G. Eleftheriou, Root of unity asymptotics for Schur indices of 4d Lagrangian theories, J. High Energy Phys. 01 (2023) 081.

[65] S. Choi, S. Kim, E. Lee, and J. Lee, From giant gravitons to black holes, J. High Energy Phys. 11 (2023) 086.

[66] C.-M. Chang and Y.-H. Lin, Words to describe a black hole, J. High Energy Phys. 02 (2023) 109.

[67] Q. Chen, Y. Liu, Y. Tian, X. Wu, and H. Zhang, Quench dynamics in holographic first-order phase transition, Phys. Rev. D **108**, 106017 (2023).

[68] J. T. Liu and N. J. Rajappa, Finite N indices and the giant graviton expansion, J. High Energy Phys. 04 (2023) 078.

[69] H. Lin, Coherent state operators, giant gravitons, and gauge-gravity correspondence, Ann. Phys. **451**, 169248 (2023).

[70] D. S. Eniceicu, Comments on the giant-graviton expansion of the superconformal index, arXiv:2302.04887 [hep-th].

[71] M. Beccaria and A. Cabo-Bizet, On the brane expansion of the Schur index, J. High Energy Phys. 08 (2023) 073.

[72] C.-M. Chang, L. Feng, Y.-H. Lin, and Y.-X. Tao, Decoding stringy near-supersymmetric black holes, SciPost Phys. **16**, 109 (2024).

[73] Y. Chen, M. Heydeman, Y. Wang, and M. Zhang, Probing supersymmetric black holes with surface defects, J. High Energy Phys. 10 (2023) 136.

[74] S. Ekhammar, J. A. Minahan, and C. Thull, The asymptotic form of the Hagedorn temperature in planar super Yang-Mills, J. Phys. A **56**, 435401 (2023).

[75] F. Bigazzi, T. Canneti, and A. L. Cotrone, Higher order corrections to the Hagedorn temperature at strong coupling, J. High Energy Phys. 10 (2023) 056.

[76] M. Beccaria and A. Cabo-Bizet, Large black hole entropy from the giant brane expansion, J. High Energy Phys. 04 (2024) 146.

[77] A. A. Ardehali, M. Martone, and M. Rosselló, High-temperature expansion of the Schur index and modularity, arXiv:2308.09738 [hep-th].

[78] C.-M. Chang, Y.-H. Lin, and J. Wu, On $\frac{1}{8}$-BPS black holes and the chiral algebra of $\mathcal{N}=4$ SYM, arXiv:2310.20086 [hep-th].

[79] J. H. Lee, Trace relations and open string vacua, J. High Energy Phys. 02 (2024) 224.

[80] G. Eleftheriou, S. Murthy, and M. Rosselló, The giant graviton expansion in $AdS_5 \times S^5$, arXiv:2312.14921 [hep-th].

[81] J. Polchinski and M. J. Strassler, Hard scattering and gauge/string duality, Phys. Rev. Lett. **88**, 031601 (2002).

[82] Z. Bern, L. J. Dixon, and D. A. Kosower, $\mathcal{N}=4$ super-Yang-Mills theory, QCD and collider physics, C. R. Phys. **5**, 955 (2004).

[83] S. Caron-Huot and G. D. Moore, Heavy quark diffusion in QCD and $\mathcal{N}=4$ SYM at next-to-leading order, J. High Energy Phys. 02 (2008) 081.

[84] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, Adv. Theor. Math. Phys. **2**, 505 (1998).

[85] D. J. Gross and E. Witten, Possible third order phase transition in the large N lattice gauge theory, Phys. Rev. D **21**, 446 (1980).

[86] S. R. Wadia, $N=\infty$ phase transition in a class of exactly soluble model lattice gauge theories, Phys. Lett. B **93**, 403 (1980).

[87] S. R. Wadia, A study of U(N) lattice gauge theory in 2-dimensions, arXiv:1212.2906 [hep-th].

[88] T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič, Dynamics of Loschmidt echoes and fidelity decay, Phys. Rep. **435**, 33 (2006).

[89] A. del Campo, J. Molina-Vilaplana, L. F. Santos, and J. Sonner, Decay of a thermofield-double state in chaotic quantum systems, Eur. Phys. J. Spec. Top. **227**, 247 (2018).

[90] A. Chenu, I. L. Egusquiza, J. Molina-Vilaplana, and A. Del Campo, Quantum work statistics, Loschmidt echo and information scrambling, Sci. Rep. **8**, 12634 (2018).
[91] A. Chenu, J. Molina-Vilaplana, and A. Del Campo, Work statistics, Loschmidt echo and information scrambling in chaotic quantum systems, Quantum **3**, 127 (2019).
[92] A. Almheiri and H. W. Lin, The entanglement wedge of unknown couplings, J. High Energy Phys. 08 (2022) 062.
[93] D. Pérez-García, L. Santilli, and M. Tierz, Dynamical quantum phase transitions from random matrix theory, Quantum **8**, 1271 (2024).
[94] E. H. Lieb, T. Schultz, and D. Mattis, Two soluble models of an antiferromagnetic chain, Ann. Phys. (NY) **16**, 407 (1961).
[95] N. M. Bogoliubov, A. G. Pronko, and J. Timonen, Scaling of many-particle correlations in a dissipative sandpile, arXiv:1102.5639.
[96] D. Pérez-García and M. Tierz, Mapping between the Heisenberg XX spin chain and low-energy QCD, Phys. Rev. X **4**, 021050 (2014).
[97] J.-M. Stéphan, Emptiness formation probability, Toeplitz determinants, and conformal field theory, J. Stat. Mech. (2014) P05010.
[98] B. Pozsgay, The dynamical free energy and the Loschmidt echo for a class of quantum quenches in the Heisenberg spin chain, J. Stat. Mech. (2013) P10028.
[99] D. Pérez-García and M. Tierz, Chern-Simons theory encoded on a spin chain, J. Stat. Mech. (2016) 013103.
[100] J. Viti, J.-M. Stéphan, J. Dubail, and M. Haque, Inhomogeneous quenches in a free fermionic chain: Exact results, Europhys. Lett. **115**, 40011 (2016).
[101] P. L. Krapivsky, J. M. Luck, and K. Mallick, Quantum return probability of a system of N non-interacting lattice fermions, J. Stat. Mech. (2018) 023104.
[102] J.-M. Stéphan, Return probability after a quench from a domain wall initial state in the spin-1/2 XXZ chain, J. Stat. Mech. (2017) 103108.
[103] L. Santilli and M. Tierz, Phase transition in complex-time Loschmidt echo of short and long range spin chain, J. Stat. Mech. (2020) 063102.
[104] J.-M. Stéphan, Exact time evolution formulae in the XXZ spin chain with domain wall initial state, J. Phys. A **55**, 204003 (2022).
[105] G. Parez, Symmetry-resolved Rényi fidelities and quantum phase transitions, Phys. Rev. B **106**, 235101 (2022).
[106] W. L. Vleeshouwers and V. Gritsev, Unitary matrix integrals, symmetric polynomials, and long-range random walks, J. Phys. A **56**, 185002 (2023).
[107] P. Jordan and E. P. Wigner, About the Pauli exclusion principle, Z. Phys. **47**, 631 (1928).
[108] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, An Index for 4 dimensional super conformal theories, Commun. Math. Phys. **275**, 209 (2007).
[109] C. Romelsberger, Counting chiral primaries in $\mathcal{N} = 1$, $d = 4$ superconformal field theories, Nucl. Phys. B **747**, 329 (2006).
[110] C. Romelsberger, Calculating the superconformal index and Seiberg duality, arXiv:0707.3702.
[111] L. Rastelli and S. S. Razamat, The supersymmetric index in four dimensions, J. Phys. A **50**, 443013 (2017).
[112] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. **17**, 661 (1982).
[113] G. 't Hooft, On the phase transition towards permanent quark confinement, Nucl. Phys. B **138**, 1 (1978).
[114] A. M. Polyakov, Thermal properties of gauge fields and quark liberation, Phys. Lett. B **72**, 477 (1978).
[115] S. A. Hartnoll and S. P. Kumar, Higher rank Wilson loops from a matrix model, J. High Energy Phys. 08 (2006) 026.
[116] L. Santilli, Phases of five-dimensional supersymmetric gauge theories, J. High Energy Phys. 07 (2021) 088.
[117] M. Suzuki, Relationship among exactly soluble models of critical phenomena. I: 2D Ising model, dimer problem and the generalized *XY*-model, Prog. Theor. Phys. **46**, 1337 (1971).
[118] O. Derzhko, Jordan-Wigner fermionization and the theory of low-dimensional quantum spin models. Dynamic properties, in *Condensed Matter Physics in the Prime of the 21st Century: Phenomena, Materials, Ideas, Methods* (World Scientific, Singapore, 2008), pp. 35–87.
[119] G. Álvarez, L. M. Alonso, and E. Medina, Complex saddles in the Gross-Witten-Wadia matrix model, Phys. Rev. D **94**, 105010 (2016).
[120] L. Santilli and M. Tierz, Multiple phases and meromorphic deformations of unitary matrix models, Nucl. Phys. B **976**, 115694 (2022).
[121] G. Szegő, On certain Hermitian forms associated with the Fourier series of a positive function, Comm. Sém. Math. Univ. Lund Tome Supplémentaire, 228 (1952).
[122] H. J. Korsch, A. Klumpp, and D. Witthaut, On two-dimensional Bessel functions, J. Phys. A **39**, 14947 (2006).
[123] J. A. Minahan and K. Zarembo, The Bethe ansatz for $\mathcal{N} = 4$ super Yang-Mills, J. High Energy Phys. 03 (2003) 013.
[124] N. Beisert and M. Staudacher, The $\mathcal{N} = 4$ SYM integrable super spin chain, Nucl. Phys. B **670**, 439 (2003).
[125] E. Gesteau and L. Santilli, Explicit large N von Neumann algebras from matrix models, arXiv:2402.10262.
[126] S. Leutheusser and H. Liu, Causal connectability between quantum systems and the black hole interior in holographic duality, Phys. Rev. D **108**, 086019 (2023).
[127] S. Leutheusser and H. Liu, Emergent times in holographic duality, Phys. Rev. D **108**, 086020 (2023).
[128] P. Basteiro, G. Di Giulio, J. Erdmenger, and Z.-Y. Xian, Entanglement in interacting Majorana chains and transitions of von Neumann algebras, Phys Rev. Lett **132**, 161604 (2024).
[129] E. Gesteau, Emergent spacetime and the ergodic hierarchy, arXiv:2310.13733 [hep-th].