Quantum mechanics of composite fermions

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We establish the quantum mechanics of composite fermions based on the dipole picture initially proposed by Read. It comprises three complimentary components: a wave equation for determining the wave functions of a composite fermion in ideal fractional quantum Hall states and when subjected to external perturbations, a wave-function *Ansatz* for mapping a many-body wave function of composite fermions to a physical wave function of electrons, and a microscopic approach for determining the effective Hamiltonian of the composite fermion. The wave equation resembles the ordinary Schrödinger equation but has drift velocity corrections that are not present in the Halperin-Lee-Read theory. The wave-function *Ansatz* constructs a physical wave function of electrons by projecting a state of composite fermions onto a half-filled bosonic Laughlin state of vortices. Remarkably, Jain's wave-function *Ansatz* can be reinterpreted as the new *Ansatz* in an alternative wave-function representation of composite fermions. The dipole picture and the effective Hamiltonian can be derived from the microscopic model of interacting electrons confined in a Landau level, with all parameters determined. In this framework, we can construct the physical wave function of a fractional quantum Hall state deductively by solving the wave equation and applying the wave-function *Ansatz*, based on the effective Hamiltonian derived from first principles, rather than relying on intuition or educated guesses. For ideal fractional quantum Hall states in the lowest Landau level, the approach reproduces the well-established results of the standard theory of composite fermions. We further demonstrate that the reformulated theory of composite fermions can be easily generalized for flat Chern bands.

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I. INTRODUCTION

Exotic correlated states of electrons emerge in fractional quantum Hall systems, where strong magnetic fields completely quench the kinetic energy of electrons, rendering conventional many-body techniques inadequate in addressing the effects of correlations between electrons [1]. The theory of composite fermions, proposed by Jain in 1989, offers a comprehensive framework for understanding these exotic states [2]. It introduces a new paradigm, interpreting correlated states of electrons as noncorrelated or weakly correlated states of fictitious particles called composite fermions, which are assumed to be the bound states of electrons and quantum vortices. Based on the insight, the theory prescribes an *Ansatz* for constructing many-body wave functions that achieve nearly perfect overlaps with those determined by exact diagonalizations for various fractional quantum Hall states in the lowest Landau level [3]. On the other hand, for predicting the responses of these states to external perturbations, one usually employs the effective theory proposed by Halperin, Lee, and Read (HLR) [4], which has been shown to make predictions that align well with experimental observations [5]. The two components of the theory, namely the wave-function *Ansatz* and the effective theory, complement each other, forming a versatile framework for understanding the rich physics of the fractional quantum Hall systems.

Despite the remarkable success, the theory still lacks a concrete foundation. On the one hand, one usually relies on intuition or educated guesses when constructing composite fermion wave functions. This is in sharp contrast to the deductive approach typically employed for ordinary particles like electrons, for which one can confidently write down a Hamiltonian for a given physical circumstance, and obtain wave functions by solving the Schrödinger equation. On the other hand, the conjecture of the HLR theory—a composite fermion also obeys the ordinary Schrödinger equation—is only justified heuristically. Lopez-Fradkin's theory [6] is often cited as the rationale behind both the wave function *Ansatz* and the effective theory [2]. However, the theory can only be viewed as a tentative argument rather than a rigorous foundation for the theory of composite fermions due to two obvious issues.

First, Jain's *Ansatz* prescribes wave functions of electrons in the form [3]

$$\Psi(\{z_i\}) = \hat{P}_{\mathrm{LLL}} J(\{z_i\}) \tilde{\Psi}_{\mathrm{CF}}(\{z_i\}), \tag{1}$$

which differs from the form suggested by Lopez-Fradkin's theory based on the singular Chern-Simons (CS)

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transformation [6]

$$\Psi(\{z_i\}) = \frac{J(\{z_i\})}{|J(\{z_i\})|}\tilde{\Psi}_{\mathrm{CF}}(\{z_i\}), \tag{2}$$

where Ψ and $\tilde{\Psi}_{\mathrm{CF}}$ represent the wave function of electrons and composite fermions, respectively,

$$J(\{z_i\}) \equiv \prod_{i<j}(z_i - z_j)^2 \tag{3}$$

is the Bijl-Jastrow factor, which presumably attaches a vortex with two flux quanta to each electron, $\hat{P}_{\mathrm{LLL}}$ is the projection operator to the lowest-Landau-level, and $\{\mathbf{z}_i \equiv (x_i, y_i)\}$ and $\{z_i \equiv x_i + \mathrm{i}y_i\}$ denote the coordinates of electrons in the vector and complex forms, respectively. Equation (2) is formulated in the full Hilbert space of free electrons, while Eq. (1) is defined in the restricted Hilbert space of a single Landau level. Reconciling the two is nontrivial [7].

Secondly, the picture of composite fermions implied by Lopez-Fradkin's theory, which is also inherited by the HLR theory, differs from that obtained by directly inspecting the *Ansatz* wave function Eq. (1). For the latter, Read's analysis indicates that the electron and the vortex in a composite fermion are spatially separated [8]. The finding contradicts the picture implied by Eq. (2), which suggests that a composite fermion is a point particle, consisting of an electron and δ-function flux tubes. More recently, Son points out that the HLR theory lacks particle-hole symmetry [9], whereas the *Ansatz* wave function is shown to preserve the symmetry well [10]. These observations raise doubts about whether the HLR theory accurately describes the composite fermions implied by the *Ansatz* wave function. It prompts the need for an alternative effective theory, or ideally, an alternative foundation from which both the wave-function *Ansatz* and the effective theory can be inferred.

The dipole picture of composite fermions, initially proposed by Read, offers an alternative picture for the fictitious particle [8,11]. The picture differs from the HLR view in two fundamental ways. First, instead of being a point particle, the composite fermion has a dipole structure with the electron and vortex spatially separated. Second, the electron and the vortex are confined in two separated Landau levels created by the physical magnetic field and an emergent CS magnetic field, respectively, as opposed to moving in a free space [12,13]. The dipole picture is shown to yield low-energy and long-wavelength electromagnetic responses that are identical to those predicted by the Dirac theory of composite fermions [14], indicating that it satisfies the general requirements of particle-hole symmetry. The feature, along with the fact that the picture is inferred directly from microscopic wave functions, sets it apart from other alternatives.

Pasquier and Haldane investigate the dipole picture of a system of bosons in an isolated Landau level at filling factor 1 [15]. Their work, along with subsequent developments by other researchers [16–18], sheds light on the construction of physical wave functions. They propose interpreting vortices as auxiliary degrees of freedom that extend the physical Hilbert space to a larger Hilbert space of composite fermions. To obtain a physical state from a state of composite fermions in the enlarged Hilbert space, one must eliminate the auxiliary degrees of freedom by projecting the state into a physical subspace defined by a pure state of the vortices. In this context, the Bijl-Jastrow factor is interpreted as the complex conjugate of the wave function of the vortex state, rather than the numerator of the singular gauge factor in Eq. (2). The interpretation naturally leads to a wave-function *Ansatz* alternative to Eq. (1), and it avoids the difficulties associating with the singular CS transformation [16].

In this paper, we present a theory of quantum mechanics for composite fermions based on the dipole picture and Pasquier-Haldane's interpretation. The theory comprises three complimentary components: a wave equation for determining the wave functions of a composite fermion in ideal fractional quantum Hall states and when subjected to external perturbations, a wave-function *Ansatz* for mapping a many-body wave function of composite fermions to a physical wave function of electrons, and a microscopic approach for determining the effective Hamiltonian of the composite fermion. In our theory, the state of a composite fermion is represented by a bivariate wave function that is holomorphic (antiholomorphic) in the coordinate of its constituent electron (vortex), and is defined in a Bergman space with a weight determined by the spatial profiles of the physical and the emergent CS magnetic fields. The wave equation is derived by applying the rules of quantization in the Bergman space to the phenomenological dipole model proposed in Ref. [14]. It resembles the ordinary Schrödinger equation but has drift velocity corrections that are not present in the HLR theory. The wave-function *Ansatz* constructs a physical wave function of electrons by projecting a state of composite fermions onto a half-filled bosonic Laughlin state of vortices. Remarkably, Jain's wave-function *Ansatz*, which underlies the success of the theory of composite fermions, can be recast to the form of the new *Ansatz* using an alternative wave-function representation for composite fermions. The phenomenological dipole model can be derived from the microscopic model of interacting electrons confined in a Landau level by applying a Hartree-like approximation, with its parameters determined from first principles. In this framework, we can construct the physical wave function of a fractional quantum Hall state deductively by solving the wave equation and applying the wave-function *Ansatz*, rather than relying on intuition or educated guesses. We further demonstrate that the reformulated theory of composite fermions can be easily generalized for flat Chern bands, which are also predicted to host the fractional quantum Hall states [19,20].

The remainder of the paper is organized as follows. In Sec. II, we introduce the dipole model, which is the basis of our discussions, and we give an overview of the main results of this work. In Sec. III, we develop the theory for ideal fractional quantum Hall systems, which are subjected only to uniform magnetic fields. In Sec. IV, the theory is established for general systems that could be subjected to spatially and temporarily fluctuating external perturbations and have inhomogeneous densities. In Sec. V, we provide a microscopic underpinning for our theory by deriving the phenomenological dipole model from the microscopic Hamiltonian of interacting electrons confined in a Landau level. In Sec. VI, as an application of our reformulation, we generalize the theory of composite fermions for flat Chern bands.

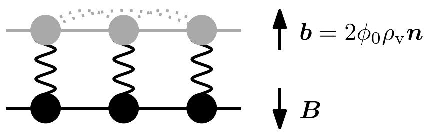

FIG. 1. Dipole model of composite fermions. A composite fermion consists of an electron (black) and a vortex (gray). The electron is confined in the Landau level induced by the physical magnetic field $\boldsymbol{B}$, while the vortex belongs to a bosonic liquid of vortices in the $\nu = 1/2$ Laughlin state. Under the mean-field approximation, the vortex is considered as an independent particle confined in the Landau level induced by an emergent CS magnetic field $\boldsymbol{b}$. The electron and the vortex are bound together by a binding potential, which can be modeled as the harmonic potential Eq. (23) in a lowest Landau level.

In Sec. VII, we summarize and discuss our results. Certain details of derivations are presented in Appendixes.

II. OVERVIEW

A. Dipole model

Our theory is based on the dipole picture which was originally proposed by Read for the half-filled Landau level [8]. The picture can be generalized to a dipole model of composite fermions which can be applied to arbitrary filling factors [13,14,21].

The model is illustrated in Fig. 1. According to the model, a composite fermion consists of an electron and a vortex confined in two separate Landau levels: the one for the electron is the Landau level induced by the physical magnetic field $\boldsymbol{B} = -B\boldsymbol{n}$, while the one for the vortex is the fictitious Landau level induced by an emergent CS magnetic field $\boldsymbol{b} = b\boldsymbol{n}$ with its strength determined by the CS self-consistent condition $b = (2h/e)\rho_v$, where ρ_v is the density of vortices, and $\boldsymbol{n}$ denotes the normal vector of the two-dimensional plane of the system. In general, both the physical magnetic field and the CS magnetic field can be nonuniform. The two particles are bounded by a binding potential, which can be shown to be well approximated by a harmonic potential for the lowest Landau level (see Sec. V B).

The noninteracting dipole model is actually a mean-field approximation for an underlying correlated system of composite fermions. First of all, electrons confined in the physical Landau level are interacting. On the other hand, in Pasquier-Haldane's interpretation [15,16], vortices are auxiliary degrees of freedom introduced to extend the physical Hilbert space of electrons to a larger Hilbert space of composite fermions. They are assumed to form a collective half-filled bosonic Laughlin state (see Sec. III B). In Sec. V, we will show how the noninteracting dipole model emerges after applying a Hartree-like approximation in the enlarged Hilbert space. We note that the standard interpretation of the vortex, namely, an entity consisting of two quantized microscopic vortices, is actually a property derived from the particular collective state assumed for the vortices (see Sec. IV B).

It is also possible to interpret the composite fermion as a point-particle by defining its momentum $\boldsymbol{p}$ and coordinate $\boldsymbol{x}$. A definition of the momentum, as pointed out by Read [8], could be

$$\boldsymbol{p} = \frac{\hbar}{l_B^2}\boldsymbol{n} \times (\boldsymbol{z} - \boldsymbol{\eta}), \tag{4}$$

where $\boldsymbol{z}$ and $\boldsymbol{\eta}$ are the coordinates of the electron and the vortex, respectively, and $l_B \equiv \sqrt{\hbar/eB}$ is the magnetic length of the B-field. We can define $\boldsymbol{x} = \boldsymbol{\eta}$, as suggested in Refs. [13,14]. The composite fermion can then be interpreted as a particle that is subjected to a uniform momentum-space Berry-curvature [13,14,22] and obeys the Sundaram-Niu dynamics [23]. It is notable that such a particle has a modified phase space measure [24]. Consequently, for a Landau level with the particle-hole symmetry, the kinetic energy of the composite fermion should be modeled as [14]

$$T = D\frac{p^2}{2m^*}, \tag{5}$$

where m^* denotes the effective mass of the composite fermion, and $D = b/B$ is the density-of-states correction factor due to the modified phase space measure [14]. In Sec. V B, we will show that the microscopic derivation of the dipole model gives rise to the peculiar form of the kinetic energy.

Finally, we note that one is actually free to choose the definition of $(\boldsymbol{x}, \boldsymbol{p})$ and have a different interpretation. The physical results do not depend on the interpretation. This is demonstrated in Ref. [14], where two different choices of the definition and their interpretations are compared.

B. Summary of results

We summarize the main results of this work as follows:

(A) The state of the composite fermion can be represented by a bivariate wave function that is holomorphic (antiholomorphic) in the coordinate of its constituent electron (vortex) (see Sec. III A). The Hilbert space is the tensor product of two Bergman spaces, one for the electron and one for the vortex, with weights determined by the spatial profiles of the physical and CS magnetic fields, respectively (see Sec. IV A).

(B) A new wave-function *Ansatz* can be logically inferred from the dipole model (see Sec. III B):

$$\Psi(\{z_i\}) = \hat{P}_v \Psi_{CF}(\{z_i, \bar{\eta}_i\}), \tag{6}$$

where $\hat{P}_v$ denotes the projection onto the collective state assumed for vortices. Remarkably, the new *Ansatz* and the standard *Ansatz* Eq. (1) are equivalent, although they use two different wave-function representations for composite fermions. The two representations can be related by a transformation shown in Eq. (20) for ideal states and Eq. (48) generally.

(C) A general wave equation can be established for composite fermions, valid not only for ideal systems, but also when external perturbations are present. The wave equation has a biorthogonal form, shown in Eqs. (73) and (74), and its Hamiltonian in the long-wavelength limit has corrections from the drift velocities of the electron and the vortex, shown in Eq. (71). For ideal systems, the wave equation yields wave functions identical to those prescribed by the standard theory. However, the responses to external perturbations predicted by the wave equation will differ from those predicted by the HLR theory because of the drift-velocity corrections. It has been shown that the dipole model yields long-wavelength responses identical to those predicted by the Dirac theory of composite fermions with a dipole correction [14].

(D) The noninteracting dipole model can be derived from the underlying microscopic model of a set of interacting electrons confined in a Landau level by applying a Hartree-like approximation in the enlarged Hilbert space of composite fermions. The origin of the fictitious Chern-Simons fields is clarified. They are introduced to impose the requirement of consistency of orthonormalities between the physical Hilbert space and the enlarged Hilbert space (see Sec. V A). The derivation also confirms that the kinetic energy, which is basically the Coulomb attraction energy between the electron and the charge void induced by the vortex, is indeed proportional to $D = 2\nu$, where ν is the filling factor of the system, and it can be well approximated by the parabolic form Eq. (5) for a lowest-Landau-level (see Sec. V B).

(E) The reformulated theory of composite fermions can be generalized for a flat Chern band with a Chern number $|C| = 1$ by substituting the Chern band in place of the physical Landau level in the dipole model shown in Fig. 3. We find that the effective band dispersion experienced by a composite fermion is renormalized by the combination of the quantum metric and the Berry curvature of the band that gives rise to the heuristic "trace condition" [25]. The observation rationalizes the "trace condition" for the stability of a fractional Chern insulator state. It also suggests the possibility of stabilizing a fractional Chern insulator state in a nonflat Chern band when the renormalization cancels the dispersion of the Chern band (see Sec. VI).

III. THEORY FOR IDEAL SYSTEMS

In this section, we develop the quantum mechanics of composite fermions in systems that are subjected only to uniform external magnetic fields and have homogeneous densities. We will establish a new wave-function *Ansatz* and a set of wave equations for composite fermions. Remarkably, our approach can be shown to reproduce the well-established results of the standard theory. The principles established in this section will serve as the foundation for developing a general theory.

A. Hilbert space

The Hilbert space of a composite fermion shown in Fig. 1 is the tensor product of two Hilbert spaces with respect to the two Landau levels. It differs from that of an ordinary quantum particle in a free space as assumed in the HLR theory.

The Hilbert space spanned by a Landau level is a weighted Bergman space [26,27]. For a disk geometry, the space includes all holomorphic polynomials in the complex electron coordinate $z = x + \mathrm{i}y$. The inner product between two states $\psi_1(z)$ and $\psi_2(z)$ in the space is defined by $\langle\psi_1|\psi_2\rangle = \int \mathrm{d}\mu_B^{(0)}(\boldsymbol{z})\psi_1^*(z)\psi_2(z)$ with the integral measure

$$\mathrm{d}\mu_B^{(0)}(\boldsymbol{z}) \equiv \frac{\mathrm{d}\boldsymbol{z}}{2\pi l_B^2} e^{-|z|^2/2l_B^2}. \tag{7}$$

A Bergman space with the Gaussian weight is also known as the Segal-Bargmann space [28].

The Hilbert space of a vortex is also a Segal-Bargmann space consisting of all antiholomorphic polynomials in the complex-conjugate vortex coordinate $\bar{\eta} = \eta_x - \mathrm{i}\eta_y$, where η_x and η_y are the Cartesian components of the vortex coordinate $\boldsymbol{\eta} \equiv (\eta_x, \eta_y)$. Note that because the direction of the b-field is opposite to that of the B-field, wave functions for the vortex are antiholomorphic functions. The corresponding integral measure is

$$\mathrm{d}\mu_b^{(0)}(\boldsymbol{\eta}) \equiv \frac{\mathrm{d}\boldsymbol{\eta}}{2\pi l_b^2} e^{-|\eta|^2/2l_b^2}, \tag{8}$$

where $l_b = \sqrt{eb/\hbar}$ is the magnetic length of the b-field.

The Hilbert space of a composite fermion is the tensor product of the two Segal-Bargmann spaces for the electron and the vortex, respectively. The state of a composite fermion can thus be naturally represented by a bivariate function:

$$\psi(z, \bar{\eta}), \tag{9}$$

which is holomorphic (antiholomorphic) in the complex coordinate z ($\bar{\eta}$) of the electron (vortex). Unlike the wave function $\psi(\boldsymbol{z}) \equiv \psi(\bar{z}, z)$ for an ordinary particle, the two coordinates of the wave function Eq. (9) belong to different particles.

For a Bergman space, we can define a reproducing kernel, which is basically the coordinate representation of the identity operator of the space [28]. For the spaces of the electron and the vortex, their reproducing kernels are

$$K_B^{(0)}(z, \bar{z}') = e^{z\bar{z}'/2l_B^2}, \tag{10}$$

$$K_b^{(0)}(\bar{\eta}, \eta') = e^{\bar{\eta}\eta'/2l_b^2}, \tag{11}$$

respectively. The kernels transform wave functions in the respective Bergman spaces back to themselves:

$$\psi(z) = \int \mathrm{d}\mu_B^{(0)}(\boldsymbol{z}')K_B^{(0)}(z, \bar{z}')\psi(z'), \tag{12}$$

$$\varphi(\bar{\eta}) = \int \mathrm{d}\mu_b^{(0)}(\boldsymbol{\eta}')K_b^{(0)}(\bar{\eta}, \eta')\varphi(\bar{\eta}'). \tag{13}$$

The kernels can also be used to project nonholomorphic functions into the Segal-Bargmann spaces [28]. Actually, $\hat{P}_{\mathrm{LLL}}$ in Eq. (1), the projection operator to the lowest Landau level, can be written as an integral transform using the reproducing kernel:

$$\hat{P}_{\mathrm{LLL}} f(\boldsymbol{z}) \equiv \int \mathrm{d}\mu_B^{(0)}(\boldsymbol{\xi})K_B^{(0)}(z, \bar{\xi})f(\boldsymbol{\xi}), \tag{14}$$

where $f(\boldsymbol{z})$ is shorthand notation of a nonholomorphic function $f(\bar{z}, z)$. We will use the notations interchangeably in this paper. The projection into the η-space can be defined similarly using the reproducing kernel $K_b^{(0)}(\bar{\eta}, \eta')$.

B. Wave-function *Ansatz*

The wave-function *Ansatz* Eq. (1) maps a many-body wave function in the fictitious world of composite fermions to a physical wave function of interacting electrons in the real world. Although the *Ansatz* is customarily expressed in a form that suggests its connection with the singular CS transformation Eq. (2), it can actually be more naturally inferred from the dipole model, as we will demonstrate in this subsection.

Pasquier and Haldane presented an alternative approach of constructing the many-body wave functions of fractional quantum Hall states [15]. The approach was further developed by Read [16] and Dong and Senthil [17]. They investigate a system of bosons at filling factor 1. Vortices of one flux quantum, which are fermions, are introduced as auxiliary degrees of freedom for extending the physical Hilbert space of bosons to a Hilbert space of composite fermions. It is envisioned that in the enlarged Hilbert space, it may become feasible to apply mean-field approximations for the composite fermions. To obtain physical wave functions, on the other hand, one needs to eliminate auxiliary degrees of freedom by projecting states of composite fermions into a physical subspace. This leads to a relation between a wave function of composite fermions $\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\})$ and its physical counterpart $\Psi(\{z_i\})$ [16]:

$$\Psi(\{z_i\}) = \hat{P}_{\rm v}\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\}) \tag{15}$$

$$\equiv \int\prod_i {\rm d}\mu_b^{(0)}(\boldsymbol{\eta}_i)\Psi_{\rm v}^*(\{\bar{\eta}_i\})\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\}), \tag{16}$$

where $\Psi_{\rm v}(\{\bar{\eta}_i\})$ is the wave function of a vortex state that defines the physical subspace in the enlarged Hilbert space, and $\hat{P}_{\rm v}$ denotes the projection into the subspace. For a system of bosons, the vortex state is assumed to be a $\nu = 1$ incompressible state of fermions with $\Psi_{\rm v}(\{\bar{\eta}_i\}) = \prod_{i<j}(\bar{\eta}_i - \bar{\eta}_j)$. The corresponding physical wave function describes a Fermi-liquid-like state of bosons.

The general idea of Pasquier-Haldane-Read's approach can be adapted for a system of electrons. We can introduce bosonic vortices as the auxiliary degrees of freedom. We assume that the vortices form a $\nu = 1/2$ bosonic Laughlin state with the wave function

$$\Psi_{\rm v}(\{\bar{\eta}_i\}) = J^*(\{\eta_i\}). \tag{17}$$

By substituting the vortex wave function into Eq. (16), we obtain an *Ansatz* for constructing physical wave functions of electrons:

$$\Psi(\{z_i\}) = \int\prod_i {\rm d}\mu_b^{(0)}(\boldsymbol{\eta}_i)J(\{\eta_i\})\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\}). \tag{18}$$

Remarkably, the *Ansatz* can be shown to be equivalent to the standard (Jain's) *Ansatz* Eq. (1). To see this, we express Eq. (1) in an integral form by using Eq. (14):

$$\begin{aligned}\Psi(\{z_i\}) &= \int\prod_i {\rm d}\mu_B^{(0)}(\boldsymbol{\xi}_i)e^{\sum_i z_i\bar{\xi}_i/2l_B^2}J(\{\xi_i\})\tilde{\Psi}_{\rm CF}(\{\boldsymbol{\xi}_i\})\\ &= \int\prod_i {\rm d}\mu_b^{(0)}(\boldsymbol{\eta}_i)J(\{\eta_i\})\int\prod_i {\rm d}\mu_B^{(0)}(\boldsymbol{\xi}_i)\\ &\quad\times e^{\sum_i(z_i\bar{\xi}_i/2l_B^2+\bar{\eta}_i\xi_i/2l_b^2)}\tilde{\Psi}_{\rm CF}(\{\boldsymbol{\xi}_i\}),\end{aligned} \tag{19}$$

where we insert the reproducing kernel Eq. (11) for each of the composite fermions. Comparing it with the new *Ansatz* Eq. (18), we have

$$\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\}) = \int\prod_i {\rm d}\mu_B^{(0)}(\boldsymbol{\xi}_i)\ e^{\sum_i(z_i\bar{\xi}_i/2l_B^2+\xi_i\bar{\eta}_i/2l_b^2)}\tilde{\Psi}_{\rm CF}(\{\boldsymbol{\xi}_i\}). \tag{20}$$

We see that the two *Ansätze* are equivalent but use different wave-function representations for a state of composite fermions. In the following, we will refer to the two representations as the dipole representation ($\Psi_{\rm CF}$) and the standard representation ($\tilde{\Psi}_{\rm CF}$), respectively.

C. Wave equation: The dipole representation

The theory of composite fermions often relies on intuition or educated guesses when selecting wave functions for composite fermions. The resulting *Ansatz* wave functions are then justified *a posteriori* by showing high overlaps with wave functions obtained from exact diagonalizations [3]. Is it possible to determine appropriate wave functions for composite fermions *a priori*, as we do for ordinary electrons? In this subsection, we take the first step towards demonstrating the possibility by developing a wave equation for composite fermions in ideal systems.

The wave equation can be derived from the variational principle $\delta L = 0$, with the Lagrangian L defined by

$$L = \int {\rm d}\mu_B^{(0)}(\boldsymbol{z}){\rm d}\mu_b^{(0)}(\boldsymbol{\eta})[\epsilon - T(\boldsymbol{z},\boldsymbol{\eta})]|\psi(z,\bar{\eta})|^2, \tag{21}$$

where ϵ is the Lagrange multiplier for the normalization constraint of the wave function

$$\int {\rm d}\mu_B^{(0)}(\boldsymbol{z})\int {\rm d}\mu_b^{(0)}(\boldsymbol{\eta})|\psi(z,\bar{\eta})|^2 = 1, \tag{22}$$

and T is the binding energy of a composite fermion modeled as a harmonic potential,

$$T = \frac{\hbar^2}{2m^* l_B^2 l_b^2}|\boldsymbol{z}-\boldsymbol{\eta}|^2. \tag{23}$$

It becomes the kinetic energy Eq. (5) in the point-particle interpretation discussed in Sec. II A.

Differentiating the Lagrangian with respect to $\psi^*(z,\bar{\eta})$ gives rise to the wave equation $\epsilon\psi = \hat{H}\psi$, with the Hamiltonian defined by

$$[\hat{H}\psi](z,\bar{\eta}) \equiv \hat{P}T(\boldsymbol{z},\boldsymbol{\eta})\psi(z,\bar{\eta}), \tag{24}$$

where $\hat{P}$ denotes the projection into the Hilbert space of the composite fermion defined in Sec. III A. Applying the rule of the projection into Landau levels, we map $\bar{z}$ and η to the operators $\hat{\bar{z}} \equiv 2l_B^2\partial_z$ and $\hat{\eta} \equiv 2l_b^2\partial_{\bar{\eta}}$, respectively [2]. The stationary-state wave equation of the composite fermion can then be written as

$$\epsilon\psi(z,\bar{\eta}) = -\frac{\hbar^2}{2m^*}\left(2\partial_z - \frac{\bar{\eta}}{l_B^2}\right)\left(2\partial_{\bar{\eta}} - \frac{z}{l_b^2}\right)\psi(z,\bar{\eta}), \tag{25}$$

where an unimportant constant term in the Hamiltonian due to the ordering of operators is ignored.

We can transform the wave equation to an ordinary Schrödinger equation for a charged particle subjected to a uniform magnetic field by applying the transformation

$$\psi(z,\bar{\eta}) = \sqrt{2\pi}\,l_B\exp\left[\frac{1}{4}\left(\frac{1}{l_B^2}+\frac{1}{l_b^2}\right)z\bar{\eta}\right]\varphi(z,\bar{\eta}). \tag{26}$$

For $\varphi(\boldsymbol{\xi}) \equiv \varphi(z,\bar{\eta})|_{z\to\xi,\bar{\eta}\to\bar{\xi}}$, we have

$$\epsilon\varphi(\boldsymbol{\xi}) = -\frac{\hbar^2}{2m^*}\left(2\partial_\xi - \frac{\sigma\bar{\xi}}{2l^2}\right)\left(2\partial_{\bar{\xi}} + \frac{\sigma\xi}{2l^2}\right)\varphi(\boldsymbol{\xi}), \tag{27}$$

with $l \equiv \sqrt{\hbar/e|\mathcal{B}|}$ being the magnetic length of the effective magnetic field $\mathcal{B} = B - b$, and $\sigma \equiv \mathrm{sgn}(\mathcal{B})$ indicating its direction.

D. Wave equation: The standard representation

We can also have a wave equation for the standard representation. In the case of noninteracting composite fermions, both Ψ_{CF} and $\tilde{\Psi}_{\mathrm{CF}}$ are Slater determinants of single-particle wave functions. The single-particle counterpart of the transformation Eq. (20) is

$$\psi(z,\bar{\eta}) = \int \mathrm{d}\mu_B^{(0)}(\boldsymbol{\xi}) e^{z\bar{\xi}/2l_B^2 + \xi\bar{\eta}/2l_b^2} \tilde{\psi}(\boldsymbol{\xi}), \tag{28}$$

where $\tilde{\psi}(\boldsymbol{\xi})$ denotes a single-particle wave function in the standard representation. Substituting it into Eq. (25), we obtain the wave equation (see Appendix E 1)

$$\epsilon\tilde{\varphi}(\boldsymbol{\xi}) = -\frac{\hbar^2}{2m^*}\left(2\partial_\xi - \frac{\sigma\bar{\xi}}{2l^2}\right)\left(2\partial_{\bar{\xi}} + \frac{\sigma\xi}{2l^2}\right)\tilde{\varphi}(\boldsymbol{\xi}) \tag{29}$$

and

$$\tilde{\psi}(\boldsymbol{\xi}) = \sqrt{2\pi}\, l_B \exp\left(\sigma\frac{|\xi|^2}{2l^2}\right)\tilde{\varphi}(\boldsymbol{\xi}). \tag{30}$$

We see that the wave equation for $\tilde{\varphi}(\boldsymbol{\xi})$ is just the ordinary Schrödinger equation for a charge particle in the uniform effective magnetic field.

Our theory reproduces the well-established results of the standard theory for ideal states. The eigensolutions of the wave equation (see Appendix A) are exactly the Λ orbits of the standard theory of composite fermions [2]. The interpretation of the fractional series is also the same: a fractional state of electrons corresponds to an integer filling state of composite fermions. The filling factor $\nu = n/(2n+1) < 1/2$, $n \in Z$ corresponds to n filled Λ-levels, and $\nu = (n+1)/(2n+1) > 1/2$ corresponds to $n+1$ filled Λ-levels with $\sigma = -1$. For the special case of $\nu = 1/2$, the effective magnetic field vanishes. The eigensolutions of the wave equation become the plane-wave states, and composite fermions will form a Fermi sea. The resulting physical wave function is exactly the Rezayi-Read wave function of the composite Fermi-liquid [11].

We note that the states $\{\varphi_i\}$and $\{\tilde{\varphi}_i\}$ are dual to each other, and form a biorthogonal system. This can be seen by applying Eq. (28) to rewrite the orthonormal condition $\int \mathrm{d}\mu_B^{(0)}(\boldsymbol{z}) \int \mathrm{d}\mu_b^{(0)}(\boldsymbol{\eta}) \psi_i^*(z,\bar{\eta})\psi_j(z,\bar{\eta}) = \delta_{ij}$ as

$$\int \mathrm{d}\boldsymbol{\xi}\, \tilde{\varphi}_i^*(\boldsymbol{\xi})\varphi_j(\boldsymbol{\xi}) = \delta_{ij}. \tag{31}$$

IV. GENERAL THEORY

In this section, we generalize our theory to systems that are subjected not only to strong uniform magnetic fields, but also to spatial and temporal fluctuations of electromagnetic fields, and generally have inhomogeneous densities. In the HLR theory, this can be done trivially by assuming that the composite fermion obeys the ordinary Schrödinger equation. In the dipole model, however, the composite fermion is far from being an ordinary particle, as we see in Sec. II A. We need to derive the general quantum theory of composite fermions in a logical way, as we demonstrate in the last section for the ideal systems.

A. Bergman space

In this subsection, we show that the Hilbert space of a particle confined in a Landau level by a nonuniform magnetic field is generally a Bergman space with its weight determined by the spatial profile of the magnetic field. Consequently, the Hilbert space of a composite fermion is the tensor product of two Bergman spaces with their weights determined by the spatial profiles of the physical and the CS magnetic fields, respectively.

We consider a nonrelativistic electron confined in the lowest Landau level by a nonuniform magnetic field $\boldsymbol{B}(\boldsymbol{z}) = -B(\boldsymbol{z})\boldsymbol{n}$, and we assume $B(\boldsymbol{z}) = B_0 + B_1(\boldsymbol{z}) > 0$, $|B_1(\boldsymbol{z})|/B_0 \ll 1$. The Hamiltonian of the system, in complex coordinates, is given by [2]

$$\hat{H} = -\frac{\hbar^2}{2m_{\mathrm{e}}}\left(2\partial_z + \mathrm{i}\frac{e}{\hbar}\bar{A}\right)\left(2\partial_{\bar{z}} + \mathrm{i}\frac{e}{\hbar}A\right) + \frac{e\hbar B(\boldsymbol{z})}{2m_{\mathrm{e}}}, \tag{32}$$

with $A \equiv A_x(\boldsymbol{z}) + \mathrm{i}A_y(\boldsymbol{z})$ and $\bar{A} \equiv A^*$ being the complex components of the vector potential of the magnetic field. The first term of the Hamiltonian yields zero-energy for a state with the wave function $\varphi(\boldsymbol{z})$ satisfying the constraint

$$\left[2\partial_{\bar{z}} + \mathrm{i}\frac{e}{\hbar}A(\boldsymbol{z})\right]\varphi(\boldsymbol{z}) = 0. \tag{33}$$

All such states form the lowest Landau level in the nonuniform magnetic field [29], and define the physical Hilbert space of the electron in the zero-electron-mass limit $m_{\mathrm{e}} \to 0$. The second term of the Hamiltonian, on the other hand, can be interpreted as the orbital magnetization energy of the electron, and will become a part of the scalar potential experienced by composite fermions [30]. We note that for a two-dimensional massless Dirac particle, Eq. (33) is an exact constraint for the zero-energy Landau level, and there is no orbital magnetization energy.

To fulfill the constraint, a wave function in Hilbert space must have the form

$$\varphi(\boldsymbol{z}) = \psi(z)\exp\left[-\tfrac{1}{2}f_B(\bar{z},z)\right], \tag{34}$$

where $\psi(z)$ is a holomorphic function in z, and $f_B(\bar{z},z)$ is determined by the equation

$$\partial_{\bar{z}} f_B(\bar{z},z) = \mathrm{i}\frac{e}{\hbar}A(\bar{z},z). \tag{35}$$

Fixing the vector potential in the Coulomb gauge, we have $\partial_z A = -\partial_{\bar{z}}\bar{A} = -\mathrm{i}B(\boldsymbol{z})/2$, and

$$\partial_z\partial_{\bar{z}} f_B(\bar{z},z) = \frac{e}{2\hbar}B(\boldsymbol{z}). \tag{36}$$

We can then choose $f_B(\bar{z},z)$ to be a real solution of the equation.

The Hilbert space of the electron is therefore a weighted Bergman space consisting of all holomorphic polynomials that are normalized by the condition $\int \mathrm{d}\mu_B(\boldsymbol{z})|\psi(z)|^2 = 1$, where the integral measure is modified to

$$\mathrm{d}\mu_B(\boldsymbol{z}) = w_B(\boldsymbol{z})\mathrm{d}\boldsymbol{z} \equiv \frac{\mathrm{d}\boldsymbol{z}}{2\pi l_B^2}\exp[-f_B(\bar{z},z)], \tag{37}$$

with $w_B(z)$ being the weight of the Bergman space, and $l_B \equiv \sqrt{\hbar/eB_0}$. We can choose the constant of integration for f_B to normalize the measure: $\int d\mu_B(z) = 1$.

Similarly, for a vortex in a nonuniform CS magnetic field $\boldsymbol{b}(\boldsymbol{\eta}) = b(\boldsymbol{\eta})\boldsymbol{n}$, $b(\boldsymbol{\eta}) > 0$, its Hilbert space is a Bergman space consisting of all antiholomorphic polynomials in $\bar{\eta}$ with the modified integral measure

$$d\mu_b(\boldsymbol{\eta}) = w_b(\boldsymbol{\eta})d\boldsymbol{\eta} \equiv \frac{d\boldsymbol{\eta}}{2\pi l_b^2}\exp[-f_b(\bar{\eta}, \eta)], \tag{38}$$

where $f_b(\bar{\eta}, \eta)$ is a real solution of the equation

$$\partial_{\bar{\eta}}\partial_\eta f_b(\bar{\eta}, \eta) = \frac{e}{2\hbar}b(\boldsymbol{\eta}). \tag{39}$$

The counterpart of Eq. (35) for the vortex is

$$\partial_\eta f_b(\bar{\eta}, \eta) = i\frac{e}{\hbar}\bar{a}(\bar{\eta}, \eta), \tag{40}$$

where $\bar{a} \equiv a_x - ia_y$ denotes the complex-conjugate component of the vector potential (a_x, a_y) of the CS magnetic field.

As in the ideal case, the state of a composite fermion is represented by a bivariate wave function that is holomorphic in the coordinate of the electron and antiholomorphic in the coordinate of the vortex, defined in the Hilbert space that is the tensor product of the two Bergman spaces.

We can also define the reproducing kernels $K_B(z, \bar{z}')$ and $K_b(\bar{\eta}, \eta')$ for the weighted Bergman spaces of electrons and vortices, respectively [28]. K_B transforms a wave function defined in the B-Bergman spaces back to itself:

$$\psi(z) = \int d\mu_B(\boldsymbol{z}')K_B(z, \bar{z}')\psi(z'). \tag{41}$$

It also defines the projection into the space:

$$\hat{P}_{\rm LLL}f(\boldsymbol{z}) \equiv \int d\mu_B(\boldsymbol{\xi})K_B(z, \bar{\xi})f(\boldsymbol{\xi}). \tag{42}$$

In general, we do not have a closed form of the reproducing kernel like Eq. (10). We formally express the reproducing kernels as

$$K_B(z, \bar{\xi}) \equiv e^{F_B(\bar{\xi}, z)} \tag{43}$$

by introducing the function $F_B(\bar{\xi}, z)$. In the long-wavelength limit, F_B and f_B can be related approximately; see Appendix C. The reproducing kernel of the b-space has a similar set of properties.

B. Wave-function *Ansatz*

Using the modified integral measure Eq. (38), we can generalize the wave-function *Ansatz* Eq. (16) straightforwardly:

$$\Psi(\{z_i\}) = \int \prod_i d\mu_b(\boldsymbol{\eta}_i)J(\{\eta_i\})\Psi_{\rm CF}(\{z_i, \bar{\eta}_i\}), \tag{44}$$

where we only change the integral measures for $\{\boldsymbol{\eta}_i\}$, and we assume that the wave function of the vortices defining the physical subspace remains the same as in the ideal case.

Due to the change of the weight of the $\boldsymbol{\eta}$-Bergman space, vortices are actually in a deformed bosonic Laughlin state with an inhomogeneous density. The joint density distribution of vortices is proportional to

$$e^{\sum_{i<j} 4\ln|\eta_i - \eta_j| - \sum_i f_b(\bar{\eta}_i, \eta_i)}. \tag{45}$$

Using Laughlin's plasma analogy [31], we can interpret it as the distribution function of a set of classical particles, each of which carries two unit "charges," on a nonuniform neutralizing background with the "charge" density $\partial_\eta\partial_{\bar{\eta}}f_b(\bar{\eta}, \eta)/\pi$. Such a system is expected to be nearly "charge-neutral" everywhere. It implies that the single-particle density of vortices should be

$$2\rho_{\rm v}(\boldsymbol{\eta}) \simeq \frac{1}{\pi}\partial_\eta\partial_{\bar{\eta}}f_b(\bar{\eta}, \eta) = \frac{e}{h}b(\boldsymbol{\eta}), \tag{46}$$

where we make use of Eq. (39). We see that the CS self-consistent condition, which relates the vortex density to the strength of the CS magnetic field, arises as a result of the constraint of the physical subspace.

The standard *Ansatz* can be generalized and shown to be equivalent to the new *Ansatz*. By using the reproducing kernel of the electron Bergman space, the standard *Ansatz* Eq. (1) can be written as

$$\Psi(\{z_i\}) = \int \prod_i d\mu_B(\boldsymbol{\xi}_i) \prod_i K_B(z_i, \bar{\xi}_i)\ J(\{\xi_i\})\tilde{\Psi}_{\rm CF}(\{\boldsymbol{\xi}_i\}). \tag{47}$$

The general relation between the dipole representation and the standard representation reads

$$\Psi_{\rm CF}(\{z_i, \bar{\eta}_i b\}) = \int \prod_i d\mu_B(\boldsymbol{\xi}_i) \times \left[\prod_i K_B(z_i, \bar{\xi}_i)K_b(\bar{\eta}_i, \xi_i)\right]\tilde{\Psi}_{\rm CF}(\{\boldsymbol{\xi}_i\}). \tag{48}$$

C. General wave equation

In this subsection, we generalize the wave equation (25) for systems that are subjected to external perturbations. We assume that the external magnetic field has a strong uniform component B_0 and a small fluctuating component $B_1(z)$ which varies slowly over space with $|B_1(z)|/B_0 \ll 1$, $|\nabla B(z)|l_B/B_0 \ll 1$, and the strength of the external electric field is weak and does not induce inter-Landau-level transitions. The resulting theory will be adequate for predicting long-wavelength responses to electromagnetic fields [14]. In this limit, we can establish a general wave equation while not obscured by excessive microscopic details. A more general theory would require taking into account microscopic details, which will be elucidated in Sec. V.

The Lagrangian of the dipole model for a set of composite fermions, in terms of the single-particle wave functions

$\{\psi_i(z,\bar{\eta})\}$, can be generally written as

$$L = \sum_i \int \mathrm{d}\mu_B(z)\mathrm{d}\mu_b(\boldsymbol{\eta})\bigg\{\epsilon_i|\psi_i(z,\bar{\eta})|^2 - \frac{\hbar^2}{2m^*} \times \int \mathrm{d}\mu_b(\boldsymbol{\eta}')\psi_i^*(z,\bar{\eta})\frac{(\bar{z}-\bar{\eta})(z-\eta')}{l_b^2(z)l_B^2(z)}K_b(\bar{\eta},\eta')\psi_i(z,\bar{\eta}')\bigg\} - \int \mathrm{d}z\Phi(z)\rho_{\mathrm{e}}(z) - E_{\mathrm{xc}}[\rho_{\mathrm{e}}] - \frac{e^2}{8\pi\varepsilon}\int \mathrm{d}z\mathrm{d}z'\frac{[\rho_{\mathrm{e}}(z)-\rho_0][\rho_{\mathrm{e}}(z')-\rho_0]}{|z-z'|} - \int \mathrm{d}\boldsymbol{\eta}\phi(\boldsymbol{\eta})\Big[\rho_{\mathrm{v}}(\boldsymbol{\eta}) - \frac{e}{2h}b(\boldsymbol{\eta})\Big], \quad (49)$$

where the summation is over the occupied states of composite fermions, and ϵ_i is the Lagrange multiplier for the normalization constraint of the wave functions. The second term in the curly brackets is the kinetic energy, which is basically the harmonic binding potential Eq. (23) written in a form that implies antinormal ordering when quantizing $\boldsymbol{\eta}$ (see Appendix B), with a space-dependent coefficient parametrized in the local magnetic lengths of the external field $l_B(z) \equiv \sqrt{\hbar/eB(z)}$ and the CS field $l_b(z) \equiv \sqrt{\hbar/eb(z)}$. The third term is the energy due to the single-body scalar potential $\Phi(z)$ experienced by electrons, which includes the scalar potential of the external electromagnetic field as well as the orbital magnetization energy discussed in Sec. IV A, and

$$\rho_{\mathrm{e}}(z) = w_B(z)\sum_i \int \mathrm{d}\mu_b(\boldsymbol{\eta})|\psi_i(z,\bar{\eta})|^2 \quad (50)$$

is the local density of electrons. The next two terms are the Coulomb energy and an exchange-correlation energy functional $E_{\mathrm{xc}}[\rho_{\mathrm{e}}]$ which accounts for the exchange and correlation effects of composite fermions. The last term imposes the CS constraint, which relates the local density of vortices

$$\rho_{\mathrm{v}}(\boldsymbol{\eta}) = w_b(\boldsymbol{\eta})\sum_i \int \mathrm{d}\mu_B(z)|\psi_i(z,\bar{\eta})|^2 \quad (51)$$

to the local strength of the CS magnetic field $b(\boldsymbol{\eta})$, with $\phi(\boldsymbol{\eta})$ serving as the Lagrange multiplier. The effects of the nonuniform physical and CS magnetic fields are included implicitly in the integral measures $\mathrm{d}\mu_B$ and $\mathrm{d}\mu_b$, respectively. In Sec. V, we will derive the Lagrangian from first principles.

Differentiating the Lagrangian with respect to ψ_i^*, we obtain a generalized wave equation for the stationary state of a composite fermion:

$$\epsilon\psi(z,\bar{\eta}) = \int \mathrm{d}\mu_B(\boldsymbol{\xi})\mathrm{d}\mu_b(\boldsymbol{\eta}')K_B(z,\bar{\xi})K_b(\bar{\eta},\eta') \times\mathcal{E}(\boldsymbol{\xi};\bar{\eta},\eta')\psi(\xi,\bar{\eta}'), \quad (52)$$

$$\mathcal{E}(\boldsymbol{\xi};\bar{\eta},\eta') = \frac{\hbar^2}{2m^*}\frac{(\bar{z}-\bar{\eta})(z-\eta')}{l_b^2(z)l_B^2(z)} + \Phi_{\mathrm{eff}}(z) + \phi(\boldsymbol{\eta}), \quad (53)$$

where we drop the state index subscripts for brevity, Φ_{eff} is the effective scalar potential experienced by electrons, defined by

$$\Phi_{\mathrm{eff}}(z) = \Phi(z) + \frac{e^2}{4\pi\varepsilon}\int \mathrm{d}z'\frac{\rho_{\mathrm{e}}(z')-\rho_0}{|z-z'|} + v_{\mathrm{xc}}[\rho_{\mathrm{e}}](z), \quad (54)$$

with the exchange-correlation potential $v_{\mathrm{xc}}[\rho_{\mathrm{e}}] \equiv \delta E_{\mathrm{xc}}[\rho_{\mathrm{e}}]/\delta\rho_{\mathrm{e}} + \tau_{\mathrm{xc}}(z)$, and

$$\tau_{\mathrm{xc}}(z) = \frac{2\pi\hbar^2}{m^*}\int \mathrm{d}\mu_b(\boldsymbol{\eta})\sum_i \left|\frac{z-\hat{\eta}}{l_B(z)}\psi_i(z,\bar{\eta})\right|^2, \quad (55)$$

which is obtained by differentiating the kinetic energy with respect to $\rho_{\mathrm{e}}(z) \approx \rho_{\mathrm{v}}(z) = 1/4\pi l_b^2(z)$ and applying the quantization (see below), and $\phi(\boldsymbol{\eta})$ can be interpreted as the scalar potential experienced by vortices. The orthonormal condition between two eigen-states is

$$\int \mathrm{d}\mu_B(z)\int \mathrm{d}\mu_b(\boldsymbol{\eta})\psi_i^*(z,\bar{\eta})\psi_j(z,\bar{\eta}) = \delta_{ij}. \quad (56)$$

Applying the rules of quantization defined in Appendix B, we can write the wave equation as

$$\epsilon\psi(z,\bar{\eta}) = \hat{H}_\psi\psi(z,\bar{\eta}), \quad (57)$$

with the effective Hamiltonian operator

$$\hat{H}_\psi = \frac{\hbar^2}{2m^*}(\hat{\bar{z}}-\bar{\eta})\frac{1}{l_b^2(\bar{\eta},z)l_B^2(\bar{\eta},z)}(z-\hat{\eta}) + N_+[\Phi_{\mathrm{eff}}(\hat{\bar{z}},z)+\phi(\bar{\eta},\hat{\eta})], \quad (58)$$

where $\hat{\bar{z}}$ and $\hat{\eta}$ are defined by (see Appendix B 1)

$$[\hat{\bar{z}}\psi](z,\bar{\eta}) \equiv \int \mathrm{d}\mu_B(\boldsymbol{\xi})K_B(z,\bar{\xi})\bar{\xi}\psi(\xi,\bar{\eta}), \quad (59)$$

$$[\hat{\eta}\psi](z,\bar{\eta}) \equiv \int \mathrm{d}\mu_b(\boldsymbol{\zeta})K_b(\bar{\eta},\zeta)\zeta\psi(z,\bar{\zeta}), \quad (60)$$

and $N_+[\cdots]$ denotes the normal ordering that places $\hat{\bar{z}}$ and $\hat{\eta}$ on the left of all z's and η's. We apply the approximations $l_B^2(z) \approx l_B^2(\bar{\eta},z)$ and $l_b^2(z) \approx l_b^2(\bar{\eta},z)$ for the coefficient of the kinetic energy.

The wave equation is complemented by a set of CS self-consistent conditions, which are obtained by differentiating the Lagrangian Eq. (49) with respect to $\boldsymbol{a}$ and ϕ. We have

$$b(\boldsymbol{\eta}) = \frac{2h}{e}\rho_{\mathrm{v}}(\boldsymbol{\eta}), \quad (61)$$

$$\boldsymbol{E}_{\mathrm{v}}(\boldsymbol{\eta}) = \frac{2h}{e}\boldsymbol{n}\times\boldsymbol{j}_{\mathrm{v}}(\boldsymbol{\eta}), \quad (62)$$

where $\boldsymbol{E}_{\mathrm{v}}$ and b are the CS electric and magnetic fields, respectively, $\boldsymbol{j}_{\mathrm{v}}(\boldsymbol{\eta})$ denotes the current density of vortices, which can be written as (see Appendix F)

$$\boldsymbol{j}_{\mathrm{v}}(\boldsymbol{\eta}) = \frac{\rho_{\mathrm{v}}(\boldsymbol{\eta})}{b(\boldsymbol{\eta})}\boldsymbol{E}_{\mathrm{v}}(\boldsymbol{\eta})\times\boldsymbol{n} + \frac{\hbar}{m^*}w_b(\boldsymbol{\eta}) \times\sum_i \int \mathrm{d}\mu_B(z)\psi_i^*(z,\bar{\eta}) \times\frac{\boldsymbol{n}\times(\boldsymbol{z}-\boldsymbol{\eta})}{l_B^2(z)}\psi_i(z,\bar{\eta}). \quad (63)$$

D. Biorthogonal quantum mechanics

As in the theory for ideal systems, we can determine the wave equation for the standard representation, and define a biorthogonal system of wave functions. In general, the single-particle wave functions of the dipole representation and the standard representation are related by the transformation

$$\psi(z,\bar{\eta}) = \int \mathrm{d}\mu_B(\boldsymbol{\xi}) K_B(z,\bar{\xi}) K_b(\bar{\eta},\xi)\tilde{\psi}(\boldsymbol{\xi}). \tag{64}$$

The operators in the dipole representation can be mapped to their counterparts in the standard representation accordingly; see Appendix E 2.

We further introduce the transformations

$$\psi(\boldsymbol{\xi}) = \sqrt{2\pi} l_B \exp\left[\frac{f_B(\boldsymbol{\xi}) + f_b(\boldsymbol{\xi})}{2}\right]\varphi(\boldsymbol{\xi}), \tag{65}$$

$$\tilde{\psi}(\boldsymbol{\xi}) = \sqrt{2\pi} l_B \exp\left[\frac{f_B(\boldsymbol{\xi}) - f_b(\boldsymbol{\xi})}{2}\right]\tilde{\varphi}(\boldsymbol{\xi}), \tag{66}$$

which are the counterparts of the transformations Eqs. (26) and (30), respectively. The orthonormal condition Eq. (56) can then be rewritten as

$$\int \mathrm{d}\boldsymbol{\xi} \tilde{\varphi}_i^*(\boldsymbol{\xi})\varphi_j(\boldsymbol{\xi}) = \delta_{ij}. \tag{67}$$

We see that as in ideal systems, $\{\varphi_i\}$ and $\{\tilde{\varphi}_i\}$ are dual to each other and form a biorthogonal system.

The wave equations for φ and $\tilde{\varphi}$ have the form of the biorthogonal quantum mechanics [32]. In general, we can show that the Hamiltonian for $\tilde{\varphi}(\boldsymbol{\xi})$ is the complex conjugate of that for $\varphi(\boldsymbol{\xi})$(see Appendix E 3). Therefore, we have

$$\epsilon\varphi(\boldsymbol{\xi}) = \hat{H}\varphi(\boldsymbol{\xi}), \tag{68}$$

$$\epsilon\tilde{\varphi}(\boldsymbol{\xi}) = \hat{H}^\dagger\tilde{\varphi}(\boldsymbol{\xi}), \tag{69}$$

where the first equation is transformed from Eq. (57) with

$$\hat{H} \equiv e^{-\frac{f_B+f_b}{2}} \hat{H}_\psi e^{\frac{f_B+f_b}{2}}. \tag{70}$$

Note that $\hat{H}$ is non-Hermitian in general.

In the long-wavelength limit, the effective Hamiltonian of a composite fermion can be written as (see Appendix C)

$$\hat{H} = -\frac{\hbar^2}{2m^*}\left[2\partial_\xi + \mathrm{i}\frac{e}{\hbar}\bar{\mathcal{A}}(\boldsymbol{\xi}) + \mathrm{i}\frac{m^*}{\hbar}\bar{v}(\boldsymbol{\xi})\right] \times \left[2\partial_{\bar{\xi}} + \mathrm{i}\frac{e}{\hbar}\mathcal{A}(\boldsymbol{\xi}) + \mathrm{i}\frac{m^*}{\hbar}V(\boldsymbol{\xi})\right] + \Phi_{\mathrm{eff}}(\boldsymbol{\xi}) + \phi(\boldsymbol{\xi}), \tag{71}$$

where $(\bar{\mathcal{A}},\mathcal{A})$ denotes the effective vector potential experienced by composite fermions:

$$\boldsymbol{\mathcal{A}} = \boldsymbol{a} + \boldsymbol{A}, \tag{72}$$

and $V(\bar{\eta},z) \equiv 2\mathrm{i}\partial_{\bar{\eta}}\Phi_{\mathrm{eff}}(\bar{\eta},z)/eB(\bar{\eta},z)$ and $\bar{v}(\bar{\eta},z) \equiv 2\mathrm{i}\partial_z\phi(\bar{\eta},z)/eb(\bar{\eta},z)$ are the complex components of the drift velocities $\boldsymbol{V} = \boldsymbol{E}\times\boldsymbol{B}/B^2$ and $\boldsymbol{v} = \boldsymbol{E}_\mathrm{v}\times\boldsymbol{b}/b^2$ in the presence of the electric fields $\boldsymbol{E} \equiv e^{-1}\nabla\Phi_{\mathrm{eff}}$ and $\boldsymbol{E}_\mathrm{v} \equiv e^{-1}\nabla\phi$ for electrons and vortices, respectively.

The set of wave equations can be further generalized for time-dependent systems. We have (see Appendix D)

$$\mathrm{i}\hbar\frac{\partial\varphi(\boldsymbol{\xi},t)}{\partial t} = \hat{H}\varphi(\boldsymbol{\xi},t), \tag{73}$$

$$\mathrm{i}\hbar\frac{\partial\tilde{\varphi}(\boldsymbol{\xi},t)}{\partial t} = \hat{H}^\dagger\tilde{\varphi}(\boldsymbol{\xi},t), \tag{74}$$

where $\hat{H}$ is formally identical to the stationary state Hamiltonian Eq. (71) in the long-wavelength limit, but the electric fields $\boldsymbol{E}$ and $\boldsymbol{E}_\mathrm{v}$, which determine the drift velocities, are replaced by their gauge-invariant forms $\boldsymbol{E} = e^{-1}\nabla\Phi_{\mathrm{eff}} - \partial_t\boldsymbol{A}$ and $\boldsymbol{E}_\mathrm{v} = e^{-1}\nabla\phi - \partial_t\boldsymbol{a}$.

The wave equations (74) and (73), together with the CS self-consistent conditions (61) and (62) and the self-consistent equation for the effective potential Eq. (54), define the effective theory of composite fermions in the presence of long-wavelength external perturbations. It is evident that the effective theory differs from the heuristic HLR theory because of the corrections from the drift velocities in Eq. (71). The corrections have previously been identified as either anomalous velocity corrections [14] or side-jump corrections [33] in the context of the semiclassical theory of composite fermions. Equation (71) shows how these corrections are manifested in the quantum mechanics of composite fermions.

V. MICROSCOPIC UNDERPINNING

In this section, we derive the phenomenological dipole model, which underlies our derivation of the quantum mechanics of composite fermions, from the microscopic model of interacting electrons confined in a Landau level in the zero-electron-mass limit. The microscopic Lagrangian of such a system can be written as

$$L_{\mathrm{M}} = \langle\Psi\,|\,E - V_{\mathrm{ee}} - \Phi\,|\,\Psi\rangle, \tag{75}$$

where Ψ denotes the many-body wave function of electrons, E is a Lagrange multiplier for the normalization constraint $\langle\Psi|\Psi\rangle = 1$, $V_{\mathrm{ee}} = (e^2/4\pi\varepsilon)\sum_{i<j}|z_i - z_j|^{-1} + V_{\mathrm{B}}$ denotes the Coulomb interaction between electrons with V_{B} being the potential from a uniform neutralizing positive charge background, and $\Phi \equiv \sum_i \Phi(z_i)$ denotes the energy of an externally applied scalar potential. The kinetic energy of electrons is ignored since it is completely quenched in a Landau level.

Our derivation is based on the general variational principle of quantum mechanics. By introducing the fictitious degrees of freedom of the vortices, we basically embed the physical Hilbert space into a larger Hilbert space of composite fermions, in the hope that the strongly correlated state of electrons can be viewed as the projection of a noncorrelated state of composite fermions onto a lower-dimensional subspace. Therefore, we choose the trial electron wave functions for $|\Psi\rangle$ to be the *Ansatz* form Eq. (44), with Ψ_{CF} being the Slater determinant of a set of single-body trial wave functions $\{\psi_i\}$ of composite fermions. We will show that the Lagrangian Eq. (49) in terms of $\{\psi_i\}$ can be derived from the microscopic Lagrangian Eq. (75). The set of the single-body trial wave functions should be determined by applying the variational principle

$$\delta L = 0, \tag{76}$$

which gives rise to the wave equations and the CS self-consistent conditions.

A. Chern-Simons constraints

A notable feature of the theory of composite fermions is the presence of the fictitious CS fields, which are determined self-consistently by Eqs. (61) and (62). In this subsection, we show how the CS fields and the self-consistent conditions emerge in a microscopic theory.

It is easy to see that for the Slater determinant wave function

$$\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\}) = \frac{1}{\sqrt{N!}}\det[\psi_j(z_i,\bar{\eta}_i)], \tag{77}$$

two sets of single-particle trial wave functions that are related by a nonsingular linear transformation yield the same physical wave function after applying Eq. (15) or Eq. (44). To eliminate the redundancy, it is necessary to impose the orthonormal condition:

$$\int d\mu_B(\boldsymbol{\xi})d\mu_b(\boldsymbol{\eta})\psi_i^*(\xi,\bar{\eta})\psi_j(\xi,\bar{\eta}) = \delta_{ij}. \tag{78}$$

We note that the orthonormality depends on the weight in $d\mu_b$, which is not yet defined at this point.

To proceed, we adopt an approximation analog to the Hartree approximation. Basically, we determine the state of a composite fermion in an effective medium formed by other composite fermions. In the spirit of the Hartree approximation [34], we introduce a test particle that is distinguishable from other composite fermions but interacts and correlates just like them. The physical wave function of a system with N composite fermions plus such a test particle can be written as

$$\Psi^{\rm t}(z,\{z_i\}) = \int d\mu_b(\boldsymbol{\eta})\Psi^{\rm v}_\eta(\{z_i\})\psi(z,\bar{\eta}), \tag{79}$$

$$\Psi^{\rm v}_\eta(\{z_i\}) = \int\prod_{i=1}^N d\mu_b(\boldsymbol{\eta}_i)\prod_{i=1}^N(\eta-\eta_i)^2 \times J(\{\eta_i\})\Psi_{\rm CF}(\{z_i,\bar{\eta}_i\}), \tag{80}$$

where the test particle has the wave function $\psi(z,\bar{\eta})$, and it correlates with other composite fermions via the Bijl-Jastrow factor. Because the test particle has no exchange symmetry with other composite fermions, it can occupy any state, including those already occupied in $\Psi_{\rm CF}$. Our approximation is to assume that the set of single-particle trial wave functions for constructing $\Psi_{\rm CF}$ can be chosen from eigen-wave-functions of the test particle.

With the approximation, we can determine the weight of $d\mu_b$ self-consistently by requiring that the orthonormality Eq. (78) in the Hilbert space of composite fermions is consistent with that of the physical Hilbert space. This requires

$$\langle\Psi^{\rm t}_i\,|\,\Psi^{\rm t}_j\rangle = \delta_{ij}, \tag{81}$$

where $|\Psi^{\rm t}_i\rangle$ and $|\Psi^{\rm t}_j\rangle$ denote two physical states obtained by setting $\psi=\psi_i$ and $\psi=\psi_j$ in Eq. (79), respectively, and ψ_i and ψ_j satisfy the orthonormal condition Eq. (78). Equation (81) can be rewritten as

$$\int d\mu_B(\boldsymbol{z})d\mu_b(\boldsymbol{\eta})d\mu_b(\boldsymbol{\eta}')\psi_i^*(z,\bar{\eta})\ K_b(\bar{\eta},\eta')\psi_j(z,\bar{\eta}') = \delta_{ij}, \tag{82}$$

with

$$K_b(\bar{\eta},\eta') \equiv \langle\Psi^{\rm v}_\eta\,|\,\Psi^{\rm v}_{\eta'}\rangle. \tag{83}$$

To make Eq. (82) consistent with Eq. (78), we can adjust the weight of $d\mu_b$ so that $K_b(\bar{\eta},\eta')$ is the corresponding reproducing kernel. Equation (82) can then be reduced to Eq. (78) by integrating out $\boldsymbol{\eta}'$.

The requirement that $K_b(\bar{\eta},\eta')$ is the reproducing kernel of the $\boldsymbol{\eta}$-space gives rise to the CS constraint Eq. (61) in the long-wavelength limit. To see this, we rewrite Eq. (83) as $K_b(\bar{\eta},\eta') = \langle e^{\mathcal{F}}\rangle$, with

$$\langle e^{\mathcal{F}}\rangle \equiv \int\prod_{i=1}^N d\mu_B(\boldsymbol{z}_i)d\mu_b(\boldsymbol{\eta}_i)d\mu_b(\boldsymbol{\eta}'_i)e^{\mathcal{F}(\bar{\eta},\eta',\{\bar{\eta}_i,\eta'_i\})} \times J^*(\{\eta_i\})J(\{\eta'_i\})\Psi^*_{\rm CF}(\{z_i,\bar{\eta}_i\})\Psi_{\rm CF}(\{z_i,\bar{\eta}'_i\}) \tag{84}$$

and $\mathcal{F}(\bar{\eta},\eta',\{\bar{\eta}_i,\eta'_i\}) \equiv 2\sum_i\ln(\bar{\eta}-\bar{\eta}_i)(\eta'-\eta'_i)$. Using the cumulant expansion, we can approximate $F_b(\bar{\eta},\eta')\equiv\ln K_b(\bar{\eta},\eta')$ as

$$F_b(\bar{\eta},\eta') \approx \langle\mathcal{F}\rangle + \tfrac{1}{2}\langle(\mathcal{F}-\langle\mathcal{F}\rangle)^2\rangle + \cdots. \tag{85}$$

To the lowest order, we ignore the fluctuation and higher-order corrections, and we have $F_b\approx\langle\mathcal{F}\rangle = 2\sum_i\langle\ln(\bar{\eta}-\bar{\eta}_i)(\eta'-\eta'_i)\rangle$. To evaluate the ith term of the summation, we expand the Slater determinant Eq. (77) along its ith row, substitute the expansion into Eq. (84), and ignore contributions involving particle exchanges. We obtain

$$F_b(\bar{\eta},\eta') \approx \sum_i\int d\mu_B(\boldsymbol{z}_i)d\mu_b(\boldsymbol{\eta}_i)d\mu_b(\boldsymbol{\eta}'_i) \times 2[\ln(\bar{\eta}-\bar{\eta}_i)+\ln(\eta'-\eta'_i)] \times\frac{1}{N}\sum_a\psi_a^*(z_i,\bar{\eta}_i)K_b^{(a)}(\bar{\eta}_i,\eta'_i)\psi_a(z_i,\bar{\eta}'_i), \tag{86}$$

where $K_b^{(a)}$ is defined by Eq. (83) but with one composite fermion in the state ψ_a being removed from Eq. (77). We assume that the effect of removing a composite fermion from the effective medium of N composite fermions is negligible, thus we have

$$K_b^{(a)}(\bar{\eta}_i,\eta'_i) \approx K_b(\bar{\eta}_i,\eta'_i). \tag{87}$$

After integrating out $\boldsymbol{\eta}'_i$, we obtain

$$F_b(\bar{\eta},\eta) \approx 2\int d^2\boldsymbol{\eta}_1\ln(|\eta-\eta_1|^2)\rho_{\rm v}(\boldsymbol{\eta}_1), \tag{88}$$

where $\rho_{\rm v}(\boldsymbol{\eta}_1)$ is the vortex density defined in Eq. (51). Applying the identity $\partial_\eta\partial_{\bar{\eta}}\ln(|\eta-\eta_1|^2) = \pi\delta(\boldsymbol{\eta}-\boldsymbol{\eta}_1)$, we have

$$\partial_\eta\partial_{\bar{\eta}}F_b(\bar{\eta},\eta) = 2\pi\rho_{\rm v}(\boldsymbol{\eta}). \tag{89}$$

In the long-wavelength limit, we have $F_b(\bar{\eta},\eta)\approx f_b(\boldsymbol{\eta}) - \ln[l_b^2(\boldsymbol{\eta})/l_b^2]$ (see Appendix C). Substituting the relation into Eq. (89), ignoring the spatial gradient of the magnetic length, and applying Eq. (39), we obtain the CS constraint Eq. (61).

We can then replace the normalization constraint $\langle\Psi|\Psi\rangle = 1$ in Eq. (75) with normalization constraints of the single-body wave functions as well as the CS constraint Eq. (61), and introduce ϵ_i and $\phi(\boldsymbol{\eta})$ as the respective Lagrange multipliers.

The Lagrangian becomes

$$L = \int d\mu_B(z)d\mu_b(\boldsymbol{\eta}) \sum_i \epsilon_i |\psi_i(z,\bar{\eta})|^2 - \int d\boldsymbol{\eta}\phi(\boldsymbol{\eta}) \times \left[\rho_v(\boldsymbol{\eta}) - \frac{e}{2h}b(\boldsymbol{\eta})\right] - \langle\Psi\,|\,V_{ee} + \Phi\,|\,\Psi\rangle, \quad (90)$$

B. Energy

In this subsection, we determine the expectation value $\langle\Psi\,|\,V_{ee} + \Phi\,|\,\Psi\rangle$. We shall show how the kinetic energy of a composite fermion, i.e., the electron-vortex binding potential, as well as its peculiar density-of-states correction factor (see Sec. III C), would emerge.

We first determine the expectation value of the scalar potential $\langle\Psi|\Phi|\Psi\rangle$. Similar to Eq. (86), we have

$$\langle\Psi\,|\,\Phi\,|\,\Psi\rangle \approx \sum_i \int d\mu_B(z_i)d\mu_b(\boldsymbol{\eta}_i)d\mu_b(\boldsymbol{\eta}_i')\Phi(z_i) \times \frac{1}{N}\sum_a \psi_a^*(z_i,\bar{\eta}_i)K_b^{(a)}(\bar{\eta}_i,\eta_i')\psi_a(z_i,\bar{\eta}_i'). \quad (91)$$

Applying the approximation Eq. (87), we obtain

$$\langle\Psi\,|\,\Phi\,|\,\Psi\rangle \approx \int dz\Phi(z)\rho_e(z). \quad (92)$$

Next, we determine the expectation value of the Coulomb interaction energy. It can be written as

$$\langle V_{ee}\rangle = \frac{e^2}{8\pi\varepsilon}\int dz dz' \frac{\rho_2(z,z') - 2\rho_e(z)\rho_0 + \rho_0^2}{|z-z'|}, \quad (93)$$

where $\rho_2(z,z') = \langle\Psi|\sum_{i\neq j}\delta(z-z_i)\delta(z'-z_j)|\Psi\rangle$ is the two-particle reduced density matrix of electrons. We decompose $\langle V_{ee}\rangle$ into two parts. The first part is the mean-field contribution of the Coulomb interaction

$$\bar{V}_{ee} = \frac{e^2}{8\pi\varepsilon}\int dz dz' \frac{[\rho_e(z) - \rho_0][\rho_e(z') - \rho_0]}{|z-z'|}, \quad (94)$$

which gives rise to the Coulomb energy term of the Lagrangian Eq. (49). The second part is the correlation contribution

$$T = \frac{e^2}{8\pi\varepsilon}\int dz dz' \frac{\rho_2(z,z') - \rho_e(z)\rho_e(z')}{|z-z'|}, \quad (95)$$

which gives rise to the binding energy between electrons and vortices.

We determine the two-particle reduced density matrix by applying the Hartree-like approximation introduced in the last subsection. We have $\rho_2(z,z') = N(N-1)w_B(z)w_B(z')\int\prod_{i=3}^N d\mu_B(z_i)|\Psi(\{z_i\}|^2$ with $z_1 = z$ and $z_2 = z'$. We treat the first particle (z_1) as a test particle, and the ensemble of other $N-1$ particles as an effective medium. By expanding the Slater determinant Eq. (77) along its first row, ignoring exchange terms in $|\Psi(\{z_i\})|^2$, and replacing the $N-1$ particle effective medium with the N-particle one as in Eq. (79), we can approximate ρ_2 as

$$\rho_2(z,z_1) \approx w_B(z)\int d\mu_b(\boldsymbol{\eta})d\mu_b(\boldsymbol{\eta}')K_b(\bar{\eta},\eta') \times \sum_a \psi_a^*(z,\bar{\eta})\psi_a(z,\bar{\eta}')\rho_c(z_1;\bar{\eta},\eta'), \quad (96)$$

$$\rho_c(z_1;\boldsymbol{\eta}) = w_B(z_1)N\int\prod_{i=2}^N d\mu_B(z_i)\frac{|\Psi_\eta^v(\{z_i\})|^2}{\langle\Psi_\eta^v\,|\,\Psi_\eta^v\rangle}, \quad (97)$$

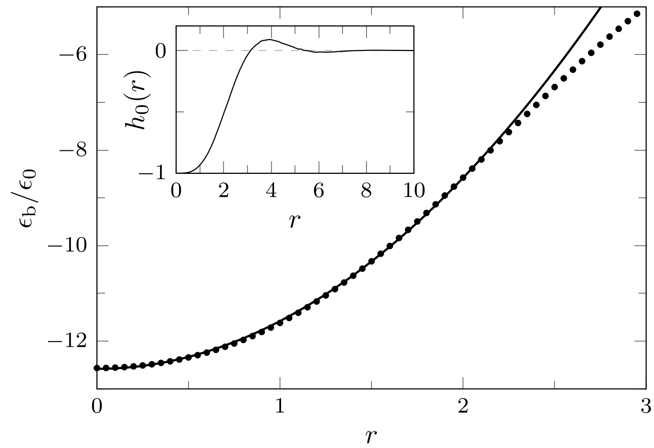

FIG. 2. Electron-vortex binding potential vs $r \equiv |z-\eta|/l_B$ for the $\nu = 1/3$ Laughlin state. Dots are numerical results, and the solid line shows the fitting $\epsilon_b(r)/\epsilon_0 = -12.6 + r^2$, $\epsilon_0 \equiv \nu e^2/16\pi^2\varepsilon l_B$. Inset: the electron-vortex pair correlation function $h_0(r)$. Calculated by H. Jin.

and $\rho_c(z_1;\bar{\eta},\eta') \equiv \rho_c(z_1,\boldsymbol{\eta})|_{\eta\to\eta'}$. $\rho_c(z_1;\boldsymbol{\eta})$ is the density profile of electrons surrounding a vortex at $\boldsymbol{\eta}$, which suppresses the electron density in its vicinity, resulting in a void of electrons.

The Coulomb attraction between the test (first) electron and the void gives rise to the binding energy of a composite fermion. Substituting Eqs. (96) and (50) into Eq. (95), we obtain

$$T \approx \int d\mu_B(z)d\mu_b(\eta)d\mu_b(\eta')K_b(\bar{\eta},\eta')\ \epsilon_b^\star(z;\bar{\eta},\eta') \times \sum_a \psi_a^*(z,\bar{\eta})\psi_a(z,\bar{\eta}'), \quad (98)$$

with $\epsilon_b^\star(z;\bar{\eta},\eta') \equiv \epsilon_b(z;\boldsymbol{\eta})|_{\eta\to\eta'}$,

$$\epsilon_b(z;\boldsymbol{\eta}) = \frac{e^2}{8\pi\varepsilon}\int dz_1 \frac{\Delta\rho_e(z_1;\boldsymbol{\eta})}{|z-z_1|}, \quad (99)$$

and $\Delta\rho_e(z_1;\boldsymbol{\eta}) \equiv \rho_c(z_1;\boldsymbol{\eta}) - \rho_e(z_1)$.

The form of $\Delta\rho_e(z_1;\boldsymbol{\eta})$ is constrained [35]. The electron density is suppressed near the center of the vortex, and recovers in a lengthscale $\sim l_B$ (see the inset of Fig. 2). Thus we have $\Delta\rho_e(\boldsymbol{z_1};\boldsymbol{\eta}) < 0$ for $z_1 \to \boldsymbol{\eta}$ and $\Delta\rho_e(z_1;\boldsymbol{\eta}) \to 0$ for $|z_1 - \boldsymbol{\eta}| \gg l_B$. Moreover, because the insertion of a $2h/e$ vortex should induce a charge void with a total charge of $2\nu e$, where ν is the filling factor, we have the sum rule

$$\int \Delta\rho_e(z_1;\boldsymbol{\eta})dz_1 = -2\nu(\boldsymbol{\eta}), \quad (100)$$

where $\nu(\boldsymbol{\eta})$ denotes the local value of the filling factor, and we assume that the electron density is nearly homogeneous.

The fact that the binding energy of a composite fermion originates from the Coulomb attraction between an electron and a void with a total charge of $2\nu e$ suggests that it should be proportional to 2ν, which is exactly the density-of-states correction factor $D \equiv b/B = 2\nu$ appearing in Eq. (5). The peculiar factor in the kinetic energy turns out to be a natural consequence of the interaction origin of the binding energy.

In the long-wavelength limit, we can approximate $\Delta\rho_e(z_1;\boldsymbol{\eta})$ as $h(z_1;\boldsymbol{\eta}) \approx \rho_e(\boldsymbol{\eta}) h_0(|z_1-\boldsymbol{\eta}|/l_B(\boldsymbol{\eta}))$, where $h_0(r) \equiv \Delta\rho_e(r)/\rho_0$ is the electron-vortex correlation function of a homogeneous system, with $\Delta\rho_e(r)$ being the change of electron density relative to the average density ρ_0 in the vicinity of a vortex at the origin of space. We expect that the binding energy, apart from the density-of-states correction factor, should depend only weakly on the density since $h_0(r)$ is constrained by the density-independent sum rule $\int_0^\infty \mathrm{d}r h_0(r) r = -2$ and the overall form. We thus estimate the binding energy using the Laughlin state at $\nu = 1/3$, for which we can complete the integrals with respect to $\{\boldsymbol{\eta}_i\}$ in Eq. (80), and obtain

$$\Psi^{\mathrm{v}}_{\eta=0}(\{z_i\}) = \prod_i z_i^2 \prod_{i<j}(z_i - z_j)^3. \tag{101}$$

The density profile of the electrons near the origin can be determined numerically using the Monte Carlo method. The result is shown in Fig. 2. We find that the binding energy can be well fitted by a quadratic function for $|z-\eta| \lesssim 2l_B$ [36], and approximated as

$$\epsilon_b(z;\boldsymbol{\eta}) \approx -g_0 \frac{e^2 l_B(z)}{\varepsilon}\rho_e(z) + \frac{\hbar^2}{2m^*}\frac{|z-\boldsymbol{\eta}|^2}{l_B^2(z) l_b^2(z)}, \tag{102}$$

with $g_0 \approx 0.5$ and

$$\frac{\hbar^2}{m^*} \approx 0.08 \frac{e^2 l_B(z)}{4\pi\varepsilon}. \tag{103}$$

The estimated effective mass is about four times larger than the one usually assumed in the literature ($\hbar^2/m^* \approx 0.3e^2 l_B/4\pi\varepsilon$) [4,37]. On the other hand, the effective masses determined in experiments vary with the measurement methods [2]. Our estimate is actually close to the cyclotron effective mass measured by Kukushkin *et al.* [38]. There is also a theoretical proposal that the effective mass should be four times larger [39].

We can define an exchange-correlation functional

$$E_{\mathrm{xc}}[\rho_e] = -g_0 \frac{e^2}{\varepsilon}\int \mathrm{d}z l_B(z)\rho_e^2(z) + \cdots, \tag{104}$$

which includes the contribution of the first term of Eq. (102), as well as contributions that are ignored in our derivation, in particular the effect of particle exchanges. In the spirit of the Kohn-Sham approach of the density functional theory, we could define $E_{\mathrm{xc}}[\rho_e]$ as the difference between the exact ground-state energy of a system with a uniform density ρ_e and the total kinetic energy of noninteracting composite fermions at the same density [37,40].

Combining all of these, we obtain the Lagrangian Eq. (49).

VI. GENERALIZATION FOR FLAT CHERN BANDS

The fractional quantum Hall effect is also predicted to emerge in flat Chern bands, i.e., Bloch bands that are nearly dispersionless and have nonzero Chern numbers [19,20]. A flat Chern band is considered as a generalized "Landau level" which possesses essential properties for hosting the fractional quantum Hall effect. Conversely, a Landau level could be interpreted as an ideal flat Chern band with a Chern number $|C| = 1$ [41]. One expects that interacting electrons confined in a flat Chern band behave similarly as in an ordinary Landau level. The expectation was recently confirmed in experiments [42–45].

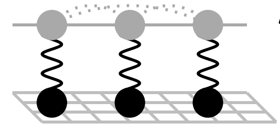

FIG. 3. Dipole model of composite fermions for a flat Chern band. Compared to the model presented in Fig. 1 for a Landau level, the electron is now confined in a Bloch band characterized by a Chern number C and other parameters such as the Berry curvature $\Omega_{\boldsymbol{k}}$ and the quantum metric $\mathbb{G}_{\boldsymbol{k}}$. A Landau level can actually be interpreted as an ideal flat Chern band with $C = -1$, a constant Berry curvature, and vanishing $\mathrm{Tr}\mathbb{G}_{\boldsymbol{k}} - |\Omega_{\boldsymbol{k}}|$. The Landau level can be continuously evolved to a flat Chern band with the same Chern number. One expects that the continuous evolution should not induce a topological phase transition to the state of vortices. The possibility that the vortices adopt other topological collective states, in particular for flat Chern bands with $|C| \neq 1$, is not yet considered in this work.

The generalization of our approach for flat Chern bands with $|C| = 1$ is straightforward. A dipole model is shown in Fig. 3, where we replace the electron Landau level in Fig. 1 with a flat Chern band. The general idea presented in Sec. III B for constructing many-body wave functions of electrons is still applicable. We introduce vortices as auxiliary degrees of freedom which should be projected out in the end, and require that electrons always reside in their original and physical Hilbert space. We thus have the wave-function *Ansatz* for flat bands with $C = -1$ [46]:

$$\Psi(\{\boldsymbol{r}_i\}) = \int \prod_i \mathrm{d}\mu_b(\boldsymbol{\eta}_i) J(\{\eta_i\})\Psi_{\mathrm{CF}}(\{\boldsymbol{r}_i, \bar{\eta}_i\}), \tag{105}$$

where $\{\boldsymbol{r}_i\}$ denotes the set of electron coordinates. For a flat Chern band, unlike a Landau level, the wave functions Ψ and Ψ_{CF} are generally not holomorphic in the electron coordinates. Instead, they should be expanded in the Bloch states of the flat band which span the physical Hilbert space. Thus, the single-body wave function of a composite fermion can be written as

$$\psi(\boldsymbol{r}, \bar{\eta}) = \sum_{\boldsymbol{k}\in \mathrm{BZ}} \varphi_{\boldsymbol{k}}(\bar{\eta}) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} u_{\boldsymbol{k}}(\boldsymbol{r}), \tag{106}$$

where $u_{\boldsymbol{k}}(\boldsymbol{r})$ denotes the periodic part of the Bloch wave function at the quasi-wave-vector $\boldsymbol{k}$ of the flat band, and the state of the composite fermion is represented by the wave function $\varphi_{\boldsymbol{k}}(\bar{\eta})$.

We can introduce an effective Hamiltonian for composite fermions. In the enlarged Hilbert space of composite fermions, each electron in the flat band is bound to a vortex. While the binding potential could be derived microscopically, as we have demonstrated for a Landau level in Sec. V B, it is reasonable to assume that the harmonic form Eq. (23) is a good first approximation. Therefore, the effective Hamiltonian of a composite fermion can be written as

$$\hat{H}_{\mathrm{CF}} = \hat{T}_e + \frac{\hbar^2}{2m^* l_B^2 l_b^2}|\boldsymbol{r} - \hat{\boldsymbol{\eta}}|^2, \tag{107}$$

where $\hat{T}_e$ is the electron kinetic energy, and we define the effective magnetic length

$$l_B^2 \equiv \frac{V_p}{2\pi}, \tag{108}$$

with V_p being the area of the primitive cell of the system.

We can obtain the effective Hamiltonian for $\varphi_{\boldsymbol{k}}(\bar{\eta})$ by determining the expectation value $\langle\psi|\hat{H}_{CF}|\psi\rangle$ for ψ defined by Eq. (106). It is easy to prove the identities:

$$\langle\psi\,|\,\boldsymbol{r}\,|\,\psi\rangle = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}^*(\bar{\eta})(\mathrm{i}\partial_{\boldsymbol{k}} + \boldsymbol{A}_{\boldsymbol{k}})\varphi_{\boldsymbol{k}}(\bar{\eta}), \tag{109}$$

$$\langle\psi\,|\,r^2\,|\,\psi\rangle = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}^*(\bar{\eta})(|\mathrm{i}\partial_{\boldsymbol{k}} + \boldsymbol{A}_{\boldsymbol{k}}|^2 + \mathrm{Tr}\mathbb{G}_{\boldsymbol{k}})\varphi_{\boldsymbol{k}}(\bar{\eta}), \tag{110}$$

where $\boldsymbol{A}_{\boldsymbol{k}}$ and $\mathbb{G}_{\boldsymbol{k}}$ are the Berry connection and quantum metric tensor of the flat band, respectively, defined by [25]

$$\boldsymbol{A}_{\boldsymbol{k}} = \mathrm{i}\langle u_{\boldsymbol{k}}\,|\,\partial_{\boldsymbol{k}} u_{\boldsymbol{k}}\rangle, \tag{111}$$

$$\mathbb{G}_{\boldsymbol{k}}^{ab} = \mathrm{Re}\langle\partial_{k_a} u_{\boldsymbol{k}}\,|\,\partial_{k_b} u_{\boldsymbol{k}}\rangle - A_{\boldsymbol{k}}^a A_{\boldsymbol{k}}^b. \tag{112}$$

Applying the identities, we obtain

$$\langle\psi\,|\,\hat{H}_{CF}\,|\,\psi\rangle = \sum_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}^*(\bar{\eta})\hat{H}\varphi_{\boldsymbol{k}}(\bar{\eta}) \tag{113}$$

and

$$\hat{H} = \epsilon_{\boldsymbol{k}} + \frac{\hbar^2}{2m^* l_B^2 l_b^2}(\mathrm{Tr}\mathbb{G}_{\boldsymbol{k}} + \Omega_{\boldsymbol{k}}) + \frac{\hbar^2}{2m^* l_B^2 l_b^2}(2\mathrm{i}\partial_k + \bar{A}_{\boldsymbol{k}} - \bar{\eta})(2\mathrm{i}\partial_{\bar{k}} + A_{\boldsymbol{k}} - \hat{\eta}), \tag{114}$$

where $\epsilon_{\boldsymbol{k}}$ and $\Omega_{\boldsymbol{k}}$ are the dispersion and Berry curvature of the flat band, respectively. The form of the $\hat{\eta}$ operator depends on the spatial profile of the vortex density. As a first approximation, one could assume a homogeneous vortex density, thus $\hat{\eta} = 2l_b^2\partial_{\bar{\eta}}$.

We could predict the stability of a fractional Chern insulator by determining the eigenspectrum of the single-body effective Hamiltonian Eq. (114). For an ideal flat band with a uniform Berry curvature, only the last term remains, and it is easy to show that the Hamiltonian gives rise to the ordinary Λ-levels (see Appendix A). For the more general cases, however, we expect that the first two terms, which could be interpreted as the renormalized band dispersion experienced by composite fermions, will make Λ-levels nondegenerate and suppress excitation gaps. When the gaps are closed, the fractional Chern insulator state will be destroyed. The application of the effective Hamiltonian to real materials is left for future investigation.

The form of the effective Hamiltonian seems to justify the heuristic trace condition, which requires $\mathrm{Tr}\mathbb{G}_{\boldsymbol{k}} - |\Omega_{\boldsymbol{k}}| \approx 0$ everywhere in the Brillouin zone for the emergence of a fractional Chern insulator [25,47]. We see that the second term, which is proportional to $\mathrm{Tr}\mathbb{G}_{\boldsymbol{k}} + \Omega_{\boldsymbol{k}} = \mathrm{Tr}\mathbb{G}_{\boldsymbol{k}} - |\Omega_{\boldsymbol{k}}|$ for $\Omega_{\boldsymbol{k}} < 0$ [48], renormalizes the dispersion $\epsilon_{\boldsymbol{k}}$ of electrons. As the renormalization tends to make a flat electron band nonflat, it destabilizes a fractional Chern insulator. On the other hand, it could be possible to engineer the correction to compensate the electron dispersion of a nonflat electron band and make it flatter after the renormalization. The latter suggests a novel possibility that fractional Chern insulators could be stabilized in nonflat Chern bands.

VII. SUMMARY AND DISCUSSION

In summary, we present a reformulation of the theory of composite fermions based on the dipole model. Some new insights emerge.

(A) The states of composite fermions can be determined by solving a wave equation, with an effective Hamiltonian that can be derived from first principles. Such a deductive approach can reproduce the well-established results of the standard theory of composite fermions, namely the wave functions of the ideal fractional quantum Hall states in the lowest Landau level. It may also provide an alternative to intuition and educated guesses for understanding more complex states such as those observed in higher Landau levels [49].

(B) A wave-function *Ansatz* $\Psi = \hat{P}_v\Psi_{CF}$, or equivalently Jain's wave-function *Ansatz* in an alternative wave-function representation of composite fermions, can be naturally inferred from the dipole model. The Bijl-Jastrow factor in Jain's *Ansatz* can be interpreted as the complex conjugate of the wave function of the collective state of vortices, rather than the numerator of the singular CS transformation.

(C) The effective theory specified by Eqs. (73) and (74) differs from the HLR theory due to the drift-velocity corrections in the effective Hamiltonian Eq. (71).

(D) The wave-function *Ansatz* and the effective theory can be unified on the common basis of the dipole model, and logically connected.

(E) The Hilbert space of composite fermions has a simple structure, i.e., the tensor product of two separate Hilbert spaces for the physical and fictitious degrees of freedom, respectively. The simple structure makes it much easier and less prone to arbitrariness to generalize the composite fermion theory, e.g., for the flat Chern bands.

ACKNOWLEDGMENTS

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APPENDIX A: Λ-LEVELS OF THE FRACTIONAL QUANTUM HALL STATES

The wave equation (27) is the same as that for an ordinary charge particle in an effective magnetic field $\boldsymbol{\mathcal{B}}$ except for an unimportant constant. Therefore, the wave functions of Λ-levels are just those for ordinary Landau levels, which can be written as [2]

$$\varphi_{n,m}(\boldsymbol{\xi}) \propto \frac{e^{-|\xi|^2/4l^2}}{\sqrt{2\pi}\,l} f_{n,m}(\boldsymbol{\xi}), \tag{A1}$$

with

$$f_{n,m}(\boldsymbol{\xi}) = c_{nm} l^{2n+m} e^{|\xi|^2/2l^2} \quad \times \begin{cases} \partial_{\xi}^{n} \partial_{\bar{\xi}}^{m+n} e^{-|\xi|^2/2l^2}, & \nu < \frac{1}{2}, \\ \partial_{\bar{\xi}}^{n} \partial_{\xi}^{m+n} e^{-|\xi|^2/2l^2}, & \nu > \frac{1}{2}, \end{cases} \tag{A2}$$

and $c_{nm} \equiv \sqrt{2^{2n+m}/n!(m+n)!}$, $m \geqslant -n$. $\psi_{mn}(z, \bar{\eta})$ is related to $\varphi_{n,m}(z, \bar{\eta}) \equiv \varphi_{n,m}(\boldsymbol{\xi})|_{\xi\to z, \bar{\xi}\to\bar{\eta}}$ by Eq. (26), and normalized by Eq. (22). We have

$$\psi_{n,m}(z, \bar{\eta}) = f_{n,m}(z, \bar{\eta}) \begin{cases} \frac{l_B}{l}\left(\frac{l_B}{l_b}\right)^n e^{z\bar{\eta}/2l_b^2}, & \nu < \frac{1}{2}, \\ \frac{l_b}{l}\left(\frac{l_b}{l_B}\right)^n e^{z\bar{\eta}/2l_B^2}, & \nu > \frac{1}{2}. \end{cases} \tag{A3}$$

The corresponding eigenenergies are

$$\epsilon_{n,m} = \hbar\omega_c^* \begin{cases} n, & \nu < 1/2, \\ n+1, & \nu > 1/2, \end{cases} \tag{A4}$$

with $\omega_c^* \equiv e|\mathcal{B}|/m^*$.

For the special case $\nu = 1/2$, $B = b$, we have $\mathcal{B} = 0$. The wave function is plane-wave-like:

$$\psi_{\boldsymbol{k}}(z, \bar{\eta}) = \frac{l_B}{\sqrt{2\pi}} e^{\mathrm{i}\frac{\bar{k}z + k\bar{\eta}}{2} + \frac{z\bar{\eta}}{2l_B^2} - \frac{|k|^2 l_B^2}{4}}, \tag{A5}$$

where $\boldsymbol{k} \equiv (k_x, k_y)$ denotes the wave vector of the state, and $k \equiv k_x + \mathrm{i}k_y$, $\bar{k} = k^*$. The wave function is normalized by $\int \mathrm{d}\mu_B(\boldsymbol{z})\mathrm{d}\mu_b(\boldsymbol{\eta})\psi_{\boldsymbol{k}}^*(z, \bar{\eta})\psi_{\boldsymbol{k}'}(z, \bar{\eta}) = \delta(\boldsymbol{k} - \boldsymbol{k}')$.

It is easy to show that the wave function describes a bound state of an electron and a vortex. Its spatial distribution can be written as

$$|\psi_{n,m}(z, \bar{\eta})|^2 e^{-|z|^2/2l_B^2 - |\eta|^2/2l_b^2} \propto \begin{cases} e^{-|z|^2/2l^2 - |z-\eta|^2/2l_b^2}, & \nu < 1/2, \\ e^{-|\eta|^2/2l^2 - |z-\eta|^2/2l_B^2}, & \nu > 1/2. \end{cases} \tag{A6}$$

We see that the electron and the vortex are bound by a Gaussian factor with a lengthscale l_b (l_B) for $\nu < 1/2$ ($\nu > 1/2$).

We can also solve Eq. (29) and obtain wave functions in the standard representation:

$$\tilde{\varphi}_{n,m}(\boldsymbol{\xi}) = \frac{e^{-\frac{|\xi|^2}{4l^2}}}{\sqrt{2\pi}\,l} f_{n,m}(\boldsymbol{\xi}) \begin{cases} \left(\frac{l_b}{l_B}\right)^n, & \nu < \frac{1}{2}, \\ \left(\frac{l_B}{l_b}\right)^{n+1}, & \nu > \frac{1}{2}, \end{cases} \tag{A7}$$

where the normalization constants are fixed using Eq. (67). $\tilde{\psi}_{n,m}(\boldsymbol{\xi})$ is related to $\tilde{\varphi}_{n,m}$ via Eq. (30). It is straightforward to verify that $\psi_{n,m}(z, \bar{\eta})$ and $\tilde{\psi}_{n,m}(\boldsymbol{\xi})$ are related by Eq. (28).

An alternative way of solving the wave equation (25) is to define a set of ladder operators [2]. For the filling factor $\nu < 1/2$, the ladder operators are

$$\hat{a} = \frac{1}{\sqrt{2}} \frac{l}{l_B l_b}(z - \hat{\eta}), \tag{A8}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}} \frac{l}{l_B l_b}(\hat{\bar{z}} - \bar{\eta}), \tag{A9}$$

$$\hat{b} = \frac{l}{\sqrt{2}}\left(\frac{\hat{\bar{z}}}{l_B^2} - \frac{\bar{\eta}}{l_b^2}\right), \tag{A10}$$

$$\hat{b}^\dagger = \frac{l}{\sqrt{2}}\left(\frac{z}{l_B^2} - \frac{\hat{\eta}}{l_b^2}\right). \tag{A11}$$

It is easy to verify the commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{b}, \hat{b}^\dagger] = 1$, and $[\hat{a}, \hat{b}] = 0$. For $\nu > 1/2$, the ladder operators can be obtained by exchanging $l_b \leftrightarrow l_B$ and $z \leftrightarrow \bar{\eta}$ in the definitions.

APPENDIX B: QUANTIZATION IN BERGMAN SPACES

In this Appendix, we discuss quantization in weighted Bergman spaces. Two alternative forms of the quantization, corresponding to the normal ordering and the antinormal ordering of operators, respectively, can be defined and related to each other.

1. Quantization: Normal ordering

We can quantize an arbitrary function $H(\bar{z}, z)$ (e.g., the energy of a composite fermion) to an operator $\hat{H}$. In analogy to Eq. (24), the action of $\hat{H}$ on a wave function $\psi(z)$ can be defined as

$$[\hat{H}\psi](z) = \int \mathrm{d}\mu_B(\boldsymbol{\xi}) K_B(z, \bar{\xi}) H(\bar{\xi}, \xi)\psi(\xi), \tag{B1}$$

where we use the reproducing kernel to project the nonholomorphic function into the Bergman space. As we see in Sec. III C, varying the Lagrangian with respect to a wave function defined in a Bergman space gives rise to a Hamiltonian operator.

The operator can be written in general as

$$\hat{H} = N_+[H(\hat{\bar{z}}, z)], \tag{B2}$$

where $\hat{\bar{z}}$ is defined in Eq. (59). It is sufficient to show the relation for $H(\boldsymbol{\xi}) = \bar{\xi}^m \xi^n$, $m, n \in Z$. Substituting the function into Eq. (B1), we can combine ξ^n with $\psi(\xi)$ (i.e., put it on the right), and show that $\bar{\xi}^m$ is mapped to $\hat{\bar{z}}^m$. For the latter, we have

$$\hat{\bar{z}}^2\psi(z) \equiv \int \mathrm{d}\mu_B(\boldsymbol{\xi}_1) K_B(z, \bar{\xi}_1)\bar{\xi}_1 \times \int \mathrm{d}\mu_B(\boldsymbol{\xi}) K_B(\xi_1, \bar{\xi})\bar{\xi}\psi(\xi) \tag{B3}$$

$$= \int \mathrm{d}\mu_B(\boldsymbol{\xi}) K_B(z, \bar{\xi})\bar{\xi}^2\psi(\xi), \tag{B4}$$

where we apply the complex conjugate of Eq. (41) when completing the integral with respect to $\boldsymbol{\xi}_1$.

We thus only need to determine the quantization of $\bar{z}$. In general, the operator $\hat{\bar{z}}$ can be written as a function of ∂_z and z satisfying

$$\hat{\bar{z}}(\partial_z, z) K_B(z, \bar{\xi}) = \bar{\xi} K_B(z, \bar{\xi}). \tag{B5}$$

For the case of a uniform magnetic field with the reproducing kernel Eq. (10), it is easy to see that $\hat{\bar{z}} = 2l_B^2\partial_z$ does satisfy the equation. For a general space, we substitute Eq. (43) into

Eq. (B5), and we obtain

$$\hat{\bar{z}} = 2l_B^2\left(\partial_z + \frac{\mathrm{i}e}{\hbar}N_-[\bar{A}_1^\star(\hat{\bar{z}}, z)]\right), \tag{B6}$$

where we introduce the starred vector potential $\bar{A}^\star$ defined by

$$\partial_z F_B(\bar{\xi}, z) \equiv -\mathrm{i}\frac{e}{\hbar}\bar{A}^\star(\bar{\xi}, z), \tag{B7}$$

and decompose as $\bar{A}^\star(\bar{\xi}, z) = \mathrm{i}B_0\bar{\xi}/2 + \bar{A}_1^\star(\bar{\xi}, z)$, and $N_-[\cdots]$ denotes the antinormal ordering that puts $\hat{\bar{z}}$'s to the right of all z's.

The starred vector potential $\bar{A}^\star$ can be related to the physical vector potential $\bar{A}$, as we will show in the next subsection. Consequently, the quantization can also be written as

$$\hat{\bar{z}} = 2l_B^2\left(\partial_z + \frac{\mathrm{i}e}{\hbar}N_+[\bar{A}_1(\hat{\bar{z}}, z)]\right). \tag{B8}$$

These equations can be solved iteratively.

The vortex degree of freedom can be quantized similarly. We express the reproducing kernel as $K_b(\bar{\eta}, \eta') \equiv \exp[F_b(\bar{\eta}, \eta')]$, and we define the corresponding starred vector potential with $\partial_{\bar{\eta}}F_b(\bar{\eta}, \eta') = -\mathrm{i}ea^\star(\bar{\eta}, \eta')/\hbar$. Making substitutions $z \to \bar{\eta}$, $\hat{\bar{z}} \to \hat{\eta}$, $B \to b$, and $\bar{A}^\star \to a^\star$, we have

$$\hat{\eta} = 2l_b^2\left(\partial_{\bar{\eta}} + \frac{\mathrm{i}e}{\hbar}N_-[a_1^\star(\bar{\eta}, \hat{\eta})]\right) \tag{B9}$$

$$= 2l_b^2\left(\partial_{\bar{\eta}} + \frac{\mathrm{i}e}{\hbar}N_+[a_1(\bar{\eta}, \hat{\eta})]\right). \tag{B10}$$

The normal (antinormal) ordering should be reinterpreted accordingly to put $\hat{\eta}$'s to the left (right) of all $\bar{\eta}$'s.

2. Quantization: Antinormal ordering

Alternatively, we can define

$$[\hat{H}\psi](z) = \int \mathrm{d}\mu_B(\boldsymbol{\xi})K_B(z, \bar{\xi})H^\star(z, \bar{\xi})\psi(\xi), \tag{B11}$$

where $H^\star$ is defined by the transformation

$$H^\star(z, \bar{\xi}) = \frac{1}{K_B(z, \bar{\xi})} \times \int \mathrm{d}\mu_B(\boldsymbol{\zeta})K_B(z, \bar{\zeta})H(\bar{\zeta}, \zeta)K_B(\zeta, \bar{\xi}). \tag{B12}$$

It is easy to verify that Eq. (B11) reduces to Eq. (B1) after substituting Eq. (B12).

It is not difficult to see that the operator can be written in the form of the antinormal ordering

$$\hat{H} = N_-[H^\star(z, \hat{\bar{z}})]. \tag{B13}$$

We thus have two alternative forms of the $\hat{H}$ operator in the normal ordering and the antinormal ordering, respectively, which are related by the transformation Eq. (B12).

The vector potential $\bar{A}$ and its starred counterpart $\bar{A}^\star$ are also related by the transformation. To see that, we note that $\partial_z\psi(z)$ can be written in two alternative forms:

$$\partial_z\psi(z) = \int \mathrm{d}\mu_B(\xi)K_B(z, \bar{\xi})\partial_\xi\psi(\xi) \tag{B14}$$

$$= \int \mathrm{d}\mu_B(\xi)[\partial_z K_B(z, \bar{\xi})]\psi(\xi). \tag{B15}$$

Applying an integral by parts to the first form, we have

$$\int \mathrm{d}\mu_B(\xi)K_B(z, \bar{\xi})\bar{A}(\bar{\xi}, \xi)\psi(\xi) = \int \mathrm{d}\mu_B(\xi)K_B(z, \bar{\xi})\bar{A}^\star(\bar{\xi}, z)\psi(\xi). \tag{B16}$$

The two sides of the equation correspond to the two alternative quantization forms of the vector potential. They are thus related by Eq. (B12).

APPENDIX C: LONG-WAVELENGTH LIMIT

We can relate f_B with F_B when the nonuniform component of the magnetic field is small and varies slowly over space. In this limit, Eq. (41) can be approximated as

$$\psi(z, \bar{\eta}) \approx \int \mathrm{d}\mu^{(0)}(\boldsymbol{\xi})K_B^{(0)}(z, \bar{\xi}) \times \left[1 - f_B^{(1)}(\boldsymbol{\xi}) + F_B^{(1)}(\bar{\xi}, z)\right]\psi(\xi, \bar{\eta}), \tag{C1}$$

where $f_B^{(1)}$ and $F_B^{(1)}$ denote the corrections to f_B and F_B due to the spatial fluctuation of the magnetic field, respectively. Since an electron is always bound to a vortex in a composite fermion with a lengthscale of the magnetic length (see Appendix A), which is much smaller than the wavelength of the fluctuating magnetic field, we expand $f_B^{(1)}$ and $F_B^{(1)}$ to the linear order of $\bar{\xi}$ around the vortex coordinate $\bar{\eta}$, and we complete the integral. We obtain

$$\psi(z, \bar{\eta}) \approx \left[1 - f_B^{(1)}(\bar{\eta}, z) + F_B^{(1)}(\bar{\eta}, z) \quad - 2l_B^2\partial_z\partial_{\bar{\eta}}f_B^{(1)}(\bar{\eta}, z)\right] \times \psi(z, \bar{\eta}). \tag{C2}$$

We thus have

$$F_B(z, \bar{\eta}) \approx \left[f_B(\boldsymbol{z}) - \ln\frac{l_B^2(\boldsymbol{z})}{l_B^2}\right]\Bigg|_{\bar{z}\to\bar{\eta}}. \tag{C3}$$

Similarly, we can relate f_b and F_b in the long-wavelength limit.

Differentiating the relation with respect to z, and ignoring the gradient of the local magnetic length, we have

$$\bar{A}^\star(\bar{\eta}, z) \approx \bar{A}(\bar{z}, z)|_{\bar{z}\to\bar{\eta}}. \tag{C4}$$

We can have the approximate forms of $\hat{\bar{z}}$ and $\hat{\eta}$ in the long-wavelength limit as well. To do that, we apply the expansion $N_+[\bar{A}_1(\hat{\bar{z}}, z)] \approx \bar{A}_1(\bar{\eta}, z) + (\hat{\bar{z}} - \bar{\eta})[\partial_{\bar{\eta}}\bar{A}_1(\bar{\eta}, z)]$ and the similar one for $N_+[a_1(\bar{\eta}, \hat{\eta})]$, and we substitute them into Eqs. (B8) and (B10), respectively. We obtain

$$\hat{\bar{z}} - \bar{\eta} \approx 2\left[\partial_z + \mathrm{i}\frac{e}{\hbar}\bar{A}(\bar{\eta}, z)\right]l_B^2(\bar{\eta}, z), \tag{C5}$$

$$\hat{\eta} - z \approx 2\left[\partial_{\bar{\eta}} + \mathrm{i}\frac{e}{\hbar}a(\bar{\eta}, z)\right]l_b^2(\bar{\eta}, z), \tag{C6}$$

with $l_B^2(z, \bar{\eta}) \equiv l_B^2(\boldsymbol{z})|_{\bar{z}\to\bar{\eta}}$ and $l_b^2(z, \bar{\eta}) \equiv l_b^2(\boldsymbol{z})|_{\bar{z}\to\bar{\eta}}$.

The scalar potentials can similarly be approximated as

$$N_+[\Phi_{\rm eff}] \approx \Phi_{\rm eff}(\bar\eta, z) - V(\bar\eta, z)[\mathrm{i}\hbar\partial_z - e\bar A(\bar\eta, z)], \quad \text{(C7)}$$

$$N_+[\phi] \approx \phi(\bar\eta, z) - \bar v(\bar\eta, z)[-\mathrm{i}\hbar\partial_{\bar\eta} - ea(\bar\eta, z)], \quad \text{(C8)}$$

where V and $\bar v$ are the complex components of the drift velocities defined in the main text.

We can then obtain the approximate form of $\hat H_\psi$ by applying these approximations to Eq. (58), and ignoring all corrections proportional to the gradients of the magnetic and electric fields as well as contributions beyond the linear order of $B_1(\boldsymbol z)$, $b_1(\boldsymbol\eta)$, and the electric fields. Equation (71) can be obtained by applying Eq. (70).

APPENDIX D: TIME-DEPENDENT SYSTEMS

We first note that the stationary-state wave equation can also be obtained from the variational principle $\delta L_{\rm eff} = 0$, where $L_{\rm eff}$ is the effective Lagrangian for a composite fermion:

$$L_{\rm eff} \equiv \int \mathrm{d}\mu_B(\boldsymbol z)\mathrm{d}\mu_b(\boldsymbol\eta)\psi^*(z, \bar\eta)(\epsilon - \hat H_\psi)\psi(z, \bar\eta). \quad \text{(D1)}$$

For time-dependent systems, the term proportional to ϵ in the Lagrangian should be replaced by

$$\int \mathrm{d}\mu_B(\boldsymbol z)\mathrm{d}\mu_b(\boldsymbol\eta)\psi^*(z, \bar\eta; t) \times \left\{\mathrm{i}\hbar\frac{\partial}{\partial t} - \frac{\mathrm{i}\hbar}{2}\left[\frac{\partial f_B(\boldsymbol z, t)}{\partial t} + \frac{\partial f_b(\boldsymbol\eta, t)}{\partial t}\right]\right\}\psi(z, \bar\eta; t), \quad \text{(D2)}$$

where the second term in the curly brackets originates from the exponential factor in Eq. (34) and its counterpart for vortices.

After making the substitution, and applying Eqs. (64)–(66), we obtain the Schrödinger action of a composite fermion:

$$S_{\rm CF} = \int \mathrm{d}t \int \mathrm{d}\boldsymbol\xi \tilde\varphi^*(\boldsymbol\xi, t)\left(\mathrm{i}\hbar\frac{\partial}{\partial t} - \hat H_1\right)\varphi(\boldsymbol\xi, t), \quad \text{(D3)}$$

with

$$\hat H_1 \equiv \hat H + \frac{\mathrm{i}\hbar}{2}N_+[\partial_t f_B(\hat z, \xi, t) - \partial_t f_B(\boldsymbol\xi, t)] + \frac{\mathrm{i}\hbar}{2}N_+[\partial_t f_b(\bar\xi, \hat\eta, t) - \partial_t f_b(\boldsymbol\xi, t)]. \quad \text{(D4)}$$

In the long-wavelength limit, we can approximate the correction terms in the same way as in Eqs. (C7) and (C8). It is not difficult to show that $\hat H_1$ has the same form as $\hat H$ shown in Eq. (71) but with the gauge-invariant definitions of the electric fields.

APPENDIX E: OPERATORS IN THE STANDARD REPRESENTATION

1. Ideal systems

In ideal systems, wave functions of composite fermions in the dipole representation and the standard representation are related by Eq. (28). Transforming to the standard representation, we have

$$\begin{aligned}\bar z\psi(z, \bar\eta) &\to 2l_B^2\partial_z\psi(z, \bar\eta)\\ &= \int \mathrm{d}\mu_B^{(0)}(\boldsymbol\xi)e^{z\bar\xi/2l_B^2 + \xi\bar\eta/2l_b^2}\bar\xi\tilde\psi(\boldsymbol\xi),\end{aligned} \quad \text{(E1)}$$

$$\begin{aligned}z\psi(z, \bar\eta) &= \int \mathrm{d}\mu_B^{(0)}(\boldsymbol\xi)\big[2l_B^2\partial_{\bar\xi}e^{z\bar\xi/2l_B^2 + \xi\bar\eta/2l_b^2}\big]\tilde\psi(\boldsymbol\xi)\\ &= \int \mathrm{d}\mu_B^{(0)}(\boldsymbol\xi)e^{z\bar\xi/2l_B^2 + \xi\bar\eta/2l_b^2}\big(-2l_B^2\partial_{\bar\xi} + \xi\big)\tilde\psi(\boldsymbol\xi),\end{aligned} \quad \text{(E2)}$$

and similar expressions for η and $\bar\eta$. Therefore, in the standard representation, the operators should be mapped to

$$z \to -2l_B^2\partial_{\bar\xi} + \xi, \quad \text{(E3)}$$

$$\bar z \to \bar\xi, \quad \text{(E4)}$$

$$\eta \to \xi, \quad \text{(E5)}$$

$$\bar\eta \to -2l_b^2\partial_\xi + \big(l_b^2/l_B^2\big)\bar\xi. \quad \text{(E6)}$$

It is straightforward to apply the mappings to Eq. (23) and obtain a wave equation for $\tilde\psi(\boldsymbol\xi)$. After applying the transformation Eq. (30), we obtain Eq. (29).

2. General systems

For general systems, wave functions in the dipole representation and the standard representation are related by Eq. (64). For the $\hat{\bar z}$ operator defined in Eq. (B5), we have

$$\hat{\bar z}\psi(z, \bar\eta) = \int \mathrm{d}\mu_B(\boldsymbol\xi)K_B(z, \bar\xi)K_b(\bar\eta, \xi)\bar\xi\tilde\psi(\boldsymbol\xi). \quad \text{(E7)}$$

On the other hand, applying complex conjugation to Eq. (B5) and exchanging $\boldsymbol z$ and $\boldsymbol\xi$, we obtain

$$\hat{\bar z}^*(\partial_\xi, \xi)K_B(z, \bar\xi) = zK_B(z, \bar\xi). \quad \text{(E8)}$$

Here, we make use of the relation $[K_B(z, \bar\xi)]^* = K_B(\xi, \bar z)$. We thus have

$$\begin{aligned}z\psi(z, \bar\eta) &= \int \mathrm{d}\mu_B(\xi)[\hat{\bar z}^*(\partial_\xi, \xi)K_B(z, \bar\xi)] \times K_b(\bar\eta, \xi)\tilde\psi(\boldsymbol\xi) \quad \text{(E9)}\\ &= \int \mathrm{d}\mu_B(\xi)K_B(z, \bar\xi)K_b(\bar\eta, \xi) \times \hat{\bar z}^\dagger\left(\partial_\xi - \frac{\mathrm{i}e}{\hbar}\bar A(\boldsymbol\xi), \xi\right)\tilde\psi(\boldsymbol\xi).\end{aligned} \quad \text{(E10)}$$

Here we apply integral by parts and make use of Eq. (35). Note that $(\partial_\xi)^\dagger = -\partial_{\bar\xi}$. The mappings of η and $\bar\eta$ can be obtained similarly.

All summarized, we have the following mapping rules in the standard representation:

$$\bar z \to \bar\xi, \quad \text{(E11)}$$

$$\eta \to \xi, \quad \text{(E12)}$$

$$z \to \hat{\bar z}^\dagger\left(\partial_\xi - \frac{\mathrm{i}e}{\hbar}\bar A(\boldsymbol\xi), \xi\right), \quad \text{(E13)}$$

$$\bar\eta \to \hat\eta^\dagger\left(\partial_{\bar\xi} + \frac{\mathrm{i}e}{\hbar}A(\boldsymbol\xi), \xi\right). \quad \text{(E14)}$$

3. Transformations of Hamiltonians

We introduce an operator $\hat{K}$ to represent the transformation Eq. (64):

$$\psi = \hat{K}\tilde{\psi}. \tag{E15}$$

We further define

$$\hat{K}_1 \equiv \hat{K}e^{f_B(\boldsymbol{\xi})}. \tag{E16}$$

Using the operator, the mapping rules Eqs. (E11)–(E14) can be written as

$$\hat{K}_1^{-1}\hat{\bar{z}}\hat{K}_1 = \bar{\xi}, \tag{E17}$$

$$\hat{K}_1^{-1}\hat{\eta}\hat{K}_1 = \xi, \tag{E18}$$

$$\hat{K}_1^{-1}z\hat{K}_1 = \hat{\bar{z}}^\dagger, \tag{E19}$$

$$\hat{K}_1^{-1}\bar{\eta}\hat{K}_1 = \hat{\eta}^\dagger. \tag{E20}$$

Applying the transformation to $\hat{H}_\psi$, we have

$$\hat{K}_1^{-1}\hat{H}_\psi\hat{K}_1 = \hat{H}_\psi^\dagger. \tag{E21}$$

The Hamiltonians $\hat{H}$ and $\hat{\tilde{H}}$ that govern the wave equations for φ and $\tilde{\varphi}$. respectively, can be identified to be

$$\hat{H} \equiv e^{-\frac{f_B+f_b}{2}}\hat{H}_\psi e^{\frac{f_B+f_b}{2}}, \tag{E22}$$

$$\hat{\tilde{H}} \equiv e^{\frac{f_B+f_b}{2}}\hat{K}_1^{-1}\hat{H}_\psi\hat{K}_1 e^{-\frac{f_B+f_b}{2}}. \tag{E23}$$

Applying Eq. (E21), we obtain

$$\hat{\tilde{H}} = \hat{H}^\dagger. \tag{E24}$$

APPENDIX F: CURRENT DENSITIES

1. Current density in a Landau level

In a Landau level, the particle density of a state ψ can be defined as

$$\rho(\boldsymbol{z}, t) = w(\boldsymbol{z}, t)|\psi(z, t)|^2, \tag{F1}$$

where $w(\boldsymbol{z})$ denotes the weight of the Bergman space. The wave equation in the space can generally be written as

$$i\hbar\frac{\partial\psi}{\partial t} = \int d\mu(\boldsymbol{\xi})K(z, \bar{\xi})H^\star(\bar{\xi}, z)\psi(\xi, t). \tag{F2}$$

We can determine the current density by establishing a continuity equation for $\rho(\boldsymbol{z})$. We have

$$\frac{\partial\rho(\boldsymbol{z}, t)}{\partial t} = \frac{1}{i\hbar}\int d\mu(\boldsymbol{\xi}_1)d\mu(\boldsymbol{\xi}_2)\psi^*(\xi_1, t)K(\xi_1, \bar{\xi}_2) \times H^\star(\bar{\xi}_2, \xi_1)\psi(\xi_2, t)[\delta(\boldsymbol{z} - \boldsymbol{\xi}_1) - \delta(\boldsymbol{z} - \boldsymbol{\xi}_2)]. \tag{F3}$$

We then substitute the expansion

$$\delta(\boldsymbol{z} - \boldsymbol{\xi}_1) - \delta(\boldsymbol{z} - \boldsymbol{\xi}_2) = -\frac{1}{2}\sum_{n,m=0}^{\infty}{}' \left[\frac{1}{m!n!}(\xi_1 - \xi_2)^m \times (\bar{\xi}_1 - \bar{\xi}_2)^n\partial_z^m\partial_{\bar{z}}^n\delta(\boldsymbol{z} - \boldsymbol{\xi}_1) - (1 \leftrightarrow 2)\right], \tag{F4}$$

where the summation excludes $(m, n) = (0, 0)$. We obtain the continuity equation

$$\frac{\partial\rho(\boldsymbol{z}, t)}{\partial t} + \nabla\cdot\boldsymbol{j}(\boldsymbol{z}, t) = 0, \tag{F5}$$

with the current density $j \equiv j_x + ij_y$:

$$j(\boldsymbol{z}, t) = j_0(\boldsymbol{z}, t) - 2i\partial_{\bar{z}}m(\boldsymbol{z}, t), \tag{F6}$$

$$j_0(\boldsymbol{z}, t) = \frac{1}{\hbar}w(\boldsymbol{z}, t)\int d\mu(\boldsymbol{\xi})(z - \xi) \times \mathrm{Im}[\psi^*(z, t)K(z, \bar{\xi})H^\star(\bar{\xi}, z)\psi(\xi, t)], \tag{F7}$$

$$m(\boldsymbol{z}, t) = \frac{1}{4\hbar}\mathrm{Re}\sum_{m,n=1}^{\infty}\frac{\partial_z^{m-1}\partial_{\bar{z}}^{n-1}}{m!n!}w(\boldsymbol{z}, t)\int d\mu(\xi) \times\psi^*(z, t)K(z, \bar{\xi})H^\star(\bar{\xi}, z)\psi(\xi, t) \times(z - \xi)^m(\bar{z} - \bar{\xi})^n, \tag{F8}$$

where $m(\boldsymbol{z}, t)$ is the orbital magnetization density.

Applying the quantization rules shown in Appendix B, we can rewrite the equations in operator forms,

$$\boldsymbol{j}_0(\boldsymbol{z}, t) = w(\boldsymbol{z}, t)\mathrm{Re}\{\psi^*(z, t)[\hat{\boldsymbol{v}}\psi](z, t)\}, \tag{F9}$$

$$m(\boldsymbol{z}, t) = \mathrm{Re}\sum_{m,n=1}^{\infty}\frac{\partial_z^{m-1}\partial_{\bar{z}}^{n-1}}{m!n!}w(\boldsymbol{z}, t)\psi^*(z, t) \times [\hat{m}_{mn}\psi](z, t), \tag{F10}$$

with

$$\hat{\boldsymbol{v}} \equiv \frac{1}{i\hbar}[\hat{\boldsymbol{z}}, \hat{H}], \tag{F11}$$

$$\hat{m}_{mn} \equiv \frac{1}{4\hbar}[\underbrace{\hat{\bar{z}}\ldots, [\hat{\bar{z}},}_{n} \underbrace{[\hat{z}, \ldots [\hat{z},}_{m} \hat{H}]\ldots]]\ldots]. \tag{F12}$$

In the long-wavelength limit, we can keep only $j_0(\boldsymbol{z}, t)$, and we ignore the magnetization current.

2. Current densities of a composite fermion system

The result derived in the preceding subsection can be applied to composite fermions with a straightforward generalization. The electron and vortex current densities for a state $\psi(z, \bar{\eta})$ can be written as

$$\boldsymbol{j}_e(\boldsymbol{z}) \approx w_B(\boldsymbol{z})\mathrm{Re}\int d\mu_b(\boldsymbol{\eta})\psi^*(z, \bar{\eta})[\hat{\boldsymbol{U}}\psi](z, \bar{\eta}), \tag{F13}$$

$$\boldsymbol{j}_v(\boldsymbol{\eta}) \approx w_b(\boldsymbol{z})\mathrm{Re}\int d\mu_B(\boldsymbol{z})\psi^*(z, \bar{\eta})[\hat{\boldsymbol{u}}\psi](z, \bar{\eta}), \tag{F14}$$

where $\hat{\boldsymbol{U}} \equiv [\hat{\boldsymbol{z}}, \hat{H}_\psi]/i\hbar$ and $\hat{\boldsymbol{u}} \equiv [\hat{\boldsymbol{\eta}}, \hat{H}_\psi]/i\hbar$ are the electron and vortex velocity operators, respectively, $\hat{H}_\psi$ is the effective Hamiltonian shown in Eq. (58), and we ignore the orbital magnetization contribution.

In the long-wavelength limit, we can apply the approximate commutators $[\hat{\bar{z}}, \hat{z}] \approx 2l_B^2(\boldsymbol{z})$, $[\hat{\eta}, \hat{\bar{\eta}}] \approx 2l_b^2(\boldsymbol{z})$. The velocity operators are then approximated as

$$\hat{\boldsymbol{U}} \approx \frac{\hbar}{m^*}\frac{\boldsymbol{n}\times(\hat{\boldsymbol{z}} - \hat{\boldsymbol{\eta}})}{l_b^2(\boldsymbol{z})} + \boldsymbol{V}, \tag{F15}$$

$$\hat{\boldsymbol{u}} \approx \frac{\hbar}{m^*}\frac{\boldsymbol{n}\times(\hat{\boldsymbol{z}} - \hat{\boldsymbol{\eta}})}{l_B^2(\boldsymbol{z})} + \boldsymbol{v}. \tag{F16}$$

Substituting Eq. (F16) into Eq. (F14) and summing over occupied states of composite fermions, we obtain the current density of vortices Eq. (63). The current density of electrons can be obtained similarly using Eq. (F15).

3. Dipole approximation

We can also obtain approximate expressions for the particle and current densities of electrons and vortices by differentiating the action Eq. (D3) with respect to $(\Phi_{\rm eff}, \boldsymbol{A})$ and $(\phi, \boldsymbol{a})$, respectively. The approximation corresponds to the multipole expansion discussed in Ref. [14]. We have

$$\rho_{\rm e}(\boldsymbol{\xi}, t) \approx \sum_i \tilde{\varphi}_i^*(\boldsymbol{\xi}, t)\varphi_i(\boldsymbol{\xi}, t) - \partial_{\bar{\xi}}\bar{P}(\boldsymbol{\xi}, t), \tag{F17}$$

$$\rho_{\rm v}(\boldsymbol{\xi}, t) \approx \sum_i \tilde{\varphi}_i^*(\boldsymbol{\xi}, t)\varphi_i(\boldsymbol{\xi}, t) + \partial_{\xi}P(\boldsymbol{\xi}, t), \tag{F18}$$

and

$$\bar{j}_{\rm e}(\boldsymbol{\xi}, t) \approx \sum_i \left[\frac{-2{\rm i}\hbar\partial_{\bar{\xi}} + e\mathcal{A} + m^* v}{m^*}\tilde{\varphi}_i\right]^* \varphi_i + \partial_t\bar{P}, \tag{F19}$$

$$j_{\rm v}(\boldsymbol{\xi}, t) \approx \sum_i \tilde{\varphi}_i^* \left[\frac{-2{\rm i}\hbar\partial_{\bar{\xi}} + e\mathcal{A} + m^* V}{m^*}\varphi_i\right] - \partial_t P, \tag{F20}$$

where P and $\bar{P}$ are the complex components of the dipole density, approximated as

$$P(\boldsymbol{\xi}, t) \approx -l_b^2(\boldsymbol{\xi}) \sum_i \tilde{\varphi}_i^* \Big(2\partial_{\bar{\xi}} + {\rm i}\frac{e}{\hbar}\mathcal{A}\Big)\varphi_i. \tag{F21}$$

4. Vanishing dipole density

We can show that a system of composite fermions always has a vanishing dipole density in the long-wavelength limit. To see this, we apply the self-consistent condition Eq. (61), and find that the first term of the current density Eq. (63) becomes an anomalous Hall current with a half-quantized Hall conductance $\sigma_{xy}^{(\rm v)} = -e^2/2h$ [14]. Comparing the current density to the second CS constraint Eq. (62), we have

$$\boldsymbol{P}(\boldsymbol{\eta}) \equiv w_b(\boldsymbol{\eta}) \sum_i \int {\rm d}\mu_B(z)(z - \boldsymbol{\eta})|\psi_i(z, \bar{\eta})|^2 \approx 0, \tag{F22}$$

where we ignore the slow spatial variation of $1/l_B^2(z)$.

The vanishing dipole density suggests that, on average, the coordinate of an electron, always coincides with the coordinate of the vortex to which it is bound. The same identity has also been found in the semiclassical theory [14]. In Ref. [16], it was considered that this condition could replace the CS self-consistent conditions and serve as the basis for a composite fermion theory without the CS fields.

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