Classification of same-gate quantum circuits and their space-time symmetries with application to the level-spacing distribution

Urban Duh and Marko Žnidarič

Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia

(Received 23 January 2024; accepted 27 March 2024; published 19 April 2024)

We study Floquet systems with translationally invariant nearest-neighbor two-site gates. Depending on the order in which the gates are applied on an N-site system with periodic boundary conditions, there are factorially many different circuit configurations. We prove that there are only $N-1$ different spectrally equivalent classes, which can be viewed either as a generalization of the brick wall or of the staircase configuration. Every class, characterized by two integers, has a nontrivial space-time symmetry with important implications for the level-spacing distribution—a standard indicator of quantum chaos. Namely, in order to study chaoticity one should not look at eigenphases of the Floquet propagator itself, but rather at the spectrum of an appropriate root of the propagator.

DOI: 10.1103/PhysRevResearch.6.023068

I. INTRODUCTION

Chaoticity and integrability are important theoretical notions. Integrability can allow for analytical results, while chaotic systems, in spite of unpredictability of trajectories, adhere to statistical laws. So-called toy models—the simplest models with a given property—play an important role. Classical single-particle H in one dimension (1D) is always integrable; one needs at least a 3D phase space for chaos to be possible. This can also be achieved already in 1D [1] by a time-dependent $H(t)$, the simplest case being a "kicked" system of form $H(t)=\frac{p^2}{2}+V(q)\tau\delta_\tau(t)$, where $\delta_\tau(t)$ is a train of delta functions. A canonical example is the standard map [2]. Similar logic of taking a Floquet propagator (map) works for quantum systems as well. For instance, one can exemplify single-particle quantum chaos with a kicked top [3,4]. Going to many-body quantum systems a plethora of possibilities opens up, one choice being a Floquet propagator that is a product of two simpler propagators, e.g., Ref. [5]. In light of experimental advances in noisy quantum simulations [6–8] it pays off to consider systems where the basic building block is a nearest-neighbor gate rather than the local Hamiltonian (applying two-site gates is also simpler in classical numerical simulations [9]).

We therefore focus on circuits where a one-unit-of-time Floquet propagator F is composed of applying *the same* two-site unitary gate V on all nearest-neighbor pairs of qubits in 1D—we call such systems *simple circuits*. Simple circuits have translational and temporal invariance, and, depending on the chosen gate V, span all dynamical regimes from integrability to chaos. Needles to say, such simple circuits have been extensively studied, a non-exhaustive list of only few of recent papers includes Refs. [10–20]. We show that any such circuit has a very simple form: it is a product of a local two-site transformation and a free propagation (translation), and that there are only $N-1$ spectrally inequivalent classes.

There are many indicators of quantum chaos with perhaps the most frequently used one being the so-called level-spacing distribution (LSD) $P(s)$ of nearest-neighbor eigenenergy spacing s [3]. According to the quantum chaos conjecture [21] Hamiltonian systems with a chaotic classical limit are expected to display $P(s)$ given by the random matrix theory (RMT) [22]. RMT LSD has also been observed in nonintegrable generic systems without a classical limit, of which simple circuits are an example, where it is sometimes even taken as a defining property of quantum chaos [23]. For Floquet systems checking for quantum chaos via $P(s)$ is even simpler: writing eigenvalues of F as $e^{i\phi_j}$ the density of eigenphases ϕ_j should be uniform and therefore taking for $s=(\phi_{j+1}-\phi_j)\mathcal{N}/2\pi$, where $\mathcal{N}$ is the Hilbert space dimension, there is no need for the unfolding that is required when studying spectra of H [3]. It is therefore rather surprising that, while there are hundreds of papers using $P(s)$ to study chaoticity in many-body Hamiltonian systems, there are essentially none studying LSD in simple (same-gate) quantum circuits (exceptions are recent Refs. [24,25]). The reason is that, surprisingly, LSD for chaotic simple circuits seemingly does not adhere to the RMT expectation. In our paper we will show that the reason behind it is a space-time symmetry that all such circuits posses.

Let us demonstrate that by a simple generalized brick-wall (BW) circuit with three layers (Fig. 1 inset), where we translate each two-site gate by three sites (instead of two as in BW). For periodic boundary conditions and N divisible by three the Floquet propagator can be written as $F=f_2f_3f_1$, where $f_j=\prod_{k=0}^{N/3-1}V_{j+3k,j+3k+1}$ is one layer beginning at site j and $V_{i,j}$ denotes the unitary two-site gate V acting on qubits i and j. Indices are taken modulo N, with sites $j=1,\dots,N$. Taking a two-qubit gate V to be some fixed generic unitary, and therefore having a system that should be quantum chaotic, we

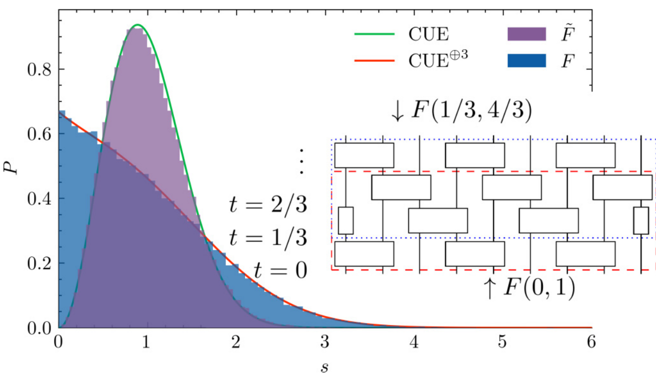

FIG. 1. A chaotic three-layer BW circuit (inset) and the eigenphases level-spacing distribution $P(s)$. Eigenphases of the propagator F (blue) do not follow the RMT expectation, while after resolving the space-time symmetry the eigenphases of the root $\tilde{F} = (S^6 F)^{1/3}$ (purple) do agree with the CUE RMT (green curve). Red curve is the theory for a direct sum of three CUE matrices [see Eq. (9) and Appendix B]. Data is for $N = 12$ in the eigenspace with momentum 0, and Haar random gate V.

can see in Fig. 1 that, after resolving the obvious translational symmetry, the LSD of F is far from the expected RMT result for a circular unitary ensemble (CUE) [3]. If anything, it is closer to a Poisson statistics typical of integrable systems, as if there would be some unresolved symmetry [26].

Indeed, each layer f_j, and thereby also F, is invariant under translation by three sites, $S^{-3} f_j S^3 = f_j$, where S is the translation operator by one site to the left,

$$V_{i+1,j+1} = S^{-1} V_{i,j} S. \tag{1}$$

We can also easily see (Fig. 1) that translating F by two sites is the same as a shift in time by one layer. Denoting propagator from time t_1 to t_2 by $F(t_1, t_2)$, e.g., $F = F(0, 1)$, the three-layer BW circuits has a space-time symmetry $S^{-2} F(0, 1) S^2 = F(1/3, 4/3)$ (application of each gate advances time by $1/N$). This symmetry is also reflected in the structure of F, which can be written as $F = S^{-4} f_1 S^4 S^{-2} f_1 S^2 f_1 = S^{-6}(S^2 f_1)^3 = S^{-6} \tilde{F}^3$. We now see where the crux of the problem lies. Since F as well as f_1 have translational symmetry by three sites the momentum k labeling eigenvalues of S^3 (also referred to as the quasimomentum, since it takes a discrete set of values) is a good quantum number. In each momentum eigenspace S^{-6} is just an overall phase, and therefore F is, up to this irrelevant phase, equal to the third power of $\tilde{F} = S^2 f_1$. The quantum chaos conjecture should therefore be applied to $\tilde{F}$ rather than to F. Doing that one recovers perfect agreement with the RMT (Fig. 1). It also tells us that, provided $\tilde{F}$ and therefore the circuit is "chaotic", the LSD of F will be equal to that of the third power of a CUE matrix, which is equal to a direct sum of three independent CUE matrices, Eq. (9).

The above example is just one possible simple circuit—in our classification it is of type $(q, r) = (3, 2)$. We shall classify symmetries of all possible simple quantum circuits, showing that all posses an appropriate space-time symmetry. Space-time symmetries have been discussed before in the solid-state physics context and time-periodic $H(t)$ [27–29]. An important offshoot will be expressing F essentially as a power of simpler matrix $\tilde{F}$, meaning that in order to probe dynamic (chaotic) properties one needs to study $\tilde{F}$ and not F. Expressions of that form have appeared before for special cases of the BW circuit in Ref. [30] [class (2,1)], and for $r = 1$ in Ref. [25], see also Ref. [24] for preliminary results.

II. CLASSIFICATION OF SIMPLE CIRCUITS

Having an N-site 1D system with periodic boundary conditions there are N nearest-neighbor gates $V_{j,j+1}$ that can be ordered in $N!$ different ways (configurations) to make a one-step propagator [31]. However, it is clear that many of those configurations have equal F since two-site gates acting on nonoverlapping nearest-neighbor sites commute. Furthermore, a lot of configurations lead to the same spectra, for instance, under cyclic permutations of gates spectra do not change [32]. Because we want to study spectra of F we will call two circuits *equivalent* if they have the same spectrum. It is clear that there are much less than $N!$ non-equivalent simple circuit classes. For open boundary conditions there is in fact just one class [32].

For periodic boundary conditions this is not the case. One can show (see Appendix, Theorem 2) that there are $(N - 1)$ different equivalence classes. A canonical representative circuit of a given equivalence class follows a similar logic as the three-layer BW example in the Introduction: A canonical circuit is characterized by two integers q and r, where the first layer of gates f_1 is made by repeatedly translating $V_{1,2}$ by q sites ($q = 3$ in the example), whereas r determines the shift of the second layer with respect to the first one ($r = 2$ in the example). The following theorem embodies the precise statement.

Theorem 1. Any simple qubit circuit on N sites with periodic boundary conditions is equivalent to exactly one of the $N - 1$ canonical simple qubit circuits having Floquet propagator

$$F_{q,r} = S^{-qr} (S^r (S^q V_{1,2})^{N/q})^q, \tag{2}$$

where q is larger than 1 and divides N, and q and r are coprime,

$$\begin{aligned} 2 \leqslant q \leqslant N, \quad \gcd(q, N) = q, \\ 1 \leqslant r < q, \quad \gcd(q, r) = 1, \end{aligned} \tag{3}$$

where gcd denotes the greatest common divisor.

The proof can be found in Appendix C. While it is not constructive in the sense of providing an explicit procedure of transforming a given F to its canonical form $F_{q,r}$, the transformation is in practice easily achieved by hand for small N, or one can in linear time calculate the invariant p introduced in Lemma 1 in Appendix C, thereby obtaining the correct (q, r). The integer invariant p is defined for an alternative simple circuit representative of a given class, and characterizes the circuit as a concatenation of two staircase sections with opposite chirality (Fig. 5 in Appendix), the length of the second being p.

The canonical form of $F_{q,r}$ in Eq. (2) has a simple geometric interpretation: the term in the inner bracket is a single layer f_1 that is composed of N/q gates, which is then with appropriate shifts repeated in altogether q layers, explicitly

TABLE I. List of $(N-1)$ allowed shift parameters (q,r) for few small N, classifying all possible spectrally equivalent N qubit circuits with periodic boundary conditions.

	Allowed (q,r)	
N	Generalized S	Generalized BW
6	(6,1),(6,5)	(2,1),(3,1),(3,2)
7	(7,1),(7,2),(7,3),(7,4),(7,5),(7,6)	
8	(8,1),(8,3),(8,5),(8,7)	(2,1),(4,1),(4,3)
9	(9,1),(9,2),(9,4),(9,5),(9,7),(9,8)	(3,1),(3,2)
10	(10,1),(10,3),(10,7),(10,9)	(2,1),(5,1),(5,2),(5,3),(5,4)

written as

$$F_{q,r} = \prod_{j=0}^{q-1} \prod_{i=0}^{N/q-1} V_{1+jr+iq,2+jr+iq}. \tag{4}$$

The $(N-1)$ classes described in Theorem 1 account for all possible circuits of which the standard brick wall with (2,1), and the staircase [32–34] (also called convolutional codes [35,36]) with $(N,1)$ are just two cases. The allowed values of (q,r) for few small N are listed in Table I, while their pictures are shown for $N=6$ in Fig. 2, and for $N=10$ in Fig. 4 in Appendix. Note that while the allowed set of (q,r) for a given N depends on the factors of N, the allowed Ns for a given (q,r) are simpler: taking any coprime $q>r$ a circuit is possible for all N that are multiples of q.

The set of allowed (q,r) naturally splits into two categories. Because q divides N, with the maximal value being N, one group is composed of the largest possible $q=N$, while the other has smaller $2 \leqslant q \leqslant N/2$. Group (i) are *generalized S* circuits with $q=N$. Because the translation by N is equivalent to the identity this group could be equivalently described by $(N,r) \equiv (r,0)$, that is, by a single integer r that gives the shift of the next gate. As r is coprime with N all nearest-neighbor gates are obtained by just this translation modulo N. Group (ii) can be viewed as *generalized BW* circuits and needs two integers. Because q divides N, translation by q alone does not generate all nearest-neighbor gates and one needs subsequent layers characterized by r. Altogether one has a q-layer BW circuit, each layer consisting of N/q gates. For a generic gate V the spectra of all $(N-1)$ propagators $F_{q,r}$ are different. If V would be symmetric with respect to the exchange of the two qubits the circuits with (q,r) and $(q,q-r)$ would have the same spectra (spatial reflection symmetry), and therefore one would have only $\lfloor \frac{N}{2} \rfloor$ different spectral classes [32]. Theorem 1 also shows that for prime N only generalized S circuits exist. For odd N there are no standard BW circuits having (2,1), but there are generalized BW circuits with $q>2$ (see Table I). It is interesting to note that circuits with more complex multisite update rules have been used before, for instance the (3,1) case [37] as well as (4,1) has been used to construct integrable models [38] (although with a three-site transformation). One interesting question is possible integrability of different canonical configurations for specific V. While for smaller $N<10$ possible circuits are straightforward generalizations of the S or BW configurations with left or right chirality, for larger N less intuitive circuits are also possible. For instance, for $N=10$ one can have (5,2) (see Fig. 4) that can be further compressed in time direction (e.g., all gates in the first two layers f_1 and f_3 commute), reducing the number of noncommuting layers from $q=5$ to just 3. Each of those compressed layers has two idle qubits (that are not acted upon), such that the compressed circuit $F^t_{5,2}$ has two separate diagonally slanted lines of idle qubits, each of width 1. Integers (q,r) therefore also determine the filling fraction, i.e., the number and the pattern of idle qubits in maximally compressed $F^t_{q,r}$ (see Appendix A). The only circuit with no idle qubits is the standard BW with (2,1).

III. SPACE-TIME SYMMETRIES

In order to understand space-time symmetries of any simple circuit it suffices to study the canonical equivalence class representatives $F_{q,r}$. Denoting the inner term in Eq. (2) by $\tilde{F}_{q,r}$, calling it a root of $F_{q,r}$,

$$\tilde{F}_{q,r} = S^r (S^q V_{1,2})^{N/q} = S^r f_1, \tag{5}$$

where $f_1 = \prod_{k=0}^{N/q-1} S^{-kq} V_{1,2} S^{kq}$, we can write Eq. (2) as

$$F_{q,r} = S^{-qr} (\tilde{F}_{q,r})^q. \tag{6}$$

Equation (6) appeared in Ref. [30] in the open-systems context for the special case of a BW circuit with (2,1). The root connection is especially simple for the generalized S case: for $q=N$ the translation S^q is identity, resulting in $\tilde{F}_{q,r} = S^r V_{1,2}$ and $F_{q,r} = (S^r V_{1,2})^N$.

The root has a translational symmetry by q sites

$$S^{-q} \tilde{F}_{q,r} S^q = \tilde{F}_{q,r}. \tag{7}$$

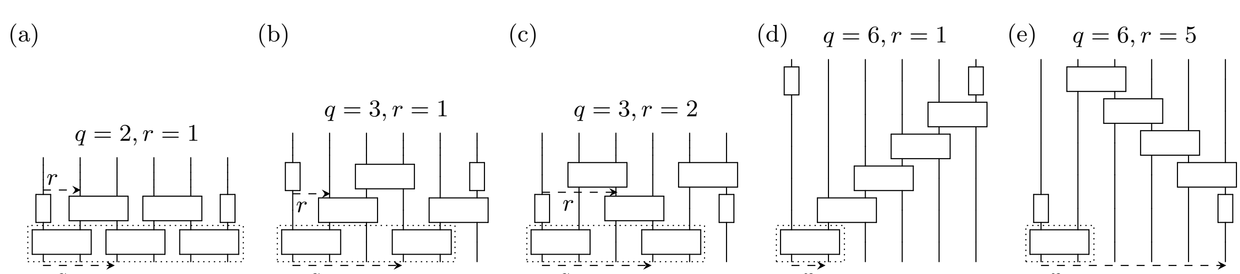

FIG. 2. Any same-gate nearest-neighbor circuit is spectrally equivalent to one of the canonical circuits $F_{q,r}$ shown here for $N=6$, see also Table I. Dashed rectangles denote f_1, one layer occurring in the root $\tilde{F}_{q,r}$ in Eq. (5). Circuits (a), (b), and (c) are generalized BW while (d) and (e) are generalized S. Circuits (b) and (c), as well as (d) and (e), are chiral pairs.

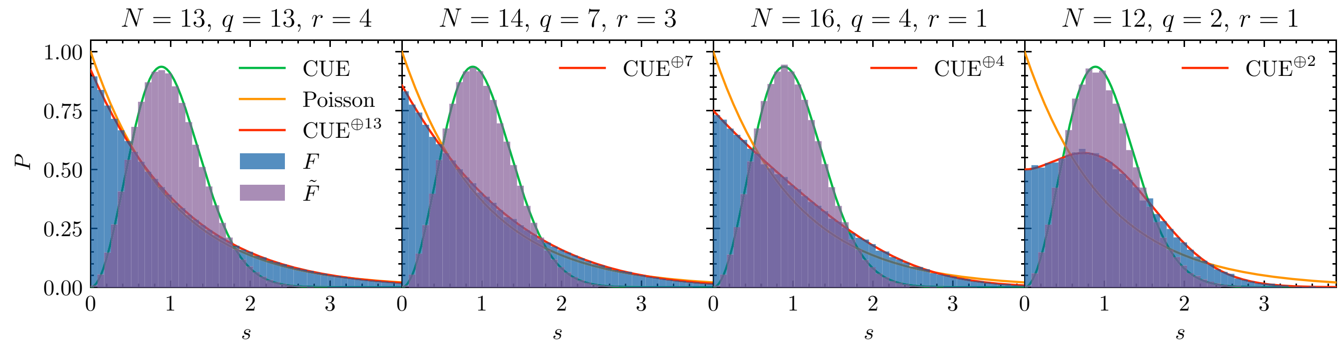

FIG. 3. Level-spacing distribution $P(s)$ of eigenphases of $F_{q,r}$ [Eq. (2)] in blue are not chaotic and are equal to a direct sum of q CUE matrices (red curve, Appendix B), while the LSD of the eigenphases of root $\tilde{F}_{q,r}$ [Eq. (5), violet] agrees with the CUE RMT prediction (green curve). For $q \neq N$ we show data from the momentum eigenspace of S^q with $k = 0$, and averaging is done over $10 - 50$ circuits each having an independent two-site Haar random gate V.

This is trivially true if $q = N$, otherwise the translated root is equal to $S^r \prod_{k=0}^{N/q-1} S^{-(k+1)q} V_{1,2} S^{(k+1)q}$, where, since all the gates in the product commute, we can relabel the index $k + 1 = k'$, yielding $S^r \prod_{k'=0}^{N/q-1} S^{-k'q} V_{1,2} S^{k'q} = \tilde{F}_{q,r}$. Because $F_{q,r}$ is a power of $\tilde{F}_{q,r}$ multiplied by some power of S^q, $F_{q,r}$ also has translational symmetry by q sites.

Furthermore, $F_{q,r}$ also has a space-time symmetry when translating by r sites

$$S^{-r} F_{q,r}(0,1) S^r = F_{q,r}(1/q, 1 + 1/q). \tag{8}$$

This can be easily seen by rewriting the left-hand side (LHS) of Eq. (8) as $S^{-qr}(S^r S^{-r}(S^q V_{1,2})^{N/q} S^r)^q$, which then equals to $S^{-qr}(S^r(S^q V_{1+r,2+r})^{N/q})^q$. The final expression can be understood as a circuit beginning with the second layer, thus justifying the equality to the right-hand side (RHS) of Eq. (8). Equation (8) appeared in Ref. [25] for the special case of $r = 1$, along with a figure showing numerically computed LSD of $F_{q,r}$ for $q = 2, r = 1$.

IV. LEVEL-SPACING STATISTICS

An immediate application of the above results is in quantum chaos for the statistics of spacings of closest eigenphases of F. Looking at the root connection in Eq. (6) and the fact that S^q commutes with all terms, one can focus on a given common momentum eigenspace of S^q with eigenvalues $e^{2\pi i k q/N}$, $k \in \{0, 1, \ldots, N/q - 1\}$. There S^{-qr} is just an overall phase factor $e^{-2\pi i q k r/N}$. Therefore $F_{q,r}$ is up to this phase equal to an appropriate power of $\tilde{F}_{q,r}$.

The eigenphases of $F_{q,r}$ are therefore simple qth multiples of eigenphases of $\tilde{F}_{q,r}$ modulo 2π (and adding the momentum phase factor). For high q such an operation will results in an uncorrelated Poisson statistics of eigenphases of $F_{q,r}$ [39], i.e., an exponential distribution of $P(s)$, and to infer possible quantum chaos one should not look at the eigenphases of $F_{q,r}$. Rather, if the circuit is quantum chaotic one would expect that the spectral statistics of the root $\tilde{F}_{q,r}$ (5) will adhere to the RMT theory. In particular, if the two-site gate V does not have any anti-unitary (time-reversal) symmetry, which is the case for our numerics where V is randomly picked according to the Haar measure, the appropriate ensemble for $\tilde{F}_{q,r}$ is the CUE (i.e., the unitary Haar measure). We can see in Fig. 3 that the LSD of $\tilde{F}_{q,r}$ indeed agrees with CUE Wigner surmise $P(s) = \frac{32}{\pi^2} s^2 e^{-s^2 4/\pi}$ for all canonical classes.

If one is on the other hand interested in the eigenphases of $F_{q,r}$ one has to take into account the nontrivial modulo 2π operation. The theorem by Rains [40] tells us what

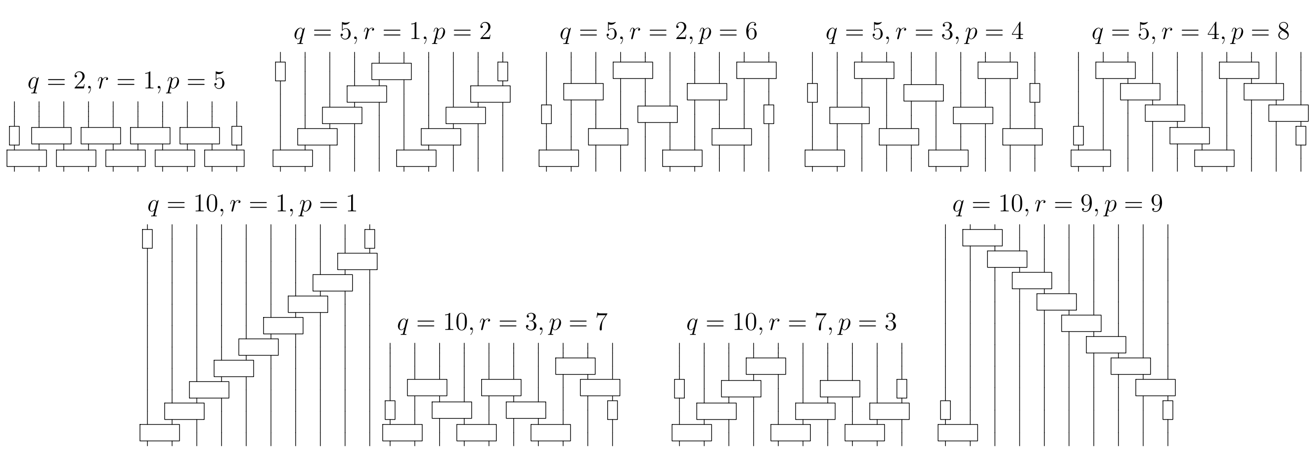

FIG. 4. All allowed $F_{q,r}$ defined in Eq. (2) for $N = 10$. The parameter p of the equivalent F_p [Eq. (C7)] is also included. Here $C(F_{q,r}) = p$, see discussion in Sec. 2 of Appendix C.

is the distribution of eigenvalues of a power of a matrix from the unitary Haar measure. For $M \in \mathrm{U}_{\mathcal{N}}$, where $\mathrm{U}_{\mathcal{N}}$ denotes the Haar distribution of $\mathcal{N} \times \mathcal{N}$ unitary matrices, the eigenvalues of its power M^q are a union of q independent eigenvalue sets distributed according to $\mathrm{U}_{\mathcal{N}_j}$ of smaller matrices,

$$\lambda(M^q) \sim \bigcup_{j=0}^{q-1} \lambda(\mathrm{U}_{\mathcal{N}_j}), \quad \mathcal{N}_j = \left\lceil \frac{\mathcal{N}-j}{q} \right\rceil, \tag{9}$$

for any $q < \mathcal{N}$, $\lceil\bullet\rceil$ denotes the ceiling function, and $\lambda(\mathrm{U}_{\mathcal{N}})$ is the distribution of eigenvalues of Haar-random unitary matrices of size $\mathcal{N}$. Note that $\mathcal{N} = \sum_{j=0}^{q-1} \mathcal{N}_j$, and therefore, as far as the eigenvalue distribution is concerned, it is as if M^q would have a block diagonal structure with blocks of smaller CUE matrices. In our case of large $\mathcal{N}$ and small $q \ll \mathcal{N}$ all dimensions in the union are approximately equal to $\mathcal{N}_j \approx \mathcal{N}/q$, which means that the level-spacing distribution in each eigenspace of S^q of size $\mathcal{N} \approx 2^N q/N$ of $F_{q,r}$ behaves in the same way as if it had another unitary symmetry with q distinct sectors [26]. Theoretical LSD in such a case of a sum of q independent spectra is known and has been studied long time ago [41], see also Appendix B (or Appendix A in Ref. [22]). We can see in Fig. 3 that this theory agrees with numerical LSD for $F_{q,r}$ [42].

V. DISCUSSION

We have classified all different quantum circuits in one dimension with periodic boundary conditions and translationally invariant nearest-neighbor two-site gates. There are $(N-1)$ different spectral classes, being generalizations of the familiar brick-wall and the staircase configurations. Each class can be characterized by two integers (q, r), such that the Floquet propagator is essentially a qth power of $\tilde{F} = S^r f_1$, where for generalized S circuits one has $f_1 = V_{1,2}$, while for generalized BW circuits f_1 is one layer of gates. We have therefore come full circle: similarly as in classical single-particle kicked models where one interchangeably applies a simple map in real space [e.g., potential $V(q)$] and a simple map in momentum space (e.g., free evolution), any quantum many-body translationally invariant Floquet system has the same basic structure. The elementary building block $\tilde{F}$ is a product of simple local transformation, like $V_{1,2}$, and of "free evolution" described by the translation operator S (that is diagonal in the Fourier basis).

We have explicitly shown how that affects the level-spacing statistics of simple circuit Floquet systems—to detect quantum chaos one must look at the spectrum of the qth root $\tilde{F}$ of the propagator. Effects of the underlying space-time symmetry on other quantifiers of quantum chaos remain to be explored. Reducing all circuits to just few canonical classes makes it possible to study chaoticity for different (q, r); are some configurations more chaotic than others, does that depend on the filling fraction (see Appendix A)? While we focused on systems without any symmetry, i.e., the unitary case, orthogonal and symplectic cases can be treated along the same lines. Generalizing classification to more than one dimension is also an open problem.

ACKNOWLEDGMENT

We acknowledge support by Grants No. J1-4385 and No. P1-0402 from the Slovenian Research Agency (ARIS).

APPENDIX A: CANONICAL CIRCUITS

In the main text in Fig. 2 we have shown canonical circuits $F_{q,r}$ for $N = 6$. Here we show in Fig. 4 all nine canonical $F_{q,r}$ for $N = 10$, where a variety of configurations is richer. We can for instance notice that even though circuits are constructed as a q-layered circuit in some cases consecutive layers commute and can therefore be compressed, thus reducing the number of layers. For $N = 10$ this is the case for $(q, r) = (5, 2)$ and its chiral pair (5,3). For (5,2) the first two layers f_1 and f_3 commute and can be compressed to a single layer; likewise for the next two layers. Therefore in the compressed form $(F_{5,2})^2$ consists of only five layers instead of 10. In this compressed form there are still two idle qubits in each layer on which no gate acts. One can say that the filling fraction of gates for the circuit (5,2) is 8/10, i.e., 20% of qubits is idle. The class (5,1) on the other hand cannot be compressed any further and has the filling fraction of only 4/10. The only circuit with filling fraction 1 is the standard BW with (2,1).

APPENDIX B: LEVEL-SPACING DISTRIBUTION OF A DIRECT SUM OF INDEPENDENT RMT MATRICES

The general formula for the LSD of a direct sum of independent RMT matrices of arbitrary dimensions was derived in Ref. [41]. The matrices in the union in the theorem by Rains [40], Eq. (9), are approximately of equal dimensions in the large N limit, which is why we use the formula for the direct sum of equally dimensional matrices throughout this paper.

For the LSD $P(s)$ of an RMT ensemble we define

$$R(y) = \int_0^\infty P(x+y)\,dx, \tag{B1}$$

$$D(y) = \int_0^\infty xP(x+y)\,dx, \tag{B2}$$

where $R(y)$ is nothing but 1 minus the cumulative distribution of P. The LSD for a direct sum of m independent equally dimensional matrices from the same RMT ensemble is then

$$P_m(s) = D^m(s/m)\left[\frac{1}{m}\frac{P(s/m)}{D(s/m)} + \left(1 - \frac{1}{m}\right)\frac{R^2(s/m)}{D^2(s/m)}\right]. \tag{B3}$$

In figures in the paper, we plot $P_m(s)$ obtained by using the Wigner surmise for $P(s)$, which is of acceptable accuracy for our application. For better approximations of $P(s)$ see Ref. [46].

APPENDIX C: PROOF OF THEOREM 1 FROM THE MAIN TEXT

In this Appendix, we set to prove Theorem 1, the main result of this paper. The proof is divided into subsections containing more theorems and lemmas. First, we introduce another canonical configuration in Sec. 1, which can be geometrically interpreted as a concatenation of two staircase circuits with opposite chiralities. It is useful for determining that the number of non-equivalent circuits is $(N-1)$ and later used in the proof of Theorem 1. After that in Sec. 2, we introduce the quantity $C(F)$ invariant for equivalent circuits and state its important properties. This invariant is crucial in later proofs but is also useful on its own to efficiently determine the canonical form of a given circuit. Finally, we focus on the generalized S/BW circuits $F_{q,r}$ in Sec. 3, where we combine the results from previous sections to prove Theorem 1.

To keep our notation clearer, we identify a product of N two-site gates acting on nearest-neighbor sites with a sequence of N numbers in the following way:

$$V_{i_N,i_N+1}\cdots V_{i_2,i_2+1}V_{i_1,i_1+1}\equiv(i_1,i_2,\dots,i_N). \tag{C1}$$

We will refer to i_l as gate numbers and l as time indices. When talking about the gates appearing before/after a gate i_l in a given F, we will call i_{l+1} the time successor and i_{l-1} the time predecessor, whereas when referring to gates with neighboring numbers, we will call i_l-1 the left neighbor and i_l+1 the right neighbor of i_l. By definition, the sequences corresponding to simple circuits are permutations of the first N natural numbers. For the canonical simple circuits defined in the paper in Eq. (2),

$$F_{q,r}\equiv(1,1+q,\dots,1+(N/q-1)q,\\ 1+r,1+r+q,\dots), \tag{C2}$$

where all gate numbers are taken modulo N (from 1 to N).

We are interested in (spectrally) equivalent circuits, as defined in the paper. The equivalence will be denoted with $\cong$. The two important equivalence operations are time predecessor/successor commutation of non-neighboring gates (here the Floquet operators are actually equal, not only equivalent)

$$|i_k-i_{k+1}|>1 \pmod N \\ \Rightarrow V_{i_N,i_N+1}\cdots V_{i_{k+1},i_{k+1}+1}V_{i_k,i_k+1}\cdots V_{i_1,i_1+1} \\ =V_{i_N,i_N+1}\cdots V_{i_k,i_k+1}V_{i_{k+1},i_{k+1}+1}\cdots V_{i_1,i_1+1}, \tag{C3}$$

in the new notation written as

$$|i_k-i_{k+1}|>1 \pmod N \quad\Rightarrow\\ (i_1,\dots,i_k,i_{k+1},\dots,i_N)\cong(i_1,\dots,i_{k+1},i_k,\dots,i_N), \tag{C4}$$

and cyclic permutation

$$V_{i_N,i_N+1}\cdots V_{i_2,i_2+1}V_{i_1,i_1+1}\\ \cong V_{i_1,i_1+1}V_{i_N,i_N+1}\cdots V_{i_2,i_2+1}, \tag{C5}$$

or in the new notation

$$(i_1,i_2,\dots i_N)\cong(i_2,\dots,i_N,i_1). \tag{C6}$$

While the equivalence of cyclically permuted circuits was motivated by spectral equivalence, a different, perhaps more general, motivation is also possible. If we have dynamics in mind, i.e., F^t with possibly large t, the definition of a starting time of our period is arbitrary, e.g., the first operator in a period could just as well have been defined as the last operator in the previous period. Therefore, it would also make sense to define cyclically permuted circuits to be equivalent in this case.

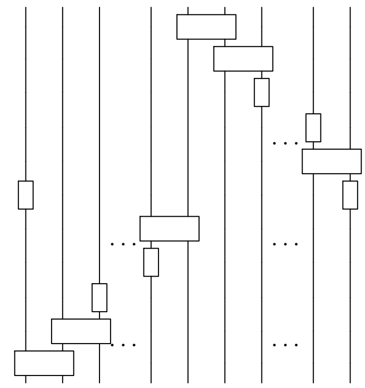

FIG. 5. Diagram of F_p defined in Eq. (C7).

1. Double staircase canonical circuits F_p

We now introduce a new canonical form, different from the one in the main paper, its diagram is shown in Fig. 5. It consists of two staircase sections with opposite chirality, the second having length p. To show that it is indeed a canonical form, i.e., that every simple circuits is equivalent to some circuit in this form, we can prove the following Theorem.

Theorem 2. Any simple circuit is equivalent to a simple circuit with the Floquet operator given by

$$F_p=V_{N-p+1,N-p+2}\cdots V_{N-1,N}V_{N,1}\\ \times V_{N-p,N-p+1}\cdots V_{2,3}V_{1,2}\\ \equiv(1,2,\dots,N-p,N,N-1,\dots,N-p+1), \tag{C7}$$

for some $p\in\{1,2,\dots,N-1\}$.

Proof. Let $F=(i_1^{(0)},i_2^{(0)},\dots,i_N^{(0)})$ be any simple circuit. This means that $i_k^{(0)}\neq i_l^{(0)}$ for $k\neq l$. First, we will bring gate 1 to index 1 (*step 1*). After that, we will try to bring gate 2 to index 2 (*step 2*). This will either be possible, in which case we will continue to try bring gate k to index k in the *general step*, or it won't be possible, in which case our gate will be equivalent to F_{N-1}.

Step 1: By means of cyclic permutations (C6) we can always transform F into an equivalent simple circuit beginning with gate 1 ($V_{1,2}$ in the standard notation)

$$F\cong\big(1,i_2^{(1)},\dots,i_N^{(1)}\big), \tag{C8}$$

where we have appropriately relabeled the indices.

Step 2: We now try to bring gate $2=i_K^{(1)},K\in\{2,\dots N\}$ to the second position by doing time predecessor/successor commutations (C4). Gate 2 does not commute only with gates 1 and 3, which means that we must consider two possibilities:

(i) Gate 3 appears after gate 2, $3=i_L^{(1)}$ for $L>K$:
By applying (C4), we can bring gate 2 to index 2

$$F\cong\big(1,2,i_3^{(2)},\dots,i_N^{(2)}\big), \tag{C9}$$

where we have again relabelled the indices. We can continue with the *general step*.

(ii) Gate 3 appears before gate 2, $3=i_L^{(1)}$ for $L<K$:

By applying (C4) (and relabelling the indices), we can bring gate 2 to be the time successor of gate 3,

$$F \cong \left(1, i_2^{\prime(1)}, \ldots, 3, 2, \ldots, i_N^{\prime(1)}\right). \tag{C10}$$

In the context of equivalence transformations, we can now think of the sequence of gates $(3, 2) = (i_{K'}^{\prime(1)}, i_{K'+1}^{\prime(1)})$ as a gate that does not commute only with gates 1 and 4. Again, we have two similar cases:

(a) Gate 4 appears after the sequence of gates (3, 2), $4 = i_{L'}^{\prime(1)}$ for $L' > K' + 1$:
By applying (C4), we can bring gate 3 to index 1 and gate 2 to index 3,

$$F \cong (3, 1, 2, \ldots) \tag{C11}$$

and by applying (C6), we can cyclically permute gate 3 to index N,

$$F \cong \left(1, 2, i_3^{(2)}, \ldots, i_{N-1}^{(2)}, 3\right). \tag{C12}$$

We can now continue with the *general step*.

(b) Gate 4 appears before (3, 2), $4 = i_{L'}^{\prime(1)}$ for $L' < K'$:
By applying (C4)

$$F \cong (1, \ldots, 4, 3, 2, \ldots). \tag{C13}$$

We can again think of (4, 3, 2) as a gate that does not commute only with gates 5 and 1 and again consider two cases similar to (a) and (b). By repeating this process, we either get to the point where case (a) arises, and we can continue with the *general step*, or our circuit is equivalent to

$$F \cong (1, N, N-1, \ldots, 2). \tag{C14}$$

Here, the obtained equivalent circuit is thus precisely F_{N-1}.

General step: Let us suppose we have already shown that F is equivalent to

$$F \cong \left(1, 2, \ldots, k, i_{k+1}^{(k)}, \ldots, i_N^{(k)}\right), \tag{C15}$$

where $i_l^{(k)}$ are arbitrary indices relabelled in a convenient way. We can treat the sequence of gates $(1, 2, \ldots, k)$ as a gate, that does not commute only with N and $k+1$, which means that we can repeat step 2 by trying to bring gate $k+1$ to the right of k (index $k+1$). Thus, either

(i) $F \cong (1, \ldots, k, k+1, \ldots)$ or

(ii) $F \cong (1, \ldots, k, N, N-1, \ldots, k+1) = F_{N-k}$.

In (i) we can repeat the *general step* until case (ii) arises, or we end up with $F \cong (1, 2, \ldots, N) = F_1$. ■

Since there are $(N-1)$ allowed p, it is clear that there are (at most) $(N-1)$ non-equivalent simple circuits. The proof given is constructive, which means that we can use it as an algorithm to convert a given F to some F_p.

2. Circuit invariant

An alternative way to determine the equivalent F_p for a given F is to calculate some quantity, which is invariant in equivalence operations and is different for all F_p. A convenient choice is the length of the second staircase p, which is clearly equal to the number of gates, for which their right neighbor (modulo N) appears in the Floquet operator before them. Let us thus define $C(F)$ to be exactly that

$$\begin{aligned} C(i_1, \ldots, i_N) = |\{i_l, l \in \{1, \ldots, N\}; \exists i_k, \quad k < l : \\ i_k - i_l \equiv 1 \pmod N\}|, \end{aligned} \tag{C16}$$

where $|\bullet|$ denotes the cardinality of a set. As stated, $C(F_p)$ is clearly equal to the length of the second staircase

$$C(F_p) = |\{N, N-1, \ldots, N-p+1\}| = p. \tag{C17}$$

Let us now show that $C(F)$ is indeed invariant under circuit equivalence operations.

Lemma 1. For simple circuits $C(F)$ is invariant under circuit equivalence operations (C4) and (C6).

Proof. Let

$$\begin{aligned} \tilde{C}(i_1, \ldots, i_N) = \{i_l, l \in \{1, \ldots, N\}; \exists i_k, \quad k < l : \\ i_k - i_l \equiv 1 \pmod N\}, \end{aligned} \tag{C18}$$

be the set of gates the invariant is counting, which means $C(i_1, \ldots, i_N) = |\tilde{C}(i_1, \ldots i_N)|$.

As stated previously, for simple circuits $(i_1, \ldots, i_N)$ is just a permutation of $(1, \ldots, N)$, which means

$$i_1 \notin \tilde{C}(i_1, \ldots, i_N), \text{ because nothing is before } i_1, \tag{C19}$$

$$i_1 - 1 \in \tilde{C}(i_1, \ldots, i_N), \text{ because } i_1 - 1 \in \{i_j\}_{j=2}^N, \tag{C20}$$

where indices are taken modulo N. But in the equivalent cyclically permuted circuit (C6)

$$i_1 \in \tilde{C}(i_2, \ldots, i_N, i_1), \text{ because } i_1 + 1 \in \{i_j\}_{j=2}^N, \tag{C21}$$

$$i_1 - 1 \notin \tilde{C}(i_2, \ldots, i_N, i_1). \tag{C22}$$

The membership of all other gates in $\tilde{C}$ stays the same in both cases, since the position of i_1 can change only the membership of $i_1 - 1$ and i_1 in $\tilde{C}$ and all other relative positions stay the same. Thus

$$C(i_1, \ldots, i_N) = C(i_2, \ldots, i_N, i_1), \tag{C23}$$

which means that C is conserved under cyclical permutations (C6).

A transposition of time successive gates can only change C if their gate number difference is ± 1, which does not happen if they commute [Eq. (C4)]. Therefore, C is conserved there

$$\begin{aligned} &|i_k - i_{k+1}| > 1 \pmod N \quad \Rightarrow \\ &C(i_1, \ldots, i_k, i_{k+1}, \ldots, i_N) = C(i_1, \ldots, i_{k+1}, i_k, \ldots, i_N). \end{aligned} \tag{C24}$$

■

An obvious consequence of Theorem 2 is the following lemma.

Lemma 2. For simple circuits $C(F) \in \{1, \ldots, N-1\}$.

Proof. According to Theorem 2, any simple circuit F is equivalent to some F_p for $p \in \{1, \ldots, N-1\}$. Since $C(F)$ is conserved under equivalence transformations (Lemma 1) and $C(F_p) = p, C(F) \in \{1, \ldots, N-1\}$. ■

3. Generalized S/BW canonical circuits $F_{q,r}$

We now turn to the canonical form $F_{q,r}$. For a given $N \in \mathbb{N}$, define the set of allowed (q, r) pairs (see Theorem 1)

$$Q_N = \{(q, r);\ 2 \leqslant q \leqslant N, \quad 1 \leqslant r < q, \quad \gcd(q, N) = q, \quad \gcd(q, r) = 1\}. \tag{C25}$$

The allowed (q, r) generate either a generalized S circuit

$$Q_N^{(\mathrm{S})} = \{(N, r); 1 \leqslant r < N, \quad \gcd(N, r) = 1\} \tag{C26}$$

or a generalized BW circuit

$$Q_N^{(\mathrm{BW})} = \{(q, r);\ 2 \leqslant q < N, \quad 1 \leqslant r < q, \quad \gcd(q, N) = q, \quad \gcd(q, r) = 1\}, \tag{C27}$$

which means that $Q_N = Q_N^{(\mathrm{S})} \cup Q_N^{(\mathrm{BW})}$, where $Q_N^{(\mathrm{S})} \cap Q_N^{(\mathrm{BW})} = \emptyset$.

The first thing to check is that $F_{q,r}$ is well defined, i.e., if Floquet operators $F_{q,r}$ actually correspond to simple circuits.

Lemma 3. $F_{q,r}$ for $N \in \mathbb{N}$ and $(q, r) \in Q_N$ are simple circuits, i.e., contain every gate on neighboring sites exactly once.

Proof. To show that $F_{q,r}$ are simple circuits, we thus have to check that the sequence (C2) contains every positive integer less than or equal to N exactly once.

Let us first consider the generalized S case, where $q = N$ and $\gcd(N, r) = 1$. Because of that, it is intuitively clear that translations by r must be "ergodic" and thus generate all gates. More formally this means that the least common multiple $\mathrm{lcm}(N, r) = rN/\gcd(N, r) = rN$. Since $F_{N,r}$ is generated by translating gate 1 by r sites, the lowest number of translations after which gate 1 (modulo N) is generated again must be N [since the smallest solution to $kr = 0 \pmod N$ is exactly $k = \mathrm{lcm}(N, r)/r$]. Thus, before looping back to gate 1 we generate N gates. By an analogous argument, any gate is generated again only after N translations, which means that all the generated gates are different.

Let us now consider the generalized BW case where $q \neq N$. In this case $\mathrm{lcm}(q, N) = N$, which means that with translations by q we generate N/q different gates. These are exactly gates 1 plus all the multiples of q less than or equal to N, so gates $1 + qk$, $k = 0, \dots, N/q - 1$. By translating them by any r, such that $1 \leqslant r < q$, we generate all gates $1 + r + qk$, which since $r < q$ cannot be equal to any gates generated with $r = 0$. More formally, this is true because $1 + r + qk \not\equiv 1 + qk \pmod q$ necessarily implies $1 + r + qk \neq 1 + qk$. In order to generate all gates, $1 + kr$ for $k \in \{0, 1, 2, \dots, q - 1\}$ must now take all the possible values modulo q. Analogous to the generalized S case, this indeed does happen when $\gcd(q, r) = 1$. ■

Since Theorem 2 implies that there are $(N - 1)$ nonequivalent simple circuits, a required condition for $F_{q,r}$ to be a canonical form is that there are $(N - 1)$ allowed (q, r).

Lemma 4. $|Q_N| = N - 1$.

Proof. Let $\varphi(N) = |\{k; 1 \leqslant k < N, \gcd(N, k) = 1\}|$ denote the number of natural numbers less than N and coprime with N, also called the Euler's totient or Euler's phi function.

We can now express the number of generalized staircase circuit with Euler's phi function

$$\left|Q_N^{(\mathrm{S})}\right| = |\{(N, r); 1 \leqslant r < N, \quad \gcd(N, r) = 1\}| = \varphi(N) \tag{C28}$$

and also the number of generalized BW circuits

$$\begin{aligned}\left|Q_N^{(\mathrm{BW})}\right| &= |\{(q, r);\ 2 \leqslant q < N, \quad 1 \leqslant r < q, \quad \gcd(q, N) = q, \quad \gcd(q, r) = 1\}| \\ &= \sum_{q=2, q|N}^{N-1} |\{(q, r); 1 \leqslant r < q, \quad \gcd(q, r) = 1\}| \\ &= \sum_{q=2, q|N}^{N-1} \varphi(q),\end{aligned} \tag{C29}$$

where $q|N$ denotes that q divides N [equivalently $\gcd(q, N) = q$].

Thus, since $Q_N^{(\mathrm{S})} \cap Q_N^{(\mathrm{BW})} = \emptyset$,

$$\begin{aligned}|Q_N| &= \left|Q_N^{(\mathrm{S})}\right| + \left|Q_N^{(\mathrm{BW})}\right| = \sum_{q=2, q|N}^{N} \varphi(q) \\ &= \sum_{q|N} \varphi(q) - \varphi(1) = N - 1,\end{aligned} \tag{C30}$$

where we used the well-known theorem that $\sum_{q|N} \varphi(q) = N$ [47] and $\varphi(1) = 1$. ■

We now wish to show that the invariant $C(F_{q,r})$ is different for all allowed (q, r), which is the last important statement required for the proof of Theorem 1. We do this in two steps, first for the generalized S in Lemma 5 and then for generalized BW in Lemma 6, finally combining both in Lemma 7.

Lemma 5. In the generalized S case, $(q, r) \in Q_N^{(\mathrm{S})}$,

$$C(F_{q,r}) = 1 + |\{\text{gates appearing after gate } N\}| \in \{p; p < N, \quad \gcd(p, N) = 1\}, \tag{C31}$$

and is different for different r. Here all the values in the set (C31) are taken for some allowed r.

Proof. Since we are considering the generalized S case: $q = N$, $r < N$ and $\gcd(N, r) = 1$.

We first want to consider if we know that a certain gate is a member of $\tilde{C}(F_{N,r})$, what can we say about the membership of its time successor and predecessor. Let $F_{N,r} = (i_1, \dots, i_N)$. Let $i_l \in \tilde{C}(F_{N,r})$, which means that its right neighbor appears in $F_{N,r}$ before it, $\exists i_k = i_l + 1, k < l$ (gate numbers are taken modulo N, but indices are not). Here $l \neq 1$, since in that case, such $k < l = 1$ never exists. Let us now consider the membership of i_l's successor and predecessor:

(a) If $i_{l+1} = i_l + r$ is a valid gate (i.e., its index is valid, which means $l < N$), then $i_{k+1} = i_k + r = i_l + 1 + r = i_{l+1} + 1$, is the right neighbor of i_{l+1} and $k + 1 < l + 1$, which implies $i_{l+1} \in \tilde{C}(F_{N,r})$. In other words: *If i_l's time successor exists (i.e., i_l is not the last gate in F), it is also a member of $\tilde{C}(F_{N,r})$.*

(b) $i_{l-1} = i_l - r$, then if $i_{k-1} = i_k - r = i_l + 1 - r = i_{l-1} + 1$ is a valid gate (i.e., $k > 1$), it is the right neighbor of i_{l-1} and $k - 1 < l - 1$, so clearly $i_{l-1} \in \tilde{C}(F_{N,r})$. In other words: *If i_l's right neighbor is not i_1, then it's time predecessor is also a member of $\tilde{C}(F_{N,r})$.*

In case (b) i_{k-1} is not valid only if $k = 1$. In the case of $F_{q,r}$, $i_1 = 1$ and thus $i_l = N$. Since always $N \in \tilde{C}(F_{N,r})$ (its right neighbor is $1 = i_1$), by iterating (a), all gates appearing after gate N are also members of $\tilde{C}(F_{N,r})$. If any gate appearing before gate N would be a member of $\tilde{C}(F_{N,r})$, iterating (b) would eventually lead to $i_1 = 1 \in \tilde{C}(i_1, \ldots, i_N)$, which cannot happen, thus leading to a contradiction. We have shown

$$C(F_{N,r}) = 1 + |\{\text{gates appearing after gate } N\}| = = N - k_N + 1, \tag{C32}$$

where k_N is the index of gate N, $i_{k_N} = N$. We have thus shown the first part of the lemma.

We now want to determine what are the possible values of $C(F_{N,r})$. In order to do that, we must only find the possible values of k_N. By definition, we get gate N after $k_N - 1$ translations of gate 1 by r,

$$1 + (k_N - 1)r \equiv 0 \pmod N, \quad \Rightarrow -(k_N - 1)r \equiv 1 \pmod N. \tag{C33}$$

Thus, $-r$ is a modular multiplicative inverse of $k_N - 1$. According to a well known theorem [47], a requirement for modular multiplicative inverses to exists is $\gcd(k_N - 1, N) = 1$. Therefore

$$\gcd(C(F_{N,r}), N) = \gcd(N - k_N + 1, N) = \gcd(k_N - 1, N) = 1. \tag{C34}$$

In other words, $C(F_{N,r})$ is coprime with N. According to Lemma 2, $C(F) \leqslant N - 1$, which means that $C(F_{N,r})$ can only be a number less than N and coprime with N,

$$C(F_{N,r}) \in \{p; p < N, \gcd(p, N) = 1\}. \tag{C35}$$

Let us now show that k_N is different for different $F_{N,r}$ and $F_{N,\tilde{r}}$ by contradiction. Let us suppose otherwise, this means that $k_N - 1$ translations by r and by $\tilde{r}$ must generate the same gate number modulo N,

$$(k_N - 1)r \equiv (k_N - 1)\tilde{r} \pmod N. \tag{C36}$$

Since $\gcd(k_N - 1, N) = 1$, we can divide the equation by $k_N - 1$ [47],

$$r \equiv \tilde{r} \pmod N. \tag{C37}$$

By definition of allowed $F_{N,r}$ circuits, $r, \tilde{r} < N$, which means that $r = \tilde{r}$, leading to a contradiction.

We have thus shown that for different allowed r, $C(F_{N,r})$ are different. Moreover, in Eq. (C35) we have shown that the value of $C(F_{N,r})$ is a member of a set with exactly the same number of elements as allowed r (see $|Q_N^{(\mathrm{S})}|$ in the proof of Lemma 4). This means that $C(F_{N,r})$ takes all the values from the set in Eq. (C35) if we let r be all the allowed r values. ■

Lemma 6. In the generalized BW case, $(q, r) \in Q_N^{(\mathrm{BW})}$, for fixed q,

$$C(F_{q,r}) \in \left\{\frac{N}{q}p; p < q, \gcd(p, q) = 1\right\}. \tag{C38}$$

The values are different for different r and are all taken for some allowed r.

Proof. Since we are considering the generalized BW case: $q|N$, $r < q$, $\gcd(q, r) = 1$.

In this case, we can divide the circuit in q blocks (layers) of N/q time consecutive gates. The first block is generated by translating gate 1 by q (modulo N) until we reach gate 1 again. The second block consists of all the gates from the first block translated by r. This means that the first block can be mapped to the subgroup of the cyclic group $\mathbb{Z}_N = \{0, \ldots, N-1\}$ of order N/q, where group addition is taken modulo N. The other blocks are cosets of this subgroup (since according to Lemma 3, they must be disjoint), implying that the first q gate numbers are contained in different blocks. We can therefore denote the blocks with the minimal gate number contained in it (and a tilde for clarity) $\tilde{1}, \tilde{2}, \ldots, \tilde{q}$.

We now wish to consider a similar thing as in the proof of Lemma 5, but in the context of a block, i.e., given that a gate is a member of $\tilde{C}$, what can we say about the membership of its time successor and predecessor. We will see that the main difference is that this time, the notion time successor/predecessor can be taken *inside a given block*, so with indices modulo N/q.

Let us suppose that a gate with index l (this index is now taken in the context of a block, meaning modulo N/q) in block $\tilde{l}$ (modulo q) is a member of $\tilde{C}(F_{q,r})$, $i_l^{(\tilde{l})} \in \tilde{C}(F_{q,r})$. This means that its right neighbor must appear before it, $\exists i_k^{(\tilde{k})} = i_l^{(\tilde{l})} + 1, \tilde{k} < \tilde{l}$ (if such k would exists in block $\tilde{k} = \tilde{l}$, the block would contain all N gate numbers, which is not possible in the generalized BW case). Analogously to the generalized S case, $i_{l+1}^{(\tilde{l})} = i_l^{(\tilde{l})} + q$ and $i_{k+1}^{(\tilde{k})} = i_k^{(\tilde{k})} + q = i_l^{(\tilde{l})} + q + 1 = i_{l+1}^{(\tilde{l})} + 1$ and thus $i_{l+1}^{(\tilde{l})} \in \tilde{C}(F_{q,r})$. Crucially, indices k and l here are taken modulo N/q because translating the last gate of the block by q (modulo N) yields the first element of the block. Iterating this, we can conclude that if a gate from a block is contained in $\tilde{C}(F_{q,r})$, then all the gates from this block are contained in $\tilde{C}(F_{q,r})$.

We have therefore shown that by relabelling the considered circuit with $\tilde{1}, \tilde{2}, \ldots, \tilde{q}$, the considered circuit behaves exactly the same as is in the generalized S case. Using Lemma 5, for a fixed q and N,

$$C(F_{q,r}) \in \left\{\frac{N}{q}p; p < q, \gcd(p, q) = 1\right\}, \tag{C39}$$

where we took into the account that the number of gates in a block is N/q. Also according to Lemma 5, all the values from the set in Eq. (C39) are taken for some r and are different for different r. ■

Lemma 7. $C(F_{q,r})$ is different for all $(q, r) \in Q_N$ for a given N.

Proof. We have calculated all the allowed values of $C(F_{q,r})$ and shown that they are different for different r and a fixed q in Lemma 5 and Lemma 6. The only thing left to do is to show that for different q, the sets (C31) and (C38) are disjoint.

From Eq. (C38), we can see that in the generalized BW case $\gcd(C(F_{q,r}), N) \geqslant N/q$, meaning that all the values of $C(F_{q,r})$ must be different than in the generalized S case ($q = N$) where $\gcd(C(F_{N,r}), N) = 1$ [Eq. (C31)].

The only thing left is to check that the sets in the BW case [Eq. (C38)] are disjoint for two different $q_1, q_2|N$. We can do this by finding a contradiction in the converse case. Let us suppose that we would find the same value of C for different q_1, q_2, this means that for some p_1, p_2, $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$ we must have

$$\frac{N}{q_1} p_1 = \frac{N}{q_2} p_2, \tag{C40}$$

$$\Rightarrow q_2 p_1 = q_1 p_2. \tag{C41}$$

This implies that $p_1|q_1 p_2$, but since $\gcd(p_1, q_1) = 1$, we see that $p_1|p_2$. Symmetrically, we can show $p_2|p_1$, which then implies $p_1 = p_2$. From Eq. (C41) we finally get $q_1 = q_2$, a contradiction. Therefore, $C(F_{q,r})$ must also be different for different q in the BW case. ■

Finally, we can put all the previous lemmas together and prove the main theorem.

Theorem 1 (restated from the main text). Any simple circuit on N sites with periodic boundary conditions is equivalent to exactly one of the $N-1$ canonical simple circuits having Floquet propagator $F_{q,r}$, where $(q, r) \in Q_N$.

Proof. The fact that $F_{q,r}$ are simple circuits follows from Lemma 3 and the number of them follows from Lemma 4.

According to Lemma 7, $C(F_{q,r})$ is different for all allowed (q, r) and a given N. Since according to Lemma 1 C is conserved under equivalence transformations and, according to Eq. (C17), is different for all F_p, different $F_{q,r}$ are equivalent to different F_p (according to Theorem 2, they must be equivalent to some F_p). Moreover, since there are $(N-1)$ $F_{q,r}$ and $(N-1)$ F_p, every F_p must be equivalent to some $F_{q,r}$, i.e., there exists a bijective equivalence mapping between $F_{q,r}$ and F_p.

Given any simple circuit F, according to Theorem 2, we know $F \cong F_p$ for some p. We have shown that $F_p \cong F_{q,r}$ for some q, r. Therefore, $F \cong F_{q,r}$ for some q, r. ■

In contrast to the proof of Theorem 2, the proof of Theorem 1 is not constructive, i.e., it cannot be used as an algorithm to transform a given F to the equivalent $F_{q,r}$. The most convenient way to do this in practice is via the invariant $C(F)$ defined in Eq. (C16), which can be numerically calculated in linear time in N. One can then calculate $C(F)$ and $C(F_{q,r})$ for the all $(N-1)$ allowed (q, r) in quadratic time. Finally, F must be equivalent to the $F_{q,r}$ with the same value of C. As an example, the values of $C(F_{q,r})$ are included in Fig. 4.

[1] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, 1993).

[2] G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, *Stochastic Behavior of a Quantum Pendulum under a Periodic Perturbation*, Lecture Notes in Physics, Vol. 93 (Springer, Berlin, 1979), p. 334.

[3] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 2010).

[4] F. Haake, M. Kuś, and R. Scharf, Classical and quantum chaos for a kicked top, Z. Phys. B **65**, 381 (1987).

[5] T. Prosen, Ergodic properties of a generic nonintegrable quantum many-body systems in the thermodynamic limit, Phys. Rev. E **60**, 3949 (1999).

[6] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. **80**, 885 (2008).

[7] X. Mi, P. Roushan, C. Quintana, S. Mandrà, J. Marshall, C. Neill, F. Arute, K. Arya, J. Atalaya, R. Babbush *et al.*, Information scrambling in quantum circuits, Science **374**, 1479 (2021).

[8] Y. Kim, A. Eddins, S. Anand, K. X. Wei, E. van den Berg, S. Rosenblatt, H. Nayfeh, Y. Wu, M. Zaletel, K. Temme, and A. Kandala, Evidence for the utility of quantum computing before fault tolerance, Nature (London) **618**, 500 (2023).

[9] U. Schollwöck, The density-matrix renormalization group in the age of matrix product states, Ann. Phys. **326**, 96 (2011).

[10] M. Akila, D. Waltner, B. Gutkin, and T. Guhr, Particle-time duality in the kicked Ising spin chain, J. Phys. A. **49**, 375101 (2016).

[11] V. Gritsev and A. Polkovnikov, Integrable Floquet dynamics, SciPost Phys. **2**, 021 (2017).

[12] M. Ljubotina, L. Zadnik, and T. Prosen, Ballistic spin transport in a periodically driven integrable quantum system, Phys. Rev. Lett. **122**, 150605 (2019).

[13] S. Gopalakrishnan and A. Lamacraft, Unitary circuits of finite depth and infinite width from quantum channels, Phys. Rev. B **100**, 064309 (2019).

[14] T. Zhou and A. Nahum, Entanglement membrane in chaotic many-body systems, Phys. Rev. X **10**, 031066 (2020).

[15] K. Klobas, B. Bertini, and L. Piroli, Exact thermalization dynamics in the "Rule 54" quantum cellular automaton, Phys. Rev. Lett. **126**, 160602 (2021).

[16] B. Bertini, P. Kos, and T. Prosen, Exact correlation functions for dual-unitary lattice models in $1+1$ dimensions, Phys. Rev. Lett. **123**, 210601 (2019).

[17] A. Chan, S. Shivam, D. A. Huse, and A. De Luca, Many-body quantum chaos and space-time translational invariance, Nat. Commun. **13**, 7484 (2022).

[18] A. Nahum, S. Roy, S. Vijay, and T. Zhou, Real-time correlators in chaotic quantum many-body systems, Phys. Rev. B **106**, 224310 (2022).

[19] P. W. Claeys, J. Herzog-Arbeitman, and A. Lamacraft, Correlations and commuting transfer matrices in integrable unitary circuits, SciPost Phys. **12**, 007 (2022).

[20] M. Žnidarič, Two-step relaxation in local many-body Floquet systems, J. Phys. A **56**, 434001 (2023).

[21] O. Bohigas, M. J. Giannoni, and C. Schmit, Characterization of chaotic quantum spectra and universality of level fluctuation Laws, Phys. Rev. Lett. **52**, 1 (1984).

[22] M. L. Mehta, *Random Matrices*, 3rd ed. (Academic Press, Amsterdam, 2004).

[23] For many-body quantum systems a word of caution is in place. Because the level spacing is exponentially small in system size N the LSD probes properties on an exponentially large timescale. There can be situations where the relevant physics is not chaotic at all despite the RMT LSD, see e.g. Refs. [43,44].

[24] U. Duh, Space-time symmetries in chaotic qubit circuits, Master's thesis, University of Ljubljana, 2023.

[25] K. Huang, X. Li, D. A. Huse, and A. Chan, Out-of-time-order correlator, many-body quantum chaos, light-like generators, and singular values, arXiv:2308.16179.

[26] O. Giraud, N. Macé, É. Vernier, and F. Alet, Probing symmetries of quantum many-body systems through gap ratio statistics, Phys. Rev. X **12**, 011006 (2022).
[27] S. Xu and C. Wu, Space-time crystal and space-time group, Phys. Rev. Lett. **120**, 096401 (2018).
[28] T. Morimoto, H. C. Po, and A. Vishwanath, Floquet topological phases protected by time glide symmetry, Phys. Rev. B **95**, 195155 (2017).
[29] Y. Peng and G. Refael, Floquet second-order topological insulators from nonsymmorphic space-time symmetries, Phys. Rev. Lett. **123**, 016806 (2019).
[30] L. Sá, P. Ribeiro, and T. Prosen, Integrable nonunitary open quantum circuits, Phys. Rev. B **103**, 115132 (2021).
[31] Single-site Hilbert space can have any dimension (qudits) although we will frequently speak about qubits because all our numerical examples will be of that kind.
[32] J. Bensa and M. Žnidarič, Fastest local entanglement scrambler, multistage thermalization, and a non-Hermitian phantom, Phys. Rev. X **11**, 031019 (2021).
[33] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, Phys. Rev. X **8**, 021014 (2018).
[34] S.-H. Lin, R. Dilip, A. G. Green, A. Smith, and F. Pollmann, Real- and imaginary-time evolution with compressed quantum circuits, PRX Quantum **2**, 010342 (2021).
[35] D. Poulin, J.-P. Tillich, and H. Ollivier, Quantum serial turbo-codes, IEEE Trans. Inf. Theory **55**, 2776 (2009).
[36] G. Anikeeva, I. H. Kim, and P. Hayden, Recycling qubits in near-term quantum computers, Phys. Rev. A **103**, 042613 (2021).
[37] B. Pozsgay, A Yang–Baxter integrable cellular automaton with a four site update rule, J. Phys. A: Math. Theor. **54**, 384001 (2021).
[38] T. Gombor and B. Pozsgay, Integrable spin chains and cellular automata with medium-range interaction, Phys. Rev. E **104**, 054123 (2021).
[39] Provided the eigenphases $\tilde{\phi}_j$ of M are incommensurate with 2π this will always hold for large q regardless of the properties of M because the map $\phi_j = q \cdot \tilde{\phi}_j \pmod{2\pi}$ is ergodic (sawtooth map) and will destroy any correlations in $\tilde{\phi}_j$.
[40] E. M. Rains, Images of eigenvalue distributions under power maps, Probab. Theory Relat. Fields **125**, 522 (2003).
[41] N. Rosenzweig and C. E. Porter, "Repulsion of energy levels" in complex atomic spectra, Phys. Rev. **120**, 1698 (1960).
[42] For small N one of course has non-negligible finite-size deviations from RMT distributions; we have checked that they are within expectations [45].
[43] L. F. Santos, Integrability of a disordered Heisenberg spin-1/2 chain, J. Phys. A: Math. Gen. **37**, 4723 (2004).
[44] M. Brenes, E. Mascarenhas, M. Rigol, and J. Goold, High-temperature coherent transport in the XXZ chain in the presence of an impurity, Phys. Rev. B **98**, 235128 (2018).
[45] T. Prosen and M. Robnik, Energy level statistics in the transition region between integrability and chaos, J. Phys. A: Math. Gen. **26**, 2371 (1993).
[46] B. Dietz and F. Haake, Taylor and Padé analysis of the level spacing distributions of random-matrix ensembles, Z. Phys. B **80**, 153 (1990).
[47] K. H. Rosen, *Elementary Number Theory and its Applications* (Addison-Wesley, Boston, 2011).