Nonspreading relativistic electron wavepacket in a strong laser field

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A solution of the Dirac equation in a strong laser field presenting a nonspreading wave packet *in the rest frame* of the electron is derived. It consists of a generalization of the self-accelerating free electron wavepacket [I. Kaminer *et al.*, Nat. Phys. **11**, 261 (2015)] to the case with the background of a strong laser field. Built upon the notion of nonspreading for an extended relativistic wavepacket, the concept of Born rigidity for accelerated motion in relativity is the key ingredient of the solution. At its core, the solution comes from the connection between the self-accelerated free electron wavepacket and the eigenstate of a Dirac electron in a constant and homogeneous gravitational field via the equivalence principle. The solution is an essential step towards the realization of the laser-driven relativistic collider [S. Meuren *et al.*, Phys. Rev. Lett. **114**, 143201 (2015)], where the large spreading of a common Gaussian wavepacket during the excursion in a strong laser field strongly limits the expectable yields.

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I. INTRODUCTION

Recent advances in ultrastrong laser technology [1–3] provide bright prospects for laser-driven particle acceleration techniques. Especially successful are laser-driven plasma-based accelerators [4], which raised hopes to further develop the technique to compete with conventional electron-positron colliders [5,6], reducing the scale of the accelerating device. Even more dramatic scale change promises the idea of the laser-driven coherent microscopic collider [7–10], where the electron and positron generation, acceleration, and collision are realized within a single stage in a microscopic scale, providing high luminosity due to the coherently controlled electron-positron recollision. The bottleneck of this idea is the large spreading of a single-electron wavepacket in the rest frame of the electron during the excursion in the laser field within one laser period, which significantly restrains the luminosity of the collision. Thus, the covet is the overriding of the wavepacket spreading for the electron motion in the continuum. Nonspreading free electron wavepackets via interference of different momentum components in the wavepacket, so-called particle Airy beams, are known for the Schrödinger equation [11,12], which generalizes the similar idea for optical beams [13–17]. However, Airy beams are not normalizable, i.e., span the whole space. Because of the infinite extension of such wavepackets in space, they are not applicable for a laser-driven collider, as the luminosity of the collider should be quenched.

In the nonspreading wavepacket, the distance between two points remains constant during the motion. While the latter has a well-defined meaning in nonrelativistic mechanics, in the relativistic case, surprises arise, particularly involving Bell's paradox [18]. In this gedanken experiment, two points connected by a thread move with a constant acceleration, keeping a constant distance between them in the laboratory frame; however, the thread between the points is broken because of the contracted length of the thread in the laboratory frame [19]. Then, how do the two points have to move to avoid breaking the thread connecting them? This question is resolved by the Born rigidity concept [20], defining the notion of a rigid body in a relativistic setting: The wordlines of the rigid body points have to be equidistant curves in space-time. Or, in more simple terms, the space distance between two infinitesimally close points measured simultaneously in the comoving inertial frame (rest frame) should be constant. In particular, this will be the case, and the thread will not break in Bell's paradox, if the points move with different constant accelerations along hyperbolic trajectories [21]. In the laboratory frame, the space distance between the infinitesimally close points will decrease, fitting to the Lorentz contraction, while the distance between them in the rest frame will remain constant. Note that for the luminosity of the laser-driven collider, the rest frame size of the electron and positron wavepackets matters at the recollision. Relativistic wave packets of a decaying particle in different reference frames is discussed in Ref. [22].

Although seemingly unrelated, generating nonspreading wavepackets in relativistic quantum mechanics shares a common thread with the resolution of Bell's paradox through the concept of Born rigidity and hyperbolic motion. In both cases, the motion of different points of the objects is crucial, whether it is the motion of the points of the rigid body or the dynamics of interference fringes of the electron wavepacket along hyperbolic trajectories in the case of quantum mechanics.

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In this paper, inspired by the geometrical concept of Born rigidity, we use the covariant relativistic dynamical inversion (CRDI) technique [23] to demonstrate the existence of nonspreading wavepackets in a laser field fulfilling the Born rigidity requirements. These wavepackets in the local rest frame of the electron feature interference fringes with a constant distance between them due to the fringes' dynamics along the hyperbolic trajectories [24]. Employing the CRDI technique, we develop a procedure to transform the wavepacket in the laser field to the local rest frame of the electron, where it evolves into a free electron wavepacket. To impose nonspreading property on the wavepacket fringes, we invoke the equivalence principle, which tells us that the hyperbolic trajectories, i.e., trajectories corresponding to a motion with constant acceleration, are similar to those in a constant gravitational field. The latter allows us the construction of the nonspreading free electron wavepacket via mimicking locally the exact solution of the Dirac equation for the electron in a constant and homogeneous gravitational field [25]. We have identified the finite lifetime of the nonspreading wavepacket because of the leaking from the Rindler space and proved that it is sufficient to allow recollision in a laser-driven collider.

Our main aim is to create relativistic nonspreading wavepackets in a sense that the distance between the wavepacket fringes remains constant with time in the electrons' local rest frame. The Born rigidity concept tells us that this aim will be realized if the dynamics of the fringes of the wavepacket manifests hyperbolic trajectories along the so-called Rindler coordinates [21].

The structure of the paper is the following. In Sec. II, a family of rigid relativistic coordinate systems is constructed. The quantum dynamics of an accelerating electron is discussed in Sec. III. A nonspreading wavepacket in a strong laser field is constructed in Sec IV. The lifetime of the nonspreading wavepacket is discussed in Sec. V. The application of the nonspreading wavepacket for a laser-driven collider is considered in Sec. VI, and our conclusion is given in Sec. VII.

II. CONSTRUCTION OF A FAMILY OF RIGID RELATIVISTIC COORDINATE SYSTEMS

In line with the concept of Born rigidity, here we show how to construct a rigid reference system. Consider a particle in arbitrary motion relative to an inertial system I; the particle's coordinates with respect to I are $X^\alpha = (T, X, Y, Z)$. In what follows, Greek indices run from 0 to 3, while Latin indices run from 1 to 3. The particle's time track may be described by the equations $X^\alpha = f^\alpha(\tau)$, with τ being the proper time of the particle. Consider now another reference system R attached to the particle which is uniformly accelerated with respect to I; the axis of R should always be parallel to that of I, the particle being always situated at its origin. Let the coordinates following the particle in its motion relative to R be $x^\alpha = (t, x, y, z)$. At any moment, there exists an inertial coordinate system I', momentarily at rest with respect to the particle, whose coordinate axes coincide with those of R. Hence, we have $x'_i = x_i, x'_0 = 0$ and $\tau = t$. The transformation connecting the coordinates X^α with x^α is

$$X^\alpha = f^\alpha(t) + x^i e_i^\alpha(t), \tag{1}$$

where $e_\nu^\alpha(t)$ is an orthonormal frame for an accelerated observer that obeys the following equation:

$$\frac{de_\mu^\nu}{dt} = \Omega^\nu{}_\beta e_\mu^\beta,$$

where $\Omega^\nu{}_\beta = u^\nu \dot{u}_\beta - u_\nu \dot{u}^\beta$ and $u_\mu = \dot{f}_\mu(t) = df_\mu(t)/dt$. Differentiation of X^α gives $dX^\alpha = [u^\alpha + x^i \dot{e}_i^\alpha(t)]dt + dx^i e_i^\alpha(t)$. From the properties $u^\alpha u_\alpha = 1$, $\dot{u}_\beta \dot{u}^\beta = -g_i g^i$, and $\dot{u}_\beta u^\beta = 0$, we get

$$ds^2 = dt^2(1 + g_i x^i)^2 - (dx^2 + dy^2 + dz^2), \tag{2}$$

where $g_i(t) = e_i^\alpha \dot{u}_\alpha$ are functions of t only, being completely determined by the motion of the origin of the system of coordinates x^α relative to the system X^α. Considering the line element (2), the corresponding system of reference is rigid since the distance between two reference points (x, y, z) and $(x + dx, y + dy, z + dz)$ is given by $d\sigma^2 = dx^2 + dy^2 + dz^2$. In fact, the space geometry is even Euclidean; thus, (x, y, z) are Cartesian space coordinates.

Now consider that the origin O of the system x^α is moving in the Z-axis direction of the X^α system. From Eqs. (1), we have

$$X = x,\ Y = y,\ Z = \int_0^t \sinh[\theta(t)]dt + z\cosh[\theta(t)],$$

$$T = \int_0^t \cosh[\theta(t)]dt + z\sinh[\theta(t)]. \tag{3}$$

For the vector g_i, we get $\mathbf{g} = [0, 0, g(t)]$, $g(t) = d\theta/dt$. Hence, Eq. (2) becomes

$$ds^2 = dt^2(1 + gz)^2 - (dx^2 + dy^2 + dz^2). \tag{4}$$

In particular, if the motion of the origin O is hyperbolic, then $\theta(t) = gt$, thus making g a constant. The transformation equations then reduce to the well-known *Rindler coordinates* [21]:

$$X = x,\ Y = y,\ Z = \frac{1}{g}[\cosh(gt) - 1] + z\cosh(gt),$$

$$T = \frac{1}{g}\sinh(gt) + z\sinh(gt). \tag{5}$$

Let us examine if the reference system R corresponding to the coordinates x^α will appear as rigid with respect to the observer A in the inertial frame I. By elimination of the variable t from Eqs. (5), we obtain

$$Z = \frac{1}{g}\Big[\sqrt{(1 + gz)^2 + g^2T^2} - 1\Big],\ X = x,\ Y = y.$$

The velocity of the reference points relative to I at time T is thus

$$v = \frac{dZ}{dT} = \frac{gT}{\sqrt{(1 + gz)^2 + g^2T^2}} = \tanh(gt). \tag{6}$$

Since the velocity v of the frame R from the point of view of A also depends on z, R will not appear as rigid with respect to I. In fact, the difference between two reference points (x, y, z) and $(x, y, z + dz)$ measured by A is found to be

$$\begin{aligned} dZ &= \frac{(1 + gz)dz}{\sqrt{(1 + gz)^2 + g^2T^2}} = \frac{dz}{\cosh(gt)} \\ &= \sqrt{1 - v^2}dz. \end{aligned}$$

Hence, from A's perspective, each part of the accelerated frame R undergoes a Lorentz contraction.

Going back to the motion of the particle in the accelerated frame R from the point of view of A, consider the velocity v for the particle located permanently at position z, that is, it is at rest with respect to R. By definition, the proper time of the particle can be calculated from Eq. (4) as

$$d\tau = \sqrt{dt^2(1+gz)^2 - dx^2 - dy^2 - dz^2} \tag{7}$$

$$= dt(1+gz), \tag{8}$$

$$\tau = (1+gz)\int_0^t dt = (1+gz)t, \tag{9}$$

given that $dx/dt = dy/dt = dz/dt = 0$.

III. QUANTUM DYNAMICS OF AN ACCELERATING ELECTRON

The idea of the nonspreading wavepacket is closely related to the confined Dirac solution of Greiner [25] for the electron in a constant gravitational field due to the equivalence principle and can be deduced from the latter. In the chiral representation [26], the eigenspinor of the Greiner's solution for a spin-up electron reads

$$\psi_R = \frac{2\sqrt{2}\mathcal{N}e^{\pi\Omega/2}}{i\pi}e^{i\gamma^5\pi/4}\begin{pmatrix} K_{i\Omega+1/2}(mu) \\ 0 \\ K_{i\Omega-1/2}(mu) \\ 0 \end{pmatrix}e^{-i\Omega\eta}, \tag{10}$$

where $K_\nu(x)$ is a Bessel function, $\mathcal{N}$ is a normalization constant, m is the electron mass, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and Ω is the eigenenergy. (η, u) are defined as the comoving coordinates of an inertial observer momentarily at rest with respect to the electron. Hence, using the Rindler coordinates, we have $\eta \equiv gt$ and $u \equiv z + 1/g = \sqrt{(Z+1/g)^2 - T^2}$. Let us massage Eq. (10) a bit more to gain intuition on how to build it from a wavepacket. First, note that

$$K_\nu(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh t + \nu t}dt,$$

along with $\cosh(t) = i\sinh(t - i\pi/2)$. Combining both identities, (10) becomes

$$\psi_R = \frac{\sqrt{2}\mathcal{N}}{i\pi}\int_{-\infty}^{\infty} dt\, e^{-imu\sinh t}\begin{pmatrix} e^{(i\Omega+1/2)t} \\ 0 \\ e^{(i\Omega-1/2)t} \\ 0 \end{pmatrix}e^{-i\Omega\eta}. \tag{11}$$

Since $Z = u\cosh\eta$, $T = u\sinh\eta$, by defining the momentum $p = m\sinh b$ for b real and making the change of coordinates $t = \eta - b$ in (11), we finally have

$$\psi_R = \frac{i\sqrt{2}\mathcal{N}}{\pi}e^{-\frac{\gamma^0\gamma^3}{2}\tanh^{-1}\left(\frac{T}{Z}\right)}\int_{-\infty}^{\infty}\begin{pmatrix} e^{-b/2} \\ 0 \\ e^{b/2} \\ 0 \end{pmatrix} \times e^{-im(T\cosh b - Z\sinh b) - i\Omega b}db. \tag{12}$$

The spinor (12) is the desired result.

The wave function of Eq. (10) is an eigenstate and is confined in the coordinate u. Note that only for gravitational fields can an accelerated electron be described as a superposition of plane waves. This is a direct consequence of the equivalence principle. In fact, only gravity-induced acceleration can be transformed away by a coordinate transformation in the immediate vicinity of the particle.

A. The Dirac equation in the accelerated frame

Here we show how the spinor discussed in the previous section is connected with the self-accelerated spinor in the laboratory frame by a Lorentz transformation. From the coordinate relations $Z = u\cosh\eta$ and $T = u\sinh\eta$, we have

$$\frac{\partial}{\partial T} - \frac{\partial}{\partial Z} = e^{\eta}\left(\frac{1}{u}\frac{\partial}{\partial\eta} - \frac{\partial}{\partial u}\right),$$
$$\frac{\partial}{\partial T} + \frac{\partial}{\partial Z} = e^{-\eta}\left(\frac{1}{u}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial u}\right),$$

leading to

$$0 = \left[-m + i\left(\gamma^0\frac{\partial}{\partial T} + \gamma^3\frac{\partial}{\partial Z}\right)\right]\psi$$
$$= \left[-um + ie^{\eta\gamma^0\gamma^3}\left(\gamma^0\frac{\partial}{\partial\eta} + \gamma^3 u\frac{\partial}{\partial u}\right)\right]\psi. \tag{13}$$

Equation (13) is exactly the Dirac equation in the laboratory frame. It can be transformed to the accelerated frame as follows:

$$e^{\gamma^0\gamma^3\frac{\eta}{2}}\left[-um + i\left(\gamma^0\frac{\partial}{\partial\eta} + \gamma^3\left\{u\frac{\partial}{\partial u} + \frac{1}{2}\right\}\right)\right]\psi_R = 0,$$

which can be rewritten in the more compact form of

$$\left[-mu + i\left(\gamma^0\frac{\partial}{\partial\eta} + \gamma^3\left\{u\frac{\partial}{\partial u} + \frac{1}{2}\right\}\right)\right]\psi_R = 0, \tag{14}$$

with $\psi_R = e^{-\gamma^0\gamma^3\frac{\eta}{2}}\psi$, where ψ is the solution of the free Dirac equation in the laboratory frame, while ψ_R is the solution in the Rindler (also known as accelerated) reference frame.

B. Constructing the superposition for a free particle

Equipped with the spinor (12) and the relationship $\psi_R = e^{-\gamma^0\gamma^3\frac{\eta}{2}}\psi$, here we will build the self-accelerating wavepacket in the laboratory frame. In the Chiral representation, the Dirac spinor for a spin-up electron in its rest frame with respect to a global inertial frame is

$$\psi = h(0)\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}e^{-imT}, \tag{15}$$

where $h(0)$ is some momentum-dependent envelope function. Let us apply a boost to a frame moving along the Z axis with momentum p,

$$\psi_p = h(p)\begin{pmatrix} \frac{m-p+E_p}{2\sqrt{m(m+E_p)}} \\ 0 \\ \frac{m+p+E_p}{2\sqrt{m(m+E_p)}} \\ 0 \end{pmatrix} e^{-i(E_pT-pZ)}, \tag{16}$$

with $E_p = \sqrt{m^2+p^2}$. Now we build a wavepacket by integrating over p the spinor (16),

$$\psi(t,z) = \int_{-\infty}^{\infty} \frac{h(p)dp}{2E_p}\begin{pmatrix} \frac{m-p+E_p}{2\sqrt{m(m+E_p)}} \\ 0 \\ \frac{m+p+E_p}{2\sqrt{m(m+E_p)}} \\ 0 \end{pmatrix} e^{-i(E_pT-pZ)}, \tag{17}$$

in which $\frac{dp}{2E_p}$ renders the integral Lorentz invariant.

Let us now choose the following envelope function:

$$h(p) = \mathcal{N}e^{-aE_p}, \tag{18}$$

where $\mathcal{N}$ is a normalization constant and $a > 0$ is a constant with units of length. Upon making the variable substitution $p = m\sinh(b)$ and including the phase factor $e^{i\alpha b}$, with α being a arbitrary real number in (17), one ends up with the desired superposition,

$$\psi = \mathcal{N}\int_{-\infty}^{\infty} dbe^{ib\alpha}\begin{pmatrix} e^{-b/2} \\ 0 \\ e^{b/2} \\ 0 \end{pmatrix} \times e^{-im[\cosh(b)(T-ia)-\sinh(b)Z]}. \tag{19}$$

Due to the particular form of $h(b)$, the b integration in (19) can be performed exactly. In order to see this, first note that for $a > 0$,

$$\int_{-\infty}^{+\infty} dt e^{iy\cosh(t)+i\zeta\sinh(t)-\nu t} = i\pi e^{\frac{i\nu\pi}{2}}\left(\frac{y+\zeta}{y-\zeta}\right)^{\frac{\nu}{2}} H_\nu^{(1)}(x),$$

$$\int_{-\infty}^{+\infty} dt e^{iy\cosh(t)+i\zeta\sinh(t)+\nu t} = i\pi e^{\frac{i\nu\pi}{2}}\left(\frac{y-\zeta}{y+\zeta}\right)^{\frac{\nu}{2}} H_\nu^{(1)}(x),$$

with $x = \sqrt{y^2-\zeta^2}$. Then, defining

$$m(ia-T) = y, \quad \zeta = mZ,$$
$$x = m\sqrt{(ia-T)^2 - Z^2}$$

leads to

$$\psi = \mathcal{N}\int_{-\infty}^{\infty} dbe^{ib\alpha}\begin{pmatrix} e^{-b/2} \\ 0 \\ e^{b/2} \\ 0 \end{pmatrix} e^{iy\cosh(b)+i\zeta\sinh(b)}. \tag{20}$$

Before continuing, note that $ix = im\sqrt{(a+iT)^2+Z^2}$ and $H_\nu^{(1)}(ix) = \frac{2K_\nu(x)}{\pi i^{1+\nu}}$. Hence, by performing the b integration, one gets

$$\psi(T,X) = \mathcal{N}\begin{pmatrix} F_{i\alpha-1/2}(\bar{\zeta}) \\ 0 \\ F_{i\alpha+1/2}(\bar{\zeta}) \\ 0 \end{pmatrix}, \tag{21}$$

where $\bar{\zeta} = i\sqrt{(\bar{a}+i\bar{T})^2+\bar{Z}^2}$ and $F_{i\alpha\pm1/2}(\bar{\zeta}) = 2(\frac{i\bar{a}-\bar{T}-\bar{Z}}{i\bar{a}-\bar{T}+\bar{Z}})^{\pm1/4+i\alpha/2}K_{\pm1/2+i\alpha}(\bar{\zeta})$.

In summary, the confined solution for the eigenstate ψ_R to the free Dirac equation with respect to the accelerated frame (η,u) of Eq. (11) can be mimicked by a superposition ψ of the Dirac solutions for a free electron with respect to the laboratory frame (T,Z),

$$\psi_R = e^{-\frac{\gamma^0\gamma^3}{2}\tanh^{-1}(\frac{T}{Z})}\psi, \tag{22}$$

where the free wavepacket ψ should have a momentum chirp via the phase $\varphi(b) = -\alpha b$, with $\alpha = \Omega$, according to Eq. (11). When, additionally, we use the momentum distribution in the free wavepacket $h(p) = e^{-aE_p}$, with the constant $a > 0$ characterizing the momentum spread of the wavepacket, we get the dispersionless free spinorial wavepacket giving by the spinor (21). While the wavepacket (21) was already discussed in Ref. [24], the emphasis here is its direct relation to the nonspreading concept.

IV. NONSPREADING WAVEPACKET IN A STRONG LASER FIELD

Our aim is to construct a solution of the Dirac equation in a laser field in the form of a wavepacket and to show, using the CRDI technique, that it represents a nonspreading spinor in the local rest frame of the electron. We construct the desired wavepacket from the Volkov solutions $\psi_p(T,X)$ for an electron in a plane-wave laser field $eA^\mu = [0, f_1(\xi), f_2(\xi), 0]$,

$$\psi_L(T,X) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{dp}{2E_p}f(p)\psi_p(T,X), \tag{23}$$

where

$$\psi_p(T,X) = \left(\mathbf{1} + \frac{\not{n}\wedge\not{A}}{n^\mu p_\mu}\right)\mathbf{u}e^{-i(E_pT-pZ-\Phi)}, \tag{24}$$

with $n^\mu = (1,0,0,1)$, $p_\mu = (E_p,0,0,-p)$, $E_p = \sqrt{m^2+p^2}$, $\not{A} = \gamma^\mu eA_\mu$, $\not{n}\wedge\not{A} = (\not{n}\not{A}-\not{A}\not{n})/2$, $\mathbf{u}$ is the leftmost column of the boost matrix $\mathcal{B} = \sqrt{\frac{E_p+m}{2E_p}}(\mathbf{1}+\frac{\gamma_0\gamma^3 p}{E_p+m})$, $\Phi = -\frac{1}{2\omega(E_p-p)}\int_0^\xi[\dot{f}_1(\phi)^2+\dot{f}_2(\phi)^2]d\phi$, and the laser field phase $\xi = \omega(T-Z)$. For the superposition coefficients in Eq. (23), we use those which yield the free wavepacket ψ; see Eqs. (11)–(22). After performing the change of variables $p = m\sinh b$,

$$\psi_L = \int_{-\infty}^{\infty} db\mathcal{N}f(b)\begin{pmatrix} e^{-b/2} \\ e^{\frac{b}{2}}[\dot{f}_1(\xi)+i\dot{f}_2(\xi)]/m \\ e^{b/2} \\ 0 \end{pmatrix} \times e^{-im(T\cosh b - Z\sinh b)-ie^b\Phi}, \tag{25}$$

the closed expression for the integral in (25) is

$$\psi_L(T,X) = \mathcal{N}\begin{pmatrix} F_{i\alpha-1/2}(\bar{\zeta}') \\ F_{i\alpha+1/2}(\bar{\zeta}')[\dot{f}_1(\xi)+i\dot{f}_2(\xi)]/m \\ F_{i\alpha+1/2}(\bar{\zeta}') \\ 0 \end{pmatrix}, \tag{26}$$

where $\bar{\zeta}' = i\sqrt{(\bar{a}+i\bar{T}')^2+\bar{Z}'^2}$, $Z' = Z-\Phi$, $T' = T+\Phi$, and $F_{i\alpha\pm1/2}(\bar{\zeta}') = 2(\frac{i\bar{a}-\bar{T}'-\bar{Z}'}{i\bar{a}-\bar{T}'+\bar{Z}'})^{\pm1/4+i\alpha/2}K_{\pm1/2+i\alpha}(\bar{\zeta}')$.

Let us transform the spinorial wavepacket (26) to the rest frame, which is defined as the space-time-dependent frame in which the spatial components of the electron's four-current vanish at the given space-time point, and demonstrate its nonspreading property. In the free-electron case, such Lorentz transformation is the matrix $e^{-\frac{\gamma^0\gamma^3}{2}\tanh^{-1}(\frac{T}{Z})}$ on the left of the spinor (11). An equivalent transformation is now needed for the case in which the electron is interacting with a plane-wave field. In order to construct the desired Lorentz transformation, we make use of the CRDI technique. In the Hestenes formulation (see, for instance, Sec. 3 of Ref. [27]), the spinor (24) can be written as

$$\Psi = e^{\frac{\not{n}\wedge\not{A}}{n^\mu p_\mu}}\mathcal{B}e^{-\gamma_2\gamma_1(E_pT-pZ-\Phi)}. \tag{27}$$

As discussed in [23] (see, also, Appendix A), the matrix $e^{\frac{\not{n}\wedge\not{A}}{n^\mu p_\mu}} \equiv \mathcal{R}$ is, in fact, a Lorentz transformation. In the chiral representation, it is given by

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{[\dot{f}_1(\xi)+i\dot{f}_2(\xi)]}{E_p-p} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{[\dot{f}_1(\xi)-i\dot{f}_2(\xi)]}{E_p-p} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

However, the Lorentz transformation (28) is valid only for the wave function (27), but not (26). Moreover, it does not account for the transformation to the Rindler (accelerated) frame. In order to encompass both transformations, we start with the following ansatz:

$$\bar{\mathcal{R}} = e^{-\frac{\gamma^0\gamma^3\eta'}{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -d^*\omega[\dot{f}_1(\xi)+i\dot{f}_2(\xi)] & 1 & 0 & 0 \\ 0 & 0 & 1 & d\omega[\dot{f}_1(\xi)-i\dot{f}_2(\xi)] \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{29}$$

where d^* and η' are free functions to be found by the requirements that the resulting spinor is of the same form as Eq. (11) and that the electron's current vanishes. These requirements are fulfilled by the following functions:

$$d^* = \sqrt{\frac{a+i(cT'+Z')}{a+i(cT'-Z')}}\frac{K_{i\alpha-\frac{1}{2}}(\kappa\bar{\zeta}')}{2m\omega K_{i\alpha+\frac{1}{2}}(\kappa\bar{\zeta}')}$$

and

$$\eta' = \frac{1}{2}\ln\left(q\frac{K_{1/2-i\alpha}(\kappa\bar{\zeta}'^*)K_{1/2+i\alpha}(\kappa\bar{\zeta}')}{K_{1/2-i\alpha}(\kappa\bar{\zeta}')K_{1/2+i\alpha}(\kappa\bar{\zeta}'^*)}\right),$$

$$q = \sqrt{\frac{a^2+(cT'+Z')^2}{a^2+(-cT'+Z')^2}},$$

with superscript * standing for complex conjugation. Applying $\bar{\mathcal{R}}$ to (23), $\bar{\mathcal{R}}\psi_L$, leads to the following spinor describing the electron in its rest frame:

$$\bar{\mathcal{R}}\psi_L \equiv \bar{\psi}_R = \mathcal{N}e^{-\frac{\gamma^0\gamma^3}{2}\eta'}\begin{pmatrix} F_{i\alpha-1/2}(\bar{\zeta}') \\ 0 \\ F_{i\alpha+1/2}(\bar{\zeta}') \\ 0 \end{pmatrix}. \tag{30}$$

The final step of the transformation consists of the coordinate transformation $Z' = Z-\Phi$, $T' = T+\Phi$.

One now needs to prove that the constructed wave function $\bar{\psi}_R$ obeys the Dirac equation in the plane-wave field given by $\not{A}$. In order to do so, one must do the following: First construct the vierbein $e^\alpha_\mu = \frac{1}{4}\mathrm{Tr}\,[\bar{\mathcal{R}}^{-1}\gamma^\alpha\bar{\mathcal{R}}\gamma_\mu]$, $e^\mu_\alpha = \frac{1}{4}\mathrm{Tr}\,[\bar{\mathcal{R}}^{-1}\gamma^\mu\bar{\mathcal{R}}\gamma_\alpha]$, which transform the γ matrices in the laboratory frame to the new γ matrices $\tilde{\gamma}^\alpha = e^\alpha_\mu\gamma^\mu$, $\tilde{\gamma}_\alpha = e^\mu_\alpha\gamma_\mu$. After the transformation, the wavepacket $\bar{\psi}_R$ satisfies the following Dirac equation:

$$i\tilde{\gamma}^\mu\nabla_\mu\bar{\psi}_R - \tilde{\gamma}_\mu eA^\mu\bar{\psi}_R - m\bar{\psi}_R = 0,$$

where $\nabla_\mu = \partial/\partial X^\mu + \mathbf{\Omega}_\mu$, $X^\mu = (T,X,Y,Z)$, which describes the electron in its rest frame. The matrix $\mathbf{\Omega}_\mu$ is the spinor connection and is given by $2\mathbf{\Omega}_\mu = \Omega_{ij\mu}\boldsymbol{\sigma}^{ij}$, $2\boldsymbol{\sigma}_{ij} = \gamma_i\gamma_j$, and $\Omega^i_{j\mu} = -e^\nu_j e^i_\sigma e^\sigma_a\partial_\mu e^a_\nu$. The wavepacket in the rest frame will be nonspreading when the spinor of Eq. (23) is represented in exactly the same form as the free spinor of Eq. (21). This is achieved by the consecutive application of the coordinate transformation $Z' = Z-\Phi$, $T' = T+\Phi$. With the change of vierbein $e'^\alpha_\mu = \frac{\partial X'^\alpha}{\partial X^\mu}$, $e'^\mu_\alpha = \frac{\partial X^\mu}{\partial X'^\alpha}$, and with the new gamma matrices $\gamma'^\alpha = e'^\alpha_\mu\tilde{\gamma}^\mu$, $\gamma'_\alpha = e'^\mu_\alpha\tilde{\gamma}_\mu$, the transformed spinor (30) satisfies the Dirac equation in the new frame,

$$i\gamma'^\mu\nabla'_\mu\bar{\psi}_R - \tilde{\gamma}_\mu eA^\mu(\xi')\bar{\psi}_R - m\bar{\psi}_R = 0,$$

with $\nabla'_\mu = \partial/\partial X'^\mu + \mathbf{\Omega}_\mu$. Note, also, that $eA^\mu(\xi') = eA^\mu[\xi(\xi')]$ since $\xi' = \xi + 2\omega\Phi(\xi)$. The variable $\xi(\xi')$ is then given by inverting the coordinate transformation.

Thus, the constructed wavepacket of the electron in a laser field in the form of Eq. (23) [or Eq. (26)] coincides, up to a boost [see Eq. (30)], with the free self-accelerating nonspreading wavepacket [cf. Eq. (21)] in the local rest frame of the electron at each (Z', T'). Note that the exact Lorentz transformation of the electron Dirac wave function in a laser field to the electron rest frame is essentially facilitated by application of the CRDI technique [23,28].

V. LIFETIME OF THE NONSPREADING WAVE PACKET

There is an important deviation of the nonspreading wave packet given by Eq. (30) from the accelerating electron solution given by Eq. (10). While in the latter $a = 0$, the former has a finite size of the wavepacket $a \neq 0$, which has essential implications. To discuss this, consider the spinor (30) in the laboratory frame (hereafter, we drop the primes in order to

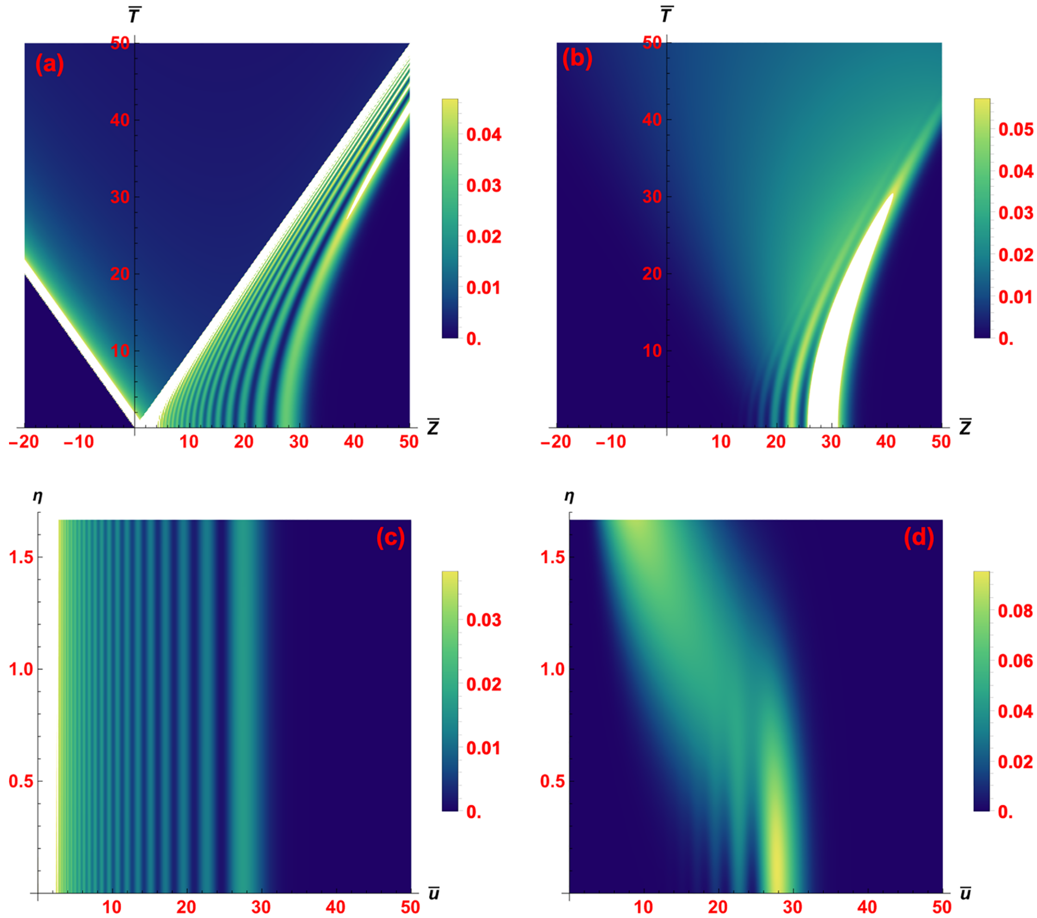

FIG. 1. Space-time profile of the electron density for the free electron modulated wavepacket of Eq. (21): (a),(b) in the laboratory frame, (c),(d) in the accelerated frame (i.e., in Rindler coordinates), (a),(c) $\alpha = 30$ and $\bar{a} = 0.005$, (b),(d) $\alpha = 30$ and $\bar{a} = 2$. Note that the Rindler coordinates only cover the region outside the light cone to the right.

simplify the notation),

$$\psi = e^{\frac{\gamma^0\gamma^3}{2}\eta}\bar{\psi}_R = \mathcal{N}\begin{pmatrix} F_{i\alpha+1/2}(\bar{\zeta}) \\ 0 \\ F_{i\alpha-1/2}(\bar{\zeta}) \\ 0 \end{pmatrix}, \tag{31}$$

The impact of the wavepacket size a is given by the prefactor in $F_{i\alpha\pm1/2}(\bar{\zeta})$, which we analyze next.

The space-time profile of the electron wavepacket is presented in Fig. 1. Figures 1(a) and 1(b) show the distribution in the laboratory-frame coordinates (T, Z), while Figs. 1(c) and 1(d) show it in the accelerated rest frame with Rindler coordinates (η, u). In the laboratory frame, the wavepacket is separated into two parts: inside the light cone with normal spreading and outside the light cone (defined as region I in Ref. [25]) representing the nonspreading wavepacket. The latter shows interference fringes, each lobe corresponding to a hyperbolic trajectory. Such a feature is entirely due to the chirping parameter α. From the rest frame perspective [Figs. 1(c) and 1(d)], the wavepacket is nonspreading (the width of the wavepacket at each instant of the Rindler's time η remains constant), while in the laboratory frame, the width of the wavepacket is contracting with time T according to the Lorentz transformation.

There is a significant effect stemming from the value of the wavepacket size parameter a; cf. Figs. 1(a) and 1(c) with Figs. 1(b) and 1(d). During evolution, the nonspreading part of the wavepacket is gradually leaking out into the normal one. The parameter a controls the balance between the nonspreading and normal parts of the wavepacket and determines the lifetime of the nonspreading part of the wavepacket. Such leaking is responsible for washing out the interference fringes: the smaller the value of a, the slower is the interference fringes' extinction (Fig. 1). There is no extinction in the case of the accelerating electron solution of Eq. (10) with $a = 0$. We can estimate the lifetime of the nonspreading wave packet as follows. Let us begin with the spinor in the accelerated frame, which is related to (21) in the same way as the spinor (12) is related to the one in the laboratory frame. We have

$$\psi = \mathcal{N}\begin{pmatrix} e^{\frac{\eta}{2}}G_{i\alpha-1/2}(\bar{\zeta}) \\ 0 \\ e^{-\frac{\eta}{2}}G_{i\alpha+1/2}(\bar{\zeta}) \\ 0 \end{pmatrix},$$

$$G_{i\alpha\pm1/2}(\bar{\zeta}) = \left(\frac{e^{\eta}(-\bar{u}e^{\eta} + i\bar{a})}{\bar{u} + i\bar{a}e^{\eta}}\right)^{\pm\frac{1}{4}+\frac{i\alpha}{2}} K_{\pm1/2+i\alpha}(\bar{\zeta}),$$

$$\bar{\zeta} = \sqrt{\bar{a}^2 + 2i\bar{a}\bar{u}\sinh(\eta) + \bar{u}^2}. \tag{32}$$

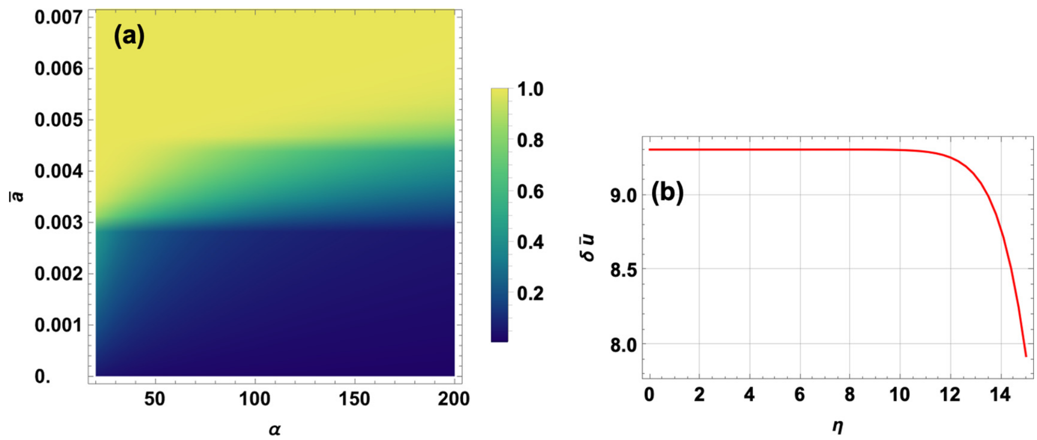

FIG. 2. (a) The asymmetry ratio $\mathcal{A} = (|\bar{\psi}_R|^2 - |\psi|^2|)/(|\bar{\psi}_R|^2 + |\psi|^2|)$ vs the wavepacket chirping parameter α and the size parameter $\bar{a}$ for $T = 1$ fs. (b) The wavepacket size $\delta\bar{u}$ vs η for $\alpha = 40$ and $\bar{a} = 10^{-6}$.

Now, for $\eta \gg 1$, we have

$$\bar{\zeta} \approx e^{i\pi/4}\sqrt{2\bar{a}\bar{u}}e^{\eta/2},\ K_{\pm 1/2+i\alpha}(\bar{\zeta}) \approx \sqrt{\frac{\pi}{2\bar{\zeta}}}e^{-\bar{\zeta}},$$

$$\left(\frac{e^{\eta}(-\bar{u}e^{\eta} + i\bar{a})}{\bar{u} + i\bar{a}e^{\eta}}\right)^{\pm\frac{1}{4}+\frac{i\alpha}{2}} \approx \left(\frac{i\bar{u}e^{\eta}}{\bar{a}}\right)^{\pm\frac{1}{4}+\frac{i\alpha}{2}}.$$

Thus,

$$\psi^{\dagger}\psi \approx \frac{e^{-2(e^{\eta}\bar{a}\bar{u})^{1/2} - \frac{\pi\alpha}{2}}}{\sqrt{2\bar{u}}}. \qquad (33)$$

The same estimation with the Rindler spinor (10) leads to

$$\psi_R^{\dagger}\psi_R \approx \frac{e^{-2\bar{u} + \frac{\pi\Omega}{2}}}{\sqrt{2\bar{u}}}. \qquad (34)$$

Using asymptotic expressions of the wave functions at $\eta \gg 1$: $|\psi|^2 \approx \exp[-2(e^{\eta}\bar{a}\bar{u})^{1/2} - \frac{\pi\alpha}{2}]/\sqrt{2\bar{u}}$, $|\psi_R|^2 \approx \exp[-2\bar{u} + \frac{\pi\Omega}{2}]/\sqrt{2\bar{u}}$. Both asymptotic expansions will coincide if $e^{\eta}\bar{a} \sim \bar{u}$ or $\bar{Z} - \bar{T} \gtrsim \bar{a}$. Taking into account that the equation for the rightmost hyperbolic trajectory ($\bar{Z}_0 \approx \alpha$) as a function of time is $\bar{Z}(\bar{T}) \approx \sqrt{\alpha^2 + \bar{T}^2}$, we have an estimate for the lifetime of the nonspreading wavepacket,

$$T_l \lesssim (\alpha^2 - \bar{a}^2)/(2\bar{a}m), \qquad (35)$$

which indicates that large α and small $\bar{a}$ are beneficial for the extension of the lifetime. For instance, $T_l \lesssim 1$ fs when using $\alpha = 30$ and $\bar{a} = 0.001$.

We also analyze the balance of the nonspreading and normal parts of the wavepacket, introducing the asymmetry parameter via $\mathcal{A} = (|\bar{\psi}_R|^2 - |\psi|^2|)/(|\bar{\psi}_R|^2 + |\psi|^2|)$ and calculating the densities of these parts via Eqs. (11) and (21); see Appendix B for a definition. The example of the dependence of $\mathcal{A}$ on the parameters α and $\bar{a}$ is shown in Fig. 2(a) for $T = 1$ fs. The wavepacket is nonspreading if $|\psi|^2 \approx |\bar{\psi}_R|^2|$, i.e., at $\mathcal{A} \to 0$, while at $\mathcal{A} \to 1$, the nonspreading wave packet is fully extinguished, $|\psi|^2 \to 0$. $\mathcal{A}$ is very sensitive to $\bar{a}$. Smaller $\bar{a}$ is preferred for $\mathcal{A} \to 0$; however, the larger α allows larger $\bar{a}$ at a given $\mathcal{A}$ [Fig. 2(a)].

We numerically evaluated the wavepacket spatial size via the accelerated frame spinor (see Appendix B), i.e., considering only the parts outside the light cone. The standard deviation $\delta\bar{u} = \sqrt{\langle\bar{u}^2\rangle - \langle\bar{u}\rangle^2}$ is calculated with $\bar{a} = 10^{-6}$ and $\alpha = 40$ [Fig. 2(b)]. With the rightmost hyperbolic trajectory, the transformation between the time in the accelerated frame and in the laboratory frame is then $\bar{T} = \alpha \sinh\eta$. As seen in Fig. 2(b), the wavepacket spreading $\delta\bar{u}$ stays constant up to $\eta \approx 11$, which corresponds to the laboratory time $T \approx 2$ fs.

Considering that the fringes of the self-accelerating part of the wavepacket must last for at least one period of the laser field, let us estimate the maximum value for the parameter $\bar{a}$. For a full cycle of the laser field in the electron's rest frame, one has $\omega(T - Z) = 2\pi$. The latter, combined with the condition $|\bar{Z} - \bar{T}| \gtrsim \bar{a}$, gives $a \lesssim \lambda'$, where λ' is the laser wavelength in the electron rest frame.

VI. LASER-DRIVEN COLLIDER

We consider applications of the nonspreading relativistic wavepackets to a laser-driven collider [10]. Here electrons and positrons are created from vacuum by high-energy gamma photons counterpropagating an ultrastrong laser field. They are accelerated by the laser field and collide within a cycle of the field. The rest frame of the created pairs depends on the γ-photon energy (Ω_0) and the laser strong field parameter [$a_0 \equiv eE_0/(m\omega)$, with the laser field amplitude E_0]. Let us estimate the velocity (or the Lorentz γ factor) of the average rest frame (RF) of the created pair at the threshold of the process for the given gamma-photon energy Ω_0 in the laboratory frame (LF), and the laser field strength a_0. Here, $a_0 \equiv eE_0/m\omega$ is the classical strong field parameter of the laser field, where the quasimomentum (momentum averaged over the laser period) of the electron and positron is vanishing, $\mathbf{q} = 0$, $q_0 = m_*$, and where $m_* = m\sqrt{(1 + a_0^2/2)}$ is the dressed mass of the electron in a linearly polarized laser field. As the pair is created by absorbing one gamma photon of the energy Ω_0' (in the RF) and n counterpropagating laser photons with an energy Ω_0' (in the RF), the energy-momentum conservation law in the RF at the threshold of the process yields $\Omega_0' = n\omega' = m_*$. In an ultrastrong laser field $a_0 \gg 1$, the average number of laser photons involved in the pair production process is $n \sim a_0^3$. We choose the gamma-photon energy in the LF to fulfill the condition $\Omega_0 > a_0^3\omega$. In this case, the RF propagates along the gamma photon and the RF's γ factor is determined from the Doppler-shifted momentum conservation condition: $\Omega_0/(2\gamma) = 2\gamma n\omega \approx ma_0/\sqrt{2}$. Thus, with the given a_0, the RF's γ of the most probable pair production is determined from the condition

$$2\sqrt{2}a_0^2\gamma\omega/m = 1, \tag{36}$$

which will require the gamma-photon energy

$$\Omega_0 \approx \sqrt{2}m\gamma a_0. \tag{37}$$

Assuming an infrared laser field with $\omega/m = 10^{-6}$, $a_0 = 10^2$ (the laser intensity of 10^{22} W/cm^2), we have, from Eqs. (36) and (37), $\gamma \approx 30$ and $\Omega_0 \approx 2$ GeV.

The laser period in the rest frame of the pair, $T' = T_L/\gamma \sim 3 \times 10^{-2}$ fs (with the laser period T_L in the laboratory frame), which is less than the wavepacket leaking time $T_l \sim 1$ fs (for $\alpha = 30$ and $\bar{a} = 0.001$), i.e., the recollision time, is short enough to maintain the nonspreading character of the wavepacket.

The next point is how to create the nonspreading wavepacket. For the latter, the specially tailored momentum chirping of the wavepacket given by the phase $\varphi(p) = \alpha b = \alpha \sinh^{-1}(p/m)$ is essential. This chirping induces a spatial shift of each momentum component in the laser field, $\delta x(p) = \partial\varphi(p)/\partial p$. The created wavepacket of the electron (positron) will be chirped if the particle with the corresponding momentum value is created with the corresponding spatial delay $\delta x(p)$. The particle in the LF moves with the momentum $p = m\gamma$. From Eqs. (36) and (37), γ is determined either by the laser field intensity a_0^2 or by the gamma-photon energy. Specifically tailoring the laser intensity in space according to the function $\delta x(p)$, one can achieve chirping of the created wavepackets of the electron and positron. Another possibility is to use a chirped γ-photon beam.

A. Role of radiation reaction

The approach based on the Dirac equation can be valid if the radiation reaction does not much disturb the electron dynamics. We can formulate it as a restriction on the laser and electron parameters. The condition for negligible radiation reaction can be formulated as the radiation energy loss ($\Delta\varepsilon$) being negligibly small compared to the electron energy (ε): $\Delta\varepsilon \ll \varepsilon$. In the laser-driven collider (Refs. [8,10]), the electron acceleration takes place during the excursion in a half cycle of the laser field. As the radiation formation length is a_0-times smaller than the electron trajectory period at $a_0 \gg 1$ (see, e.g., Ref. [6]), the number of the radiation formation lengths during one laser period is a_0. As the probability for a photon emission on a formation length is of the order of the fine-structure constant α_f, $\Delta\varepsilon \sim \alpha_f a_0\omega_c$, with the characteristic energy of the emitted photon $\omega_c \sim \chi\varepsilon$, where the quantum strong field parameter $\chi \equiv E'/E_{cr}$ describes the photon recoil (see, e.g., Ref. [6]). Here, E' is the background field strength in the rest frame of the electron and E_{cr} is the Schwinger critical field. In the laser collider setup, one can estimate $\chi \sim 2\gamma_0(\omega/m)a_0$, when the gamma photon with an energy $m\gamma_0$ counterpropagates the laser field as in Ref. [10]. Thus, the condition to neglect radiation reaction in the laser collider will read $\alpha_f a_0\chi \sim 2\alpha_f a_0^2\gamma_0(\omega/m) \ll 1$. For instance, in the case of an infrared laser field $\omega/m \sim 10^{-6}$ GeV, gamma photon $\gamma_0 \sim 10^3$, and an ultrastrong laser field of the intensity of 10^{22} W/cm^2 ($a_0 \sim 10^2$), this condition can be fulfilled, $\Delta\varepsilon/\varepsilon \sim 10^{-1}$. Thus, the approach based on the Dirac equation can still be valid for typical parameters of a laser-driven collider.

B. Role of spreading in the transversal directions

We discussed the possibility of generating a dispersionless Dirac electron wavepacket in a *plane* laser field. While this is an approximation, it is relevant to the discussed setup and employed parameters. This approximation can be valid when the electron transverse oscillation amplitude is much less than the laser beam waist size. We can estimate the electron transverse oscillation amplitude as $x_0 \sim (a_0/\gamma_0)(c/\omega)$ (see, e.g., [29]), with the laser field nonlinear parameter $a_0 = eE_0/mc\omega$, the laser frequency ω, and the electron gamma factor γ_0 after switching off the laser field. Using the parameters $a_0 = 10^2$ (10^{22} W/cm^2), $\lambda = 1$ µm, and $\gamma_0 = 30$, we have $x \approx 0.5$ µm. Assuming the laser waist size of $w_0 \sim 10$ µm, the condition $x_0 \ll w_0$ is fulfilled, which would require a strong but feasible 10 PW laser field [$10^{22} \times (10^{-3})^2$]. Moreover, the electron wavepacket size is of the order of $a \approx 10^{-2}\lambda_C$, with the Compton wavelength λ_C, and thus $a \ll w_0$ is safely fulfilled. Thus our plane-wave description is suitable for our employed realistic parameters given that the interference pattern of the wavepacket will not be affected.

VII. CONCLUSION

We have shown the existence of nonspreading relativistic wavepackets in a laser field, which, in the local rest frame of the electron, is similar to a self-accelerating nonspreading free wavepacket. We have established that there is a finite lifetime for the self-accelerating wavepacket and found that the wavepacket chirping and extension parameters impose strict restrictions on the lifetime duration. The nonspreading feature of the relativistic wavepacket represents the essential property to permit an efficient laser-driven high-energy collider.

APPENDIX A: DECOMPOSITION OF $\bar{\mathcal{R}}$ INTO BOOSTS AND ROTATIONS

In the simplified case with $a = 0$ and $\alpha = 0$, it is straightforward to see the following relationship:

$$\begin{aligned}\bar{\mathcal{R}} &= e^{-\frac{\eta'\gamma^0\gamma^3}{2}}UB,\\ U &= e^{-\theta\left(\gamma^1\gamma^3\frac{\dot{f}_1(\xi)}{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}+\gamma^2\gamma^3\frac{\dot{f}_2(\xi)}{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}\right)},\\ B &= e^{-w(V_1\gamma^0\gamma^1+V_2\gamma^0\gamma^2+V_3\gamma^0\gamma^3)},\end{aligned} \tag{A1}$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}{2mc}\right),$$

$$w = \tanh^{-1}\left(\frac{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}{2mc\sqrt{1+\left(\frac{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}{2mc}\right)^2}}\right)$$

$$
\begin{aligned}
&= \tanh^{-1}(\sin\theta),\\
V_1 &= \frac{\dot{f}_1(\xi)}{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}\cos\theta,\\
V_2 &= \frac{\dot{f}_2(\xi)}{\sqrt{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}}\cos\theta,\\
V_3 &= \sin\theta.
\end{aligned}
$$

Incidentally, the boost B leads to the following proper velocity:

$$
\begin{aligned}
\frac{\mathbf{u}}{c} &= B^2\gamma_0 = \gamma(\mathbf{1}+\gamma^0\gamma^k\beta_k),\\
\gamma &= 1+\frac{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}{2m^2c^2},\\
\vec{\beta} &= \left(\frac{\dot{f}_1(\xi)}{mc},\frac{\dot{f}_2(\xi)}{mc},\frac{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}{2m^2c^2}\right)\\
&\quad\times\left(1+\frac{\dot{f}_1(\xi)^2+\dot{f}_2(\xi)^2}{2m^2c^2}\right)^{-1}.
\end{aligned}
$$

This is expected since the solutions to the classical and quantum equations of motion for an electron in a laser field are, up to the phase factor to the right of the matrix spinor (i.e., Ψ in the main text), the same.

APPENDIX B: WAVEPACKET VARIANCE AND NORM

The wavepacket variance with respect to both the laboratory frame and the accelerated frame is defined as

$$\delta\bar{Z} = \sqrt{\langle\bar{Z}^2\rangle-\langle\bar{Z}\rangle^2},\quad \langle\bar{Z}^n\rangle = \int_{-\infty}^{+\infty} d\bar{Z}\,\bar{Z}^n\psi^\dagger\psi, \tag{B1}$$

$$\delta\bar{u} = \sqrt{\langle\bar{u}^2\rangle-\langle\bar{u}\rangle^2},\quad \langle\bar{u}^n\rangle = \int_0^{\infty} d\bar{u}\,\bar{u}^n\psi_R^\dagger\psi_R, \tag{B2}$$

while the wavepacket norm in both frames is

$$|\psi|^2 = \int_{-\infty}^{+\infty} d\bar{Z}\,\psi^\dagger\psi,\quad |\psi_R|^2 = \int_0^{\infty} d\bar{u}\,\psi_R^\dagger\psi_R. \tag{B3}$$

For the spinor (21), these integrals can be calculated exactly. The results are

$$
\begin{aligned}
\langle\bar{Z}^2\rangle &= K_1(2\bar{a})\frac{\bar{a}[4(\alpha^2-\bar{T}^2)+4\pi\bar{a}L_0(2\bar{a})(\alpha^2-\bar{T}^2)+1]}{2K_0(2\bar{a})}\\
&\quad-2\pi\bar{a}^2L_{-1}(2\bar{a})(\bar{T}-\alpha)(\alpha+\bar{T})+\frac{\pi\bar{a}(\bar{T}-\alpha)(\alpha+\bar{T})}{K_0(2\bar{a})}\\
&\quad+\bar{T}^2,
\end{aligned}\tag{B4}
$$

$$\langle\bar{Z}\rangle = \pi\alpha\left(\bar{a}L_{-1}(2\bar{a})+\frac{[2\bar{a}L_0(2\bar{a})K_1(2\bar{a})-1]}{2K_0(2\bar{a})}\right), \tag{B5}$$

where $K_n(2\bar{a})$ and $L_{-n}(2\bar{a})$ are the Bessel and the modified Struve functions, respectively. One can extract the important result for variance $\delta\bar{Z}^2 = \langle\bar{Z}^2\rangle-\langle\bar{Z}\rangle^2$,

$$
\begin{aligned}
\delta\bar{Z}(\alpha)^2-\delta\bar{Z}(0)^2 &= \pi\alpha^2\Bigg[\bar{a}^2L_{-1}(2\bar{a})[2-\pi L_{-1}(2\bar{a})]\\
&\quad-\frac{\pi[1-2\bar{a}L_0(2\bar{a})K_1(2\bar{a})]^2}{4K_0(2\bar{a})^2}+\frac{\bar{a}[\pi L_{-1}(2\bar{a})-1]}{K_0(2\bar{a})}\\
&\quad+\frac{\bar{a}\left\{\frac{2}{\pi}-2\bar{a}[\pi L_{-1}(2\bar{a})-1]L_0(2\bar{a})\right\}K_1(2\bar{a})}{K_0(2\bar{a})}\Bigg],
\end{aligned}\tag{B6}
$$

which is independent of time.

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