


Security proof for variable-length quantum key distribution

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 (Received 18 November 2023; revised 5 February 2024; accepted 12 March 2024; published 1 April 2024)

We present a security proof for variable-length QKD in the Renner framework against IID collective attacks. Our proof can be lifted to coherent attacks using the postselection technique. Our first main result is a theorem to convert a sequence of security proofs for fixed-length protocols satisfying certain conditions to a security proof for a variable-length protocol. This conversion requires no new calculations, does not require any changes to the final key lengths or the amount of error-correction information, and at most doubles the security parameter. Our second main result is the description and security proof of a more general class of variable-length QKD protocols, which does not require characterizing the honest behavior of the channel connecting the users before the execution of the QKD protocol. Instead, these protocols adaptively determine the length of the final key, and the amount of information to be used for error correction, based upon the observations made during the protocol. We apply these results to the qubit BB84 protocol, and show that variable-length implementations lead to higher expected key rates than the fixed-length implementations.

DOI: [10.1103/PhysRevResearch.6.023002](https://doi.org/10.1103/PhysRevResearch.6.023002)

I. INTRODUCTION

Security proofs for QKD protocols are typically proven in the “fixed-length” scenario, where two users Alice and Bob either produce a key of a fixed length, or abort the protocol [1–6]. Such protocols accept and produce a key of fixed length if and only if their observed statistics belong to some predetermined “acceptance set”. Otherwise, the protocol aborts. Such protocols have two main disadvantages.

First, in order to ensure that the protocol accepts with high probability for honest behavior, the acceptance set needs to be chosen carefully. Typically, the acceptance set is chosen to be the set of statistics that are close to what is expected from honest behavior [2–4]. This requires Alice and Bob to know the honest behavior of the channel connecting Alice and Bob, *before* a run of the QKD protocol. In many practical scenarios, such as ground-to-satellite QKD [7–11], it is difficult to know the behavior of the channel in advance. In fact, this can be a problem even in fibre-based setups [12–14].

Second, even if the honest behavior is known, the size of the acceptance set affects the length of the final key that can be produced. This reflects the fact that the key has to be secure for the *worst-case* event that accepts. Larger acceptance sets have a high probability of accepting on any given run of the QKD protocol, but lead to a shorter length of the final key, since they

include worse accept events. In particular, if users choose a large acceptance set, and then find that their observed statistics are much better than expected, they are *not* allowed to produce a larger key. Thus, there is a trade-off between protocols that accept with high probability, and protocols, which produce a large key on accepting.

A variable-length QKD protocol is one that allows users to adjust the length of the key generated based upon the observed statistics during the protocol [15,16]. This eliminates the trade-off described above. It also does not require the expected behavior of the channel to be known in advance. In fact, many prior studies have implemented such a variable-length protocol based upon intuition. For the qubit BB84 protocol, a rigorous treatment of variable-length protocols can be found in Ref. [16], using the phase-error approach for security proofs.

In this paper, we present a security proof for variable-length QKD protocols against IID collective attacks, which can be lifted to coherent attacks using the postselection technique [17]. Since that lift to coherent attacks is technical, and requires details of the postselection technique, it is included in Ref. [18] for pedagogical reasons. This paper differs from Ref. [16] in that it follows the lines of the Renner framework (i.e., bounding suitable entropies and applying a leftover hashing lemma) rather than the phase-error approach; in particular, it does not involve an explicit reduction to an analysis of a virtual phase-error-correction procedure.

This paper is organized as follows: In Sec. II we describe the QKD protocol steps, and setup the notation used in this paper. In Sec. III we show how, under certain conditions, a sequence of fixed-length QKD security proofs against IID collective attacks can be lifted to a variable-length QKD security proof against IID collective attacks. Our approach involves at most a doubling of the security parameter. Therefore, for the same target security parameter, the various key lengths

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for the variable-length protocol are nearly identical to the key lengths for the fixed-length protocols. In Sec. IV, we consider the scenario where the expected channel behavior is known in advance, and compute expected key rates for the qubit BB84 protocol, and show that the variable-length protocol generates better expected key rates than the best fixed-length protocol.

We then move on to study scenarios where the channel behavior is unpredictable, and *not known* in advance. In Sec. V we present another variable-length protocol, where the procedure for choosing the final key length (and length of error-correction information) is especially suited for such scenarios. Our protocol allows Alice and Bob to perform QKD without any prior knowledge about the channel connecting them. In Sec. VI we apply these results to the qubit BB84 protocol, and show that the variable-length implementation lead to higher expected key rates than the best fixed-length implementation.

In Sec. VII, we point out and remedy a gap between the theory and implementation of privacy amplification in QKD protocols. This gap exists because in implementations, privacy amplification is typically done on a register of variable length containing the raw sifted key, whereas in theory, privacy amplification is typically done on a register of fixed length containing data from all the signals. In Sec. VIII we present concluding remarks. Various technical details are delegated to the appendices.

II. NOTATION AND PROTOCOL SPECIFICATIONS

In this paper, we will either consider a *sequence of fixed-length* protocols indexed by i , or a *single variable-length* protocol where different events in the variable-length decision (see below) are indexed by i . We use the same index i since we construct the variable-length protocol from the sequence of fixed-length protocols in Sec. III. We describe the protocol steps for both fixed-length and variable-length protocols below.

A. Protocol steps

(1) *State preparation and transmission.* Alice prepares signal states and sends them to Bob, who measures them. We let N be the total number of signals sent by Alice. For prepare-and-measure protocols, the source-replacement scheme [19] can be used to equivalently describe this step as Alice and Bob receiving subsystems of the state $\rho_{A^N B^N E^N}$, followed by Alice measuring her subsystem A^N . In this case, one can assume that $\text{Tr}_{B^N E^N}(\rho_{A^N B^N E^N}) = \bar{\sigma}_A^{\otimes N}$, where $\bar{\sigma}_A^{\otimes N}$ is a fixed marginal state, which reflects the fact that Alice's system never leaves her laboratory, and that each signal is prepared independently. Furthermore, since we assume IID collective attacks, we have $\rho_{A^N B^N E^N} = \rho_{ABE}^{\otimes N}$.

(2) *Measurement.* Bob performs measurements on the received states, and stores measurement data.

(3) *Public announcements.* Alice and Bob select at random m rounds [20] out of the total N rounds, and announce their measurement outcomes for those rounds in the register C_{AT}^m . These announcements will be used to determine whether to accept or abort in the fixed-length protocol, or to determine

appropriate lengths of various strings in the variable-length protocol.

On the remaining $n := N - m$ rounds, Alice and Bob perform round-by-round announcements C^n (such as basis-choice, detect/no-detect). They store their private data in registers X^n and Y^n . The state of the protocol at this stage is given by $\rho_{X^n Y^n C^n C_{AT}^m E^n} = \rho_{XYCE}^{\otimes n} \otimes \rho_{C_{AT}^m E^m}$, where we split up the m test rounds and n key generation rounds. Note that since we assume IID collective attacks, the test round announcements C_{AT}^m and registers E^m are independent of the raw key Z^n .

From the public announcements C_{AT}^m , Alice and Bob compute \mathbf{F}^{obs} , which is the observed frequency of outcomes in the test rounds of the QKD protocol.

(4) *Acceptance test/variable-length decision.* For the i th *fixed-length* protocol, Alice and Bob accept the protocol if $\mathbf{F}^{\text{obs}} \in \tilde{Q}_i$. Here \tilde{Q}_i denotes the acceptance set for the protocol, and we use $\tilde{\Omega}_i$ to denote the event $\mathbf{F}^{\text{obs}} \in \tilde{Q}_i$. Note that in this paper, we use variables Ω and $\tilde{\Omega}$ (with subscripts) to denote the boolean variables corresponding to the occurrence of various events.

For the *variable-length* protocol, we instead use the following procedure: We have multiple disjoint sets Q_i , and use Ω_i to denote the event $\mathbf{F}^{\text{obs}} \in Q_i$. Depending on which event Ω_i is observed, Alice and Bob can choose different parameters in the processing of the data to the final key (for instance, the number of bits used in error correction, and the length of the final key).

Remark 1. It is important to note that for fixed-length protocols, the details of the acceptance test need to be determined *before* looking at \mathbf{F}^{obs} . In particular, current security proofs for such protocols do not allow users to *first* look at the observed statistics \mathbf{F}^{obs} and *then* decide the nature of the acceptance test. This is the reason why it is important to know the expected behavior of the channel *before* the QKD protocol is run, in order to design an acceptance test that accepts with high probability for honest behavior.

(5) *Key map and sifting.* Alice maps her raw data X^n to her raw key Z^n where Z is a binary variable, based on the announcements C^n . In this paper, we assume that Alice sets Z to 0 for signals that are sifted out, and let d_Z denote the dimension of the Z register.

(6) *Error correction.* Alice and Bob implement error correction by exchanging classical information in the register C_E . For the i th *fixed-length* protocol, we use λ_i^{EC} to denote the number of bits of communication during error correction, when the protocol accepts. For *variable-length* protocols we use λ_i^{EC} to denote the number of bits of communication during error correction, when event Ω_i occurs. Note that C_E may contain additional information beyond λ_i^{EC} bits, as long as the information is independent of Alice and Bob's data. For example, if the error-correction protocol randomly divides the data into blocks, then the descriptions of the randomly generated blocks can be included in C_E . Thus λ_i^{EC} actually refers to the number of bits in C_E that are computed from Alice and Bob's data.

Remark 2. It is important to note that one has to fix the exact number of bits of communication during error correction *before* the QKD protocol is run. In particular, current security proofs do not allow users to *first* count the number of bits used

during error correction and *then* adjust the length of the final key produced. For the framework described in this paper for variable-length protocols, it is still the case that the values λ_i^{EC} need to be decided before the protocol is run, i.e., for each i , the users must implement an error-correction procedure that uses a fixed number [21] of bits, rather than one that uses a randomly varying number of bits.

(7) *Error verification.* Alice chooses a two-universal hash function that hashes to $\lceil \log(1/\varepsilon_{\text{EV}}) \rceil$ bits, computes the hash of her raw key, and sends the hash value to Bob, along with the description of the hash function. Bob hashes his guess for Alice's key, compares the hash values, and announces whether the values match or not. We use C_V to denote the classical register that stores this communication. We note that since the hash function is chosen independently of Alice and Bob's data, only $\lceil \log(1/\varepsilon_{\text{EV}}) \rceil$ bits of C_V are correlated to Alice and Bob's data [22]. We use Ω_{EV} to denote the event that the hash values match, and Alice and Bob continue with the protocol. The state of the protocol at this stage is given by $\rho_{Z_B^n Z_B^m C_V E^n} \otimes \rho_{C_{AT}^m E^m}$, where Z_B^n is Bob's guess for Alice's raw key after error correction.

(8) *Privacy amplification.* For the i th *fixed-length* protocol, if event $\tilde{\Omega}_i \wedge \Omega_{\text{EV}}$ occurs, Alice chooses a two-universal hash function from n bits to l_i bits. She announces the description of the hash function in the register C_P , and Alice and Bob apply the hash function to their data to produce their final keys in registers K_A and K_B . We use $\rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}$ to denote the final state of the protocol, conditioned on the event $\tilde{\Omega}_i \wedge \Omega_{\text{EV}}$, where we use \tilde{C} to denote the registers $C^n C_{AT}^m C_E C_V C_P$ for brevity.

For the *variable-length* protocol, if event $\Omega_i \wedge \Omega_{\text{EV}}$ occurs, Alice chooses a two-universal hash function from n bits to l_i bits. She announces the description of the hash function in the register C_P , and Alice and Bob apply the hash function to their data to produce their final keys in registers K_A and K_B . We use $\rho_{K_A K_B \tilde{C} E^n | \Omega_i \wedge \Omega_{\text{EV}}}$ to denote the final state of the protocol, conditioned on the event $\Omega_i \wedge \Omega_{\text{EV}}$.

Thus for all the fixed-length QKD protocols, the details of the acceptance test, and the value of λ_i^{EC} and l_i must be fixed *before* the start of the protocol. For the variable-length QKD protocol, the details of the variable-length decision (in particular the values of λ_i^{EC} and l_i) must be fixed *before* the start of the protocol. Moreover, the events $\tilde{\Omega}_i, \Omega_i$ determine the pair of values $(l_i, \lambda_i^{\text{EC}})$ for the corresponding protocol.

III. VARIABLE-LENGTH SECURITY FROM FIXED-LENGTH SECURITY

In this section, we show how a sequence of security proofs for fixed-length protocols (against IID collective attacks) can be converted to a security proof for variable-length protocols (against IID collective attacks). Let us suppose that we have M fixed-length QKD protocols, indexed by $i \in \{1, 2, \dots, M\}$. The protocols differ only in their choice of acceptance test, the number of bits used for error correction, and length of the final key generated. In particular, the i th protocol accepts if and only if $\mathbf{F}^{\text{obs}} \in \tilde{Q}_i$, where \tilde{Q}_i is the acceptance set. Upon acceptance, it uses λ_i^{EC} bits for error correction and produces a key of fixed length l_i .

A. Fixed-length security statements

Following standard composable security definitions [23,24], the i th fixed-length QKD protocol is said to be $\varepsilon_{\text{secure}}$ -secure against some class of attacks if the following condition holds: for all attacks in that class, at the end of the protocol we have

$$\frac{1}{2} \Pr(\tilde{\Omega}_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\| \leq \varepsilon_{\text{secure}}. \quad (1)$$

Here, $\rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})}$ denotes the actual output state at the end of the protocol, while $\rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})}$ denotes an "ideal state" obtained by replacing the key registers of $\rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})}$ with perfect keys, i.e.,

$$\rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} := \sum_{k \in \{0, 1\}^{l_i}} \frac{|kk\rangle \langle kk|}{2^{l_i}} \otimes \rho_{\tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})}. \quad (2)$$

Note that $\rho_{K_A K_B \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})}$ is not a "fixed" state but rather a function of the input state.

In particular, in this paper we focus on restricting the class of attacks to IID collective attacks, which means we suppose that the input states supplied to the QKD protocol are always of the form $\rho_{ABE}^{\otimes N}$. For prepare-and-measure protocols, the input states are further constrained to satisfy $\text{Tr}_{BE}(\rho_{ABE}) = \bar{\sigma}_A$, where $\bar{\sigma}_A$ is the fixed marginal state on Alice's system that is obtained from the source-replacement scheme [19]. (For entanglement-based protocols, this constraint is not imposed; either version can be handled using the framework presented in this paper.)

Furthermore, as explained in Ref. [15,23,24], to show that a QKD protocol is $\varepsilon_{\text{secure}}$ -secure (against some class of attacks), it suffices to prove a pair of simpler conditions, namely that it is ε_1 -correct and ε_2 -secret (against that class of attacks) with $\varepsilon_1 + \varepsilon_2 \leq \varepsilon_{\text{secure}}$. Specifically, ε_1 -correctness means the output state satisfies

$$\Pr(K_A \neq K_B \wedge \Omega_{\text{EV}}) \leq \varepsilon_1, \quad (3)$$

while ε_2 -secrecy means it satisfies

$$\frac{1}{2} \Pr(\tilde{\Omega}_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^n | \tilde{\Omega}_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\| \leq \varepsilon_2, \quad (4)$$

i.e., the same condition as $\varepsilon_{\text{secure}}$ -security [Eq. (1)] but with Bob's key register omitted. (Both of the above conditions are to be implicitly understood as holding against all attacks in the considered class.) In our subsequent discussion, we shall indeed proceed by proving the above pair of conditions rather than Eq. (1) directly.

We assume that for each protocol $i \in \{1, 2, \dots, M\}$, the following statements have been shown to be true. As we shall shortly show, these statements together imply Eq. (1) with $\varepsilon_{\text{secure}} = \varepsilon_{\text{EV}} + \max\{\varepsilon_{\text{AT}}, \varepsilon_{\text{PA}}\}$, following established approaches described in, e.g., [2–4].

(1) There is a "feasible" set $\tilde{S}_i \subseteq \{\rho \in S_{\circ}(AB) | \text{Tr}_B(\rho_{AB}) = \bar{\sigma}_A\}$, where S_{\circ} denotes the set of normalized states, such that if the state ρ_{AB} is not in the set \tilde{S}_i , the protocol aborts with high

probability, and is thus secure. That is,

$$\rho_{AB} \notin \tilde{S}_i \Rightarrow \Pr(\tilde{\Omega}_i) \leq \varepsilon_{AT}. \quad (5)$$

Note that in the entirety of this paper, the statement $\rho \notin \tilde{S}_i$ is assumed to be with respect to the parent set $\{\rho \in S_o(AB) | \text{Tr}_B(\rho_{AB}) = \bar{\sigma}_A\}$ having the fixed marginal on A .

(2) The hash length l_i is given by

$$l_i = \left\lceil n \min_{\rho \in \tilde{S}_i} H(Z|CE)_\rho - \lambda_i^{\text{EC}} - \lceil \log(1/\varepsilon_{EV}) \rceil - n(\alpha - 1) \log^2(d_Z + 1) - \frac{\alpha}{\alpha - 1} \left(\log\left(\frac{1}{4\varepsilon_{PA}}\right) + \frac{2}{\alpha} \right) \right\rceil, \quad (6)$$

where H denotes the conditional von Neumann entropy, and d_Z denotes the dimension of the Z register. This choice of l_i is such that if the state $\rho_{AB} \in \tilde{S}_i$, a key of length l_i can be safely extracted from the protocol. In the entirety of this work, we choose $\alpha = 1 + \kappa/\sqrt{n}$ with $\kappa := \sqrt{\log(1/\varepsilon_{PA})/\log(d_Z + 1)}$, assuming n is large enough to

ensure that $\alpha \leq 1 + 1/\log(2d_Z + 1)$ is satisfied. This is the choice of α that maximizes Eq. (6) [up to a minor approximation that $\alpha/(\alpha - 1) \approx 1/(\alpha - 1)$], and also leads to the expected asymptotic scaling in the key rate expression.

(3) The error-verification step compares two-universal hashes of length $\lceil \log(1/\varepsilon_{EV}) \rceil$.

To see how these three statements imply Eq. (1), we first note that the protocol is ε_{EV} -correct,

$$\begin{aligned} \Pr(K_A \neq K_B \wedge \Omega_{EV}) &\leq \Pr(Z^n \neq Z_B^n \wedge \Omega_{EV}) \\ &\leq \Pr(\Omega_{EV} | Z^n \neq Z_B^n) \leq \varepsilon_{EV}, \end{aligned} \quad (7)$$

where Z_B^n denotes Bob's guess for Alice's raw key, and the second inequality follows from the fact that $K_A \neq K_B \Rightarrow Z^n \neq Z_B^n$, while the final inequality follows from that the fact that error-verification step compares hashes of length $\lceil \log(1/\varepsilon_{EV}) \rceil$.

Furthermore, we obtain the following chain of inequalities using some technical lemmas from [25–27], which we restate in Appendix A. The derivation of these inequalities is explained below. We obtain

$$\begin{aligned} &\frac{1}{2} \Pr(\tilde{\Omega}_i \wedge \Omega_{EV}) \left\| \rho_{K_A \tilde{C} E^N | \tilde{\Omega}_i \wedge \Omega_{EV}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \tilde{\Omega}_i \wedge \Omega_{EV}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\ &\leq \frac{1}{2} \Pr(\tilde{\Omega}_i \wedge \Omega_{EV}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{AT}^m C_E C_V E^n)_{\rho | \tilde{\Omega}_i \wedge \Omega_{EV}} - l_i) + \frac{2}{\alpha} - 1} \\ &\leq \frac{1}{2} \Pr(\tilde{\Omega}_i \wedge \Omega_{EV}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n E^n)_{\rho | \tilde{\Omega}_i \wedge \Omega_{EV}} - \lambda_i^{\text{EC}} - \lceil \log(1/\varepsilon_{EV}) \rceil - l_i) + \frac{2}{\alpha} - 1} \\ &\leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n E^n)_\rho - \lambda_i^{\text{EC}} - \lceil \log(1/\varepsilon_{EV}) \rceil - l_i) + \frac{2}{\alpha} - 1} \\ &= \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(nH_\alpha(Z|CE)_\rho - \lambda_i^{\text{EC}} - \lceil \log(1/\varepsilon_{EV}) \rceil - l_i) + \frac{2}{\alpha} - 1} \\ &\leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(nH(Z|CE)_\rho - n(\alpha-1) \log^2(d_Z + 1) - \lambda_i^{\text{EC}} - \lceil \log(1/\varepsilon_{EV}) \rceil - l_i) + \frac{2}{\alpha} - 1} \\ &\leq \varepsilon_{PA} \quad \forall \rho \in \tilde{S}_i, \end{aligned} \quad (8)$$

where H_α denotes the Rényi entropy (see Definition 1 and Appendix A) with α as the Rényi parameter. Here we used the leftover hashing lemma for Rényi entropy [[26], Theorem 8] (restated in Lemma 6) in the first inequality, and Lemma 11 to split off the error-correction and error-verification information for the second inequality, along with the registers C_{AT}^m, E^m (which are independent of Z^n). We further use Lemma 9 to get rid of the conditioning on acceptance events in the third inequality. The fourth equality follows from additivity of Rényi entropy (Lemma 7), and fifth inequality follows from Lemma 8. The choice of l_i from Eq. (6) is the largest possible value that guarantees the final inequality in Eq. (8). The IID assumption comes into play in the use of Lemma 7.

Since either $\rho \in \tilde{S}_i$ or $\rho \notin \tilde{S}_i$, Eqs. (5) and (8) together imply that the protocol is $\max\{\varepsilon_{AT}, \varepsilon_{PA}\}$ -secret,

$$\begin{aligned} &\frac{1}{2} \Pr(\tilde{\Omega}_i \wedge \Omega_{EV}) \left\| \rho_{K_A \tilde{C} E^N | \tilde{\Omega}_i \wedge \Omega_{EV}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \tilde{\Omega}_i \wedge \Omega_{EV}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\ &\leq \max\{\varepsilon_{AT}, \varepsilon_{PA}\}, \end{aligned} \quad (9)$$

for all states $\text{Tr}_{B^N E^N}(\rho_{ABE}^{\otimes N}) = \bar{\sigma}_A^{\otimes N}$. Finally, as previously mentioned, a fixed-length QKD protocol that is ε_{EV} -correct [Eq. (7)] and $\max\{\varepsilon_{AT}, \varepsilon_{PA}\}$ -secret, is $(\max\{\varepsilon_{AT}, \varepsilon_{PA}\} + \varepsilon_{EV})$ -secure [Eq. (1)], as desired.

B. From fixed-length security to variable-length security

For a variable-length protocol, again following composable security definitions [15,24], we say that it is $\varepsilon_{\text{secure}}$ -secure against some class of attacks [28] if the following condition holds: for all attacks in that class, at the end of the protocol we have

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A K_B \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} \right\|_1 \\ &\leq \varepsilon_{\text{secure}}. \end{aligned} \quad (10)$$

Here, $\Omega_{\text{len}=k}$ denotes the event that a final key [29] of length k is produced, while $\rho_{K_A K_B \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k)}$ denotes the actual output state at the end of the protocol conditioned on the event

$\Omega_{\text{len}=k}$, and $\rho_{K_A K_B \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})}$ denotes an ‘‘ideal state’’ obtained by replacing the key registers of $\rho_{K_A K_B \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k)}$ with perfect keys of length k [analogous to Eq. (2)]. Note that we recover the security definition of fixed-length protocols [Eq. (1)] from Eq. (10) by setting k to be a fixed value $k = l_i$ in the sum in Eq. (10), and noting that $\Omega_{\text{len}=k}$ is the same event as $\tilde{\Omega}_i \wedge \Omega_{\text{EV}}$, corresponding to a key length of l_i bits, and λ_i^{EC} bits used for error correction. Again, in this paper we focus only on IID collective attacks, in the sense previously described in Sec. III A.

Similar to fixed-length protocols, one can define correctness and secrecy for variable-length protocols. Specifically, we shall take ε_1 -correctness to be defined the same way as before [Eq. (3)], while ε_2 -secrecy is analogously defined by omitting Bob’s registers from the variable-length $\varepsilon_{\text{secure}}$ -security condition, i.e., for all attacks (in the considered class) we have

$$\sum_{k=0}^{\infty} \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A \tilde{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} \right\|_1 \leq \varepsilon_2. \quad (11)$$

Just as in the fixed-length case, for the variable-length case we also have the property that ε_1 -correctness and ε_2 -secrecy together imply $(\varepsilon_1 + \varepsilon_2)$ -security—the argument is identical to the fixed-length case [15,23,24], and we provide it in Lemma 12 of Appendix A.

In order to use the security statements from Sec. III A for (a sequence of) fixed-length protocols to prove security for a variable-length protocol, we require the acceptance sets \tilde{Q}_i of those fixed-length protocols to satisfy the following condition. We assume that the acceptance sets \tilde{Q}_i for the fixed-length protocols are ordered such that $\tilde{Q}_i \subseteq \tilde{Q}_{i+1}$. This can in principle be satisfied by suitable construction of the acceptance sets, as we show in Sec. IV (although the resulting variable-length protocol may not be suitable in all contexts, as we discuss later in Sec. V). Without loss of generality, we can then pick feasible sets \tilde{S}_i such that $\tilde{S}_i \subseteq \tilde{S}_{i+1}$, since

$$\rho \notin \tilde{S}_{i+1} \Rightarrow \Pr(\tilde{\Omega}_{i+1}) \leq \varepsilon_{\text{AT}} \Rightarrow \Pr(\tilde{\Omega}_i) \leq \varepsilon_{\text{AT}}. \quad (12)$$

Thus, the feasible set \tilde{S}_i can always be chosen to be smaller than the feasible set \tilde{S}_{i+1} .

Remark 3. Recall from Eq. (6), we have $l_i = \lfloor n \min_{\rho \in \tilde{S}_i} H(X|CE)_\rho - \lambda_i^{\text{EC}} - \text{constant correction terms} \rfloor$. Thus, $\tilde{S}_i \subseteq \tilde{S}_{i+1}$ implies that $l_i + \lambda_i^{\text{EC}}$ is a *nonincreasing sequence in i* . This property will play a crucial part in proving the security of our variable-length protocol.

We now use the acceptance sets \tilde{Q}_i from the sequence of fixed-length protocols to construct the sets Q_i (in the variable-length decision step) for a variable-length protocol. Specifically, let us define $Q_1 := \tilde{Q}_1$, and $Q_i := \tilde{Q}_i \setminus \tilde{Q}_{i-1}$. We then prove the following theorem concerning the security of variable-length QKD protocols.

Theorem 1. Let there be a sequence of fixed-length QKD protocols that vary only in their acceptance criterion (\tilde{Q}_i), length of error-correction communication (λ_i^{EC}) and final hash length l_i . Suppose that for each of these fixed-length protocols, we have a security proof against IID collective attacks, in which Eqs. (5)–(7) are true. Furthermore, suppose $\tilde{Q}_i \subseteq \tilde{Q}_{i+1}$ and $\tilde{S}_i \subseteq \tilde{S}_{i+1}$. Then, the variable-length protocol that upon the event $\Omega_i \wedge \Omega_{\text{EV}}$, generates a key of length l_i while having used

λ_i^{EC} number of bits for error correction, is $(\varepsilon_{\text{AT}} + \varepsilon_{\text{EV}} + \varepsilon_{\text{PA}})$ -secure against IID collective attacks, where the values of ε_{AT} , ε_{EV} , ε_{PA} are the same as those in the fixed-length protocol statements Eqs. (5)–(7).

Proof. As before, we will prove that the protocol is ε_{EV} -correct and $(\varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secret. This will then imply that the protocol is $(\varepsilon_{\text{EV}} + \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secure.

The proof of ε_{EV} -correctness of the protocol remains essentially the same as before [Eq. (7)],

$$\begin{aligned} \Pr(K_A \neq K_B \wedge \Omega_{\text{EV}}) &\leq \Pr(Z^n \neq Z_B^n \wedge \Omega_{\text{EV}}) \\ &\leq \Pr(\Omega_{\text{EV}} | Z^n \neq Z_B^n) \\ &\leq \varepsilon_{\text{EV}}. \end{aligned} \quad (13)$$

We now focus on proving the $(\varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secrecy of the protocol. To do so, we first note that the secrecy definition for variable-length protocols [Eq. (11)] groups together terms with the same output length of the key. However, the different events $\Omega_i \wedge \Omega_{\text{EV}}$ may correspond to the same output length of the key and different lengths of error-correction information. Nevertheless, the events $\Omega_i \wedge \Omega_{\text{EV}}$ are deterministic functions of public announcements C_{AT}^m, C_V . Thus, the states $\rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})}$ conditioned on event $\Omega_i \wedge \Omega_{\text{EV}}$ have orthogonal supports. Therefore, it is sufficient to show that

$$\begin{aligned} \sum_{i=1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\ \leq \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}}, \end{aligned} \quad (14)$$

since we can group terms with the same output key length in Eq. (14) to show that Eqs. (11) and (14) are equivalent. (An analogous argument can be conducted at the level of the security condition [Eq. (10)] directly, though here we focus on just the secrecy condition since that (together with correctness) is sufficient to imply the security condition.) We will now prove Eq. (14).

We proceed by noting that since we have the ordering $\tilde{S}_i \subseteq \tilde{S}_{i+1}$, any input state ρ_{AB} has to fall under exactly one of the following three cases:

- (1) $\rho_{AB} \in \tilde{S}_1$.
- (2) $\rho_{AB} \notin \tilde{S}_j$ but $\rho \in \tilde{S}_{j+1}$ for some j .
- (3) $\rho_{AB} \notin \tilde{S}_M$.

We prove the secrecy claim separately for each case. We start with Case 2.

Case 2. If $\rho \notin \tilde{S}_j$ but $\rho \in \tilde{S}_{j+1}$ for some j , we split up the security definition from Eq. (10) into two parts. For the first part, we show that if $\rho \notin \tilde{S}_j$, the probability of the protocol obtaining the event $\Omega_1 \cup \dots \Omega_j$ is small,

$$\begin{aligned} \sum_{i=1}^j \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\ \leq \sum_{i=1}^j \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \leq \sum_{i=1}^j \Pr(\Omega_i) = \Pr(\tilde{\Omega}_j) \leq \varepsilon_{\text{AT}}, \end{aligned} \quad (15)$$

where the final inequality follows from the fixed-length security statement [Eq. (5)].

To bound the remaining terms, we use some technical lemmas from [25–27] (which are restated in Appendix A), to obtain the following chain of inequalities:

$$\begin{aligned}
& \sum_{i=j+1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\
& \leq \sum_{i=j+1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{AT}^m C_V E^N)_\rho | \Omega_i \wedge \Omega_{\text{EV}} - l_i) + \frac{2}{\alpha} - 1} \\
& \leq \sum_{i=j+1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{AT}^m C_V E^n)_\rho | \Omega_i \wedge \Omega_{\text{EV}} - \lambda_i^{\text{EC}} - l_i) + \frac{2}{\alpha} - 1} \\
& \leq \sum_{i=j+1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{AT}^m C_V E^n)_\rho | \Omega_i \wedge \Omega_{\text{EV}} - \lambda_{j+1}^{\text{EC}} - l_{j+1}) + \frac{2}{\alpha} - 1} \\
& \leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{AT}^m C_V E^n)_\rho - \lambda_{j+1}^{\text{EC}} - l_{j+1}) + \frac{2}{\alpha} - 1} \\
& \leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n E^n)_\rho - \lambda_{j+1}^{\text{EC}} - \lceil \log(1/\varepsilon_{\text{EV}}) \rceil - l_{j+1}) + \frac{2}{\alpha} - 1} \\
& = \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(nH_\alpha(Z|CE)_\rho - \lambda_{j+1}^{\text{EC}} - \lceil \log(1/\varepsilon_{\text{EV}}) \rceil - l_{j+1}) + \frac{2}{\alpha} - 1} \\
& \leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(nH(Z|CE)_\rho - n(\alpha-1)\log^2(d_Z+1) - \lambda_{j+1}^{\text{EC}} - \lceil \log(1/\varepsilon_{\text{EV}}) \rceil - l_{j+1}) + \frac{2}{\alpha} - 1} \leq \varepsilon_{\text{PA}}. \tag{16}
\end{aligned}$$

The inequalities above are explained below, with the crucial step explained in Remark 4. We used the leftover hashing lemma for Rényi entropy in the first inequality (Lemma 6), and Lemma 11 to split off the information leakage due to error correction and the E^m register (which is independent of Z^n), in the second inequality. For the third inequality, we use the fact that $l_i + \lambda_i^{\text{EC}}$ is a nonincreasing sequence in i (Remark 3). We use Lemma 10 to get rid of the conditioning on events for the fourth inequality, and Lemma 11 to split off information leakage due to error verification and the C_{AT}^m register (which is independent of Z^n) in the fifth inequality. The sixth equality follows from the additivity of Rényi entropy (Lemma 7), while the seventh inequality follows from Lemma 8. Finally, we use the security proof statement for fixed-length protocols [Eq. (8)] and the fact that $\rho \in \tilde{\mathcal{S}}_{j+1}$ for the final inequality.

Remark 4. We highlight two critical steps in Eq. (16). The first is in the third inequality, where we replace $l_i + \lambda_i^{\text{EC}}$ with the constant value $l_{j+1} + \lambda_{j+1}^{\text{EC}}$, using Remark 3. The second is in the use of Lemma 10 in the fourth inequality, which allows us to get rid of terms involving Rényi entropies of the state conditioned on events. In particular, smooth min-entropy does not straightforwardly allow a statement analogous to Lemma 10, which is the reason for using Rényi entropy in this paper. Moreover we split off the C_{AT}^m , C_V registers after using Lemma 10, since we require the events $\Omega_i \wedge \Omega_{\text{EV}}$ to be known to Eve to use Lemma 10 for the fourth inequality.

Bringing Eqs. (15) and (16) together, we obtain that the protocol is $(\varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secret,

$$\begin{aligned}
& \sum_{i=1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\
& \leq \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}}. \tag{17}
\end{aligned}$$

Case 1 and Case 3. The analysis of Case 1 is a special case of the above analysis, and follows from choosing $j = 0$ in Eq. (16). The analysis of Case 3 is also a special case, and follows from choosing $j = M$ in Eq. (15).

The theorem claim then follows from the correctness and secrecy statements. ■

Thus, a sequence of security proofs for fixed-length protocols satisfying certain conditions can be turned into a security proof for a variable-length protocol. Moreover, the only penalty imposed by our approach is a minor increase in the security parameter of the protocol, which goes from $(\max\{\varepsilon_{\text{AT}}, \varepsilon_{\text{PA}}\} + \varepsilon_{\text{EV}})$ for the fixed-length case, to $(\varepsilon_{\text{AT}} + \varepsilon_{\text{PA}} + \varepsilon_{\text{EV}})$ for the variable-length case. In fact, this minor penalty is completely compensated by the ability to generate longer keys in the variable-length case, as we show in the next section.

IV. APPLICATION TO QUBIT BB84

In this section we will show how Theorem 1 can be utilized to improve the *expected key rate* [30] of QKD protocols. For the sake of simplicity, we consider the qubit-based BB84 protocol to illustrate our results. However, Theorem 1 can be directly applied to any fixed-length protocols whose security proof satisfies 567. We use Ref. [2] for the finite-size security proof of qubit BB84, and the numerical key rate framework from Ref. [31] to compute key rates. The signal preparation and measurement steps of the qubit BB84 protocol are described in Appendix B. The acceptance test and key rate computation is described in Sec. IV B. We start by explaining the notion of expected key rates.

A. Expected key rate

Before defining expected key rates we first set up the following notation:

(1) $R_{\text{fixed},i}$: This denotes the key rate obtained upon the event $\tilde{\Omega}_i \wedge \Omega_{\text{EV}}$ for the i th fixed-length protocol.

(2) $R_{\text{variable},i}$: This denotes the key rate obtained upon the event $\Omega_i \wedge \Omega_{\text{EV}}$ for the variable-length protocol.

(3) $\bar{R}_{\text{fixed},i}$: This denotes the expected key rate for the i th fixed-length protocol.

(4) $\bar{R}_{\text{variable}}$: This denotes expected key rate for the variable-length protocol.

(5) ρ_{hon} : This denotes the state corresponding to the honest implementation of the QKD protocol.

Note that $R_{\text{fixed},i}$ and $R_{\text{variable},i}$ are obtained from the security proofs, and are independent of honest behavior.

For the purposes of this paper, in all the expected key rate computations we assume that the probability of the event Ω_{EV} in the honest case is approximately 1. We make this simplifying assumption because the true value would depend on the (honest) probability of Bob correctly guessing Alice's key in the error-correction step, but many error-correction protocols used in practice do not have rigorous lower bounds on this probability, only heuristic estimates. We stress, however, that this in no way affects our proof that the protocol satisfies the *security* condition, which does not require any lower bound on this probability.

With this approximation, the expected key rate for the i th fixed-length protocol is given by

$$\begin{aligned} \bar{R}_{\text{fixed},i} &:= \Pr(\tilde{\Omega}_i \wedge \Omega_{\text{EV}})_{\rho_{\text{hon}}} R_{\text{fixed},i} \\ &\approx \Pr(\tilde{\Omega}_i)_{\rho_{\text{hon}}} R_{\text{fixed},i}, \end{aligned} \quad (18)$$

where $\Pr(\tilde{\Omega}_i)_{\rho_{\text{hon}}}$ is the probability of the protocol accepting during honest behavior, and $R_{\text{fixed},i}$ is the key rate upon accepting for the i th protocol. We use \bar{R}_{fixed} as a useful metric to compare the *practical key rate* of a QKD protocol.

We can generalize the notion of expected key rate to the variable-length case in a straightforward manner. We define

$$\begin{aligned} \bar{R}_{\text{variable}} &:= \sum_{i=1}^M \Pr(\Omega_i \wedge \Omega_{\text{EV}})_{\rho_{\text{hon}}} R_{\text{variable},i} \\ &\approx \sum_{i=1}^M \Pr(\Omega_i)_{\rho_{\text{hon}}} R_{\text{variable},i}, \end{aligned} \quad (19)$$

where $\Pr(\Omega_i)_{\rho_{\text{hon}}}$ is the probability of obtaining the event Ω_i for honest implementations, and $R_{\text{variable},i}$ is the key rate obtained upon the event Ω_i . Again, $\bar{R}_{\text{variable}}$ is a useful metric to compare the *practical key rate* of a QKD protocol.

Thus, the expected key rates can be computed from Eqs. (18) and (19). The values of $R_{\text{fixed},i}$ and $R_{\text{variable},i}$ can be obtained from security proofs. The probabilities $\Pr(\tilde{\Omega}_i)_{\rho_{\text{hon}}}$ and $\Pr(\Omega_i)_{\rho_{\text{hon}}}$ can be estimated numerically, by simulating the channel a large number of times, and computing the fraction of runs that lead to events $\tilde{\Omega}_i$ and Ω_i .

We now describe the acceptance test from Ref. [2], and key rate computations.

B. Acceptance test and key rates

Consider a sequence of fixed-length protocols indexed by $i \in \{1, 2, \dots, M\}$. Let Σ denote the set of outcomes that can take place in the test rounds. For qubit BB84, Σ consists of the 16 possible outcomes corresponding to Alice's choice of signal state and Bob's measurement outcome. Then following Ref. [2], we shall define the acceptance set \tilde{Q}_i for each of these fixed-length protocols as

$$\tilde{Q}_i := \{\mathbf{F}^{\text{obs}} \in \mathcal{P}(\Sigma) \mid \|\mathbf{F}^{\text{obs}} - \bar{\mathbf{F}}\|_1 \leq t_i\}, \quad (20)$$

where $\mathcal{P}(\Sigma)$ the set of probability distributions on Σ . Here $\bar{\mathbf{F}}$ is the probability vector of outcomes for the honest implementation, i.e., each entry of $\bar{\mathbf{F}}$ is the probability of obtaining some outcome in a single round of the honest implementation, as determined by Eq. (21) below. \mathbf{F}^{obs} is the observed frequency of outcomes, and the acceptance test checks whether the observed frequency of outcomes is close to the expected frequency ($\bar{\mathbf{F}}$). Note that one can easily satisfy the condition $\tilde{Q}_i \subseteq \tilde{Q}_{i+1}$, by choosing $t_i \leq t_{i+1}$. This ensures that Theorem 1 can be applied safely.

Let Γ_j be the POVM element corresponding to the j th outcome, and define

$$\Phi(\rho) := \sum_{j \in \Sigma} \text{Tr}(\Gamma_j \rho) |j\rangle \langle j| \quad (21)$$

to be the map that takes the state ρ and outputs the probability distribution over the outcomes. Given such an acceptance set \tilde{Q}_i , a feasible set \tilde{S}_i satisfying Eq. (5) is given by [[2], Theorem 8]

$$\tilde{S}_i := \{\rho \in S_o(AB) \mid \|\Phi(\rho) - \bar{\mathbf{F}}\|_1 \leq t_i + \mu\}, \quad (22)$$

where μ is given by

$$\mu := \sqrt{2} \sqrt{\frac{\ln(1/\epsilon_{\text{AT}}) + |\Sigma| \ln(m+1)}{m}}. \quad (23)$$

The construction of this set crucially uses the concentration inequality from Lemma 13 (Appendix B 1).

Therefore, the key length l_i satisfying Eq. (6) is given by

$$\begin{aligned} l_i &\leq n \min_{\substack{\rho \in \tilde{S}_i \\ \text{Tr}_B(\rho) = \bar{\sigma}_A}} H(Z|CE)_\rho - \lambda_i^{\text{EC}} - n(\alpha - 1) \log^2(2d_Z + 1) \\ &\quad - \lceil \log(1/\epsilon_{\text{EV}}) \rceil - \frac{\alpha}{\alpha - 1} \left(\log\left(\frac{1}{4\epsilon_{\text{PA}}}\right) + \frac{2}{\alpha} \right), \end{aligned} \quad (24)$$

where d_Z is the dimension of Z , λ_i^{EC} is the number of bits used for error correction, n is the number of signals for key generation, and we set the Rényi parameter to be $\alpha = 1 + \kappa/\sqrt{n}$ with $\kappa := \sqrt{\log(1/\epsilon_{\text{PA}})/\log(d_Z + 1)}$. Moreover, λ_i^{EC} can be chosen to be any number for the purposes of proving the security of the protocol. However, a careful choice of λ_i^{EC} and design of the error-correction protocol is necessary to guarantee that the protocol passes error verification with high probability for honest behavior.

C. Results

We now use the above results to compare the expected key rates for fixed-length protocols and variable-length protocols. We consider a protocol with honest behavior determined by

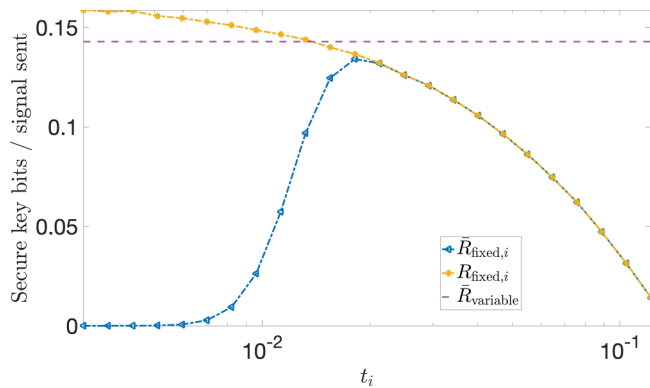


FIG. 1. Expected key rate for fixed-length protocols ($\bar{R}_{\text{fixed},i}$) for various values of t_i , key rate upon acceptance for fixed-length protocols ($R_{\text{fixed},i}$) for various values of t_i , and the expected key rate for the variable-length protocol ($\bar{R}_{\text{variable}}$) constructed from the fixed-length protocols.

a depolarization probability of 0.02, and misalignment angle about the Y axis of $\theta = 2^\circ$. We set the basis choice probabilities to $p_z = p_x = 0.5$. The total number of signals is given by $N = 10^6$, and the number of signals used for testing is given by $m = 0.05N$. The number of bits used for error correction is always taken to be $\lambda_i^{\text{EC}} = \lambda^{\text{EC}} = fnH(Z|YC)_{\rho_{\text{hon}}}$ where $f = 1.16$ is the efficiency parameter. We set $\alpha = 1 + \kappa/\sqrt{n}$ with $\kappa := \sqrt{\log(1/\epsilon_{\text{PA}})}/\log(d_Z + 1)$, and fix a range of values of t_i (horizontal axis of Fig. 1) such that $t_i \leq t_{i+1}$. This determines the acceptance sets \tilde{Q}_i for the fixed-length protocols, and thus also the sets Q_i in the variable-length decision of the variable-length protocol we construct (Sec. III). We set a target security parameter of $\epsilon_{\text{secure}} = 10^{-12}$. We plot various key rates in Fig. 1, which are explained below.

(1) $R_{\text{fixed},i}$: This is the key rate upon acceptance for fixed-length protocols for various values of t_i . Since $\epsilon_{\text{secure}} = \max\{\epsilon_{\text{PA}}, \epsilon_{\text{AT}}\} + \epsilon_{\text{EV}}$ for fixed-length protocols, we set $\epsilon_{\text{AT}} = \epsilon_{\text{PA}} = \epsilon_{\text{EV}} = \epsilon_{\text{secure}}/2$. We use Eqs. (24) and (22) to compute l_i for various values of t_i , and plot $R_{\text{fixed},i} = l_i/N$. We see that $R_{\text{fixed},i}$ decreases monotonically on increasing values of t_i , which reflects the fact that larger acceptance sets lead to lower key rate upon acceptance.

(2) $\bar{R}_{\text{fixed},i}$: This is the expected key rate for fixed-length protocols, for various values of t_i . We use the values of $R_{\text{fixed},i}$ obtained above, along with Eq. (18) to compute $\bar{R}_{\text{fixed},i}$. The probability of accepting the protocol $\Pr(\tilde{\Omega}_i)_{\rho_{\text{hon}}}$ is computed as follows:

(1) We first compute the probability vector $\bar{\mathbf{F}}$ corresponding to the honest behavior ρ_{hon} , by setting $\bar{\mathbf{F}} = \Phi(\rho_{\text{hon}})$, where Φ is defined in Eq. (21).

(2) We sample m times from $\bar{\mathbf{F}}$, and obtain the observed frequency of outcomes \mathbf{F}^{obs} . We check whether $\mathbf{F}^{\text{obs}} \in \tilde{Q}_i$.

(3) We estimate $\Pr(\tilde{\Omega}_i)_{\rho_{\text{hon}}}$ by repeating (b) 100 000 times, and computing the fraction of times we obtained $\mathbf{F}^{\text{obs}} \in \tilde{Q}_i$.

We see that the $\bar{R}_{\text{fixed},i}$ is small at low values of t_i , since the probability of the protocol accepting is small. We also see that $\bar{R}_{\text{fixed},i}$ is small at larger values of t_i , since the key rate upon acceptance is small. The expected key rate thus captures

the trade-off between accepting with large probability, versus producing a large key upon acceptance.

(3) $\bar{R}_{\text{variable}}$: This is the expected key rate for variable-length protocols. Note that this is a fixed value and *not* plotted as a function of t_i (since we obtain a single variable-length protocol from a sequence of fixed-length protocols determined by the t_i s). Anticipating the use of Theorem 1, we set $\epsilon_{\text{AT}} = \epsilon_{\text{secure}}/4$, $\epsilon_{\text{PA}} = \epsilon_{\text{secure}}/4$, and $\epsilon_{\text{EV}} = \epsilon_{\text{secure}}/2$. We compute l_i using Eq. (24) for various values of t_i , and set $R_{\text{variable},i} = l_i/N$. Using Theorem 1, we obtain that the variable-length protocol constructed from the sequence of fixed-length protocols, for the given set of t_i s, is $\epsilon_{\text{secure}} = (\epsilon_{\text{AT}} + \epsilon_{\text{PA}} + \epsilon_{\text{EV}})$ -secure. We compute the various probabilities $\Pr(\tilde{\Omega}_i)_{\rho_{\text{hon}}}$ in the same manner as (2) above (by simulating 100000 runs of the QKD protocol), and compute $\bar{R}_{\text{variable}}$ using Eq. (19).

Crucially, we find that the variable-length protocol has higher expected key rate than the *best fixed-length protocol*. Since the variable-length protocol consists of exactly the same steps as the fixed-length protocol, and only differs in the parameters of the classical processing of the data, the implementation of the variable-length protocol does not impose *any* additional difficulties, and is accompanied by an *increase* in the expected key rate. In fact, we expect that implementing a variable-length protocol will almost always lead to an improvement in the expected key rate, as we argue in the following remark.

Remark 5. Consider any fixed-length protocol where the honest behavior is given by ρ_{hon} . Suppose that Alice and Bob choose to implement a fixed-length protocol, with parameters l , λ^{EC} and t . Now, consider the variable-length protocol for the same honest behavior ρ_{hon} , for the same choice of ϵ_{PA} , ϵ_{AT} , ϵ_{EV} , obtained by choosing l_i according to $t_1 \leq t_2 \dots \leq t$, and choosing $\lambda_i^{\text{EC}} = \lambda^{\text{EC}}$. Then since $l_1 \geq l_2 \dots \geq l$, using Theorem 1, one is guaranteed to improve upon the expected key rate by switching to a variable-length protocol (albeit with a small increase in the security parameter). This is because the variable-length protocol always has some nonzero probability of producing keys of larger length when compared to the fixed-length protocol. We believe that in almost all cases, this improvement will remain even after choosing the same security parameter for both fixed-length and variable-length protocols (as we saw in Fig. 1).

V. A TRUE VARIABLE-LENGTH PROTOCOL

In the preceding section, we considered a scenario where the honest implementation of the protocol is fixed and known beforehand. However, in scenarios where the honest implementation varies unpredictably between each run of the protocol, it is not clear how Theorem 1 can be used to obtain good key rates. For instance, suppose that the channel has a 50% chance of having honest behavior ρ_{hon} (leading to statistics $\bar{\mathbf{F}}$) and ρ'_{hon} (leading to statistics $\bar{\mathbf{F}}'$), and $\bar{\mathbf{F}}$ and $\bar{\mathbf{F}}'$ are very different frequencies. Then it is not clear how to choose suitable acceptance sets that: (a) give good key rates and (b) on which Theorem 1 can be applied. This is because the size of the acceptance test that includes both $\bar{\mathbf{F}}$ and $\bar{\mathbf{F}}'$, is of the order of $\|\bar{\mathbf{F}} - \bar{\mathbf{F}}'\|_1$, which can be quite large. This leads to low (or in many cases zero) key rate upon acceptance. Thus, we have not yet resolved the problem of unpredictable channels, which we

shall now address in this section. We note that one potential solution to this problem is to coarse grain the acceptance data, and set the acceptance condition to be “QBER is less than some fixed value”. However, coarse graining involves throwing away information, and has been shown to lead to suboptimal key rates [32]. Note that an unpredictable channel connecting Alice and Bob is already a significant issue for experimental implementations, which is sometimes incorrectly resolved by choosing the acceptance test $(\bar{\mathbf{F}}, t)$ after seeing the observed statistics \mathbf{F}^{obs} (see Remark 1).

In this section, we will propose and analyze a variable-length protocol that directly uses \mathbf{F}^{obs} , the observed frequency of outcomes, to determine the length of the secret key to be produced and the number of bits to be used for error correction. Note that unlike Sec. III, we no longer need to go through a sequence of fixed-length protocols in this section. Instead, we will design the variable-length decision in a different manner that does not depend on a sequence of fixed-length acceptance tests. Crucially, this will involve the construction of a statistical estimator $b_{\text{stat}}(\mathbf{F}^{\text{obs}})$, that with high probability is a lower bound on the Rényi entropy $H_\alpha(Z^n|C^nE^n)_\rho$ of the state ρ in the QKD protocol. That is, we will first construct a b_{stat} such that for any state ρ , it is the case that

$$\Pr_{\mathbf{F}^{\text{obs}}}(b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \leq H_\alpha(Z^n|C^nE^n)_\rho) \geq 1 - \varepsilon_{\text{AT}}. \quad (25)$$

This estimator will then be used to determine the length of the output key. We start by presenting some results that allow us to construct such an estimator in Sec. V A. In Sec. V B we specify the variable-length protocol, and explain how the users use $b_{\text{stat}}(\mathbf{F}^{\text{obs}})$ to decide the length of the key and the number of bits to use for error correction. Finally, in Sec. V C we prove the security of the variable-length protocol.

We highlight that the only place we use the IID collective attacks assumption is in Sec. V A, in the construction of b_{stat} . Therefore, if alternative methods could be found that construct b_{stat} without this assumption, our proof framework would generalize to coherent attacks.

A. Constructing the estimator

In a QKD protocol, we deal with a *fixed yet unknown* ρ_{AB} . In particular ρ_{AB} is a fixed state and *not* a random variable. This ρ_{AB} then gives rise to a random variable \mathbf{F}^{obs} . Given that Alice and Bob observe \mathbf{F}^{obs} , obtained by performing measurements on $\rho_{AB}^{\otimes m}$, we would like to construct a set of states $V(\mathbf{F}^{\text{obs}})$, such that $V(\mathbf{F}^{\text{obs}})$ contains ρ_{AB} with high probability.

Remark 6. In general, one can use a variety of concentration inequalities to obtain such a set. In the following Lemma, we will use the concentration inequality from Lemma 13 (Appendix B 1). We make this choice since it is the same concentration inequality used in the construction of the feasible set Eq. (22), and we wish to make a fair comparison between the variable-length and fixed-length protocols. Thus, for our comparisons later in Sec. VI, the acceptance tests for the fixed-length protocols and the variable-length decision in the variable-length protocol are designed using the *same* concentration inequalities.

Lemma 1. For any state ρ , let $\mathbf{F}^{\text{obs}} \in \mathcal{P}(\Sigma)$ be the frequency vector obtained from measuring the state m times, where Σ is the set of possible outcomes. Let Γ_j be the POVM

element corresponding to outcome j . Define parameters

$$\mu := \sqrt{2} \sqrt{\frac{\ln(1/\varepsilon_{\text{AT}}) + |\Sigma| \ln(m+1)}{m}}, \quad (26)$$

and the map $\Phi(\rho) := \sum_{j \in \Sigma} \text{Tr}(\Gamma_j \rho) |j\rangle \langle j|$, and the set

$$V(\mathbf{F}^{\text{obs}}) := \{\rho \in \mathcal{S}_c(AB) \mid \|\Phi(\rho) - \mathbf{F}^{\text{obs}}\|_1 \leq \mu\}. \quad (27)$$

Then, $V(\mathbf{F}^{\text{obs}})$ contains ρ with probability greater than $1 - \varepsilon_{\text{AT}}$. That is,

$$\Pr_{\mathbf{F}^{\text{obs}}}(\rho \in V(\mathbf{F}^{\text{obs}})) \geq 1 - \varepsilon_{\text{AT}}. \quad (28)$$

Proof. \mathbf{F}^{obs} is sampled from the probability distribution given by $\Phi(\rho)$. The claim follows from Lemma 13, which states that if \mathbf{F}^{obs} is obtained by sampling m times from the probability distribution $\Phi(\rho)$, then

$$\Pr(\|\mathbf{F}^{\text{obs}} - \Phi(\rho)\|_1 \leq \mu) \geq 1 - \varepsilon_{\text{AT}}. \quad (29)$$

Next, we use the above result to obtain a statistical estimator of a lower bound on the Rényi entropy of the state $\rho_{ZCE}^{\otimes n}$.

Lemma 2. For any state ρ satisfying $\text{Tr}_{BE}(\rho_{ABE}) = \bar{\sigma}_A$, let $\mathbf{F}^{\text{obs}} \in \mathcal{P}(\Sigma)$ be the frequency vector obtained from measuring the state m times. Define

$$b_{\text{stat}}(\mathbf{F}^{\text{obs}}) := \min_{\substack{\rho \in V(\mathbf{F}^{\text{obs}}), \\ \text{Tr}_{BE}(\rho) = \bar{\sigma}_A}} nH(Z|CE)_\rho - n(\alpha - 1) \log^2(d_Z + 1), \quad (30)$$

where $d_Z = \dim(Z)$ and $1 < \alpha < 1 + 1/\log(2d_Z + 1)$. Then,

$$\Pr_{\mathbf{F}^{\text{obs}}}(b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \leq H_\alpha(Z^n|C^nE^n)_\rho) \geq 1 - \varepsilon_{\text{AT}}. \quad (31)$$

Proof. From Lemmas 7 and 8, we have that $H_\alpha(Z^n|C^nE^n)_\rho \geq nH(Z|CE)_\rho - n(\alpha - 1) \log^2(d_Z + 1)$. The claim then follows from Lemma 1. ■

B. Variable length decision

We will use $b_{\text{stat}}(\mathbf{F}^{\text{obs}})$ to construct the following variable-length decision procedure. Let \mathcal{F} be the (possibly infinite [33]) set of all possible observations \mathbf{F}^{obs} in the variable-length decision step. Let $\{0, 1, \dots, l_{\text{max}}\}$ denote all the possible values of output key lengths in our protocol. Let $\{0, 1, \dots, \lambda_{\text{max}}^{\text{EC}}\}$ denote all the possible values of the number of bits used for error correction in our protocol. Then, the variable-length decision is implemented as follows:

- (1) From public announcements C_{AT}^m , Alice and Bob compute \mathbf{F}^{obs} and $b_{\text{stat}}(\mathbf{F}^{\text{obs}})$.
- (2) They compute $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}})$, the number of bits to be used for error-correction information, where $\lambda^{\text{EC}}(\cdot) : \mathcal{F} \rightarrow \{0, 1, \dots, \lambda_{\text{max}}^{\text{EC}}\}$ is some predetermined function.
- (3) They compute $l(\mathbf{F}^{\text{obs}})$, the length of the final key to be produced, where $l(\cdot) : \mathcal{F} \rightarrow \{0, 1, \dots, l_{\text{max}}\}$ is a function defined as

$$l(\mathbf{F}^{\text{obs}}) := \max(0, \lfloor b_{\text{stat}}(\mathbf{F}^{\text{obs}}) - \lambda^{\text{EC}}(\mathbf{F}^{\text{obs}}) - \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}) \rfloor), \quad (32)$$

$$\theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}) := \frac{\alpha}{\alpha - 1} \left(\log\left(\frac{1}{4\varepsilon_{\text{PA}}}\right) + \frac{2}{\alpha} \right) + \lceil \log(1/\varepsilon_{\text{EV}}) \rceil.$$

Recall that Ω_i denotes the event that a key of length l_i is produced using λ_i^{EC} bits for error correction for some values $(l_i, \lambda_i^{\text{EC}})$. In other words, the index i determines the pair $(l_i, \lambda_i^{\text{EC}})$ of values of the key length and length of error-correction information. The sets \mathcal{Q}_i for the variable-length decision in our protocol are thus formally defined by

$$\mathcal{Q}_i = \{\mathbf{F}^{\text{obs}} \in \mathcal{F} | l(\mathbf{F}^{\text{obs}}) = l_i, \lambda^{\text{EC}}(\mathbf{F}^{\text{obs}}) = \lambda_i^{\text{EC}}\}. \quad (33)$$

Note that number of possible events Ω_i is always finite (unlike the set of possible observations \mathcal{F}), and we denote it by M . The remaining steps of the variable-length protocol are identical to the ones described in Sec. II.

Remark 7. Recall that in the proof of Theorem 1, the fact that $l_i + \lambda_i^{\text{EC}}$ was a nonincreasing sequence in i played a crucial role. This property is required for similar reasons in the proof of Theorem 2. Thus, without loss of generality, we label the events Ω_i such that $l_i + \lambda_i^{\text{EC}}$ forms a nonincreasing sequence in i .

Such an ordering allows us to prove the following Lemma, which we use in the next section in our security proof.

Lemma 3. Let $\mathcal{T}_p = \{i | l_i > 0\}$ be the set of values of i that lead to nontrivial length of the key. Then,

$$\sum_{\substack{j \\ i \in \mathcal{T}_p}} \Pr(\Omega_i) \leq \Pr(b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \geq l_j + \lambda_j^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}})), \quad (34)$$

for any $j \in \{1, 2, \dots, M\}$.

Proof. For any $i \in \mathcal{T}_p$, using Eq. (32) we have

$$\Omega_i \Rightarrow b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \geq l_i + \lambda_i^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}). \quad (35)$$

Therefore

$$\begin{aligned} \sum_{\substack{j \\ i \in \mathcal{T}_p}} \Pr(\Omega_i) &= \Pr\left(\bigcup_{\substack{i=1 \\ i \in \mathcal{T}_p}}^j \Omega_i\right) \\ &\leq \Pr\left(\bigcup_{\substack{i=1 \\ i \in \mathcal{T}_p}}^j (b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \geq l_i + \lambda_i^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}))\right) \\ &\leq \Pr(b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \geq l_j + \lambda_j^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}})), \end{aligned} \quad (36)$$

where we used the fact that Ω_i are disjoint events in the first equality, Eq. (35) for the second inequality, and the ordering on $l_i + \lambda_i^{\text{EC}}$ (Remark 7) in the final inequality. ■

We now have all the tools necessary to prove the security of our variable-length protocol. Before presenting the security proof, we compare the fixed-length and variable-length implementations in the following remark.

Remark 8. Note that with the way we construct the acceptance tests and variable-length decision in this section, the following property holds. Focusing on the fixed-length implementation for some specific i , the key length whenever the protocol accepts is given by Eq. (24), which is an optimization over the feasible set \tilde{S}_i [Eq. (22)] whose size is determined by $t_i + \mu$. On the other hand, the variable-length implementation determines the key length [Eq. (32)] by looking at the observed value \mathbf{F}^{obs} and optimizing over the set $V(\mathbf{F}^{\text{obs}})$

[Eq. (27)] whose size is determined by μ . Now observe that whenever \mathbf{F}^{obs} takes a value such that the fixed-length implementation would accept during the acceptance test [Eq. (20)], $V(\mathbf{F}^{\text{obs}})$ is smaller than \tilde{S}_i , and the only difference between Eq. (24) and Eq. (32) is in the optimization set. Therefore, it follows that the variable-length key rate is always higher when the same values of $\varepsilon_{\text{AT}}, \varepsilon_{\text{EV}}, \varepsilon_{\text{PA}}$ are used in the two cases (although as previously discussed, this results a minor increase in $\varepsilon_{\text{secure}}$).

C. Security proof of variable-length protocol

Theorem 2. The variable-length protocol that, on obtaining \mathbf{F}^{obs} during the variable-length decision and passing error verification, hashes to length $l(\mathbf{F}^{\text{obs}})$ using $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}})$ bits for error correction [according to Eq. (32)], is $(\varepsilon_{\text{AT}} + \varepsilon_{\text{EV}} + \varepsilon_{\text{PA}})$ -secure.

Proof. As in the proof of Theorem 1, we will show that the protocol is ε_{EV} -correct and $(\varepsilon_{\text{PA}} + \varepsilon_{\text{AT}})$ -secret, implying that the protocol is $(\varepsilon_{\text{EV}} + \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secure (Lemma 12 or Ref. [15]). First note that the proof of ε_{EV} -correctness of the protocol is the same as in the proof of Theorem 1 [Eq. (13)]. Thus we only need to prove secrecy.

Again, as in the proof of Theorem 1, it is sufficient to show that

$$\begin{aligned} \sum_{i=1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) &\left\| \rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\ &\leq \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}}, \end{aligned} \quad (37)$$

since each of the states $\rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})}$ have orthogonal supports. This is because the event $\Omega_i \wedge \Omega_{\text{EV}}$ is a deterministic function of the registers C_{AT}^m, C_V . Thus, Eqs. (37) and (11) are equivalent.

Recall that the values $l_i + \lambda_i^{\text{EC}}$ are ordered such that they form a nonincreasing sequence (Remark 7). Thus, for any ρ_{AB} that the protocol can start with, the Rényi entropy $H_\alpha(Z^n | C^n E^n)_\rho$ has to fall under at least one of the following three cases:

- (1) $H_\alpha(Z^n | C^n E^n)_\rho \geq l_1 + \lambda_1^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}})$.
- (2) $l_j + \lambda_j^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}) \geq H_\alpha(Z^n | C^n E^n)_\rho \geq l_{j+1} + \lambda_{j+1}^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}})$ for some $j \in \{1, \dots, M-1\}$.
- (3) $l_M + \lambda_M^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}) \geq H_\alpha(Z^n | C^n E^n)_\rho$.

We will prove the secrecy claim separately for each case. Suppose ρ is such that it satisfies case 2, for some value j . In this case, the secrecy bound can be obtained similar to the proof of Theorem 1, by splitting up the sum into two convenient parts. The first part groups the set of events that happen with low probability, and is given by

$$\begin{aligned} \sum_{i=1}^j \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) &\left\| \rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\ &= \sum_{\substack{j \\ i \in \mathcal{T}_p}} \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C}^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \end{aligned}$$

$$\begin{aligned}
 &\leq \sum_{\substack{i=1 \\ i \in \mathcal{T}_p}}^j \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \leq \sum_{\substack{i=1 \\ i \in \mathcal{T}_p}}^j \Pr(\Omega_i) \\
 &\leq \Pr(b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \geq l_j + \lambda_j^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}})) \\
 &\leq \Pr(b_{\text{stat}}(\mathbf{F}^{\text{obs}}) \geq H_\alpha(Z^n | C^n E^n)_\rho) \leq \varepsilon_{\text{AT}}. \quad (38)
 \end{aligned}$$

Here the first equality follows from the fact that the real and the ideal outputs are identical when the length of the key

generated is zero, the second inequality uses the fact that the trace norm is upper bounded by 2, and the third inequality follows from the properties of probabilities. The fourth inequality uses Lemma 3, the fifth inequality follows from the fact that $l_j + \lambda_j^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}) \geq H_\alpha(Z^n | C^n E^n)_\rho$, and the final inequality from Lemma 2.

For the remaining terms, we follow the same steps as Eq. (16) from the proof of Theorem 1. We obtain the following inequalities:

$$\begin{aligned}
 &\sum_{i=j+1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\
 &= \sum_{\substack{i \geq j+1 \\ i \in \mathcal{T}_p}}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\
 &\leq \sum_{\substack{i \geq j+1 \\ i \in \mathcal{T}_p}}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{\text{AT}}^m C_V E^N)_\rho | \Omega_i \wedge \Omega_{\text{EV}} - l_i) + \frac{2}{\alpha} - 1} \\
 &\leq \sum_{\substack{i \geq j+1 \\ i \in \mathcal{T}_p}}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{\text{AT}}^m C_V E^N)_\rho | \Omega_i \wedge \Omega_{\text{EV}} - \text{leak}_i - l_i) + \frac{2}{\alpha} - 1} \\
 &\leq \sum_{\substack{i \geq j+1 \\ i \in \mathcal{T}_p}}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_{\text{AT}}^m C_V E^N)_\rho | \Omega_i \wedge \Omega_{\text{EV}} - \text{leak}_{j+1} - l_{j+1}) + \frac{2}{\alpha} - 1} \\
 &\leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n C_V E^N)_\rho - \text{leak}_{j+1} - l_{j+1}) + \frac{2}{\alpha} - 1} \\
 &\leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^n | C^n E^n)_\rho - \text{leak}_{j+1} - l_{j+1} - \lceil \log(1/\varepsilon_{\text{EV}}) \rceil) + \frac{2}{\alpha} - 1} \\
 &\leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(\theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}}) - \lceil \log(1/\varepsilon_{\text{EV}}) \rceil) + \frac{2}{\alpha} - 1} \\
 &\leq \varepsilon_{\text{PA}}. \quad (39)
 \end{aligned}$$

We now explain the derivation of the expressions above, and highlight the crucial steps in Remark 9. The first equality follows from the fact that the real and the ideal outputs are identical when the length of the key generated is zero. We use the leftover hashing lemma for Rényi entropy for the second inequality (Lemma 6), and Lemma 11 to split off the information leakage due to error correction, and the E^m register (which is independent of Z^n) in the third inequality. The ordering on $l_i + \lambda_i^{\text{EC}}$ from Remark 7 allows us obtain the fourth inequality, and we use Lemma 10 to get rid of the conditioning on events for the fifth inequality. We use Lemma 11 again to split off the error-verification communication, and the C_{AT}^m register (which is independent of Z^n) in the sixth inequality. We use the fact that $H_\alpha(Z^n | C^n E^n)_\rho \geq l_{j+1} + \lambda_{j+1}^{\text{EC}} + \theta(\varepsilon_{\text{PA}}, \varepsilon_{\text{EV}})$ for the seventh inequality, and Eq. (32) for the eighth inequality.

Remark 9. As in the proof of Theorem 1, the critical steps in the above chain of inequalities are the replacement of $l_i + \lambda_i^{\text{EC}}$ with $l_{j+1} + \lambda_{j+1}^{\text{EC}}$ in the third inequality, and using

Lemma 10 in the fourth inequality to get rid of the conditioning on events in the Rényi entropies. As in the proof of Theorem 1, we split of C_{AT}^m and C_V registers *after* using Lemma 10, since we need the events $\Omega_i \wedge \Omega_{\text{EV}}$ to be known to Eve in order to use Lemma 10.

Case 1 and Case 3. The analysis of Case 1 is a special case of Case 2, and is obtained by setting $j = 1$ in Eq. (39). The analysis of Case 3 is a special case of Case 2, and is obtained by setting $j = M$ in Eq. (38).

Bringing Eqs. (38) and (39) together, we obtain

$$\begin{aligned}
 &\sum_{i=1}^M \frac{1}{2} \Pr(\Omega_i \wedge \Omega_{\text{EV}}) \left\| \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}})} - \rho_{K_A \tilde{C} E^N | \Omega_i \wedge \Omega_{\text{EV}}}^{(l_i, \lambda_i^{\text{EC}}, \text{ideal})} \right\|_1 \\
 &\leq \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}}. \quad (40)
 \end{aligned}$$

Since the protocol is ε_{EV} -correct, and $(\varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secret, it is also $(\varepsilon_{\text{EV}} + \varepsilon_{\text{AT}} + \varepsilon_{\text{PA}})$ -secure. ■

Remark 10. Since the protocol from Theorem 2 does not impose any condition on the sets Q_i , unlike Theorem 1, which requires the acceptance sets \tilde{Q}_i of the fixed-length protocols to form a nested sequence, one can use Theorem 2 in scenarios where the channel behavior is unpredictable and chaotic. This is especially desirable for ground-to-satellite QKD, where the channel behavior is difficult to predict in advance. In the next section, we show how Theorem 2 can be used to improve the expected key rate in such scenarios.

VI. APPLICATION TO QUBIT BB84

In this section, we compute expected key rates for fixed-length and variable-length qubit BB84 protocols for a scenario where the honest behavior is unpredictable. The fixed-length implementation is identical to the one from Sec. IV. The variable-length implementation is also similar, except the variable-length decision, which is implemented as described in Sec. VB above. In particular, after signal transmission, measurements, and public announcements, Alice and Bob compute \mathbf{F}^{obs} from public announcements, and determine $l(\mathbf{F}^{\text{obs}})$ and $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}})$ according to Eq. (32).

For the sake of simplicity, we consider a channel model that can take a discrete set of values for the depolarization probability and the misalignment angle. We assume that the channel is such that, on any given run, the depolarization probability is chosen randomly from $\{0.02, 0.03, 0.04, 0.05\}$ with equal probability, and the misalignment angle is chosen randomly from $\{2^\circ, 4^\circ, 6^\circ, 8^\circ, 10^\circ\}$ with equal probability. Thus the channel has $n_{\text{ch}} = 20$ possible values, which it takes with equal probability. We use $\rho_{\text{hon}}^{(j)}$ to denote the state corresponding to the j th honest behavior of the channel. We set the basis choice probabilities to $p_x = p_z = 0.5$, the total number of signals to $N = 10^6$, and the number of signals used in the public announcement to $m = 0.05N$. We estimate the number of bits to be used for error correction from \mathbf{F}^{obs} , by setting $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}}) = fnH(Z|YC)_{\mathbf{F}^{\text{obs}}}$ where $f = 1.16$ is the efficiency factor. We set a target security parameter of $\varepsilon_{\text{secure}} = 10^{-12}$.

We now explain the various key rates plotted in Fig. 2.

(1) $R_{\text{fixed},i}$: This is the key rate upon acceptance for the fixed-length protocol, plotted against t_i , the size of the acceptance set, and is identical to the plot from Fig. 1. We

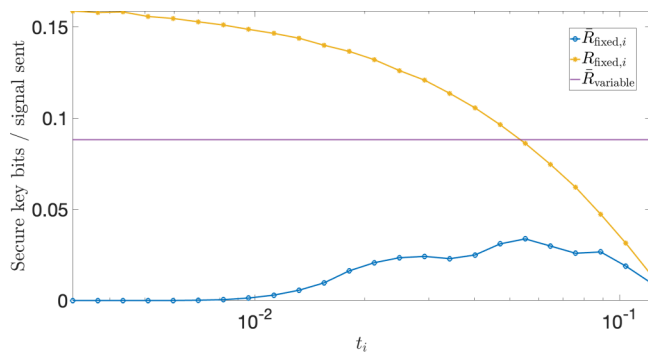


FIG. 2. Expected key rate for fixed-length protocols ($\bar{R}_{\text{fixed},i}$) for various values of t_i , key rate upon acceptance for fixed-length protocols ($R_{\text{fixed},i}$) for various values of t_i and the expected key rate for variable-length protocol ($\bar{R}_{\text{variable}}$).

set $\varepsilon_{\text{PA}} = \varepsilon_{\text{EV}} = \varepsilon_{\text{AT}} = \varepsilon_{\text{secure}}/2$, and compute $R_{\text{fixed},i} = l_i/N$, where l_i is computed according to Eq. (24) for the acceptance test in Eq. (20). We choose the center ($\bar{\mathbf{F}}$) of the fixed-length acceptance set [Eq. (20)] to be the expected frequency of outcomes corresponding to the channel with the least possible depolarization probability (0.02) and least possible misalignment angle (2°). As expected, we see that $R_{\text{fixed},i}$ decreases monotonically as we increase t_i , reflecting the fact that larger acceptance sets lead to lower key rates upon acceptance.

(2) $\bar{R}_{\text{fixed},i}$: This is the expected key rate for fixed-length protocols. Let \tilde{Q}_i be the acceptance set for the fixed-length protocol for a given value of t_i . Then, the expected key rate is given by

$$\bar{R}_{\text{fixed},i} = \frac{1}{n_{\text{ch}}} \sum_{j=1}^{n_{\text{ch}}} \left(\sum_{i=1}^M \Pr(\mathbf{F}^{\text{obs}} \in \tilde{Q}_i)_{\rho_{\text{hon}}^{(j)}} R_{\text{fixed},i} \right), \quad (41)$$

where the expression in the parenthesis represents the expected key rate for the j th channel behavior. We use the values of $R_{\text{fixed},i}$ obtained above, and numerically estimate $\Pr(\mathbf{F}^{\text{obs}} \in \tilde{Q}_i)_{\rho_{\text{hon}}^{(j)}}$ from the following steps.

(1) For each channel model, we compute the probability vector $\bar{\mathbf{F}}^{(j)}$ corresponding to the honest behavior $\rho_{\text{hon}}^{(j)}$, by setting $\bar{\mathbf{F}}^{(j)} = \Phi(\rho_{\text{hon}}^{(j)})$, where Φ is defined in Eq. (21).

(2) From each $\bar{\mathbf{F}}^{(j)}$, we sample m times to obtain \mathbf{F}^{obs} . We check whether $\mathbf{F}^{\text{obs}} \in \tilde{S}_i$ or not.

(3) We estimate $\Pr(\mathbf{F}^{\text{obs}} \in \tilde{Q}_i)_{\rho_{\text{hon}}^{(j)}}$ by repeating step (b) 50 times for each possible honest behavior, and computing the fraction of times we obtained $\mathbf{F}^{\text{obs}} \in \tilde{S}_i$.

Thus, the whole process is a simulation of 1000 runs of the QKD protocol, with each channel being used 50 times.

In Fig. 2 we see that $\bar{R}_{\text{fixed},i}$ is much smaller than $R_{\text{fixed},i}$ since the fixed-length protocol only accepts on a small number of channel behaviors. As t increases, the fixed-length acceptance set becomes larger, starts accepting on multiple values of $\bar{\mathbf{F}}^j$, and therefore has a larger probability of acceptance. Thus, the expected key rate $\bar{R}_{\text{fixed},i}$ increases slightly. However, the size of the acceptance test is already large, and the key rate upon acceptance ($R_{\text{fixed},i}$), rapidly goes to zero for large t_i . This causes $\bar{R}_{\text{fixed},i}$ to also go to zero rapidly.

(3) $\bar{R}_{\text{variable}}$: This is the expected key rate for variable-length protocols. This is given by

$$\bar{R}_{\text{variable}} = \frac{1}{n_{\text{ch}}} \sum_{j=1}^{n_{\text{ch}}} \left(\sum_{\mathbf{F}^{\text{obs}}} \Pr(\mathbf{F}^{\text{obs}})_{\rho_{\text{hon}}^{(j)}} R_{\text{variable}}(\mathbf{F}^{\text{obs}}) \right), \quad (42)$$

where $\Pr(\mathbf{F}^{\text{obs}})_{\rho_{\text{hon}}^{(j)}}$ is the probability of obtaining \mathbf{F}^{obs} when the honest behavior is $\rho_{\text{hon}}^{(j)}$, and $R_{\text{variable}}(\mathbf{F}^{\text{obs}})$ is the key rate obtained for the observed frequency \mathbf{F}^{obs} . The term in the parenthesis represents the expected key rate for the j th channel behavior. Note that $\bar{R}_{\text{variable}}$ is a fixed value and *not* plotted against t_i , and is computed as follows.

(1) For each channel model, we compute the expected statistics $\bar{\mathbf{F}}^{(j)}$ corresponding to the honest behavior $\rho_{\text{hon}}^{(j)}$.

(2) From each $\bar{\mathbf{F}}^{(j)}$, we sample m times to obtain \mathbf{F}^{obs} . We compute $l(\mathbf{F}^{\text{obs}})$ according to Eq. (32), with $\varepsilon_{\text{PA}} = \varepsilon_{\text{AT}} = \varepsilon_{\text{secure}}/4$ and $\varepsilon_{\text{EV}} = \varepsilon_{\text{secure}}/2$, and set $R_{\text{variable}}(\mathbf{F}^{\text{obs}}) = l(\mathbf{F}^{\text{obs}})/N$.

(3) For each channel model, we repeat step (b) 50 times, and compute the average value of $R_{\text{variable}}(\mathbf{F}^{\text{obs}})$. This is our estimate of $(\sum_{\mathbf{F}^{\text{obs}}} \Pr(\mathbf{F}^{\text{obs}})_{\rho_{\text{hon}}^{(j)}} R_{\text{variable}}(\mathbf{F}^{\text{obs}}))$.

(4) $\bar{R}_{\text{variable}}$ is then computed by averaging the key rate obtained in step (c), over all the possible channel models.

Thus the above procedure is a simulation of 1000 runs of the QKD protocol, with each channel behavior being used 50 times. Crucially, we find the expected key rate for variable-length protocols is much higher than the expected key rate for the fixed-length protocols.

Remark 11. Note that the degree of improvement shown by the variable-length protocol in Fig. 2 depends on n_{ch} . Larger values of n_{ch} reflect a higher variation in the channel behavior, and will lead to a bigger difference between the performance of fixed-length and variable-length protocols. In this paper, we chose the above channel model for the sake of simplicity. Detailed studies of practical QKD protocols over realistic, unpredictable channel models will be the subject of future work.

VII. VARIABLE INPUT-LENGTH PRIVACY AMPLIFICATION

So far we have studied the variable-length aspects of the *final key* that is generated after privacy amplification in QKD protocols. In this section, we will turn our attention to the variable-length aspect of the *sifted raw key* in QKD implementations, before privacy amplification. In particular, we will point out and remedy a gap between the theoretical analysis of privacy amplification and its experimental implementation. For simplicity, we only consider fixed-length QKD protocols. However, our results can be generalized to variable-length protocols in a straightforward manner.

A. Sifting in QKD

Consider the following three ways of implementing the sifting step in QKD protocols.

(1) *Map the discard outcomes to \perp .* In this case, the state prior to privacy amplification is given by $\rho_{\hat{Z}^n Y^n \tilde{C} E^N}$, where \hat{Z} is a register that takes values in $\{0, 1, \perp\}$, and $\tilde{C} = C^n C_{AT}^m C_E C_V$ for brevity. In this case, one has to implement privacy amplification using two-universal hashing from \hat{Z}^n to l bits. In particular, binary Toeplitz hashing, a widely used choice, is not possible.

(2) *Map the discard outcomes to 0.* In this case, the state prior to privacy amplification is given by $\rho_{Z^n Y^n \tilde{C} E^N}$, where Z is a register that takes values in $\{0, 1\}$. In this case, one has to implement privacy amplification using two-universal hashing from n bits to l bits. In particular, binary Toeplitz hashing, a widely used choice, is possible; however, the hash matrices must always be for input strings of a *fixed* length n .

(3) *Actually discard the discard outcomes.* In this case, the state prior to privacy amplification is given by $\rho_{Z^{\leq n} Y^n \tilde{C} E^N}$, where $Z^{\leq n}$ is a register that takes values in the set of bitstrings of length less than or equal to n , which we shall denote as $\{0, 1\}^{\leq n}$. In this case, one first looks at the number of bits in the register $Z^{\leq n}$, denoted by $\text{len}(Z^{\leq n})$, and chooses a two-universal hashing procedure from $\text{len}(Z^{\leq n})$ bits to l bits. This is what is commonly done in QKD experiments.

Practically, one would like to use binary Toeplitz hashing in this procedure. However, we will see below that this is not a valid two-universal hashing procedure from $\{0, 1\}^{\leq n}$ to l bits.

The theoretical analysis of Case 1 and Case 2 is straightforward, since they constitute valid two-universal hashing procedures from $\{0, 1, \perp\}^n$ to l bits, and n bits to l bits respectively. Thus, the leftover hashing lemma can be directly applied. However, Case 3 is *not* necessarily a two-universal hashing procedure from $\{0, 1\}^{\leq n}$ to l bits, as we now explain. Thus we cannot directly apply the leftover hashing lemma in this case.

B. The problem

For every $i \in \{0, 1, \dots, n\}$, let $\mathcal{F}_i^{\text{hash}}$ denote a two-universal hash family from i bits to l bits. Then, the procedure described in Case 3 above is equivalent to *first* randomly sampling $f_i \in \mathcal{F}_i^{\text{hash}}$ for every i , followed by computing $f_{\text{len}(Z^{\leq n})}(Z^{\leq n})$. Note that in this case, only one of the sampled f_i s is ever applied. In order for this procedure to be a valid two-universal hashing procedure from $\{0, 1\}^{\leq n}$ to l , by definition it must be the case that for any two inputs $z_1 \neq z_2$, we have

$$\Pr_{f_1, f_2, \dots, f_n} [f_{\text{len}(z_1)}(z_1) = f_{\text{len}(z_2)}(z_2)] \leq \frac{1}{2^l}. \quad (43)$$

When z_1 and z_2 are of the same length, then Eq. (43) follows from the two-universal property of $\mathcal{F}_i^{\text{hash}}$. When z_1 and z_2 are of different length, an explicit counter example can be obtained by considering z_1 and z_2 to be all-zero strings of different lengths. In this case, if $\mathcal{F}_i^{\text{hash}}$ is a two-universal *linear* hash family, then $f_{\text{len}(z_1)}(z_1) = f_{\text{len}(z_2)}(z_2) = \mathbf{0}$ with probability 1. Thus for binary Toeplitz hashing, Case 3 is *not* a valid two-universal hashing procedure. Thus we cannot directly apply the leftover hashing lemma.

Remark 12. We note that if every $\mathcal{F}_i^{\text{hash}}$ is chosen such that it is two-universal *and* has the following ‘‘uniform output’’ property,

$$\Pr_{f_i \in \mathcal{F}_i^{\text{hash}}} [f_i(\mathbf{z}) = k] \leq 1/2^l \quad \forall \quad \mathbf{z} \in \{0, 1\}^i, k \in \{0, 1\}^l, \quad (44)$$

then it is straightforward to prove that Eq. (43) holds and hence the described procedure is a valid two-universal hashing. Furthermore, in principle any two-universal hashing procedure can be modified into one that satisfies Eq. (44), via the construction we describe in the Lemma 4 proof below. However, physically implementing this conversion in an actual QKD protocol would be an undesirable additional cost, hence we instead provide a proof that shows that this is not necessary.

C. The solution

We address this issue with Lemmas 4 and 5 below. We start by proving the following modified leftover hashing lemma that is applicable to Case 3, as long as the protocol satisfies the property that the positions and values of the discarded outcomes can be determined from the public announcements \tilde{C} (we return to this point after presenting the lemmas and their proofs). Our approach is to first use Remark 12 to construct a virtual hashing procedure that is a valid two-universal hashing procedure from $\{0, 1\}^{\leq n}$ to l bits. We will then show that the

actual output states can be obtained by performing a CPTP map on the virtual output states. The required result then follows from data-processing inequalities.

Lemma 4. Let $\rho_{Z^{\leq n} \tilde{C} E^N}$ be a state classical in $Z^{\leq n} \tilde{C}$ (where the $Z^{\leq n}$ register takes values in $\{0, 1\}^{\leq n}$), with the property that conditioned on each possible value \tilde{c} on the \tilde{C} register, the resulting distribution on $Z^{\leq n}$ is only supported on values in $\{0, 1\}^{k_{\tilde{c}}}$ for some constant $k_{\tilde{c}} \in \mathbb{N}$. Let $\rho_{K_A \tilde{C} E^N}$ be the state obtained from $\rho_{Z^{\leq n} \tilde{C} E^N}$ by first computing the number of bits $\text{len}(Z^{\leq n})$ in the $Z^{\leq n}$ register, then implementing a two-universal hashing procedure from $\text{len}(Z^{\leq n})$ bits to l bits, where $\tilde{C} := \tilde{C} C_P$ with C_P being the choice of hashing function (in other words, the procedure described above in Case 3). Then for any event Ω on the classical register \tilde{C} , we have (for $\rho_{K_A \tilde{C} E^N}^{(\text{ideal})} := \frac{\mathbb{1}_{K_A}}{|K_A|} \otimes \rho_{\tilde{C} E^N}$),

$$\begin{aligned} & \frac{1}{2} \Pr(\Omega) \left\| \rho_{K_A \tilde{C} E^N | \Omega} - \rho_{K_A \tilde{C} E^N | \Omega}^{(\text{ideal})} \right\|_1 \\ & \leq \frac{1}{2} \Pr(\Omega) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^{\leq n} | \tilde{C} E^N)_{\rho | \Omega} - l) + \frac{2}{\alpha} - 1} \\ & \leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^{\leq n} | \tilde{C} E^N)_\rho - l) + \frac{2}{\alpha} - 1}. \end{aligned} \quad (45)$$

Proof. As explained in Sec. VII B, the hashing procedure described above can be thought of as *first* randomly sampling $f_i \in \mathcal{F}_i^{\text{hash}}$ for every i , and then computing $f_{\text{len}(Z^{\leq n})}(Z^{\leq n})$. However, as noted in that section, this process is not a valid two-universal hashing procedure from $\{0, 1\}^{\leq n}$ to l bits.

Consider instead the following virtual hashing process, based on new hash families [34] $\hat{\mathcal{F}}_i^{\text{hash}} := \mathcal{F}_i^{\text{hash}} \times \{0, 1\}^l$ (for every i). This virtual process first randomly samples $(f_i, u_i) \in \hat{\mathcal{F}}_i^{\text{hash}}$ for every i , i.e., f_i is sampled from the same two-universal hash family $\mathcal{F}_i^{\text{hash}}$ as before, and u_i is a random l -bit string. It then computes $f_{\text{len}(Z^{\leq n})}(Z^{\leq n}) \oplus u_{\text{len}(Z^{\leq n})}$ as its hash output. Now, this virtual hashing procedure is a valid two-universal hashing procedure from $\{0, 1\}^{\leq n}$ to l bits, because each hash family $\hat{\mathcal{F}}_i^{\text{hash}}$ is two-universal and satisfies the “uniform output” property [Eq. (44)].

Denote the output state of the virtual process (acting on $\rho_{Z^{\leq n} \tilde{C} E^N}$) as $\rho_{K_A \tilde{C} E^N}^{(\text{virtual})}$, where $\tilde{C} = \tilde{C} \hat{C}_P$ with \hat{C}_P being the description of the hash function chosen in the virtual process [in particular, all the values (f_i, u_i) from the virtual process]. Let us analogously define $\rho_{K_A \tilde{C} E^N}^{(\text{ideal, virtual})} := \frac{\mathbb{1}_{K_A}}{|K_A|} \otimes \rho_{\tilde{C} E^N}^{(\text{virtual})}$.

Now, we construct a CPTP map $\mathcal{E} : K_A \tilde{C} E^N \rightarrow K_A \tilde{C} E^N$ that will map the virtual output states to the actual output states. This map \mathcal{E} does the following operations:

- (1) Look at \tilde{C} and determine the corresponding value $k_{\tilde{c}}$ (as defined in the conditions of this lemma) [35], to be used in the subsequent steps.
- (2) Look at \hat{C}_P and determine $u_{k_{\tilde{c}}}$, to be used in the subsequent steps.
- (3) Replace K_A with $K_A \oplus u_{k_{\tilde{c}}}$.
- (4) Partial trace on the \hat{C}_P register, on everything except the $f_{k_{\tilde{c}}}$ information.

It is straightforward to verify that this map \mathcal{E} indeed satisfies

$$\begin{aligned} \mathcal{E}\left(\rho_{K_A \tilde{C} E^N}^{(\text{virtual})}\right) &= \rho_{K_A \tilde{C} E^N}, \\ \mathcal{E}\left(\rho_{K_A \tilde{C} E^N}^{(\text{ideal, virtual})}\right) &= \rho_{K_A \tilde{C} E^N}^{(\text{ideal})}, \end{aligned} \quad (46)$$

and analogously for the above states conditioned on the event Ω (since \mathcal{E} does not disturb the register \tilde{C}).

Therefore, we have

$$\begin{aligned} & \frac{1}{2} \Pr(\Omega) \left\| \rho_{K_A \tilde{C} E^N | \Omega} - \rho_{K_A \tilde{C} E^N | \Omega}^{(\text{ideal})} \right\|_1 \\ &= \frac{1}{2} \Pr(\Omega) \left\| \mathcal{E}\left(\rho_{K_A \tilde{C} E^N | \Omega}^{(\text{virtual})}\right) - \rho_{K_A \tilde{C} E^N | \Omega}^{(\text{ideal, virtual})}\right\|_1 \\ &\leq \frac{1}{2} \Pr(\Omega) \left\| \rho_{K_A \tilde{C} E^N | \Omega}^{(\text{virtual})} - \rho_{K_A \tilde{C} E^N | \Omega}^{(\text{ideal, virtual})} \right\|_1 \\ &\leq \frac{1}{2} \Pr(\Omega) 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^{\leq n} | \tilde{C} E^N)_{\rho | \Omega} - l) + \frac{2}{\alpha} - 1} \\ &\leq \frac{1}{2} 2^{-\left(\frac{\alpha-1}{\alpha}\right)(H_\alpha(Z^{\leq n} | \tilde{C} E^N)_\rho - l) + \frac{2}{\alpha} - 1}, \end{aligned} \quad (47)$$

where we used the fact that CPTP maps cannot increase trace norm in the third inequality, and leftover hashing lemma for Rényi entropies (Lemma 6) for the fourth inequality, and Lemma 9 for the final inequality.

Remark 13. While here we have focused on proving an analog of the leftover hashing lemma for Rényi entropy (Lemma 6), a similar result for the smooth min-entropy version can be obtained by exactly the same proof (except that when conditioning on the event Ω , one should use the subnormalized conditional states; see [[1], Lemma 10 and Proposition 9]).

In order to use Lemma 4, we have to compute bounds on the Rényi entropy $H_\alpha(Z^{\leq n} | \tilde{C} E^N)_\rho$, which is computed on the state just prior to privacy amplification in Case 3. However, we expect that if the registers that were discarded to produce $Z^{\leq n}$ are completely determined by the register \tilde{C} , then this entropy should be the same as the value before the discarding process, since the conditioning register \tilde{C} could be used to isometrically convert between the values before and after discarding some registers. We formalize this claim in the following Lemma and subsequent discussion.

Lemma 5. Suppose $\rho_{R \tilde{C} E^N}$, $\rho_{Z^{\leq n} \tilde{C} E^N}$ are states that are classical in \tilde{C} , and related to each other as follows: letting $R_{\tilde{c}}$ be a register containing the support of the conditional state $\rho_{R | \tilde{C} = \tilde{c}}$, there exist isometries $V_{R_{\tilde{c}} \rightarrow Z^{\leq n}}^{(\tilde{c})}$ such that [36]

$$\begin{aligned} V \rho_{R \tilde{C} E^N} V^\dagger &= \rho_{Z^{\leq n} \tilde{C} E^N}, \text{ where} \\ V &:= \sum_{\tilde{c}} V_{R_{\tilde{c}} \rightarrow Z^{\leq n}}^{(\tilde{c})} \otimes |\tilde{c}\rangle \langle \tilde{c}|. \end{aligned} \quad (48)$$

Then we have

$$H_\alpha(Z^{\leq n} | \tilde{C} E^N)_\rho = H_\alpha(R | \tilde{C} E^N)_\rho. \quad (49)$$

Proof. We intuitively expect Eq. (49) to be true, since Eq. (48) essentially states that \tilde{C} can be used to isometrically convert R to $Z^{\leq n}$. To formalize this, we first note that each isometry $V_{R_{\tilde{c}} \rightarrow Z^{\leq n}}^{(\tilde{c})}$ can always be extended to an isometry $V_{R \rightarrow Z^{\leq n}}^{(\tilde{c})}$, i.e., where the domain is the full Hilbert space of

R [padding the output space $Z^{\leq n}$ with extra dimensions if $\dim(R) > \dim(Z^{\leq n})$]. Furthermore, Eq. (48) still holds with V defined in terms of these new isometries instead, i.e., we have

$$V \rho_{RCE^N} V^\dagger = \rho_{Z^{\leq n} CE^N}, \text{ where} \\ V := \sum_{\bar{c}} V_{R \rightarrow Z^{\leq n}}^{(\bar{c})} \otimes |\bar{c}\rangle \langle \bar{c}|_{\bar{C}}. \quad (50)$$

(It does not matter how we chose the extensions, since ρ_{RCE^N} is only supported on a subspace that is unaffected by these choices of extensions.)

Furthermore, letting \bar{C}_c be a copy of the register \bar{C} , using [J25], Lemma B.7] we have

$$H_\alpha(Z^{\leq n} \bar{C}_c | \bar{C} E)_\rho = H_\alpha(Z^{\leq n} | \bar{C} E)_\rho, \\ H_\alpha(R \bar{C}_c | \bar{C} E)_\rho = H_\alpha(R | \bar{C} E)_\rho. \quad (51)$$

Thus, it is enough to show that $H_\alpha(Z^{\leq n} \bar{C}_c | \bar{C} E)_\rho = H_\alpha(R \bar{C}_c | \bar{C} E)_\rho$. This follows from Eq. (50), and the fact that the Rényi entropy is invariant under isometries on the first subsystem, since by defining the isometry $\tilde{V}_{R \bar{C}_c \rightarrow Z^{\leq n} \bar{C}_c} := \sum_{\bar{c}} V_{R \rightarrow Z^{\leq n}}^{(\bar{c})} \otimes |\bar{c}\rangle \langle \bar{c}|_{\bar{C}_c}$ we have

$$\tilde{V} \rho_{R \bar{C}_c C E} \tilde{V}^\dagger = \rho_{Z^{\leq n} \bar{C}_c C E}, \quad (52)$$

which concludes the proof [37]. \blacksquare

To apply Lemmas 4 and 5 in comparing Cases 1, 2, and 3 described previously, we can begin by viewing R as being \hat{Z}^n in Case 1 or Z^n in Case 2. If the protocol satisfies the condition that the positions and values of discarded outcomes are fixed by the public announcements \bar{C} , we can define operations $V_{R_c \rightarrow Z^{\leq n}}^{(\bar{c})}$ that simply drop the discarded outcomes specified by \bar{c} , and it is not difficult to show the state $\rho_{Z^{\leq n} \bar{C} E^N}$ in Case 3 has the following properties:

(1) These $V_{R_c \rightarrow Z^{\leq n}}^{(\bar{c})}$ operations are indeed isometries, and $\rho_{Z^{\leq n} \bar{C} E^N}$ is related to $\rho_{R \bar{C} E^N}$ in the sense expressed in Eq. (48).

(2) $\rho_{Z^{\leq n} \bar{C} E^N}$ satisfies the conditions of Lemma 4, and hence Eq. (45) is valid.

(3) $\rho_{Z^{\leq n} \bar{C} E^N}$ satisfies the conditions of Lemma 5, and hence $H_\alpha(Z^{\leq n} | \bar{C} E^N)_\rho$ in Eq. (45) can be replaced with $H_\alpha(R | \bar{C} E^N)_\rho$.

(Basically, the above statements hold because under that protocol condition, for each value \bar{c} , the output length of $V_{R_c \rightarrow Z^{\leq n}}^{(\bar{c})}$ is fixed, and all the discarded positions have fixed values so there are no ‘‘collisions’’.)

With this, we see that for Case 3 the bound in Eq. (45) holds with $H_\alpha(Z^{\leq n} | \bar{C} E^N)_\rho$ replaced by $H_\alpha(\hat{Z}^n | \bar{C} E^N)_\rho$ from Case 1 or $H_\alpha(Z^n | \bar{C} E^N)_\rho$ from Case 2 [38]; in particular, for the purposes of this work this means the third line in Eq. (39) (and similar bounds in other calculations) is valid even if we apply the procedure in Case 3 rather than Case 2. To qualitatively summarize, under that protocol condition, the bounds obtained on the privacy amplification procedure in QKD are unaffected if the actual protocol implements Case 3 in place of Case 1 or Case 2.

VIII. CONCLUSIONS

In this paper, we presented a security proof for variable-length QKD protocols in the security analysis framework of Renner, against IID collective attacks. First, we showed

how a sequence of security proofs for fixed-length protocols satisfying certain conditions can be converted to a security proof for a variable-length protocol. This conversion did not require any new calculations, or any changes to the final key lengths or the lengths of error-correction information. Moreover, the maximum penalty imposed by this approach is a doubling of the security parameter. We exemplified this result by studying the performance of variable-length and fixed-length implementations of the qubit BB84 protocol, implemented over a fixed, known channel. We showed that the variable-length implementation leads to an improvement in the expected key rate of the protocol, compared to the best fixed-length implementation. Additionally, we showed that implementing the variable-length protocol eliminates the typical trade-off in fixed-length implementations, where a larger acceptance test leads to a higher probability of accepting during honest behavior, but low key rate upon acceptance.

Next, we moved on to consider scenarios of unpredictable channels. Here, we construct the variable-length decision in a way that does not rely on a nested sequence of acceptance tests, and proved the security of the resulting class of variable-length protocols. These protocols did not require users to characterize their channel before running the QKD protocol. Instead, they include instructions for adjusting the length of the final key, and the amount of error-correction information, for every possible observation during the protocol. We exemplified this result by studying the performance of the qubit BB84 protocol implemented in this fashion. We showed that the variable-length implementation leads to a significant improvement in the expected key rate compared to fixed-length implementations, especially for scenarios where the channel is chaotic and unpredictable.

These results are a significant step towards practical QKD implementations, since they eliminate the typical trade-off from fixed-length implementations, and remove the requirement of channel characterization. Moreover, variable-length protocols have already been implemented in several works based on intuition. This paper puts such claims (under the Renner framework) on a solid mathematical footing. (We highlight that in particular, our proof approach relies on a leftover hashing lemma for Rényi entropies that was only recently developed, in Ref. [26]. It does not seem entirely straightforward to construct a similar rigorous analysis using the earlier leftover hashing lemma versions that were based on smooth min-entropy.)

In order to use the results of this paper to implement a valid variable-length QKD protocol, one can follow the following steps. First, decide $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}})$ to be *any* function of the observed statistics \mathbf{F}^{obs} . This fixes the number of bits used for error correction, for any \mathbf{F}^{obs} . Second, construct a set $V(\mathbf{F}^{\text{obs}})$ satisfying Eq. (28). This fixes $b_{\text{stat}}(\mathbf{F}^{\text{obs}})$ via Lemma 2, and $l(\mathbf{F}^{\text{obs}})$ via Eq. (32). Then, the variable-length protocol that produces a key of length $l(\mathbf{F}^{\text{obs}})$, and uses $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}})$ bits for error correction, upon obtaining \mathbf{F}^{obs} , is secure. In practice, one should choose $\lambda^{\text{EC}}(\mathbf{F}^{\text{obs}})$ such that the error-correction protocol has a high chance of succeeding. Furthermore, while we have provided one construction of $V(\mathbf{F}^{\text{obs}})$ in this paper, it is straightforward to construct $V(\mathbf{F}^{\text{obs}})$ using other concentration inequalities.

Finally, all our security proofs can be lifted to coherent attacks using the postselection technique. For pedagogical reasons, this will be included in Ref. [18], where we also fix a technical flaw in the application of postselection technique to QKD. Alternatively, we highlight that the only part of our Sec. V proof that relied on the IID collective-attacks assumption was the construction of b_{stat} in Lemmas 1 and 2. Therefore, any alternative approach that could construct a valid b_{stat} for coherent attacks would also serve to yield a security proof against such attacks for variable-length protocols.

ACKNOWLEDGMENTS

We would like to acknowledge useful discussions with R. Renner, especially for most results in Sec. VII. We would like to thank L. Kamin for helpful discussions on the finite-size security proof of QKD protocols. We would like to thank J. Burniston for help with debugging code. This work was funded by the NSERC Discovery Grant No. 341495, and was conducted at the Institute for Quantum Computing, University of Waterloo, which is funded by Government of Canada through ISED. This work was partially funded by the Mike and Ophelia Laziridis Fellowship.

APPENDIX A: TECHNICAL DEFINITIONS AND LEMMAS

We use $S_o(A)$ denote the normalized states on A . We start by defining the Rényi entropy used in this paper.

Definition 1 (Rényi entropy). For $\rho \in S_o(AB)$, and $\alpha \in (0, 1) \cup (1, \infty)$, the sandwiched Rényi entropy of A given B for a state ρ_{AB} is given by

$$H_\alpha(A|B)_\rho := \max_{\sigma_B \in S_o(B)} H_\alpha(A|B)_{\rho|\sigma}, \quad (\text{A1})$$

where

$$H_\alpha(A|B)_{\rho|\sigma} := \begin{cases} \frac{1}{1-\alpha} \log \text{Tr} \left[\left(\sigma_B^{\frac{1-\alpha}{2\alpha}} \rho_{AB} \sigma_B^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right] & \text{if } \rho \in A \otimes \text{supp}(\sigma), \\ -\infty & \text{otherwise.} \end{cases} \quad (\text{A2})$$

We will require several results regarding the Rényi entropy defined above from [25–27]. Note that the sandwiched Rényi entropy is referred to as $H_\alpha(A|B)$ in Ref. [26] (Definition 1), and $\tilde{H}_{A|B}^\uparrow$ in Ref. [27] (Definition 5.2), and $H_\alpha^\uparrow(A|B)$ in Ref. [25] (Definition B.1).

Lemma 6. (Leftover hashing lemma using Rényi entropy [[26], Theorem 8]) Let $\rho_{XE} \in S_o(XE)$ be classical on X . Let \mathcal{F} be a family of two-universal hash functions from X to $K = \{0, 1\}^l$. Let $\omega_K = \sum_{k=1}^{2^l-1} \frac{|k\rangle\langle k|}{2^l}$ be the perfectly mixed state on K , and let $\omega_F = \sum_{f \in \mathcal{F}} \frac{|f\rangle\langle f|}{|\mathcal{F}|}$. ρ_{KEF} be the state obtained from $\rho_{XE} \otimes \omega_F$ by applying the two-universal hash function in the register F to X . Then,

$$\|\rho_{KEF} - \omega_K \otimes \rho_E \otimes \omega_F\|_1 \leq 2^{-\frac{(\alpha-1)}{\alpha}(H_\alpha(X|E)_\rho - l) + \frac{2}{\alpha} - 1}, \quad (\text{A3})$$

where $\alpha \in (1, 2)$.

The following two lemmas are used to convert the Rényi entropy of an IID state to the von Neumann entropy on a single round state.

Lemma 7. (Additivity of Rényi entropy [[27], Corollary 5.2]) For any two states $\rho_{AB} \in S_o(AB)$, $\sigma_{CD} \in S_o(CD)$, and $\alpha \geq \frac{1}{2}$, we have

$$H_\alpha(AC|BD)_{\rho \otimes \sigma} = H_\alpha(A|B)_\rho + H_\alpha(C|D)_\sigma. \quad (\text{A4})$$

Lemma 8. ([[25], Lemma B.9]) For any $\rho_{AB} \in S_o(AB)$, and $1 < \alpha < 1 + 1/\log(1 + 2d_A)$, we have

$$H_\alpha(A|B)_\rho > H(A|B)_\rho - (\alpha - 1) \log^2(1 + 2d_A). \quad (\text{A5})$$

In some cases the slightly more elaborate continuity bound derived in [[39], Corollary IV 2] may perform better than Lemma 8; we leave a more detailed analysis for future work.

Lemma 9. (Conditioning on events [[25], Lemma B.5]) Let $\rho_{AB} \in S_o(AB)$ be a state of the form $\rho_{AB} = \sum_x p_x \rho_{AB|x}$, where p_x is a probability distribution. Then, for $\alpha > 1$,

$$H_\alpha(A|B)_{\rho|x} \geq H_\alpha(A|B)_\rho - \frac{\alpha}{\alpha - 1} \log \left(\frac{1}{p_x} \right). \quad (\text{A6})$$

The following Lemma plays a crucial role in getting rid of terms involving the Rényi entropy evaluated on states conditioned on events, in the proofs of Theorems 1 and 2.

Lemma 10. Let $\rho_{ABCY} = \sum_{y \in \Lambda} p(y) \rho_{ABC|y} \otimes |y\rangle\langle y| \in S_o(ABCY)$ be classical in Y, C , where $p(y)$ is a probability distribution over Λ , and Y can be generated from C (more precisely: $Y \leftrightarrow C \leftrightarrow AB$ forms a Markov chain). Let $\Lambda' \subseteq \Lambda$. Then,

$$\sum_{y \in \Lambda'} p(y) 2^{-\frac{(\alpha-1)}{\alpha} H_\alpha(A|BC)_{\rho|y}} \leq 2^{-\frac{(\alpha-1)}{\alpha} H_\alpha(A|BC)_\rho}. \quad (\text{A7})$$

Proof. We have

$$\sum_{y \in \Lambda'} p(y) 2^{-\frac{(\alpha-1)}{\alpha} H_\alpha(A|BC)_{\rho|y}} \leq \sum_{y \in \Lambda} p(y) 2^{-\frac{(\alpha-1)}{\alpha} H_\alpha(A|BC)_{\rho|y}}, \quad (\text{A8})$$

since we only add positive terms to the expression to go from the left-hand side (LHS) to the right-hand side (RHS). Now, on the RHS, $p(y)$ is a normalized probability distribution function over Λ . Therefore, we can directly use [[27], Proposition 5.1], and we obtain

$$\sum_{y \in \Lambda} p(y) 2^{-\frac{(\alpha-1)}{\alpha} H_\alpha(A|BC)_{\rho|y}} = 2^{-\frac{(\alpha-1)}{\alpha} H_\alpha(A|BCY)_\rho}. \quad (\text{A9})$$

Since Y can be generated from C , the fact that $H_\alpha(A|BCY) = H_\alpha(A|BC)$ follows by applying the data-processing inequality for Rényi entropy [[27], Corollary 5.1] in both directions $YC \rightarrow C$ and $C \rightarrow YC$. Therefore, the claim follows. ■

The following Lemma is used to split off the information leakage due to error correction and error verification.

Lemma 11. (Splitting off a classical register [27]) Let $\rho_{ABCC'} \in S_o(ABCC') = \rho_{ABC} \otimes \rho_{C'}$ be classical on CC' , then,

$$\begin{aligned} H_\alpha(A|BCC')_\rho &= H_\alpha(A|BC)_\rho \\ &\geq H_\alpha(AC|B)_\rho - \log(\dim(C)) \\ &\geq H_\alpha(A|B)_\rho - \log(\dim(C)). \end{aligned} \quad (\text{A10})$$

Proof. The equality follows from the use of data-processing inequalities [[27], Eq. 5.40]. The inequalities follow from [[27], Eq. 5.94]. ■

Although the fact that ε_1 -correctness and ε_2 -secrecy implies $(\varepsilon_1 + \varepsilon_2)$ -security for QKD protocols has been shown in

many places for fixed-length protocols [3,15,23,24], here we show that the same claim holds for variable-length protocols as well.

Lemma 12. (Correctness and secrecy imply security) Consider a variable-length QKD protocol that only produces a key if Ω_{EV} occurs (error-verification passes). Suppose that the protocol satisfies the correctness condition [Eq. (3)],

$$\Pr(K_A \neq K_B \wedge \Omega_{\text{EV}}) \leq \varepsilon_1, \quad (\text{A11})$$

and the secrecy condition [Eq. (11)],

$$\sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} \right\|_1 \leq \varepsilon_2, \quad (\text{A12})$$

where $\Omega_{\text{len}=k}$ is the event that a key of length k is produced. Then the protocol satisfies the security statement

$$\begin{aligned} & \sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} \right\|_1 \\ & \leq \varepsilon_1 + \varepsilon_2. \end{aligned} \quad (\text{A13})$$

Proof. Let

$$\begin{aligned} \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} &= \sum_{s_A, s_B \in \{0,1\}^k} \Pr(s_A, s_B | \Omega_{\text{len}=k}) \\ & \quad \otimes |s_A, s_B\rangle \langle s_A, s_B| \otimes \rho_{\bar{C} E^N}^{(k, s_A, s_B)} \\ \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{equal})} &= \sum_{s_A, s_B \in \{0,1\}^k} \Pr(s_A, s_B | \Omega_{\text{len}=k}) \\ & \quad \otimes |s_A, s_A\rangle \langle s_A, s_A| \otimes \rho_{\bar{C} E^N}^{(k, s_A, s_B)} \\ \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} &= \sum_{s_A \in \{0,1\}^k} \frac{1}{2^k} |s_A, s_A\rangle \langle s_A, s_A| \otimes \rho_{\bar{C} E^N}^{(k)}. \end{aligned} \quad (\text{A14})$$

By the triangle inequality, we have

$$\begin{aligned} & \sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} \right\|_1 \\ &= \sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{equal})} \right\|_1 \\ & \quad + \sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{equal})} - \rho_{K_A \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{ideal})} \right\|_1. \end{aligned} \quad (\text{A15})$$

We will now relate the first term on the RHS with the correctness condition, and the second term on the RHS with the secrecy condition. To bound the first term in Eq. (A15) we first obtain

$$\begin{aligned} & \frac{1}{2} \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{equal})} \right\|_1 \\ & \leq \sum_{s_A \neq s_B} \Pr(s_A, s_B | \Omega_{\text{len}=k}) \\ & = \Pr(K_A \neq K_B | \Omega_{\text{len}=k}), \end{aligned} \quad (\text{A16})$$

where the first inequality follows from the definition of the states in Eq. (A14). Therefore, the first term in Eq. (A15) can

be bounded via

$$\begin{aligned} & \sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \left\| \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k)} - \rho_{K_A K_B \bar{C} E^N | \Omega_{\text{len}=k}}^{(k, \text{equal})} \right\|_1 \\ & \leq \sum_k \frac{1}{2} \Pr(\Omega_{\text{len}=k}) \Pr(K_A \neq K_B | \Omega_{\text{len}=k}) \\ & = \sum_k \Pr(K_A \neq K_B \wedge \Omega_{\text{len}=k}) \\ & \leq \Pr(K_A \neq K_B \wedge \Omega_{\text{EV}}) \leq \varepsilon_1, \end{aligned} \quad (\text{A17})$$

where in the penultimate step, we use the fact that a key is produced only if event Ω_{EV} occurs.

The second term in Eq. (A15) is identical to the secrecy statement Eq. (A13) since $K_A = K_B$, and hence we obtain the desired result. ■

APPENDIX B: BB84 PROTOCOL

Our protocol is such that in every round Alice and Bob select their basis independently, with probabilities p_z (for Z basis) and $p_x = 1 - p_z$ (for X basis). If basis Z is selected, Alice sends the states $|0\rangle, |1\rangle$ with equal probability. If basis X is selected, Alice sends the states $|+\rangle, |-\rangle$ with equal probability. Using the source-replacement scheme [19], this process can be equivalently described as Alice creating the Bell-state $|\psi\rangle_{AA'} = |\phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, and sending A' to Bob. Eve then interacts with the A' system, and forwards the system B to Bob. This is then followed by Alice and Bob measuring their respective systems using the POVMs $\{P_{(Z,0)} = p_z |0\rangle\langle 0|, P_{(Z,1)} = p_z |1\rangle\langle 1|, P_{(X,0)} = p_x |+\rangle\langle +|, P_{(X,1)} = p_x |-\rangle\langle -|\}$. The rest of the protocol steps (such as sifting, key map etc.) are the same as in Sec. II. After n such rounds have been performed, Alice and Bob choose a uniformly random subset of size m out of the n rounds, to be publicly announced [40].

To simulate the channel statistics, we model misalignment as a rotation of angle θ about the Y axis on A' , with

$$U(\theta) = I_A \otimes \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

$$\mathcal{E}_{\text{misalign}}(\rho) = U(\theta)\rho U(\theta)^\dagger. \quad (\text{B1})$$

Depolarization is modelled as a map

$$\mathcal{E}_{\text{depol}}(\rho) = (1 - q)(\rho) + q \text{Tr}_{A'}(\rho) \otimes \frac{I_B}{2}, \quad (\text{B2})$$

where q is the depolarization probability. The state on which expected statistics are computed is given by $\rho_{\text{phon}} = \mathcal{E}_{\text{depol}}(\mathcal{E}_{\text{misalign}}(|\phi_+\rangle\langle\phi_+|))$.

1. Concentration inequality

We state the following lemma from Ref. [2], which forms the basis of our acceptance tests in the fixed-length protocols and the variable-length decision in the variable-length protocol [Lemma 2, Eq. (22)].

Lemma 13. Let $\bar{\mathbf{F}} \in \mathcal{P}(\Sigma)$ be a probability distribution, and let $\mathbf{F}^{\text{obs}} \in \mathcal{P}(\Sigma)$ be a frequency of outcomes obtained from m IID samples from $\bar{\mathbf{F}}$. Let $\mu = \sqrt{2} \sqrt{\frac{\ln(1/\varepsilon_{\text{NT}}) + |\Sigma| \ln(m+1)}{m}}$.

Then,

$$\Pr(\|\mathbf{F}^{\text{obs}} - \bar{\mathbf{F}}\|_1 \geq \mu) \leq \varepsilon_{\text{AT}}. \quad (\text{B3})$$

Proof. From [[41], Theorem 11.2.1 (Sanov's theorem)], we obtain

$$\Pr(D(\mathbf{F}^{\text{obs}} \|\bar{\mathbf{F}}) > \epsilon) \leq 2^{-m(\epsilon - |\Sigma| \frac{\log_2(m+1)}{m})}, \quad (\text{B4})$$

where D is the classical relative entropy. Furthermore, from [[41], Theorem 11.6.1], we have

$$\sqrt{2 \ln(2) D(\mathbf{F}^{\text{obs}} \|\bar{\mathbf{F}})} \geq \|\mathbf{F}^{\text{obs}} - \bar{\mathbf{F}}\|_1. \quad (\text{B5})$$

Combining (B4) and (B5) we obtain

$$\begin{aligned} \Pr(\|\mathbf{F}^{\text{obs}} - \bar{\mathbf{F}}\|_1 \geq \mu) &\leq \Pr(\sqrt{2 \ln(2) D(\mathbf{F}^{\text{obs}} \|\bar{\mathbf{F}})} \geq \mu) \\ &= \Pr\left(D(\mathbf{F}^{\text{obs}} \|\bar{\mathbf{F}}) \geq \frac{\mu^2}{2 \ln(2)}\right) \\ &\leq 2^{-m\left(\frac{\mu^2}{2 \ln(2)} - |\Sigma| \frac{\log_2(m+1)}{m}\right)}. \end{aligned} \quad (\text{B6})$$

The required result is obtained by setting $\varepsilon_{\text{AT}} = 2^{-m\left(\frac{\mu^2}{2 \ln(2)} - |\Sigma| \frac{\log_2(m+1)}{m}\right)}$. ■

2. Numerics

We use the numerical framework from [31] to compute key rates in this paper. This framework equivalently describes the steps in the QKD protocol via Kraus operators $\{K_i\}$, which represent measurements, announcements and sifting done by Alice and Bob, and $\{Z_j\}$, which implement a pinching channel on the key register. The optimization problem

$\min_{\rho \in \mathcal{S}} H(Z|CE)_\rho$ is then restated as

$$\min_{\rho \in \mathcal{S}} H(Z|CE)_\rho = \min_{\rho \in \mathcal{S}} f(\rho), \quad (\text{B7})$$

where

$$\begin{aligned} f(\rho) &= D(\mathcal{G}(\rho) \|\mathcal{Z}(\mathcal{G}(\rho))), \\ \mathcal{G}(\rho) &= \sum_i K_i \rho K_i^\dagger, \\ \mathcal{Z}(\mathcal{G}(\rho)) &= \sum_j Z_j \mathcal{G}(\rho) Z_j^\dagger, \end{aligned} \quad (\text{B8})$$

and where $D(X\|Y) = \text{Tr}(X \log(X)) - \text{Tr}(X \log(Y))$ is the quantum relative entropy where \log denotes the matrix logarithm.

The construction of the Kraus operators K_i and Z_i is specified in [31] along with improvements in [42]. The Kraus operators for qubit BB84 protocol are given by

$$\begin{aligned} K_Z &= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_Z \otimes \sqrt{p_z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_A + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_Z \otimes \sqrt{p_z} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_A \right] \\ &\quad \otimes \sqrt{p_z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_B \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_C, \\ K_X &= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_Z \otimes \sqrt{\frac{p_x}{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_A + \begin{pmatrix} 0 \\ 1 \end{pmatrix}_Z \otimes \sqrt{\frac{p_x}{2}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}_A \right] \\ &\quad \otimes \sqrt{p_x} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_B \otimes \sqrt{p_x} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_C, \end{aligned} \quad (\text{B9})$$

and

$$\begin{aligned} Z_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I}_{\dim(A) \times \dim(B) \times \dim(C)}, \\ Z_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbb{I}_{\dim(A) \times \dim(B) \times \dim(C)}. \end{aligned} \quad (\text{B10})$$

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- [20] For the purposes of this paper we take m to be a constant; with minor modifications our proof should generalize to the case where m could be a random variable.
- [21] This condition can be slightly weakened to having a fixed upper bound v_i^{EC} on the number of bits used, by noting that the number of bitstrings of length up to some value v is $2^{v+1} - 1$, so an $(v + 1)$ -bit register suffices to encode all such bitstrings. With this, it suffices to replace the λ_i^{EC} values in our subsequent key length formulas with $v_i^{\text{EC}} + 1$.
- [22] While Bob’s announcement technically constitutes an extra bit, we note that in our security proofs, when accounting for the “leakage” caused by C_V , we only consider the state conditioned on accepting in this step, in which case this extra bit takes a deterministic value and does not affect any entropies.
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- [28] A full specification of a composable security framework [23,24] also technically requires describing some *honest* ideal functionality in the case where Eve does not attack the protocol. For a variable-length protocol, we can take this to simply be a functionality that outputs perfect keys (of variable length) to Alice and Bob and nothing to Eve except the length of the key, with the distribution of key lengths being the same as that of the honest protocol behavior. (This behavior does not have to be explicitly known, for instance when considering the Sec. V protocol. We merely require this honest behavior to *exist* in principle, and (to avoid only having trivial operational implications) for it to produce some “reasonable” expected key rate.) With this choice of ideal functionality, the protocol satisfies the property of *completeness* (see [23,24] for details) with perfect completeness parameter.
- [29] In principle there is the technicality that for an arbitrary variable-length protocol, Alice and Bob might produce final keys of different lengths. For this work, we focus on protocols where the final key length is completely determined from the public announcements, and so this is not an issue.
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- [33] For protocols compatible with the specific b_{stat} construction we use in this work, \mathcal{F} would have to be finite because the formula we use in [26] requires the outcome space Σ to be finite in order to obtain nontrivial results. However, we cover a possibly infinite \mathcal{F} in this part of our analysis to accommodate potential follow-up work; in particular, for continuous-variable QKD it should be possible to construct an appropriate estimator b_{stat} (via a different concentration inequality) even if the outcome space is infinite.
- [34] This specification of $\hat{\mathcal{F}}_i^{\text{hash}}$ is not technically a set of *functions*, since each element of $\hat{\mathcal{F}}_i^{\text{hash}}$ is instead a tuple where the second term is an l -bit string. However, each such element uniquely specifies a function in a simple manner that we shall shortly specify.
- [35] \mathcal{E} cannot “directly” compute $\text{len}(Z^{\leq n})$ because the register $Z^{\leq n}$ is no longer present in the states it acts on.
- [36] Equation (48) is a well-defined expression despite the fact that V is not defined on all of $R\bar{C}$, because $\rho_{R\bar{C}E^N}$ is only supported on the subspace on which V is defined.
- [37] An alternative proof would be to instead use [[27], Proposition 5.1] to split the conditional entropies into terms conditioned on each value of \bar{C} , and note that the equality holds for each term by invariance of Rényi entropy under isometries on the first subsystem.
- [38] Note that depending on the proof method used, the bound on $H_\alpha(\hat{Z}^n|\bar{C}E^N)_\rho$ for Case 1 may not be equal to the bound on $H_\alpha(Z^n|\bar{C}E^N)_\rho$ for Case 2. For instance, with the approach we use in this work, formulas such as lemma 8 depend on the dimension of \hat{Z} versus Z . Other proof approaches (for instance, bounding the single-round Rényi entropy directly, rather than first bounding the von Neumann entropy and then applying lemma 8) may not have this feature.
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this could be increased to approximately $\sim p_x^2 n$ (see e.g., [25] Sec. 5.1); however, we leave the details for future work.

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