Steady-state quantum thermodynamics with synthetic negative temperatures

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A bath with a negative temperature is a subject of intense debate in recent times. It raises fundamental questions not only on our understanding of negative temperature of a bath in connection with thermodynamics but also on the possibilities of constructing devices using such baths. In this work, we study steady-state quantum thermodynamics involving baths with negative temperatures. A bath with a negative temperature is created synthetically using two baths of positive temperatures and weakly coupling these with a qutrit system. These baths are then coupled to each other via a working system. At steady state, the laws of thermodynamics are analyzed. We find that whenever the temperatures of these synthetic baths are identical, there is no heat flow, which reaffirms the zeroth law. There is always a spontaneous heat flow for different temperatures. In particular, heat flows from a bath with a negative temperature to a bath with a positive temperature which, in turn, implies that a bath with a negative temperature is "hotter" than a bath with a positive temperature. This warrants an amendment in the Kelvin-Planck statement of the second law, as suggested in earlier studies. In all these processes, the overall entropy production is positive, as required by the Clausius statement of the second law. We construct continuous heat engines operating between positive and negative temperature baths. These engines yield maximum possible heat-to-work conversion efficiency, that is, unity. We also study the thermodynamic nature of heat from a bath with a negative temperature and find that it is thermodynamic work but with negative entropy.

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I. INTRODUCTION

Thermodynamics constitutes a fundamental building block of the modern understanding of nature. With the advent of quantum mechanics, there have been numerous efforts to extend the framework to systems composed of a finite or large number of quantum particles while each particle has a discrete energy spectrum and the states are in a superposition of different energy levels, see for example [1]. One of the possibilities for quantum systems with bounded energy is that, in certain situations, they can assume "negative" temperatures. This arises when the population distribution of a system or

comprehensively discussed the thermodynamic implications of such negative temperatures and the inter-relation between negative and positive temperatures [3]. He states that the Clausius statement of the second law remains unchanged if one considers that negative temperature is "hotter" than a positive temperature, and there heat can only flow from a "hot" bath to a "cold" bath in absence of any external work. Further, he advocates for a modification of the Kelvin-Planck statement of the second law: "It is impossible to construct an engine that will operate in a closed cycle and produce no effect other than (1) the extraction of heat from a positive temperature reservoir with the performance of an equivalent amount of work or (2) the rejection of heat into a negativetemperature reservoir with the corresponding work being done on the engine" [3].

bath becomes an inverted Boltzmann distribution, i.e., states with higher energy are populated more than the ones with

lower energy. It was first pointed out by Purcell and Pound in

the context of nuclear spin systems [2]. Subsequently, Ramsey

Initially, Schöpf raised some foundational questions regarding the dynamics of negative temperature [4]. He claimed

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that it is impossible to transform a thermodynamic system adiabatically: from a positive finite temperature to the positive infinite temperature, then from there to a negative infinite (Boltzmann) temperature, and then subsequently to a negative finite (Boltzmann) temperature [4]. Tykodi and Tremblay [5–7] disagreed and showed that the arguments used by Schöpf are thermodynamically inconsistent as these violate the second law of thermodynamics. Nevertheless, the debates on thermodynamics with negative temperatures are not settled, see for example [8]. Recent theoretical [9,10] and experimental [11–13] studies with cold atoms have brought the debate on negative temperatures back again into the spotlight. The works in Refs. [14–16] claim that "all previous negative temperature claims and their implications are invalid as they arise from the use of an entropy definition that is inconsistent both mathematically and thermodynamically." Another study in Ref. [17] states that thermodynamic equilibrium at negative temperatures would be unstable but can be used for work storage or battery. Several researchers have come forward and systematically explained that identification of the thermodynamic entropy exclusively with the volume entropy proposed by Gibbs is the root of all doubts [18-22], and it is inconsistent with the postulates of thermodynamics [23,24]. Using Boltzmann entropy as the thermodynamic entropy, they argue that negative temperature is a valid extension of thermodynamics.

Apart from these foundational issues, there are questions on whether a negative temperature bath can be used to construct thermal devices, such as heat engines, refrigerators, heat pumps, etc. Initially, a study on Carnot engines was made by Geusic et al. [25] and, later on, by Landsberg and Nakagomi [26,27] in this context. There are also some studies about how the Carnot cycle should be modified in the presence of negative temperatures [28–30]. Further, some propositions are made to construct a quantum Otto engine [31-34] and refrigerators [35] using a bath with effective negative temperature. It is shown that the heat-to-work conversion efficiency of an engine operating between negative and positive temperatures would be greater than unity [10-12]. But many of these thermal device models either use already-existing negative temperature baths without caring how these baths can be prepared or are artificially prepared using either external driving or work. Further, a systematic study of thermodynamics in the continuous or steady-state regime in the presence of baths with negative temperatures and continuous heat engines with such baths are still missing.

In this article, we outline how to create a thermal bath with arbitrary temperature, including negative temperature, and study steady-state quantum thermodynamics. Along with this, we also propose continuous heat engines with baths at negative and positive temperatures. The bath with arbitrary temperature is synthesized by letting a quantum system interact simultaneously with two thermal baths at different positive temperatures without external driving. We study various laws of steady-state quantum thermodynamics with these synthetic baths and construct continuous heat engines. We start by verifying the zeroth law and show that there is no net heat flow whenever two such baths with identical temperatures are brought in contact with each other. This, in turn, legitimizes the notion of the temperature of a synthetic bath (namely, the



FIG. 1. A synthetic bath is created using two baths with different temperatures and letting them interact with a qutrit system. In particular, the hot bath (*H*) with inverse temperature β_H is weakly coupled to the energy eigenstates $|1\rangle$ and $|3\rangle$. The cold bath (*C*) with inverse temperature β_C weakly interacts with the energy eigenstates $|2\rangle$ and $|3\rangle$. As a result, the populations of the states $|1\rangle$ and $|2\rangle$ reach an equilibrium corresponding to a synthetic temperature β_S . By tuning the temperature of the baths and the energy spacing between the states, an arbitrary synthetic temperature can be obtained, including negative temperatures. See text for more details.

synthetic temperature). We verify the Kelvin-Planck statement of the second law for negative temperatures, which states that it is impossible to construct a device operating in a cyclic process that produces no effect other than dumping heat into a negative-temperature bath with the corresponding work being performed on the device. In the case of two different temperatures, we verify the Clausius statement of the second law and demonstrate that there is a spontaneous heat flow from a bath with a negative temperature to one with a positive temperature. Both of these corroborate with the finding of Ramsey [3]—baths with negative temperatures are "hotter" than the ones with positive temperatures. Interestingly, in such cases, the entropy flow is opposite to the direction of heat flow which is again expected for the baths with negative temperatures. We also construct continuous heat engines involving synthetic baths and find that engines operating between positive and negative temperature baths can yield unit engine efficiency. This leads us to question the physical meaning of heat flow in the presence of a bath with negative temperatures. With a systematic analysis, we show that the heat associated with a bath with a negative temperature is equivalent to work but with a negative entropy flow.

II. SYNTHETIC BATHS AND NEGATIVE TEMPERATURES

In general, naturally occurring thermal equilibrium results in nonnegative temperatures. Only in certain situations, as discussed earlier, can the temperatures be negative. Below, we introduce a method through which a bath can be synthesized. The temperatures of these synthetic baths can assume arbitrary values, including negative ones.

The method utilizes a qutrit system, a hot bath (*H*) with inverse temperature β_H , and a cold bath (*C*) at inverse temperature β_C . The energy levels of the qutrit are denoted by $|1\rangle$, $|2\rangle$, $|3\rangle$, with the corresponding Hamiltonian $H_0 = (E_H - E_C) |2\rangle\langle 2| + E_H |3\rangle\langle 3|$. As shown in Fig. 1, the hot (cold) bath weakly interacts with the levels $|1\rangle$ and $|3\rangle$ (levels $|2\rangle$ and $|3\rangle$). As convention, we consider $\beta_H < \beta_C$, Planck constant $\hbar = 1$, and the Boltzmann constant $k_B = 1$ throughout the article. The levels $|1\rangle$ and $|2\rangle$ are not directly coupled. However, they are indirectly linked through the level $|3\rangle$. When the couplings between the qutrit and the baths are weak and satisfy the Markov condition, the overall dynamics is expressed in terms of the Lindblad or Lindblad-Gorini-Kossakowski-Sudarshan (LGKS) form of master equation [36]

$$\dot{\rho} = \mathcal{L}_U(\rho) + \mathcal{L}_H(\rho) + \mathcal{L}_C(\rho). \tag{1}$$

Here ρ represents the density matrix corresponding to a state of the qutrit. The first term on the right-hand side of Eq. (1) $\mathcal{L}_U(\rho) = i \ [\rho, H_0]$ takes care of the unitary part of the evolution due to the system Hamiltonian H_0 . The second and third terms, with the Lindblad superoperators (LSOs) $\mathcal{L}_H(\cdot)$ and $\mathcal{L}_C(\cdot)$, represent the contributions due to the dissipative part of the evolution induced by the hot and cold thermal baths, respectively. The LSOs are expressed (for X = H, C) as

$$\mathcal{L}_{X}(\rho) = \Gamma_{X}(N_{X}+1)(A_{X}\rho A_{X}^{\dagger} - 1/2 \{A_{X}^{\dagger}A_{X}, \rho\}) + \Gamma_{X}N_{X}(A_{X}^{\dagger}\rho A_{X} - 1/2 \{A_{X}A_{X}^{\dagger}, \rho\}), \qquad (2)$$

where $A_H = |1\rangle\langle 3|$, $A_C = |2\rangle\langle 3|$, anticommutator $\{Y, Z\} = YZ + ZY$, and $N_X = 1/(e^{\beta_X E_X} - 1)$. The coefficient Γ_X is the Weiskopf-Wigner decay constant. The overall dynamics leads to heat exchange between the baths and the system. The heat fluxes are quantified as $\dot{Q}_X = \text{Tr}[\mathcal{L}_X(\rho) H_0]$ due to interaction with the bath X [37–39]. Heat flux $\dot{Q}_X > 0$ implies that heat is flowing into the qutrit system from the bath with inverse temperature β_X .

This dynamics always leads to a steady state, say σ , which is diagonal in the energy eigenstates. The populations $\{p_i\}$ of the states $\{|i\rangle\}$ satisfy

$$\frac{p_1}{p_3} = e^{\beta_H E_H} \text{ and } \frac{p_2}{p_3} = e^{\beta_C E_C}$$

The populations corresponding to states $|1\rangle$ and $|3\rangle$ attain thermal equilibrium with the hot bath and, similarly, the populations of $|2\rangle$ and $|3\rangle$ attain thermal equilibrium with the cold bath. In fact, the dynamics drives the overall system to reach thermal equilibrium, albeit in interactions with two baths at different temperatures. This is justified because the heat flux and entropy production vanish, i.e., $\dot{Q}_X = 0$ for X = H, *C* and $-\beta_H \dot{Q}_H - \beta_C \dot{Q}_C = 0$, respectively. As the entire system is in thermodynamic equilibrium, so are the populations of the states $|1\rangle$ and $|2\rangle$.

In general, if one introduces an interaction between the levels $|1\rangle$ and $|2\rangle$, be it time-dependent or time-independent, then the heat and entropy fluxes become nonzero [38,39] and the corresponding populations change. But once the interaction is switched off, the populations revert to their equilibrium values. This is as if the levels $|1\rangle$ and $|2\rangle$, or the subspace spanned by these two levels, are interacting with a synthetic bath at inverse temperature β_S , defined as

$$\beta_S = \frac{1}{E_S} \ln\left(\frac{p_1}{p_2}\right) = \frac{1}{E_S} \ln\left(\frac{p_1}{p_3} \frac{p_3}{p_2}\right) = \frac{\beta_H E_H - \beta_C E_C}{E_S},$$
(3)

where $E_S = E_H - E_C$. A similar concept of temperature is also introduced in Ref. [40] in the context of quantum thermal



FIG. 2. Two synthetic baths are engineered with the help of a hot and a cold bath at inverse temperatures β_H and β_C , respectively. Each synthetic bath is created by letting the baths weakly interact with one qutrit, as shown in Fig. 1. Different synthetic temperatures are engineered by tuning energy spacings between the states $|1\rangle$, $|2\rangle$, and $|3\rangle$. In addition, an interaction is introduced between the synthetic baths to study the heat and entropy flow. See text for more details.

machines. We note that a LSO cannot be given exclusively for the equilibration dynamics due to the synthetic bath. However, as we discuss in the later sections, this is a legitimate thermal bath. We call β_S the "synthetic" inverse temperature because it can be tuned to assume arbitrary values, including negative values, by changing the energy-level spacings and the β_H and β_C . In literature, there are debates on whether the temperature of a system can be continuously changed from a positive to a negative equilibrium temperature [17]. However, in this setup, the inverse temperature of the synthetic bath can be tuned continuously, e.g., from $\beta_S > 0$ to $\beta_S < 0$, including $\beta_S = 0$ (infinite temperature).

III. THERMODYNAMICS WITH SYNTHETIC BATHS

To study thermodynamics with synthetic temperatures, we consider two different qutrit systems *L* and *R* with the corresponding Hamiltonians $H_X = (E_{XH} - E_{XC}) |2\rangle\langle 2| + E_{XH} |3\rangle\langle 3|$, with X = L, *R*. We assume $E_{XH} - E_{XC} = E_S$ for both systems, i.e., the energy spacing between $|1\rangle$ and $|2\rangle$ for both *L* and *R* are same. For brevity, we denote $H_L \equiv H_L \otimes \mathbb{I}$ and $H_R \equiv \mathbb{I} \otimes H_R$. Each system couples to a hot and a cold bath with inverse temperatures β_H and β_C , respectively (see Fig. 2), and reaches an equilibrium state. Without an interaction in between, the equilibrium state of the composite *LR* becomes

$$\rho_L \otimes \rho_R = \sum_{m,n=1}^3 p_m q_n |m n\rangle \langle m n|, \qquad (4)$$

where $p_1/p_3 = e^{\beta_H E_{LH}}$, $p_2/p_3 = e^{\beta_C E_{LC}}$, $q_1/q_3 = e^{\beta_H E_{RH}}$, and $q_2/q_3 = e^{\beta_C E_{RC}}$. The population ratio between the degenerate energy states $|21\rangle$ and $|12\rangle$ is

$$p_1 q_2 / p_2 q_1 = e^{(\beta_{LS} - \beta_{RS})E_S}, \tag{5}$$

where β_{LS} and β_{RS} are the synthetic temperatures corresponding to qutrit *L* and *R*, respectively.

An interaction is introduced that only couples subspace spanned by the energy levels belonging to $|1\rangle$ and $|2\rangle$ in each qutrit, ensuring an energy exchange between L and R only through these subspaces. The most general interaction Hamiltonian that drives an energy exchange between these subspaces is given by

$$H_{\rm in} = (\lambda + i\gamma) |12\rangle\langle 21| + (\lambda - i\gamma) |21\rangle\langle 12|, \qquad (6)$$

where $\lambda, \gamma \in \mathbb{R}$. This interaction also strictly conserves energy, as $[H_{\text{in}}, H_L + H_R] = 0$. The overall dynamics of *LR* is expressed as

$$\dot{\rho}_{LR} = i \left[\rho_{LR}, H_T \right] + \mathcal{L}_L(\rho_{LR}) + \mathcal{L}_R(\rho_{LR}), \tag{7}$$

where $H_T = H_L + H_R + H_{in}$, $\mathcal{L}_L(\cdot) = \mathcal{L}_{LH}(\cdot) + \mathcal{L}_{LC}(\cdot)$, and $\mathcal{L}_R(\cdot) = \mathcal{L}_{RH}(\cdot) + \mathcal{L}_{RC}(\cdot)$. Here, $\mathcal{L}_{XH}(\cdot)$ and $\mathcal{L}_{XC}(\cdot)$ are the LSOs taking into account the dissipative part of the dynamics due to the coupling with hot and cold baths, respectively, with the qutrit *X*. Under this dynamics, the composite system *LR* reaches a steady state, say σ_{LR} . Then, the heat flux and the entropy flux, respectively, are

$$\dot{Q}_X = \operatorname{Tr}[\mathcal{L}_X(\sigma_{LR}) H_X], \quad \dot{S}_X = \beta_{XS} \dot{Q}_X,$$
 (8)

with X = L, R, and $\dot{Q}_X = \dot{Q}_{XH} + \dot{Q}_{XC}$.

In absence of any interaction between the *L* and *R*, the steady (or equilibrium) state is $\sigma_{LR} = \rho_L \otimes \rho_R$ [see Eq. (4)]. Then, the heat flux from *L* is $\dot{Q}_L = 0$, as $\dot{Q}_{LH} = \dot{Q}_{LC} = 0$. However, in presence of interaction via H_{in} , the steady state becomes $\sigma_{LR} \neq \rho_L \otimes \rho_R$, and then the $\dot{Q}_L = \dot{Q}_{LH} + \dot{Q}_{LC} \neq 0$. This means that there is heat flux through the subspace spanned by $\{|1\rangle, |2\rangle$ of *L*, which we may consider as the heat flux due to the synthetic bath associated with *L*. By convention, $\dot{Q}_L > 0$ implies a heat flux from the synthetic bath to *L*, which is then passed to *R*. With these tools at hand, we now set out to explore steady-state quantum thermodynamics with synthetic temperatures. Note that the first law is always respected at steady state as $\dot{Q}_L + \dot{Q}_R = 0$. Thus, our emphasis would be on studying the zeroth and second laws.

A. Zeroth law

In thermodynamics, the zeroth law interlinks the notion of temperature with equilibrium. It states that if two systems are in thermal equilibrium, they must have the same temperature and vice versa. Again, thermal equilibrium implies that when two systems are in contact, there is no net flux in any thermodynamic quantities (such as heat and entropy) between systems. In such situations, the overall entropy production also vanishes. Below we show that, in the two-qutrit scenario discussed above, there is no net flux of any thermodynamic quantity whenever the synthetic temperatures are identical for both qutrits.

Recall the setup we consider in Fig. 2. Without any interaction, the steady state of LR is $\rho_L \otimes \rho_R$, which is diagonal in the energy eigenstates [see Eq. (4)]. With same synthetic inverse temperatures $\beta_{LS} = \beta_{RS}$, the populations of the states $|12\rangle$ and $|21\rangle$ satisfy $p_1q_2 = p_2q_1$ [see Eq. (5)]. It means, the matrix corresponding to the state $\rho_L \otimes \rho_R$ in the subspace spanned by $\{|12\rangle, |21\rangle\}$ is proportional to an identity operator. Now an interaction between L and R introduced by H_{in} , as in Eq. (6). It is easily seen that $[H_{in}, \rho_L \otimes \rho_R] = 0$ for $\beta_{LS} = \beta_{RS}$. Thus, even after the interaction is switched on, the steady state remains unaltered, i.e., $\sigma_{LS} = \rho_L \otimes \rho_R$ and $\dot{\sigma}_{LR} = 0$. Hence, there is no exchange of heat and entropy between L and R, as $\dot{Q}_X =$ $S_X = 0$ for X = L, R. This implies that the synthetic baths are in thermal equilibrium whenever the synthetic temperatures are identical, irrespective of whether the temperatures are positive or negative. A numerical analysis also confirms this. See Fig. 3(a).

B. Second law

For $\beta_{LS} \neq \beta_{RS}$, heat and entropy flow is possible from one qutrit to the other. However, the flow cannot be arbitrary. The second law dictates the physically allowed processes given that zeroth and first laws are respected. There are various statements of the second law. Below, we verify the Clausius statement regarding the directionality of heat flow and the entropy production.

Let us consider the case for which $-\beta_{LS} > -\beta_{RS}$. When there is no interaction between the qutrits, the equilibrium state of the composite *LS* is $\rho_L \otimes \rho_R$. The state is expressed in the block-diagonal form as

$$\rho_L \otimes \rho_R = \Pi_0 \rho_L \otimes \rho_R \Pi_0 + \Pi_1 \rho_L \otimes \rho_R \Pi_1,$$

where $\Pi_0 = |12\rangle\langle 12| + |21\rangle\langle 21|$ and $\Pi_1 = \mathbb{I} - \Pi_0$. The populations corresponding to the energy eigenstates $|12\rangle$ and $|21\rangle$ satisfy $p_1q_2 < p_2q_1$ [see Eq. (5)].

Now let us disconnect the thermal baths and introduce an interaction driven by H_{in} . This, or the total Hamiltonian H_T , evolves the composite and induces a rotation onto the subspace spanned by $|12\rangle$ and $|21\rangle$ only. At the same time, the other part of the density matrix remains unchanged. As a result, there appear off-diagonal elements in this subspace. Say, the state of *LR* after any evolution becomes

$$\begin{split} \rho_{LR}' &= a \mid \! 12 \rangle \langle 12 \mid + b \mid \! 12 \rangle \langle 21 \mid + c \mid \! 21 \rangle \langle 12 \mid + d \mid \! 21 \rangle \langle 21 \mid \\ &+ \Pi_1 \rho_L \otimes \rho_R \Pi_1. \end{split}$$

The unitary nature of the evolution in the subspace $|12\rangle$ and $|21\rangle$ guarantees that $a > p_1q_2$ and $d < p_2q_1$. For this reason and as the off-diagonal elements do not contribute to the populations of the reduced state of L, i.e., $\rho'_L = \text{Tr}_R \rho'_{LR}$, we find $p'_1 = \langle 1 | \rho'_L | 1 \rangle > p_1$ and $p'_2 = \langle 2 | \rho'_L | 2 \rangle < p_2$. Similarly, for the reduced state of $\rho'_R = \text{Tr}_L \rho'_{LR}$, the modified populations becomes $q'_1 = \langle 1 | \rho'_R | 1 \rangle < q_1$ and $q'_2 = \langle 2 | \rho'_R | 2 \rangle > q_2$. Note, the populations corresponding to level $|3\rangle$ for both L and *R* remain unchanged, i.e., $\langle 3|\rho'_L|3\rangle = p_3$ and $\langle 3|\rho'_R|3\rangle = q_3$. Clearly, the qutrit L loses some energy. As the evolution respects strict energy conservation, the qutrit R gains the same amount of energy. Thus, any evolution due to H_{in} ensures that there is an energy flow from *L* to *R* for $-\beta_{LS} > -\beta_{RS}$. After this modification, if L is now exposed to its baths, then the dissipative dynamics due to $\mathcal{L}_L(\cdot)$ forces the qutrit to restore its equilibrium state, $p'_1 \rightarrow p_1$ and $p'_2 \rightarrow p_2$. This, in turn, increases the energy of L by absorbing some heat from the hot and cold baths or, equivalently, from the synthetic bath. Similarly, if R is exposed to its baths, then some of its energy is released to its synthetic bath in the form of heat and thereby attains its equilibrium. Note in this process, to reach the equilibrium, the populations of $|3\rangle$ do not remain constant throughout in both L and R.

From the above arguments, we see that the unitary evolution with the interaction between *L* and *R* drives the composite out of equilibrium leading to a spontaneous heat flow from *L* to *R* for $-\beta_{LS} > -\beta_{RS}$. At the same time, the dissipative evolution due to thermal interactions with the baths tries to restore the composite back to the initial equilibrium state $(\rho_L \otimes \rho_R)$ by pumping some heat into *L* and absorbing some heat from *R*. When both unitary and dissipative evolutions occur simultaneously, as in Eq. (7), the opposing tendencies



FIG. 3. The figures represent heat and entropy fluxes, \dot{Q}_L and \dot{S}_L , respectively, through the qutrit *L*. The numerical calculation is carried out with the parameters: $\Gamma_H = \Gamma_C = 0.001$, $\beta_H = 0.05$, $\beta_C = 1$, $\lambda = 1$, $\gamma = 0$, $E_S = 9.5$. The different synthetic inverse temperatures are obtained by tuning the energy E_H of $|3\rangle$ for the qutrits *L* and *R*. (a) The density plot represents the Clausius statement of the second law in terms of the heat flux through *L*, i.e., \dot{Q}_L . The plot shows no heat flux for $\beta_{LS} = \beta_{RS}$. This corroborates with the zeroth law. However, there is a positive (negative) heat flux \dot{Q}_L , i.e., heat flows from *L* to *R* (from *R* to *L*), whenever $-\beta_{LS} > -\beta_{RS} (-\beta_{LS} < -\beta_{RS})$ implying bath with β_{LS} (β_{RS}) is hotter than $\beta_{RS} (\beta_{LS})$. Clearly, a bath with a negative temperature is always 'hotter' than any bath with a positive temperature. (b) The density plot represents the variation of entropy flux \dot{S}_L for different synthetic temperatures. As seen from the plot, for $\beta_{LS} = \beta_{RS}$, $\dot{S}_L = 0$. For $\beta_{LS} > 0$ and $\beta_{RS} > 0$, the direction of entropy flux is same with the heat flux as expected for the baths with positive temperatures. However, the direction of heat flow is opposite to the direction of entropy flow, in general, for baths with negative inverse temperatures. See text for details.

balance each other and result in a steady state, say σ_{LR} . This steady state is again block-diagonal in total energy eigenstates and has off-diagonal elements in the eigenstates $|12\rangle$ and $|21\rangle$. Nevertheless, the steady-state dynamics generate a heat flux from *L* to *R*,

$$\dot{Q}_L = \operatorname{Tr}[\mathcal{L}_L(\sigma_{LR}) H_L] = \operatorname{Tr}[\mathcal{L}_L(\sigma_L) H_L] > 0, \qquad (9)$$

and $\dot{Q}_R = -\dot{Q}_L$. The expression of \dot{Q}_L can be given analytically, and it has complicated dependencies with all the parameters. Rather, a numerical analysis is more illuminating; we have done so in Fig. 3(a). Again for $-\beta_{LS} < -\beta_{RS}$, we find that $\dot{Q}_L < 0$. This means that there is a spontaneous heat flow from *R* to *L*.

In thermodynamics, the Clausius statement of the second law states that heat can only flow from a hot bath to a cold bath when no external work is performed. As we see above for $-\beta_{LS} > -\beta_{RS}$, there is a heat flow from *L* to *R*. This means that: (1) baths with negative inverse temperatures are "hotter" than the baths with positive inverse temperatures; (2) baths with larger negative inverse temperatures are "hotter" than those with smaller negative inverse temperatures. This is in conformation with the findings of Ramsey [3]. However, one important point to be noted here is that although there is a heat flow from a negative to a positive temperature bath, the entropy flow is the opposite. This is indeed a signature of a bath having a negative temperature.

Another version of the Clausius statement states that the overall entropy production is always positive in a thermodynamical process. For steady-state thermodynamics with synthetic baths, the overall entropy production rate is given by

$$\Sigma = \dot{S}_{LR} - \beta_{LS}\dot{Q}_L - \beta_{RS}\dot{Q}_R, \qquad (10)$$

where $S_{LR} = \partial S_{LR}/\partial t$ is the rate of change in von Neumann entropy $S_{LR} = -\text{Tr}[\sigma_{LR} \log \sigma_{LR}]$, and β_{LS} (β_{RS}) is the synthetic inverse temperature of L (R) and \dot{Q}_L (\dot{Q}_R) is the heat flux from L (R). Note at steady state, the $\partial S_{LR}/\partial t = 0$ and $\Sigma = (\beta_{RS} - \beta_{LS}) \dot{Q}_L$, as $\dot{Q}_L = -\dot{Q}_R$. For $-\beta_{LS} > -\beta_{RS}$, the heat flux from L is positive, $\dot{Q}_L > 0$. Consequently, $\Sigma > 0$. Similarly, $\dot{Q}_L < 0$ for $-\beta_{LS} < -\beta_{RS}$, and thus $\Sigma > 0$. For $\beta_{LS} = \beta_{RS}$, we have $\Sigma = 0$. Thus, the Clausius inequality in the differential form is

and it is always satisfied.

At steady state, the entropy production rate is positive, $\Sigma \ge 0$. This is mainly due to the dissipative interaction between the baths and the system. However, one may find out an entropy flow through the system *LR*, as

 $\Sigma \ge 0$,

$$S_X = -\text{Tr}[\mathcal{L}_X(\sigma_{LR})\log\sigma_{LR}],$$

for X = L, *R* (see Appendix A). At steady state, $\dot{S}_L + \dot{S}_R = 0$ as the state does not evolve over time. $\dot{S}_L > 0$ implies that there is an entropy flux from bath with inverse temperature β_{LS} to *R* via *L*, and similarly for $\dot{S}_R > 0$. In general, for a bath with positive temperature, an outflow of heat is associated to a decrease in entropy. One striking feature we must note here is that, although there is a spontaneous heat flow from a bath with negative temperature to a bath with positive temperature, the entropy flow is just opposite to that [see Fig. 3(b)]. This is also true when both baths are of negative temperatures.

For a bath with negative temperature, an outflow of heat is associated with an increase in entropy of the bath. Thus, a bath with negative temperature in general acts as an entropy sink.

Now, we focus on the Kelvin-Planck statement of the second law that applies to cyclic processes involving a single bath. Traditionally, for baths with positive temperatures, the statement says that it is impossible to devise a cyclic process that absorbs energy in the form of heat from a single thermal reservoir and delivers an equivalent amount of work as the sole effect. According to Ramsey, the statement must be appended to incorporate the cases involving baths with negative temperatures with the following. It is impossible to construct a device operating in a cyclic process that produces no effect other than dumping heat into a negative-temperature bath with the corresponding work being performed on the device [3].

To verify the Kelvin-Planck statement, we consider a cyclic process that involves one synthetic bath. We utilize a device where a qutrit interacts with a hot and a cold bath at inverse temperatures β_H and β_C , respectively, as considered in Sec. II and shown in Fig. 1. In this case, the Hamiltonian of the qutrit is $H_0 = (E_H - E_C) |2\rangle\langle 2| + E_H |3\rangle\langle 3|$ with $E_H > E_C > 0$, and the working system, composed of the energy levels $|1\rangle$ and $|2\rangle$, is interacting with the synthetic bath with the synthetic inverse temperature $\beta_S = \frac{\beta_H E_H - \beta_C E_C}{E_S}$. In addition, the levels $|1\rangle$ and $|2\rangle$ are driven by an external field, given by the driving Hamiltonian

$$H_d(t) = \alpha(|1\rangle\langle 2|e^{i\omega t} + |2\rangle\langle 1|e^{-i\omega t}).$$
(11)

This device model is widely considered in the literature; see, for example, Refs. [37–39,41]. Following the discussion in Appendix B, the total Hamiltonian can be made time-independent by moving to a rotating frame, and a steady state is attained when the qutrit is weakly coupled to the baths. As this represents a continuous device, exploiting a nonequilibrium steady state, we may consider that the device is undergoing a cyclic process where every cycle is completed instantaneously. With an explicit calculation of the steady state, the power *P* and heat fluxes \dot{Q}_H and \dot{Q}_C associated with the hot and cold baths are given by [41]

$$P = -\frac{4\alpha^2 \Gamma (N_H - N_C)(E_H - E_C)}{G},$$
(12)

$$\dot{Q}_H = \frac{4\alpha^2 \Gamma (N_H - N_C) E_H}{G},\tag{13}$$

$$\dot{Q}_C = -\frac{4\alpha^2 \Gamma (N_H - N_C) E_C}{G},\tag{14}$$

where $G = 4\alpha^2(3(N_H + N_C) + 4) + \Gamma^2(N_H + N_C)(N_H + N_C + 3N_HN_C)$ is positive constant and $N_X = 1/(e^{\beta_X E_X} - 1)$. Without loss of generality, here we consider $\Gamma_H = \Gamma_C = \Gamma > 0$. The heat flux corresponding to the synthetic bath is $\dot{Q}_S = \dot{Q}_H + \dot{Q}_C$, and the first law implies $\dot{Q}_S + P = 0$. The P < 0 represents that the device is operating in a cycle in which heat is absorbed from the synthetic bath $(\dot{Q}_S > 0)$ and work is delivered to the driving field. However, P > 0 implies that the device is operating in a cycle in which the work is performed on the device by the driving field while heat $(\dot{Q}_S < 0)$ is dumped into the bath.

We first consider the device with a bath at positive synthetic inverse temperature $\beta_S > 0$, which is equivalent to the condition $(N_H - N_C) < 0$. For all values of the parameters α and Γ , the only possibility is that P > 0 and $\dot{Q}_S < 0$. This is in full compliance with the traditional Kelvin-Planck statement, which states that, in a cyclic process, heat cannot be absorbed from a bath with positive temperature and converted into an equivalent amount of work as the sole effect.

For the device with a bath at negative synthetic inverse temperature $\beta_S < 0$, the equivalent condition $(N_H - N_C) > 0$ implies that there is only one possibility, i.e., P < 0 and $\dot{Q}_S > 0$, for all values of α and Γ . This is again in full compliance with the amendment to the Kelvin-Planck statement proposed by Ramsey, which states that in a cyclic process, heat cannot be dumped into a bath with negative temperature and converted into an equivalent amount of work as the sole effect.

With verification above, let us discuss the conceptual takeaways of the Kelvin-Planck statement in this context. From our discussion on the Clausius statement earlier, we have seen that, while the direction of the flow of heat and entropy is the same for $\beta_S > 0$, the direction of heat and entropy flow are opposite for $\beta_S < 0$. For a device operating in cycle with a bath at $\beta_S > 0$, any heat flown out of the bath cannot be converted into work entirely in the absence of an entropy sink, where work is a pure form of energy and does not contain entropy. Again, absorbing entropy into an entropy sink or a bath with positive (negative) temperature must result in an increase (decrease) in energy or heat. Thus, such a process cannot be realized without an entropy sink attached to the device or without the possibility of increasing its energy. For a device attached to a bath with $\beta_S < 0$ and operating in a cycle, a heat flow into the bath is associated with an opposite entropy flow, i.e., an entropy flow out of the bath. Again, without the possibility of dumping that entropy into a sink or a bath, such a process cannot be implemented even after performing an equivalent amount of work on the device.

IV. QUANTUM HEAT ENGINES WITH A BATH AT NEGATIVE TEMPERATURE

Now we discuss heat engines operating with a bath at synthetic temperatures, particularly at negative temperatures. A device acting as a heat engine aims to transform heat into work. A generic heat engine consists of three primary parts: two separate heat baths with different temperatures and a working system. It operates by absorbing heat from the hot bath. The working system transforms part of this heat into work, dumping the rest into the cold bath. The model of a quantum heat engine (QHE) we are concerned with utilizes a synthetic bath with negative temperature and a heat bath with positive temperature (as depicted in Fig. 4). The working system is composed of a qutrit (L) and a qubit (W). The synthetic bath with inverse temperature β_{LS} is created using two baths at different temperatures β_H and β_C and letting these weakly interact with L, similar to the one considered in Fig. 1. The Hamiltonian of L is given by $H_L = (E_H - E_C)|2\rangle\langle 2| +$ $E_H|3\rangle\langle 3|$. The qubit W is weakly coupled to a bath at inverse temperature $\beta_W \ge 0$, and its Hamiltonian is $H_W = E_W |2\rangle\langle 2|$, where $E_W = E_H - E_C$. To operate the device as a heat engine, a time-dependent interaction is introduced between L and W



FIG. 4. A schematic of a quantum heat engine operating with two baths; one with negative inverse temperature β_{LS} and the other with positive temperature β_W . The negative temperature is synthesized by weakly coupling a qutrit *L* with a hot and a cold bath, as discussed in Fig. 1. This is as if the energy levels $|1\rangle$ and $|2\rangle$ are coupled to a synthetic bath at inverse temperature β_{LS} . A qubit *W* is weakly coupled to the bath with β_W . In the engine, *L* and *W* are coupled through a time-dependent interaction as in Eq. (15). The arrows represent the direction of heat (\dot{Q}_X) and entropy (\dot{S}_X) fluxes for X = L, W, and *P* represents the power of the engine. See text for more details.

driven by the interaction Hamiltonian

$$H_{\rm in}^E(t) = \delta \left(|11\rangle \langle 22| \ e^{i\omega t} + |22\rangle \langle 11| \ e^{-i\omega t} \right). \tag{15}$$

The total Hamiltonian is then $H_T^E = H_0 + H_{in}^E(t)$, where $H_0 = H_L + H_W$. The overall dynamics will never lead to a steady state for a time-dependent interaction. However, for a periodic time dependence, there is a rotating frame in which the interaction becomes time-independent. For instance, to move from the laboratory frame to a rotating frame we may introduce a rotation, given by the unitary $U = \exp[iH_R t]$ which satisfies $[H_R, H_0] = 0$. For a suitable H_R , the interaction becomes time-independent, i.e., $V_{in} = UH_{in}^E(t)U^{\dagger}$. In this rotating frame, the overall dynamics comprising the unitary and the dissipative evolution is given by (see Appendix A)

$$\dot{\rho}_{LW}^{R} = i \left[\rho_{LW}^{R}, \bar{H}_{T}^{E} \right] + \mathcal{L}_{L} \left(\rho_{LW}^{R} \right) + \mathcal{L}_{W} \left(\rho_{LW}^{R} \right), \qquad (16)$$

for a state ρ_{LW} , with $\rho_{LW}^R = U \rho_{LW} U^{\dagger}$ and $\bar{H}_T^E = H_0 - H_R + V_{\text{in}}$. Note the LSOs remain unchanged in the rotating frame. Now that the time-dependence in the Hamiltonian is lifted, the dynamics attains a steady state σ_{LW}^R in the rotating frame.

With this, the heat flux, entropy flux, and the power in the laboratory frame are quantified as (see Appendix A)

$$\dot{Q}_X = \operatorname{Tr} \left[\mathcal{L}_X(\sigma_{LW}^R) H_0 \right], \quad \dot{S}_X = -\operatorname{Tr} \left[\mathcal{L}_X(\rho_{LW}^R) \log \sigma_{LW}^R \right],$$

and $P = i \operatorname{Tr} \left[\sigma_{LW}^R \left[V_{\mathrm{in}}, H_0 \right] \right].$ (17)

for
$$X = L, W$$
, and $[A, B] = AB - BA$. Here \dot{Q}_X
and \dot{S}_X represent the heat and entropy fluxes
through system X, respectively, and P represents
the power. The condition $\dot{Q}_L + \dot{Q}_W + P = 0$ is always
satisfied as required by the first law [42,43] and at steady
state, $\dot{S}_L + \dot{S}_W = 0$. For any negative inverse temperature
 $\beta_{LS} < 0$, we have, as confirmed by numerical analysis,
 $\dot{Q}_L > 0, \dot{Q}_W > 0$, and $P < 0$. This means the device draws

 $Q_L > 0$, $Q_W > 0$, and P < 0. This means the device draws heat from both the synthetic bath and the bath with inverse temperature β_W . For traditional engines, the efficiency is calculated as the ratio of work extracted and the heat absorbed by the engine from the hot bath. Here, the heat is absorbed from both baths. For each bath, the corresponding engine efficiency may be defined as

$$\eta_L = \frac{-P}{\dot{Q}_L}$$
, and $\eta_W = \frac{-P}{\dot{Q}_W}$



FIG. 5. Heat and entropy fluxes in the engine. The red-solid and blue-dotted traces represent the change in heat flux (\dot{Q}_L) and entropy flux (\dot{S}_L) through *L* in the engine with respect to β_{LS} , respectively. The calculation is done with the parameters: $\beta_H = 0.01$, $\beta_W = 0.1$, $\beta_C = 10$, $\delta = 5$, $\omega = 1$, and $E_S = 5$. As clearly seen, $\dot{Q}_L > 0$ and $\dot{S}_L < 0$ for all $\beta_{LS} < 0$. See text for more details.

It can be easily checked that $\eta_L > 1$ and $\eta_W > 1$. Thus, the efficiency exceeds unity for an engine operating between baths with positive and negative temperatures. This is what is also proposed in Ref. [25]. We, however, find this conclusion incomplete. The heat-to-work conversion efficiency should always be defined with respect to the total amount of heat entering the engine and the amount of work produced out of that. In that sense, the total heat flux entering the engine is $\dot{Q}_L + \dot{Q}_W$, and this entire heat is converted into work. As a result, we find

$$\eta = \frac{-P}{\dot{Q}_L + \dot{Q}_W} = 1,$$

i.e., the engine efficiency becomes unity. Thus, the efficiency of an engine can never exceed unity in any circumstance as long as the first law, i.e., the overall energy conservation, is respected.

This is, nevertheless, different from what we see in traditional heat engines. It raises a question on the physical meaning of the heat released or absorbed by a bath with a negative temperature. By definition, heat is a form of energy that is always associated with a change in the entropy of the corresponding bath. Heat flow, thus, occurs with an entropy flow. The work, however, is a pure form of energy and is not associated with any flow of entropy. In traditional engines operating with baths at positive temperatures, the direction of heat flow and the direction of entropy flow is the same. That is how a bath gets cooled down when it releases some heat. For a bath with a negative temperature, this is not true [see Fig. 3(b)]. There the bath's entropy increases as it releases heat. For the engine we have considered above, the direction of heat flux $\dot{Q}_L > 0$ is opposite to the direction of entropy flux $S_L < 0$ in L which is coupled to the negative bath (see Fig. 5). While for W, coupled to positive temperature bath, the direction of heat flux $\hat{Q}_W > 0$ is the same as the direction of entropy flux $\dot{S}_W > 0$. In fact, while heat is entering the engine from both baths, there is an entropy flow from the bath with inverse temperature β_W to the bath with β_{LS} where the latter acts as an entropy sink.

To understand the thermodynamic nature of this heat, let us make a closer inspection of the process that is happening on *L* and *W* alone in the rotating frame. To create a synthetic bath with negative temperature, *L* interacts with hot and cold baths with inverse temperatures β_H and β_C . The corresponding heat fluxes are $\dot{Q}_{LH} = \text{Tr}[\mathcal{L}_{LH}(\sigma_{LW}^R)H_L]$ and $\dot{Q}_{LC} =$ $\text{Tr}[\mathcal{L}_{LC}(\sigma_{LW}^R)H_L]$. Recall that $\mathcal{L}_L(\cdot) = \mathcal{L}_{LH}(\cdot) + \mathcal{L}_{LC}(\cdot)$. We can also quantify the power as $P_L = i\text{Tr}[\sigma_{LW}[V_{\text{in}}, H_L]]$ which is produced in *L*. For the interaction V_{in} , we always find $\dot{Q}_{LH} > 0$, $\dot{Q}_{LC} < 0$, and $P_L > 0$. For *W*, the heat flux and power can be similarly calculated, and they are $\dot{Q}_W > 0$ and $P_W < 0$. At steady state, we have

$$\dot{Q}_{LH} + \dot{Q}_{LC} + P_L = 0,$$

 $\dot{Q}_W + P_W = 0,$
 $P = P_L + P_W.$

Note $\dot{Q}_L = \dot{Q}_{LH} + \dot{Q}_{LC}$ and $\dot{Q}_L = -P_L$. This may imply that the heat flux from the synthetic bath is quantitatively equal to the power extracted from *L*. But, as mentioned above, this cannot be just power, as power is not associated with any entropy flux. For justification, we may remove the interaction between qutrit *L* and qubit *W* and drive *L* with an oscillating external field instead. This is exactly what is considered in Ref. [39], where power is quantified as the flux of pure energy being stored in the external field and is not associated with entropy flow (see Appendix B). Here, on the contrary, we see an entropy flux opposite to the power extracted (heat flux) in *L*. This leads us to conclude that heat from a bath with a negative temperature is thermodynamic work but with negative entropy.

V. CONCLUSION

In this article, we have studied steady-state quantum thermodynamics with negative temperatures. For that, we have engineered synthetic baths by utilizing two baths at different positive temperatures and letting them weakly interact with qutrit systems in a particular fashion. These synthetic baths can assume arbitrary temperatures, including negative ones. These baths with negative temperatures are exploited to study steady-state thermodynamics. We have explored the thermodynamic laws, particularly the zeroth and second laws. We have shown that whenever two synthetic baths with identical temperatures are brought in contact, there is no heat flow in between. This, in turn, legitimizes the notion of temperatures in synthetic baths. However, for nonidentical temperatures, there is a heat flow. Further, heat always flows spontaneously from a bath with a negative temperature to a positive one. This implies that a bath with a negative temperature is "hotter" than a bath with a positive temperature. Further, there is a heat flow from a bath with a less negative temperature to a bath with a more negative temperature. We have then studied the Clausius statement of the second law in case of negative temperatures and found that there is always nonnegative entropy production in the steady-state thermal processes. Then, we have introduced a heat engine model that operates between two baths, one with negative temperature and the other with positive temperature. Unlike traditional engines, the continuous engines considered above always yield unit heat-to-work conversion efficiency. A systematic analysis has revealed that the heat flow from a bath with a negative temperature is equivalent to an injection of work into the working system by an equal amount. This is exactly the reason why these engines yield unit efficiency.

The proposed continuous engine utilizes qutrit and qubit systems weakly coupled to two or three baths at different temperatures. Thus, these engines can be realized experimentally, for example, using atom-optical [44–47], NMR [48,49], and solid-state [50,51] systems. Finally, we conclude that:

(i) A thermal bath with a negative temperature can be synthesized with two baths with different positive temperatures.

(ii) In steady-state thermodynamics with negative temperatures, the zeroth law remains unchanged. The Clausius statement is to be updated to incorporate that there is a spontaneous heat flow from a bath with a negative inverse temperature to a bath with a positive inverse temperature and from a bath with smaller negative inverse temperature to a bath with a larger negative inverse temperature. Also, the Kelvin-Plack statement is to be appended to incorporate that it is impossible to construct a device operating in a cyclic process that produces no effect other than dumping heat into a negative-temperature bath with the corresponding work being performed on the device.

(iii) A continuous heat engine can be constructed using baths with negative and positive temperatures. In such engines, the heat-to-work conversion efficiency is always unity. This is maximum for any device that respects first law, i.e., conservation of total energy.

(iv) On the fundamental level, the thermodynamic nature of heat from a bath with a positive temperature is qualitatively different from the one with a negative temperature. For the former, heat flows in the same direction as entropy flow. For the latter, heat flows in the opposite direction of entropy flow. Further, heat from a bath with a negative temperature is thermodynamic work but with negative entropy.

(iv) The models we have introduced to study steady-state quantum thermodynamics and continuous heat engines with baths at negative temperatures are experimentally feasible. Thus, our work opens up possibilities for quantum technologies utilizing quantum thermal devices and engines that are highly efficient in the near future.

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APPENDIX A: ROTATING FRAME AND STEADY-STATE THERMODYNAMICS

Let us consider the setup discussed in Sec. IV of the main text. The Hamiltonian of the working systems are

$$H_L = (E_H - E_C)|2\rangle\langle 2| + E_H|3\rangle\langle 3|,$$

$$H_W = E_W|2\rangle\langle 2|,$$

$$H_{\rm in}^E(t) = \delta \ (|11\rangle\langle 22| \ e^{i\omega t} + |22\rangle\langle 11| \ e^{-i\omega t}),$$

$$H_T^E(t) = H_L + H_W + H_{\rm in}^E(t),$$

where $E_W = E_H - E_C$. The Hamiltonian H_L corresponds to a qutrit *L*. It weakly interacts with a hot (*H*) and a cold (*C*) bath at inverse temperatures β_H and β_C (as described in Sec. IV of the main text) to synthesize a bath with negative inverse temperature β_{LS} . The qubit *W*, with Hamiltonian H_W , weakly interacts with a bath with positive inverse temperature β_W . The *L* and *W* interact between them with a time-dependent interaction Hamiltonian given by $H_{in}^E(t)$. After having all these interactions, the overall dynamics of the composite *LW* is

$$\frac{\partial \rho_{LW}(t)}{\partial t} = i \big[\rho_{LW}(t), H_T^E(t) \big] + \mathcal{L}_L(\rho_{LW}(t)) + \mathcal{L}_W(\rho_{LW}(t)),$$
(A1)

for a state ρ_{LW} , where $\mathcal{L}_L(\cdot) = \mathcal{L}_{LH}(\cdot) + \mathcal{L}_{LC}(\cdot)$ is the LSO representing the dissipative dynamics due to baths *H* and *C*, and $\mathcal{L}_W(\cdot)$ is the LSO for the bath with inverse temperature β_W . For this dynamics, the heat flux and power are defined as [37,42]

$$\dot{Q} = \text{Tr}\bigg[\frac{\partial \rho_{LW}(t)}{\partial t} H_T^E(t)\bigg], \qquad (A2)$$

$$P = \text{Tr}\left[\rho_{LW} \ \frac{\partial H_T^E(t)}{\partial t}\right].$$
 (A3)

Note the heat flux \dot{Q} and the power *P* may have time dependence.

For time-dependent Hamiltonians, the dynamics generally never leads to a steady state. However, for a periodic timedependence, as in $H_{in}^E(t)$, there is a rotating frame in which the Hamiltonian can be made time-independent. For that, a counterrotation is applied on the laboratory frame by $U = e^{iH_R t}$ with $[H_R, H_0]$, where $H_0 = H_L + H_W$. In the rotating frame, an operator A in the laboratory frame transforms as $A \rightarrow (A)_R = UAU^{\dagger}$. Further, there exists a Hamiltonian H_R for which the interaction Hamiltonian reduces to a timeindependent one, given by $V_{in} = UH_{in}^E(t)U^{\dagger}$. Accordingly, the overall Hamiltonian becomes time-independent, and it is $\bar{H}_T^E = H_0 - H_R + V_{in}$. In this rotating frame, the overall dynamics is recast as

$$\frac{\partial \rho_{LW}^{\kappa}(t)}{\partial t} = \mathcal{L}_U(\rho_{LW}^{R}) + \mathcal{L}_L(\rho_{LW}^{R}) + \mathcal{L}_W(\rho_{LW}^{R}), \quad (A4)$$

for a state ρ_{LW} , with $\rho_{LW}^R = U \rho_{LW} U^{\dagger}$. Here we denote $\mathcal{L}_U(\rho_{LW}^R) = i[\rho_{LW}^R, \bar{H}_T^E]$ which is the unitary contribution to the dynamics. This dynamics can lead to a steady state, say σ_{LW}^R . It can be easily checked that the LSOs remain unchanged in this rotating frame. Given that $\text{Tr}[AB] = \text{Tr}[(A)_R(B)_R]$ for two arbitrary operators *A* and *B*, we may re-express the heat flux and power as [37]

$$\dot{Q} = \text{Tr}\left[\left(\frac{\partial \rho_{LW}(t)}{\partial t}\right)_{R} \left(H_{T}^{E}(t)\right)_{R}\right] = \text{Tr}\left[\mathcal{L}(\rho_{LW}^{R}) H_{0}\right], \quad (A5)$$

where $\mathcal{L}(\rho_{LW}^R) = \mathcal{L}_L(\rho_{LW}^R) + \mathcal{L}_W(\rho_{LW}^R)$, and the power as

$$P = \operatorname{Tr}\left[(\rho_{LW})_R \left(\frac{\partial H_T^E(t)}{\partial t}\right)_R\right] = -i\operatorname{Tr}\left[\rho_{LW}^R \left[H_0, V_{\mathrm{in}}\right]\right], \quad (A6)$$

where [A, B] = AB - BA. The heat flux \dot{Q} can be divided into two parts. One contribution comes from interaction of L with baths H and C, i.e., $\dot{Q}_L = \text{Tr}[\mathcal{L}_L(\rho_{LW}^R) H_0]$ and the other due to interaction between W with its bath, i.e., $\dot{Q}_W =$ $\text{Tr}[\mathcal{L}_W(\rho_{LW}^R) H_0]$.

Now we study the entropy flux through LW. Note the rate of change in von Neumann entropy is given by

$$\dot{S} = -\mathrm{Tr}\left[\frac{\partial \rho_{LW}(t)}{\partial t} \log \rho_{LW}(t)\right]$$
$$= -\mathrm{Tr}\left[\left(\frac{\partial \rho_{LW}(t)}{\partial t}\right)_{R} \log \left(\rho_{LW}(t)\right)_{R}\right]$$

In the rotating frame and at steady state, it reduces to

$$\dot{S} = -\mathrm{Tr}\left[\frac{\partial \sigma_{LW}^R}{\partial t}\log \sigma_{LW}^R\right] = 0.$$

The rate of change in entropy vanishes because the state does not change over time. Nevertheless, there still can be a nonvanishing flux of entropy passing through *L* and *W*. Given that unitary dynamics does not contribute to the entropy flux, i.e., $-\text{Tr}[\mathcal{L}_U(\rho_{LW}^R) \log \sigma_{LW}^R] = 0$, we can calculate the entropy

flux through L and W, respectively, as

$$\hat{S}_L = -\text{Tr}[\mathcal{L}_L(\rho_{LW}^R)\log\sigma_{LW}^R], \qquad (A7)$$

$$\dot{S}_W = -\text{Tr}\Big[\mathcal{L}_W(\rho_{LW}^R)\log\sigma_{LW}^R\Big],\tag{A8}$$

where $\dot{S}_L + \dot{S}_R = 0$.

APPENDIX B: POWER AND ASSOCIATED ENTROPY FLUX IN TRADITIONAL STEADY-STATE ENGINES

In traditional engines, unlike heat, work is a form of energy not associated with entropy. To understand that, we may reconsider the heat engine composed of a qutrit and two baths at different temperatures, which is also studied in Ref. [39]. Consider the qutrit system outlined in Sec. II where the Hamiltonian of the qutrit is $H_0 = (E_H - E_C)|2\rangle\langle 2| + E_H|3\rangle\langle 3|$, and it is weakly interacting with a hot (*H*) and cold (*C*) with inverse temperatures β_H and β_C , respectively. In addition, an external driving

$$H_d(t) = \alpha \ (|1\rangle\langle 2|e^{i\omega t} + |2\rangle\langle 1|e^{-i\omega t}) \tag{B1}$$

is introduced. As a result, the reduced dynamics becomes

$$\dot{\rho} = i \left[\rho, H_T(t)\right] + \mathcal{L}_H(\rho) + \mathcal{L}_H(\rho), \quad (B2)$$

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where $H_T(t) = H_0 + H_d(t)$, and $\mathcal{L}_H(\cdot)$ and $\mathcal{L}_C(\cdot)$ are LSOs representing the dissipation due to interactions with hot and cold baths. Again, as discussed in Appendix A, this dynamics leads to a steady state, say σ^R , in a rotating frame. The heat fluxes, entropy fluxes, and power are calculated, respectively, as

$$\dot{Q}_{H} = \operatorname{Tr} \left[\mathcal{L}_{H}(\sigma^{R})H_{0} \right],$$

$$\dot{Q}_{C} = \operatorname{Tr} \left[\mathcal{L}_{C}(\sigma^{R})H_{0} \right],$$

$$\dot{S}_{H} = -\operatorname{Tr} \left[\mathcal{L}_{H}(\sigma^{R})\log\sigma^{R} \right],$$

$$\dot{S}_{C} = -\operatorname{Tr} \left[\mathcal{L}_{C}(\sigma^{R})\log\sigma^{R} \right],$$

$$P = -i\operatorname{Tr} \left[\sigma^{R} \left[H_{0}, V \right] \right],$$

where $V = \alpha$ ($|1\rangle\langle 2| + |2\rangle\langle 1|$). Here \dot{Q}_X and \dot{S}_X represent the heat and entropy fluxes from bath X = H, C, and P is the power. As required by first law, $\dot{Q}_H + \dot{Q}_C + P = 0$ and, at steady state, $\dot{S}_H = -\dot{S}_C$. When the device operates as an engine, we have $\dot{Q}_H > 0$, $\dot{Q}_C < 0$, and P < 0. In this case, heat enters from the hot bath. Part of that heat is converted into work which is stored in the driving field, and the rest is dumped into the cold bath. Further, in this case, $\dot{S}_H > 0$ and $\dot{S}_C < 0$. Clearly, at steady state, whatever amount of entropy enters the system from the hot bath is released to the cold bath. Thus, the work-flux, i.e., power P, does not carry any entropy.

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