Symmetry-protected topological order as a requirement for measurement-based quantum gate teleportation

Zhuohao Liu, Emma C. Johnson, and David L. Feder
Institute for Quantum Science and Technology and Department of Physics and Astronomy, University of Calgary, Calgary, Alberta T2N 1N4, Canada

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All known resource states for measurement-based quantum teleportation in correlation space possess symmetry-protected topological order, but is this a sufficient or even necessary condition? This work considers two families of one-dimensional qubit states to answer this question in the negative. The first is a family of matrix-product states with bond dimension two that includes the cluster state as a special case, protected by a global non-on-site symmetry, which is characterized by a finite correlation length and a degenerate entanglement spectrum in the thermodynamic limit but which is unable to deterministically teleport a universal set of single-qubit gates. The second are states with bond dimension four that are a resource for deterministic universal teleportation of finite single-qubit gates, but which possess no symmetry.

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I. INTRODUCTION

The measurement-based model of quantum computation (MBQC) [1,2] is wholly equivalent to the quantum circuit model in its ability to effect arbitrary quantum gates [3] but is advantageous for practical implementations where the application of local entangling gates on demand is challenging. In MBQC, the entanglement is present at the outset, in the form of a specific resource state, and quantum gates are teleported by means of adaptive single-qubit measurements. A long-standing open problem has been to identify the essential characteristics required of a resource state, and accordingly much attention has been focused on one-dimensional systems which are able to perform measurement-based gate teleportation (MBQT) of arbitrary single-qubit gates. To date, all resource states for MBQT that have been identified possess symmetry-protected topological (SPT) order [4–8], which passively protects the quantum information from certain kinds of errors [9]; these include the cluster states of the original one-way quantum computation model [3,10], and Haldane-phase states [11] such as the ground states of the Affleck-Kennedy-Lieb-Tasaki (AKLT) state [12,13], its generalizations to two dimensions and higher spin [14–18], and a two-dimensional state with genuine SPT order [19].

For all resource states with SPT order, MBQT is performed in correlation space in the matrix product state (MPS) representation [20–23]. The group cohomology [4] ensures that the teleported gate in correlation space can be expressed as a rotation operator in a tensor product with an unimportant "junk" matrix [6,7]. Unfortunately, the teleported gates throughout the universal SPT phase are not strictly in the protected "wire basis," which restricts the target teleported gates to infinitesimal rotations in correlation space [8,24] except for the cluster state itself. Another key signature of SPT is the degeneracy of the entanglement spectrum (ES) [7,25,26].

While SPT order has been a powerful approach to classifying resource states for MBQT, a plethora of key questions remain, even for the simplest case of one-dimensional qubits. Are resource states with SPT order required for MBQT? If not, what other kinds of resource states are possible? Can any resources other than the cluster state effect the teleportation of universal single-qubit gates based on finite, rather than infinitesimal, unitary rotations? What is the relationship between the ability of a state to be a resource for MBQT and the structure of the teleported gates?

This work partially addresses these questions by considering two specific examples. The first is a family of SPT states with bond dimension $D = 2$ that includes the cluster state as a special case, which is protected by a global $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry that is generally neither unitary nor on-site. States within this family have finite correlation length and exhibit a degenerate ES for arbitrary boundary conditions in the thermodynamic limit but (except for the cluster state itself) are unable to deterministically teleport a universal set of protected single-qubit gates in correlation space. The second example is an extension of the cluster state to a family of non-SPT states with bond dimension $D = 4$, which are a resource for the deterministic teleportation of single-qubit gates, based on finite rotations. The results demonstrate that SPT order is neither sufficient nor necessary for a state to be an MBQT resource.

II. TECHNICAL BACKGROUND

A one-dimensional state for n qubits can be written in the MPS representation as

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} A^{[n]}[i_n]\cdots A^{[1]}[i_1]|i_1\cdots i_n\rangle. \quad (1)$$

The $A^{[k]}[i_k]$, where $i_n = \{0, 1\}$, are rectangular matrices and their product (indexed by the strings $i_1 \cdots i_n$) therefore constitute the amplitudes of $|\psi\rangle$ in the computational basis. Given that $A^{[n]}[i_n]$ ($A^{[1]}[i_1]$) is a row (column) vector, it is conventional to introduce matrices $B^{[k]}[i_k]$ and boundary vectors $|L\rangle$ and $|R\rangle$ such that $A^{[n]}[i_n] = \langle R|B^{[n]}[i_n]$, $A^{[1]}[i_1] = B^{[1]}[i_1]|L\rangle$, and $A^{[k]}[i_k] = B^{[k]}[i_k]$ otherwise, in which case Eq. (1) becomes

$$|\psi\rangle = \sum_{i_1,\ldots,i_n} \langle R|B^{[n]}[i_n]\cdots B^{[1]}[i_1]|L\rangle|i_1\cdots i_n\rangle. \quad (2)$$

In this work, the MPS matrices $B^{[k]}[0]$ and $B^{[k]}[1]$ are arbitrary complex matrices with fixed "bond" dimension D, and are normalized in (left) site canonical form, $\sum_{i_k} B^{[k]}[i_k]^\dagger B^{[k]}[i_k] = I$ for each k, where I is the identity matrix.

A general single-qubit measurement can be effected by first performing a unitary gate $\tilde{U}$ on the qubit, and then measuring it in the computational basis by projecting the result onto $|m\rangle\langle m|$, $m = 0, 1$. This is equivalent to applying the operator $|m\rangle\langle m|\tilde{U} = |m\rangle\langle\phi_m|$, where $|\phi_m\rangle = \tilde{U}^\dagger|m\rangle$ constitute a basis for the unitary:

$$|\phi_0\rangle = e^{-i\varphi_1}\cos\vartheta|0\rangle + e^{-i\varphi_2}\sin\vartheta|1\rangle;$$
$$|\phi_1\rangle = e^{i\varphi_2}\sin\vartheta|0\rangle - e^{i\varphi_1}\cos\vartheta|1\rangle. \quad (3)$$

Consider the action of this operator on the first qubit of $|\psi\rangle$ in Eq. (2). Ignoring normalization, one obtains

$$|m\rangle_1\langle\phi_m|\psi\rangle = |m_1\rangle\langle R|\sum_{i_2,\ldots,i_n} B^{[n]}[i_n]\cdots B^{[2]}[i_2] \times \left(\sum_{i_1}\langle\phi_m|i_1\rangle B^{[1]}[i_1]\right)|L\rangle|i_2\cdots i_n\rangle = |m\rangle_1\sum_{i_2,\ldots,i_n}\langle R|B^{[n]}[i_n]\cdots B^{[2]}[i_2]|L'\rangle|i_2\cdots i_n\rangle, \quad (4)$$

where the left boundary state in correlation space is transformed into $|L'\rangle = B^{[1]}[\phi_m]|L\rangle$ by the operator $B^{[1]}[\phi_m] = \sum_{i_1}\langle\phi_m|i_1\rangle B^{[1]}[i_1]$; in general, one obtains

$$B^{[k]}[\phi_0] = e^{i\varphi_1}\cos\vartheta B^{[k]}[0] + e^{i\varphi_2}\sin\vartheta B^{[k]}[1]; \quad (5)$$

$$B^{[k]}[\phi_1] = e^{-i\varphi_2}\sin\vartheta B^{[k]}[0] - e^{-i\varphi_1}\cos\vartheta B^{[k]}[1], \quad (6)$$

for measurements of 0 and 1, respectively. Successive measurements therefore apply a sequence of gates to $|L\rangle$. For MBQT, however, $B^{[k]}[\phi_m]$ must correspond to a unitary operator for all m, a severe restriction on possible resource states which are defined by the matrices $B^{[i_k]}[i_k]$.

One-dimensional cluster states of qubits with open boundary conditions provide a convenient reference for the work presented here. It is straightforward to verify that these are (nonuniquely) described by an MPS representation with matrices

$$B^{[k]}[0] = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = |+\rangle\langle 0|;$$
$$B^{[k]}[1] = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix} = |-\rangle\langle 1| \quad (7)$$

for $k = 1, \ldots, n$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, $\langle R| = \langle 0|$, and $|L\rangle = \sqrt{2}|+\rangle$. Because the matrices are independent of site, the MPS representation is said to be translationally invariant, even though the state itself is defined with open boundary conditions. One obtains

$$B^{[k]}[\phi_m] = \langle\phi_m|0\rangle|+\rangle\langle 0| + \langle\phi_m|1\rangle|-\rangle\langle 1|, \quad (8)$$

which is unitary if

$$|\langle\phi_m|0\rangle|^2|+\rangle\langle+| + |\langle\phi_m|1\rangle|^2|-\rangle\langle-| = cI;$$
$$|\langle\phi_m|0\rangle|^2|0\rangle\langle 0| + |\langle\phi_m|1\rangle|^2|1\rangle\langle 1| = cI, \quad (9)$$

where c is a constant related to the (re)normalization of $|\psi\rangle$ after the measurement. These two conditions require $|\langle\phi_m|0\rangle|^2 = |\langle\phi_m|1\rangle|^2 = c$ for $m = 0, 1$, and it is straightforward to verify that they together imply $B^{[k]}[\phi_m] = X^m H R_Z(\theta)$ and $c = 1/2$ ignoring overall phase factors, where

$$X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix};\ Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\ H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad (10)$$

and $R_Z(\theta) = \exp(iZ\theta)$.

III. SPT STATES UNABLE TO EFFECT MBQT

A. MPS matrices and the state

The (unitary) teleported gate (5) is chosen to have the form yU, where

$$U = \begin{pmatrix} e^{i\phi_1}\cos\theta & e^{i\phi_2}\sin\theta\\ e^{-i\phi_2}\sin\theta & -e^{-i\phi_1}\cos\theta\end{pmatrix} \quad (11)$$

with all parameters assumed to be real, and y is a proportionality factor to account for the renormalization of the state after measurement; this case corresponds to an MPS with bond dimension $D = 2$. Consider first the simplest case of a translationally invariant system. The derivation is straightforward but unwieldy, and is relegated to Appendix A. Choosing the MPS matrices to be in column form as in the cluster-state case, Eq. (7), and ensuring that they do not depend on the measurement angles, restricts both the measurement basis and the teleported gates; one choice corresponds to $\varphi_2 = -\varphi_1 := -\varphi$, $\vartheta = (2k+1)\pi/4$, $k \in \mathbb{Z}$, and $y = 1/\sqrt{2}$. These yield the measurement basis $\tilde{U}^\dagger = HR_Z(\varphi)$, exactly as in cluster-state teleportation. One obtains the (nonunique) expressions for the MPS matrices, Eq. (A10):

$$B[0] = \begin{pmatrix}\cos\theta & 0\\ \sin\theta & 0\end{pmatrix};\quad B[1] = \begin{pmatrix}0 & \sin\theta\\ 0 & -\cos\theta\end{pmatrix}, \quad (12)$$

which in turn yield the measurement-dependent teleported unitary gates, Eq. (A11):

$$U[0] = ZR_Y(\theta)R_Z(\varphi); \quad (13)$$

$$U[1] = R_Y(-\theta)R_Z(\varphi) = ZR_Y(\theta)ZR_Z(\varphi) = U_0 Z, \quad (14)$$

where $Y = iXZ$.

Assuming a translationally invariant MPS, consistently measuring $|0\rangle$ would yield successive rotations about Z and $R_Y(-\theta)R_Z(\varphi)R_Y(\theta)$, a measurement-dependent rotation around Z conjugated by a fixed rotation around Y. These are nonparallel axes, which allows for the teleportation of any single-qubit unitary; for the cluster state, $\theta = \theta_c := (2k+1)\pi/4$, $k \in \mathbb{Z}$, and the latter rotation axis is X. However, the byproduct operator when measuring $|1\rangle$ is not easily compensated for. Consider the teleported gates on two successive measurements:

$$\begin{aligned} U^{(2)}U^{(1)} &= ZR_Y(\theta)Z^{m_2}R_Z(\varphi_2)ZR_Y(\theta)Z^{m_1}R_Z(\varphi_1) \\ &= Z^{m_1+m_2}R_Y[(-1)^{m_1+m_2+1}\theta]R_Z(\varphi_2) \\ &\quad \times R_Y[(-1)^{m_1}\theta]R_Z(\varphi_1). \end{aligned} \tag{15}$$

If $m_1 = 1$, the second rotation corresponds instead to $R_Y[(-1)^{m_2}\theta]R_Z(\varphi_2)R_Y(-\theta)$, which if $m_2 = 0$ is a rotation around Z conjugated by a rotation around Y in the opposite direction than would be the case for $m_1 = 0$. One strategy would be to choose $\varphi_2 = 0$ on the next measurement, but if one instead obtains $m_2 = 1$ one is left with a second unwanted rotation $R_Y(-2\theta)$ that would somehow need to be compensated for on the third measurement. The MBQT protocol would therefore become nondeterministic. Another strategy might be to restrict the φ_i to infinitesimal angles, but in this case the error induced by the byproduct is $R_Y(-\theta)R_Z(\varphi)R_Y(\theta) - R_Y(\theta)R_Z(\varphi)R_Y(-\theta) \approx i\sin(\theta)\varphi X \to 0$ as $\varphi \to 0$. Because the error accumulates, the teleported state would be indistinguishable from noise after several iterations.

The inability of a state defined by the MPS (12) to effect deterministic MBQT of either finite or infinitesimal gates is in marked contrast from the "oblivious wire" protocol that ensures that SPT states have uniform computational power to effect MBQT [24,27]. In that case the gates are infinitesimally displaced from the symmetry-protected wire basis, in order to compensate for the fact that the junk matrices are generally measurement-dependent. But this is not possible for the simple $D = 2$ case under consideration here. Over all MPS matrices (12), only the cluster state can effect deterministic teleportation.

The (unnormalized) state $|\psi\rangle$ can be constructed either directly from Eq. (2) or using the machinery in Ref. [21]. With left and right boundary states

$$\langle R| = a_R\langle 0| + b_R\langle 1|;\ |L\rangle = a_L|0\rangle + b_L|1\rangle \tag{16}$$

and site-dependent values of θ, the state takes an especially simple form after some straightforward algebra:

$$\begin{aligned} |\psi\rangle = \prod_{j=1}^{n-1} C_{\theta_j}^{(j,j+1)}(x_1|0\rangle + x_2|1\rangle) \otimes |+\rangle^{\otimes n-2} \\ \otimes(a_L|0\rangle + b_L|1\rangle), \end{aligned} \tag{17}$$

$x_1 = a_R\cos\theta_n + b_R\sin\theta_n$, $x_2 = a_R\sin\theta_n - b_R\cos\theta_n$, and

$$C_{\theta_j}^{(j,j+1)} = \sqrt{2}\mathrm{diag}(\cos\theta_j, \sin\theta_j, \sin\theta_j, -\cos\theta_j)_{j,j+1} \tag{18}$$

acts on qubits j and $j+1$. With $\theta_j = \pi/4$ for $j > 1$, so that $C_{\theta_j} = C_Z = \mathrm{diag}(1, 1, 1, -1)$, the state coincides with the one-dimensional cluster state with rotated left and right physical qubits. As $C_{\theta_j}^{j,j+1}$ is only a unitary operator for $\theta_j = \theta_c\ \forall j$, Eq. (17) should be considered as an expression of the state rather than as a procedure for generating it.

The static correlation function $C_{\mathcal{O}_k,\mathcal{O}_r'} = \langle\mathcal{O}_k\mathcal{O}_r\rangle - \langle\mathcal{O}_k\rangle\langle\mathcal{O}_r\rangle$ with respect to operators $\mathcal{O}_k$ and $\mathcal{O}_r$ acting on sites k and r generically decays exponentially for a 1D MPS with finite bond dimension [28] (consistent with the parent Hamiltonian being gapped [29–32]): $\|C_{\mathcal{O}_k,\mathcal{O}_r'}\| \sim e^{-|k-r|/\xi}$, where ξ is the correlation length. For a translationally invariant MPS, $\xi = -1/\ln(\lambda_1)$, where λ_1 is the second-largest eigenvalue of the transfer matrix [33]

$$T_k = \sum_{i_k} B^{[k]}[i_k]^* \otimes B^{[k]}[i_k]. \tag{19}$$

Using Eq. (12), one obtains $\lambda_0 = 1$ and $\lambda_1 = \cos 2\theta$; thus, $\xi = -1/\ln(\cos 2\theta)$. The correlation length is zero when $\theta = \theta_c$, it diverges as $\xi \sim 1/2\theta^2$ for $\theta \to 0$,

B. SPT order

The real-space representation of the state, Eq. (17), allows for the explicit construction of the symmetry operators. As shown in Appendix B, the state possesses an exact $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry $O(g_1, g_2)|\psi\rangle = |\psi\rangle$, where $O(g_1, g_2) = X_{\mathrm{odd}}^{g_1}X_{\mathrm{even}}^{g_2}$ and $g_1, g_2 \in \{0, 1\}$. The operators X_{odd} and X_{even} are the analogs of the X symmetry operators that act on odd-labeled and even-labeled sites of the cluster state, respectively, and for an even number of sites are given by Eq. (B25):

$$\begin{aligned} X_{\mathrm{odd}} &= P_{\theta_1}^{1,2}\left(X_1\prod_{j=1} P_{\theta_{2j}}^{2j,2j+1}P_{\theta_{2j+1}}^{2j+1,2j+2}X_{2j+1}\right)Z_n; \\ X_{\mathrm{even}} &= Z_1\left(\prod_j P_{\theta_{2j-1}}^{2j-1,2j}P_{\theta_{2j}}^{2j,2j+1}X_{2j}\right)\left(P_{\theta_n}^{n-1,n}X_n\right), \end{aligned} \tag{20}$$

where

$$P_{\theta_j}^{j,j+1} = \mathrm{diag}(\cot\theta_j, \tan\theta_j, \tan\theta_j, \cot\theta_j)_{j,j+1}. \tag{21}$$

Alternatively, these can be written as $X_{\mathrm{odd}} = S_{1,2}(\prod_j S_{2j+1,2j+2})$ and $X_{\mathrm{even}} = (\prod_j S_{2j,2j+1})S_{n-1,n}$, where $S_{j,j+1} = P_{\theta_j}^{j-1,j}P_{\theta_{j+1}}^{j,j+1}Z_{j-1}X_jZ_{j+1}$ are (nonlocal) stabilizer generators for the state, Eq. (17). While these symmetry operators square to the identity and commute with one another, as shown in Appendix B, they are neither unitary nor on-site.

Consider the left boundary qubit. The X and Z gates are transformed by the C_θ operators into effective Pauli gates: $\overline{X}_1 = P_{\theta_1}^{1,2}X_1Z_2$ and $\overline{Z}_1 = Z_1$. Then one may determine the effective operators $\overline{X}_1'$ and $\overline{Z}_1'$ corresponding to $\overline{X}_1$ and $\overline{Z}_1$ conjugated by $O(g_1, g_2)$, respectively. Straightforward algebra presented in Appendix B reveals $\overline{Z}_1' = (-1)^{g_1}\overline{Z}_1$ and $\overline{X}_1' = (-1)^{g_2}\overline{X}$. The transformations on $\overline{Z}$ and $\overline{X}$ by the $\mathbb{Z}_2\times\mathbb{Z}_2$ operators are therefore equivalent to conjugation under an effective operator $O_{\mathrm{eff}}(g_1, g_2) = \overline{X}^{g_1}\overline{Z}^{g_2}$, which is the same as for the regular cluster state. A similar result holds for the right boundary. Thus the state belongs to the same maximally noncommutative phase as the cluster state [6,8].

In the cluster-state limit $\theta_j = \theta_c\ \forall j$, the symmetry operators (20) reduce to $X_{\mathrm{odd}} = \prod_{j=1} X_{2j-1}$ and $X_{\mathrm{even}} = \prod_{j=1} X_{2j}$, as expected. The on-site symmetry $U(g) = U(g_1, g_2) = X_{\mathrm{odd}}^{g_1}X_{\mathrm{even}}^{g_2}$, which acts in parallel on adjacent two-site blocks so that $U(g)^{\otimes n/2}|\psi\rangle = |\psi\rangle$, is shared by the MPS matrices themselves via [21,34,35]

$$\sum_{\mathbf{j}} U(g)_{\mathbf{i},\mathbf{j}} A[\mathbf{j}] = e^{i\phi_g} V(g) A[\mathbf{i}] V(g)^\dagger, \tag{22}$$

where $\mathbf{i}$ and $\mathbf{j}$ are bitstrings of length 2, and $A[\mathbf{i}] = A[i_1 i_2] := A[i_1] \otimes A[i_2]$. It is straightforward to verify that $V(g) = Y_1^{g_1} \otimes Y_2^{g_2}$ and $\phi_g = 0$. For any choices of $\theta_j \neq \theta_c$, however, $O(g) = O(g_1, g_2)$ is non-on-site, and there is no analog of Eq. (22) that can be expressed in block-injective form for any length smaller than n. Rather,

$$\sum_{\mathbf{j}} O(g)_{\mathbf{i},\mathbf{j}} A[\mathbf{j}] \neq e^{i\phi_g} V(\mathbf{g}) A[\mathbf{i}] V(\mathbf{g})^{-1}, \tag{23}$$

where $\mathbf{i}$, and $\mathbf{j}$ are now bitstrings of length n and $\mathbf{g} = (1010\cdots)^{g_1} \oplus (0101\cdots)^{g_2}$ ($\oplus$ is bitwise addition mod 2), for any $V(\mathbf{g})$: the only nonzero term on the left is $\mathbf{j} = \mathbf{i} \oplus \mathbf{g}$, and $|O(g)_{\mathbf{i},\mathbf{i}\oplus\mathbf{g}}| \neq 1$. Thus the symmetry of the real-space state is no longer shared by the (product of) MPS matrices for a global non-on-site symmetry.

C. Entanglement spectrum

Consistent with the SPT order of the state defined either by the MPS matrices (7) or the state (17), the ES is asymptotically degenerate in the thermodynamic limit for all choices of boundary conditions. The ES corresponds to the eigenvalues of the reduced density matrix associated with a partition of the one-dimensional state with ℓ qubits on the left and $n-\ell$ qubits on the right. It can be obtained by diagonalizing the reduced density matrix, but more efficiently from the MPS matrices. Following Prosen [36], one may express the amplitudes of the state (2) as

$$\langle R|B^{[n]}[i_n]\cdots B^{[1]}[i_1]|L\rangle = \sum_{j=1}^{D} \Phi^R_{\ell,j}\Phi^L_{\ell,j}; \tag{24}$$

here,

$$\Phi^R_{\ell,j} := \langle R|B^{[n]}_n \cdots B^{[\ell+1]}_{\ell+1}|j\rangle;\ \Phi^L_{\ell,j} := \langle j|B^{[\ell]}_\ell \cdots B^{[1]}_1|L\rangle, \tag{25}$$

and $|j\rangle$ are computational basis states. The elements of covariance matrices V^L_n and V^R_n are obtained via

$$\langle j'|V^R_\ell|j\rangle := \sum_{\ell+1,\dots,n} \Phi^{R*}_{\ell,j'}\Phi^R_{\ell,j};$$

$$\langle j'|V^L_\ell|j\rangle := \sum_{1,\dots,\ell} \Phi^L_{\ell,j'}\Phi^{L*}_{\ell,j}, \tag{26}$$

where the sum is over all internal indices. The ES coincides with the eigenvalues of $V^R_\ell V^L_\ell$.

The calculations for the solution (12) with boundary conditions specified in Eq. (16) are given in Appendix C. For a bulk bipartition where $2 < \ell < n-1$, one obtains Eq. (C5):

$$V^R_\ell = \frac{1}{2}\begin{pmatrix} 1+\alpha & 0 \\ 0 & 1-\alpha \end{pmatrix};$$

$$V^L_\ell = \frac{1}{2}\begin{pmatrix} 1+\beta\cos(2\theta_\ell) & \beta\sin(2\theta_\ell) \\ \beta\sin(2\theta_\ell) & 1-\beta\cos(2\theta_\ell) \end{pmatrix}, \tag{27}$$

where $\alpha := (|x_1|^2 - |x_2|^2)\prod_{k=\ell+1}^{n-1}\cos(2\theta_k)$ and $\beta := (|a_L|^2 - |b_L|^2)\prod_{k=1}^{\ell-1}\cos(2\theta_k)$, with x_1, x_2 defined below Eq.(17). Note that the matrix elements depend explicitly on the boundary states. If $\prod_{k=1}^{\ell-1}\cos(2\theta_k) = \prod_{k=\ell+1}^{n-1}\cos(2\theta_k) = 0$, then V^R_ℓ and V^L_ℓ are proportional to the identity and the ES is degenerate. This condition is automatically satisfied for the cluster state, $\theta_k = \theta_c, \forall k$. If $\theta_k \neq \theta_c$, however, both matrices are strictly diagonal only if α and β do not depend on the choice of ℓ, corresponding to $|x_1|^2 = |x_2|^2$ and $|a_L|^2 = |b_L|^2$, which includes the state that is fully invariant under $O(g_1, g_2)$. In general, the (unnormalized) eigenvalues of $V^R_\ell V^L_\ell$ are given by

$$\lambda_\pm = \tfrac{1}{4}\{1 + \alpha\beta\cos(2\theta_\ell) \pm\sqrt{[1+\alpha\beta\cos(2\theta_\ell)]^2 - (1-\alpha^2)(1-\beta^2)}\}. \tag{28}$$

The ES becomes asymptotically degenerate in the thermodynamic limit, for any boundary conditions. For $0 < \theta_k < \theta_c$, one has $0 < \cos(2\theta_k) < 1$ so that $\alpha, \beta \to 0$ as $n \to \infty$ for any bulk bipartition, $\ell \sim n/2$; in that case, $V^L_\ell, V^R_\ell \to I/2$. Alternatively, in the translationally invariant case $\theta_k = \theta$ one may write $\cos(2\theta) = e^{-1/\xi}$, where ξ is the correlation length. This yields $\alpha = (|x_1|^2 - |x_2|^2)e^{-(n-\ell-2)/\xi}$ and $\beta = (|a_L|^2 - |b_L|^2)e^{-(\ell-2)/\xi}$, so that $\alpha, \beta \to 0$ exponentially quickly on finite chains as long as $\ell, n-\ell \gg \xi$. In this aspect, the system behaves much like the AKLT chain [37].

To summarize the results of this section: SPT order on qubits is not a sufficient condition for the state to be a resource for deterministic MBQT with finite or infinitesimal gates.

IV. NON-SPT STATES THAT EFFECT MBQT

Consider next the case where the teleported gate in the $D = 4$ correlation space is a direct sum $U \oplus J$ of a 2×2 unitary U, given again by Eq. (11), and an arbitrary junk matrix

$$J = \begin{pmatrix} pe^{i\phi_p} & qe^{i\phi_q} \\ re^{i\phi_r} & se^{i\phi_s} \end{pmatrix}, \tag{29}$$

with all parameters real. The U at each measurement step acts on the $\{|00\rangle, |01\rangle\}$ computational subspace of the virtual two-qubit state, which can be considered as encoding a single qubit, while J acts on the complementary subspace. Assuming a direct sum is notationally convenient in what follows, but choosing any other subset of registers yields an equivalent description. For example, if U and J act on the odd-parity and even-parity subspaces $\{|01\rangle, |10\rangle\}$ and $\{|00\rangle, |11\rangle\}$ respectively, the output has the characteristic structure of a match gate [38,39], and indeed would correspond exactly to a matchgate if $\det(U) = \det(J)$.

The procedure follows closely the strategy above. Setting $\varphi_2 = -\varphi_1 := -\varphi$, $\phi_1 = \phi_p = \phi_r = \varphi$, and $\phi_2 = \phi_q = \phi_s = -\varphi$, one obtains

$$B^{[k]}[0] = y\sec\vartheta \begin{pmatrix} \cos\theta & 0 & 0 & 0 \\ \sin\theta & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & r & 0 \end{pmatrix};$$

$$B^{[k]}[1] = y\csc\vartheta \begin{pmatrix} 0 & \sin\theta & 0 & 0 \\ 0 & -\cos\theta & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & s \end{pmatrix}. \tag{30}$$

Enforcing the canonical normalization conditions requires $\sec^2\vartheta = \csc^2\vartheta$, which is satisfied again by setting $\vartheta = (2k+1)\pi/4$, $k \in \mathbb{Z}$, in which case the teleported single-qubit unitaries coincide with Eqs. (13) and (14) according to the measurement outcome. The normalization conditions also require $y = 1/\sqrt{2}$ and $p^2 + r^2 = q^2 + s^2 = 1$. These last conditions can be conveniently incorporated by setting $p = \cos\gamma$, $r = \sin\gamma$, $q = \sin\delta$, and $s = \cos\delta$, in which case the junk matrices become

$$J[0] = \begin{pmatrix} e^{i\varphi}\cos\gamma & e^{-i\varphi}\sin\delta \\ e^{i\varphi}\sin\gamma & e^{-i\varphi}\cos\delta \end{pmatrix}; \quad J[1] = J[0]Z. \tag{31}$$

A major motivation is to explore the possibility that the direct-sum format can yield new resource states for deterministic MBQT with finite rotations, so the primary focus is on the $\theta = \pi/4$ case, in which case $U[m] = X^m H R_Z(\varphi)$. Assuming site-dependent junk matrices, and general boundary states $\langle R| = \{a_R, b_R, c_R, d_R\}$ and $|L\rangle = \{a_L, b_L, c_L, d_L\}^T$, one obtains the unnormalized states

$$|\psi\rangle = \prod_{j=1}^{n-1} C_Z^{j,j+1}|\psi_1\rangle + \prod_{j=1}^{n-1} C_{\gamma_j,\delta_j}^{j,j+1}|\psi_2\rangle, \tag{32}$$

where

$$|\psi_1\rangle = (x_1|0\rangle + x_2|1\rangle) \otimes |+\rangle^{n-2} \otimes (a_L|0\rangle + b_L|1\rangle);$$
$$|\psi_2\rangle = (x_3|0\rangle + x_4|1\rangle) \otimes |+\rangle^{n-2} \otimes (c_L|0\rangle + d_L|1\rangle), \tag{33}$$

$x_1 = a_R + b_R, x_2 = a_R - b_R,$

$$x_3 = \sqrt{2}(c_R\cos\gamma_n + d_R\sin\gamma_n);$$
$$x_4 = \sqrt{2}(c_R\sin\delta_n + d_R\cos\delta_n), \tag{34}$$

and

$$C_{\gamma_j,\delta_j}^{j,j+1} = \sqrt{2}\,\mathrm{diag}(\cos\gamma_j, \sin\delta_j, \sin\gamma_j, \cos\delta_j)_{j,j+1}. \tag{35}$$

The state in Eq. (32) allows for the teleportation of deterministic single-qubit unitaries (with feed-forward) for all choices of junk-matrix angles γ_j and δ_j, because the computational subspace acts like a cluster state and is orthogonal to (and therefore remains independent of) the junk subspace.

The matrices (30) are in block-diagonal form, so that the MPS is not injective [21]. This further implies that the state (32) cannot be the unique ground state of a local frustration-free parent Hamiltonian, but rather that the ground-state degeneracy of such a parent Hamiltonian is two, corresponding to the number of blocks; this however doesn't preclude the possibility of preparing the state directly via a quantum circuit. In principle, the noninjectivity could affect readout of the final state [8]. In practice, the state can be chosen such that $C_{\gamma_{n-1},\delta_{n-1}}^{n-1,n} = I$ via $\gamma_{n-1} = \delta_{n-1} = \pi/4$ so that no entanglement is generated between the last two qubits in the junk sector. This prevents any information reaching the junk output state $c_L|0\rangle + d_L|1\rangle$, which can be defined in any convenient way, and therefore the quantum information encoded in the cluster sector remains uncontaminated.

Similar to state (17), the $C_{\gamma_j,\delta_j}^{j,j+1}$ in Eq. (32) are not generally unitary and are (potentially) site-dependent. Because (32) is described by a superposition of states, each defined by a different set of generalized stabilizers, it no longer possesses SPT order. Thus neither SPT order nor injectivity are necessary conditions for a state to be a resource for MBQT.

V. CONCLUSIONS AND DISCUSSION

The results presented in this work demonstrate that the presence of symmetry-protected topological order is neither a sufficient nor necessary condition for a quantum state to be a resource for deterministic measurement-based quantum gate teleportation. On the one hand, a family of states of one-dimensional qubits with a non-on-site SPT symmetry is unable to deterministically teleport universal one-qubit gates in correlation space, while on the other a family of states with no SPT order is able to do so. All identified states can be considered to be analogs of cluster states, but where the C_Z entangling gates in their description are generally replaced by diagonal nonunitary operators.

The family of states with non-on-site symmetries identified here belong to the same SPT phase as the cluster state, and therefore can be prepared from the cluster state via a constant-depth quantum circuit comprised of nonoverlapping k-local unitaries [40]. The fact that such a unitary transformation maps a resource state for deterministic MBQT to a nonresource state suggests that a large number of states in a given SPT phase may not be resources for MBQT. Rather, perhaps only the subset of transformations that preserve the on-site nature of the symmetry would ensure that the state remains computationally useful.

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APPENDIX A: DERIVATION OF MPS MATRICES

Express the MPS matrices as follows:

$$B[0] = \begin{pmatrix} a_{11}e^{i\phi_{11}} & a_{12}e^{i\phi_{12}} \\ a_{13}e^{i\phi_{13}} & a_{14}e^{i\phi_{14}} \end{pmatrix}; \quad B[1] = \begin{pmatrix} a_{21}e^{i\phi_{21}} & a_{22}e^{i\phi_{22}} \\ a_{23}e^{i\phi_{23}} & a_{24}e^{i\phi_{24}} \end{pmatrix}. \tag{A1}$$

Equations (5) and (6) become

$$B[\phi_0] = (e^{i\varphi_1}\cos\vartheta B[0] + e^{i\varphi_2}\sin\vartheta B[1]) := yU[0]; \tag{A2}$$

$$B[\phi_1] = (e^{-i\varphi_2}\sin\vartheta B[0] - e^{-i\varphi_1}\cos\vartheta B[1]) := yU[1], \tag{A3}$$

where

$$U[0] = \begin{pmatrix} e^{i\phi_1}\cos\theta & e^{i\phi_2}\sin\theta \\ e^{-i\phi_2}\sin\theta & -e^{-i\phi_1}\cos\theta \end{pmatrix} \tag{A4}$$

is an arbitrary target single-qubit unitary gate, and $U[1]$ and the constant y are to be determined. Solving Eq. (A2) yields constraints on the parameters appearing in $B[1]$:

$$\begin{aligned} a_{21} &= e^{-i(\phi_{21}+\varphi_2)}(-a_{11}e^{i(\phi_{11}+\varphi_1)}\cot\vartheta + ye^{i\phi_1}\cos\theta\csc\vartheta);\\ a_{22} &= e^{-i(\phi_{22}+\varphi_2)}(-a_{12}e^{i(\phi_{12}+\varphi_1)}\cot\vartheta + ye^{i\phi_2}\sin\theta\csc\vartheta);\\ a_{23} &= e^{-i(\phi_{23}+\phi_2+\varphi_2)}(-a_{13}e^{i(\phi_{13}+\varphi_1+\phi_2)}\cot\vartheta + y\sin\theta\csc\vartheta);\\ a_{24} &= e^{-i(\phi_{24}+\phi_1+\varphi_2)}(-a_{14}e^{i(\phi_{14}+\varphi_1+\phi_1)}\cot\vartheta - y\cos\theta\csc\vartheta). \end{aligned} \tag{A5}$$

The MPS matrices can be expressed in a column-oriented form similar to those of the cluster state, Eq. (7), by setting $a_{21} = a_{23} = 0$, which can be accomplished via

$$\begin{aligned} &a_{11} = y\cos\theta\sec\vartheta;\ \phi_{11} = \phi_1 - \varphi_1;\ a_{13} = y\sin\theta\sec\vartheta;\ \phi_{13} = -\phi_2 - \varphi_1;\\ &a_{12} = 0;\ \phi_{22} = \phi_2 - \varphi_2;\ a_{14} = 0;\ \phi_{24} = -\phi_1 - \varphi_2; \end{aligned} \tag{A6}$$

this yields

$$B[0] = ye^{-i\varphi_1}\sec\vartheta\begin{pmatrix} e^{i\phi_1}\cos\theta & 0 \\ e^{-i\phi_2}\sin\theta & 0 \end{pmatrix};\quad B[1] = ye^{-i\varphi_2}\csc\vartheta\begin{pmatrix} 0 & e^{i\phi_2}\sin\theta \\ 0 & -e^{-i\phi_1}\cos\theta \end{pmatrix}. \tag{A7}$$

If the MPS matrices depend explicitly on all the measurement angles then MBQC is impossible. Setting $\phi_1 = \varphi_1$, $\phi_2 = -\varphi_1$, and $\varphi_2 = -\varphi_1$ yields

$$B[0] = y\sec\vartheta\begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix};\quad B[1] = y\csc\vartheta\begin{pmatrix} 0 & \sin\theta \\ 0 & -\cos\theta \end{pmatrix}, \tag{A8}$$

with only ϑ remaining. The normalization condition (neglecting the junk sector) is

$$B[0]^\dagger B[0] + B[1]^\dagger B[1] = y^2\begin{pmatrix} \sec^2\vartheta & 0 \\ 0 & \csc^2\vartheta \end{pmatrix} = I, \tag{A9}$$

which yields $\vartheta = (2k+1)\pi/4$, $k \in \mathbb{Z}$, and $y = 1/\sqrt{2}$. Thus the measurement angle ϑ is fixed; choosing $\vartheta = \pi/4$, one obtains

$$B[0] = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{pmatrix};\quad B[1] = \begin{pmatrix} 0 & \sin\theta \\ 0 & -\cos\theta \end{pmatrix}. \tag{A10}$$

The output unitary matrices are then

$$\begin{aligned} U[0] &= \begin{pmatrix} e^{i\varphi_1}\cos\theta & e^{-i\varphi_1}\sin\theta \\ e^{i\varphi_1}\sin\theta & -e^{-i\varphi_1}\cos\theta \end{pmatrix} = ZR_Y(\theta)R_Z(\varphi);\\ U[1] &= \begin{pmatrix} e^{i\varphi_1}\cos\theta & -e^{-i\varphi_1}\sin\theta \\ e^{i\varphi_1}\sin\theta & e^{-i\varphi_1}\cos\theta \end{pmatrix} = R_Y(-\theta)R_Z(\varphi) = ZR_Y(\theta)ZR_Z(\varphi) = U[0]Z, \end{aligned} \tag{A11}$$

where $R_\alpha(\theta) := \exp(i\theta\alpha)$.

APPENDIX B: SPT ORDER

In this Appendix, we show that the states defined by Eq. (17) have nontrivial symmetry-protected topological order, by explicitly constructing the symmetry operators in real space.

1. Review of SPT order in cluster states

It is useful to review the basics of SPT order in cluster states, and the following analysis follows expands on Ref. [41]. The $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry of the cluster state with an even number of sites n and periodic boundary conditions is explicitly generated by the operators $X_{\text{even}} = \prod_j X_{2j}$ and $X_{\text{odd}} = \prod_j X_{2j+1}$. Because $(X \otimes I)C_Z = C_Z(X \otimes Z)$, applying the X_j operator on $C_Z^{j-1,j}C_Z^{j,j+1}$ returns $C_Z^{j-1,j}C_Z^{j,j+1}Z_{j-1}X_jZ_{j+1}$ so that all Z factors cancel on the application of either $X_{\rm odd}$ or $X_{\rm even}$. The symmetry operators leave the cluster state invariant because all qubits are originally set to $|+\rangle$ which is an eigenstate of X.

For a cluster state with open boundary conditions, the symmetry operators need to be slightly modified. Again assume n is even. The additional Z operators resulting from the action of X on the first and last sites, $X_1C_Z^{1,2} = C_Z^{1,2}X_1Z_2$ and $X_nC_Z^{n-1,n} = C_Z^{n-1,n}Z_{n-1}X_n$, will be canceled by other Z gates arising from the adjacent odd or even sites, respectively. But the action of X on site 2 and $n-1$, $X_2C_Z^{1,2} = C_Z^{1,2}Z_1X_2$ and $X_{n-1}C_Z^{n-1,n} = C_Z^{n-1,n}X_{n-1}Z_n$, yield Z operators on the first and last sites that aren't canceled by other X gates in $X_{\rm even}$ or $X_{\rm odd}$. The symmetry operators therefore become $X_{\rm even} = Z_1\prod_j X_{2j}$ and $X_{\rm odd} = \prod_j X_{2j+1}Z_n$.

The C_Z gates are diagonal and therefore commute with Z operators, so the cluster state is the unique $+n$ eigenstate of the n-fold sum of stabilizer generators in the bulk $S_j = Z_{j-1}X_jZ_{j+1}$ $(2 \leqslant j \leqslant n-1)$ and at the boundaries $S_1 = X_1Z_2$ and $S_n = Z_{n-1}X_n$. Pauli gates at the boundaries are transformed by the C_Z operators into effective Pauli gates $\overline{X}_1 = C_Z^{1,2}X_1C_Z^{1,2} = X_1Z_2$ (note that $C_Z^\dagger = C_Z$), $\overline{Z}_1 = Z_1$, $\overline{X}_n = Z_{n-1}X_n$, and $\overline{Z}_n = Z_n$. Defining the generators of the $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry as $U(g_1, g_2) = X_{\rm odd}^{g_1}X_{\rm even}^{g_2}$, where $g_1, g_2 \in \{0, 1\}$, the effective Pauli operators on the left site are transformed as $U(g_1, g_2)\overline{X}_1U(g_1, g_2)^\dagger = (-1)^{g_2}\overline{X}_1$ and $U(g_1, g_2)\overline{Z}_1U(g_1, g_2)^\dagger = (-1)^{g_1}\overline{Z}_1$, which is equivalent to an effective transformation $U_{\rm eff}(g_1, g_2) = \overline{X}^{g_1}\overline{Z}^{g_2}$. A similar result holds for the right edge.

SPT phases in one-dimensional systems can be classified by the second cohomology group. If $U_{\rm eff}$ is a projective representation of the symmetry group, then $U_{\rm eff}(g)U_{\rm eff}(h) = \omega(g, h)U_{\rm eff}(g \oplus h)$, where $\oplus$ here corresponds to bitwise addition (not a direct sum!) and $\omega(g, h)$ (called a 2-cocycle) must satisfy the consistency conditions [42]

$$\omega(g, h)\omega(g \oplus h, k) = \omega(h, k)\omega(g, h \oplus k) \tag{B1}$$

and

$$\omega(g, h) \sim \omega(g, h)\beta(g)\beta(h)\beta(g \oplus h)^{-1}, \tag{B2}$$

where the β are phase factors. The 2-cocycle for the left boundary of the cluster state is therefore

$$\begin{aligned}\omega(g_1, g_2; h_1, h_2) &= U_{\rm eff}(g_1, g_2)U_{\rm eff}(h_1, h_2)U_{\rm eff}^{-1}(g_1 \oplus h_1, g_2 \oplus h_2) = \overline{X}^{g_1}\overline{Z}^{g_2}\overline{X}^{h_1}\overline{Z}^{h_2}\overline{Z}^{g_2\oplus h_2}\overline{X}^{g_1\oplus h_1}\\ &= (-1)^{g_2h_1}I,\end{aligned} \tag{B3}$$

where the result is obtained by considering each case. Check that this satisfies Eq. (B1): $(-1)^{g_2h_1}(-1)^{(g_2\oplus h_2)k_1} = (-1)^{h_2k_1}(-1)^{g_2(h_1\oplus k_1)}$. If $g_2 = 0$ then the left and right sides are $(-1)^{h_2k_1}$, and if $g_2 = 1$ then one requires $(-1)^{h_1}(-1)^{\bar{h}_2k_1} = (-1)^{h_2k_1}(-1)^{h_1\oplus k_1}$. Next, if $k_1 = 0$ then both sides are $(-1)^{h_1}$, and if $k_1 = 1$ then one requires $(-1)^{h_1}(-1)^{\bar{h}_2} = (-1)^{h_2}(-1)^{\bar{h}_1}$ or $(-1)^{h_1-\bar{h}_1} = (-1)^{h_2-\bar{h}_2}$ which is true for any $\{h_1, h_2\}$. Therefore Eq. (B1) is satisfied. A sufficient condition for SPT order is that the 2-cocycles anticommute:

$$\begin{aligned}\omega(g_1, g_2; h_1, h_2)\omega(h_1, h_2; g_1, g_2)^{-1} &= U_{\rm eff}(g_1, g_2)U_{\rm eff}(h_1, h_2)U_{\rm eff}(g_1, g_2)^{-1}U_{\rm eff}(h_1, h_2)^{-1}\\ &= (X_1Z_2)^{g_1}Z_1^{g_2}(X_1Z_2)^{h_1}Z_1^{h_2}Z_1^{g_2}(Z_2X_1)^{g_1}Z_1^{h_2}(Z_2X_1)^{h_1}\\ &= X_1^{g_1}Z_1^{g_2}X_1^{h_1}Z_1^{h_2}Z_1^{g_2}X_1^{g_1}Z_1^{h_2}X_1^{h_1}\\ &= X_1^{g_1}(-1)^{g_2h_1}X_1^{h_1}Z_1^{h_2}X_1^{g_1}Z_1^{h_2}X_1^{h_1}\\ &= (-1)^{g_1h_2}(-1)^{g_2h_1}I \neq I,\end{aligned} \tag{B4}$$

consistent with Eq. (B3); thus, the algebra associated with the two effective Pauli operators at the surface is (maximally) noncommutative [19].

2. Symmetry operators for the state defined in Eq. (17)

We first show that there exist two-qubit operators P_{θ_j} and Q_{θ_j}, defined as

$$\begin{aligned}P_{\theta_j}^{j,j+1} &= |0\rangle_j\langle 0|X_{j+1}M_{\theta_j}^{j+1}X_{j+1} + |1\rangle_j\langle 1|M_{\theta_j}^{j+1};\\ Q_{\theta_j}^{j,j+1} &= |0\rangle_j\langle 1|X_{j+1}M_{\theta_j}^{j+1}X_{j+1} + |1\rangle_j\langle 0|M_{\theta_j}^{j+1} = P_{\theta_j}^{j,j+1}X_j,\end{aligned} \tag{B5}$$

where

$$M_{\theta_j}^{j+1} = \tan\theta_j|0\rangle_{j+1}\langle 0| + \cot\theta_j|1\rangle_{j+1}\langle 1|, \tag{B6}$$

such that

$$P_{\theta_j}^{j-1,j}Q_{\theta_{j+1}}^{j,j+1}C_{\theta_j}^{j-1,j}C_{\theta_{j+1}}^{j,j+1} = P_{\theta_j}^{j-1,j}P_{\theta_{j+1}}^{j,j+1}X_jC_{\theta_j}^{j-1,j}C_{\theta_{j+1}}^{j,j+1} = C_{\theta_j}^{j-1,j}C_{\theta_{j+1}}^{j,j+1}(Z_{j-1}X_jZ_{j+1}), \tag{B7}$$

i.e., that

$$P_{\theta_j}^{j-1,j} Q_{\theta_{j+1}}^{j,j+1} = C_{\theta_j}^{j-1,j} C_{\theta_{j+1}}^{j,j+1} (Z_{j-1} X_j Z_{j+1}) \Big(C_{\theta_j}^{j-1,j} C_{\theta_{j+1}}^{j,j+1} \Big)^{-1}. \tag{B8}$$

Note that the inverse is applied since C_θ is not unitary.

Neglecting the factor of $\sqrt{2}$, the two-qubit operator $C_{\theta_j}^{j,j+1}$ defined in Eq. (18) can be conveniently expressed as

$$C_{\theta_j}^{j,j+1} = |0\rangle_j \langle 0| N_{\theta_j}^{j+1} + |1\rangle_j \langle 1| X_{j+1} N_{\theta_j}^{j+1} X_{j+1} Z_{j+1}, \tag{B9}$$

where

$$N_{\theta_j}^{j+1} = \cos\theta_j |0\rangle_{j+1}\langle 0| + \sin\theta_j |1\rangle_{j+1}\langle 1|. \tag{B10}$$

To simplify the notation without loss of generality, rewrite

$$C_{\theta_j}^{j,j+1} \to C_j = |0\rangle\langle 0| \otimes N_j + |1\rangle\langle 1| \otimes (X N_j X Z), \tag{B11}$$

with

$$N_j = \cos\theta_j |0\rangle\langle 0| + \sin\theta_j |1\rangle\langle 1|, \tag{B12}$$

and furthermore assume that $j = 1$. A few lines of algebra yields

$$C_{\theta_j}^{j,j+1} C_{\theta_{j+1}}^{j+1,j+2} \to (C_1 \otimes I)(I \otimes C_2) = N_1 \otimes |0\rangle\langle 0| \otimes N_2 + X N_1 X Z \otimes |1\rangle\langle 1| \otimes X N_2 X Z \tag{B13}$$

and

$$(C_1 \otimes I)(I \otimes C_2)(Z \otimes X \otimes Z) = N_1 Z \otimes |0\rangle\langle 1| \otimes N_2 Z + X N_1 X \otimes |1\rangle\langle 0| \otimes X N_2 X. \tag{B14}$$

Next, one seeks operators $P_{\theta_j}^{j,j+1} \to P_1 \otimes I$ and $Q_{\theta_{j+1}}^{j+1,j+2} \to I \otimes Q_2$, where

$$\begin{aligned} P_1 &= |0\rangle\langle 0| \otimes \alpha_1 + |0\rangle\langle 1| \otimes \alpha_2 + |1\rangle\langle 0| \otimes \alpha_3 + |1\rangle\langle 1| \otimes \alpha_4; \\ Q_2 &= |0\rangle\langle 0| \otimes \beta_1 + |0\rangle\langle 1| \otimes \beta_2 + |1\rangle\langle 0| \otimes \beta_3 + |1\rangle\langle 1| \otimes \beta_4, \end{aligned} \tag{B15}$$

and the α_i and β_i are free parameters, such that $(P_1 \otimes I)(I \otimes Q_2)(C_1 \otimes I)(I \otimes C_2)$ returns the right hand side of Eq. (B14). Straightforward algebra yields

$$\begin{aligned} (P_1 \otimes I)(I \otimes Q_2)(C_1 \otimes I)(I \otimes C_2) = {} & \cos\theta_1 |0\rangle\langle 0| \otimes \alpha_1 |0\rangle\langle 0| \otimes \beta_1 N_2 + \cos\theta_1 |0\rangle\langle 0| \otimes \alpha_1 |1\rangle\langle 0| \otimes \beta_3 N_2 \\ & + \sin\theta_1 |0\rangle\langle 0| \otimes \alpha_1 |0\rangle\langle 1| \otimes \beta_2 X N_2 X Z + \sin\theta_1 |0\rangle\langle 0| \otimes \alpha_1 |1\rangle\langle 1| \otimes \beta_4 X N_2 X Z \\ & + \sin\theta_1 |1\rangle\langle 1| \otimes \alpha_4 |0\rangle\langle 0| \otimes \beta_1 N_2 + \sin\theta_1 |1\rangle\langle 1| \otimes \alpha_4 |1\rangle\langle 0| \otimes \beta_3 N_2 \\ & - \cos\theta_1 |1\rangle\langle 1| \otimes \alpha_4 |0\rangle\langle 1| \otimes \beta_2 X N_2 X Z - \cos\theta_1 |1\rangle\langle 1| \otimes \alpha_4 |1\rangle\langle 1| \otimes \beta_4 X N_2 X Z, \end{aligned} \tag{B16}$$

and $\alpha_2 = \alpha_3 = 0$. Comparing this expression with the right hand side of Eq. (B14), the free parameters must take the values

$$\begin{aligned} &\alpha_4 = \tan\theta_1 |0\rangle\langle 0| + \cot\theta_1 |1\rangle\langle 1|; \quad \alpha_1 = \cot\theta_1 |0\rangle\langle 0| + \tan\theta_1 |1\rangle\langle 1| = X \alpha_4 X; \\ &\beta_3 = \tan\theta_2 |0\rangle\langle 0| + \cot\theta_2 |1\rangle\langle 1|; \quad \beta_2 = \cot\theta_2 |0\rangle\langle 0| + \tan\theta_2 |1\rangle\langle 1| = X \beta_3 X; \quad \beta_1 = \beta_4 = 0. \end{aligned} \tag{B17}$$

One then obtains

$$\begin{aligned} P_1 &= |0\rangle\langle 0| \otimes X M_1 X + |1\rangle\langle 1| \otimes M_1; \\ Q_2 &= |0\rangle\langle 1| \otimes X M_2 X + |1\rangle\langle 0| \otimes M_2, \end{aligned} \tag{B18}$$

where

$$M_j = \tan\theta_j |0\rangle\langle 0| + \cot\theta_j |1\rangle\langle 1|. \tag{B19}$$

Reverting to unsimplified notation, one obtains

$$P_{\theta_j}^{j,j+1} Q_{\theta_{j+1}}^{j+1,j+2} C_{\theta_j}^{j,j+1} C_{\theta_{j+1}}^{j+1,j+2} = P_{\theta_j}^{j,j+1} P_{\theta_{j+1}}^{j+1,j+2} X_{j+1} C_{\theta_j}^{j,j+1} C_{\theta_{j+1}}^{j+1,j+2} = C_{\theta_j}^{j,j+1} C_{\theta_{j+1}}^{j+1,j+2} (Z_j X_{j+1} Z_{j+2}), \tag{B20}$$

where

$$P_{\theta_j}^{j,j+1} = |0\rangle_j \langle 0| X_{j+1} M_{\theta_j}^{j+1} X_{j+1} + |1\rangle_j \langle 1| M_{\theta_j}^{j+1} \tag{B21}$$

and

$$M_{\theta_j}^{j+1} = \tan\theta_j |0\rangle_{j+1}\langle 0| + \cot\theta_j |1\rangle_{j+1}\langle 1|, \tag{B22}$$

though the nonunitary operator $P_{\theta_j}^{j,j+1}$ is more conveniently expressed as a diagonal matrix acting on qubits j and $j+1$:

$$P_{\theta_j}^{j,j+1} = \mathrm{diag}(\cot\theta_j, \tan\theta_j, \tan\theta_j, \cot\theta_j)_{j,j+1}. \tag{B23}$$

These recover the statement Eq. (B7). Using the same procedure, one may derive similar expressions for the boundary operators:

$$P_{\theta_1}^{1,2}X_1C_{\theta_1}^{1,2}C_{\theta_2}^{2,3} = C_{\theta_1}^{1,2}C_{\theta_2}^{2,3}(X_1Z_2I_3); \quad P_{\theta_n}^{n-1,n}X_nC_{\theta_{n-1}}^{n-2,n-1}C_{\theta_n}^{n-1,n} = C_{\theta_{n-1}}^{n-2,n-1}C_{\theta_n}^{n-1,n}(I_{n-2}Z_{n-1}X_n). \tag{B24}$$

Second, we prove that the joint operator $P_{\theta_j}^{j,j+1}Q_{\theta_{j+1}}^{j+1,j+2}$ commutes with its counterpart $P_{\theta_{j+2}}^{j+2,j+3}Q_{\theta_{j+3}}^{j+3,j+4}$ two sites over. Using the simplified notation, this corresponds to proving that $[(P_1\otimes I)(I\otimes Q_2)\otimes I^{\otimes 2}, I^{\otimes 2}\otimes(P_3\otimes I)(I\otimes Q_4)] = 0$. The third qubit is the only one with support on both operators, so one need only focus on the contributions of the operators on this position: the XM_2X and M_2 from Q_2, and the $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ from P_3. And, because all of these operators are diagonal in the computational basis, they commute, and therefore the $P_{\theta_j}^{j,j+1}Q_{\theta_{j+1}}^{j+1,j+2}$ and $P_{\theta_{j+2}}^{j+2,j+3}Q_{\theta_{j+3}}^{j+3,j+4}$ operators commute. Thus $\prod_j P_{\theta_{2j}}^{2j,2j+1}Q_{\theta_{2j+1}}^{2j+1,2j+2} = \prod_j P_{\theta_{2j}}^{2j,2j+1}P_{\theta_{2j+1}}^{2j+1,2j+2}X_{2j+1}$ and $\prod_j P_{\theta_{2j-1}}^{2j-1,2j}Q_{\theta_{2j}}^{2j,2j+1} = \prod_j P_{\theta_{2j-1}}^{2j-1,2j}P_{\theta_{2j}}^{2j,2j+1}X_{2j}$ are bulk symmetry operators for the state (17). Likewise, the boundary operators in Eq. (B24) automatically commute with the bulk operators two sites over, because they have support on different qubits. The analogs of the cluster-state symmetry operators (for even n) are then

$$X_{\mathrm{odd}} = P_{\theta_1}^{1,2}X_1\left(\prod_{j=1} P_{\theta_{2j}}^{2j,2j+1}P_{\theta_{2j+1}}^{2j+1,2j+2}X_{2j+1}\right)Z_n; \quad X_{\mathrm{even}} = Z_1\left(\prod_{j=1} P_{\theta_{2j-1}}^{2j-1,2j}P_{\theta_{2j}}^{2j,2j+1}X_{2j}\right)\left(P_{\theta_n}^{n-1,n}X_n\right), \tag{B25}$$

where Z gates are added to the last and first sites of X_{odd} and X_{even}, respectively, as was necessary for the cluster state with open boundary conditions. Note that X_{odd} and X_{even} are neither unitary nor on-site global symmetries.

3. Generalized stabilizers for the state defined in Eq. (17)

We first show that the P_{θ_j} operators yield $n-2$ operators $S_{j,j+1}$, $2\leqslant j\leqslant n-1$, such that $S_{j,j+1}|\psi\rangle = |\psi\rangle$. These are the analogs of the bulk cluster-state stabilizer generators $Z_{j-1}X_jZ_{j+1}$. Given that the C_{θ_j} gates are diagonal, one may rewrite Eq. (B7):

$$P_{\theta_j}^{j-1,j}P_{\theta_{j+1}}^{j,j+1}Z_{j-1}X_jZ_{j+1}C_{\theta_j}^{j-1,j}C_{\theta_{j+1}}^{j,j+1} = S_{j,j+1}C_{\theta_j}^{j-1,j}C_{\theta_{j+1}}^{j,j+1} = C_{\theta_j}^{j-1,j}C_{\theta_{j+1}}^{j,j+1}X_j. \tag{B26}$$

Thus $S_{j,j+1} = P_{\theta_j}^{j-1,j}P_{\theta_{j+1}}^{j,j+1}Z_{j-1}X_jZ_{j+1}$ are eigenoperators for the state with unit eigenvalue, for any j in the bulk, and generalize the stabilizer operators for the cluster state. With Eq. (B21), one obtains

$$\begin{aligned}
S_{j,j+1} &= \Big[|0\rangle_{j-1}\langle 0|X_jM_{\theta_j}^jX_j + |1\rangle_{j-1}\langle 1|M_{\theta_j}^j\Big]\Big[|0\rangle_j\langle 0|X_{j+1}M_{\theta_{j+1}}^{j+1}X_{j+1} + |1\rangle_j\langle 1|M_{\theta_{j+1}}^{j+1}\Big]Z_{j-1}X_jZ_{j+1}\\
&= \big[|0\rangle_{j-1}\langle 0|X_j\big(\tan\theta_j|0\rangle_j\langle 0| + \cot\theta_j|1\rangle_j\langle 1|\big)X_j + |1\rangle_{j-1}\langle 1|\big(\tan\theta_j|0\rangle_j\langle 0| + \cot\theta_j|1\rangle_j\langle 1|\big)\big]\\
&\quad\times\big[|0\rangle_j\langle 0|X_{j+1}\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| + \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)X_{j+1} + |1\rangle_j\langle 1|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| + \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
&\quad\times Z_{j-1}X_jZ_{j+1}\\
&= \big[|0\rangle_{j-1}\langle 0|\big(\tan\theta_j|1\rangle_j\langle 1| + \cot\theta_j|0\rangle_j\langle 0|\big) + |1\rangle_{j-1}\langle 1|\big(\tan\theta_j|0\rangle_j\langle 0| + \cot\theta_j|1\rangle_j\langle 1|\big)\big]\\
&\quad\times\big[|0\rangle_j\langle 0|\big(\tan\theta_{j+1}|1\rangle_{j+1}\langle 1| + \cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big) + |1\rangle_j\langle 1|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| + \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]Z_{j-1}X_jZ_{j+1}\\
&= \big[|0\rangle_{j-1}\langle 0|\big(\cot\theta_j|0\rangle_j\langle 0|\big) + |1\rangle_{j-1}\langle 1|\big(\tan\theta_j|0\rangle_j\langle 0|\big)\big]\\
&\quad\times\big[|0\rangle_j\langle 0|\big(\tan\theta_{j+1}|1\rangle_{j+1}\langle 1| + \cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]Z_{j-1}X_jZ_{j+1}\\
&\quad+\big[|0\rangle_{j-1}\langle 0|\big(\tan\theta_j|1\rangle_j\langle 1|\big) + |1\rangle_{j-1}\langle 1|\big(\cot\theta_j|1\rangle_j\langle 1|\big)\big]\\
&\quad\times\big[|1\rangle_j\langle 1|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| + \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]Z_{j-1}X_jZ_{j+1}\\
&= \big(\cot\theta_j|0\rangle_{j-1}\langle 0| + \tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 0|\big(\tan\theta_{j+1}|1\rangle_{j+1}\langle 1| + \cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)Z_{j-1}X_jZ_{j+1}\\
&\quad+\big(\tan\theta_j|0\rangle_{j-1}\langle 0| + \cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 1|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| + \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)Z_{j-1}X_jZ_{j+1}\\
&= \big(\cot\theta_j|0\rangle_{j-1}\langle 0| - \tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1| + \cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\\
&\quad+\big(\tan\theta_j|0\rangle_{j-1}\langle 0| - \cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| - \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big).
\end{aligned} \tag{B27}$$

Note that the $S_{j,j+1}$ are nonseparable three-local operators that are neither unitary nor Hermitian. These operators square to the identity, as required for generalized stabilizers:

$$\begin{aligned}
S_{j,j+1}^2 = \big[&\big(\cot\theta_j|0\rangle_{j-1}\langle 0| - \tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1| + \cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\\
&+\big(\tan\theta_j|0\rangle_{j-1}\langle 0| - \cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0| - \cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&\times\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\\
&+\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
=&\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]\\
&\times\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]\\
&+\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]\\
&\times\big[\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
&+\big[\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
&\times\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]\\
&+\big[\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
&\times\big[\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
=&\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]\\
&\times\big[\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
&+\big[\big(\tan\theta_j|0\rangle_{j-1}\langle 0|-\cot\theta_j|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 0|\big(\tan\theta_{j+1}|0\rangle_{j+1}\langle 0|-\cot\theta_{j+1}|1\rangle_{j+1}\langle 1|\big)\big]\\
&\times\big[\big(\cot\theta_j|0\rangle_{j-1}\langle 0|-\tan\theta_j|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 1|\big(-\tan\theta_{j+1}|1\rangle_{j+1}\langle 1|+\cot\theta_{j+1}|0\rangle_{j+1}\langle 0|\big)\big]\\
=&\big[\big(|0\rangle_{j-1}\langle 0|+|1\rangle_{j-1}\langle 1|\big)|0\rangle_j\langle 0|\big(|1\rangle_{j+1}\langle 1|+|0\rangle_{j+1}\langle 0|\big)\big]\\
&+\big[\big(|0\rangle_{j-1}\langle 0|+|1\rangle_{j-1}\langle 1|\big)|1\rangle_j\langle 1|\big(|0\rangle_{j+1}\langle 0|+|1\rangle_{j+1}\langle 1|\big)\big]=I_{j-1,j,j+1}, \qquad \text{(B28)}
\end{aligned}
$$

as desired.

Second, internal consistency also requires that the generalized stabilizer operators for different j values commute. In simplified notation, one needs to only verify that $[S_{1,2}\otimes I, I\otimes S_{2,3}]=0$, where using Eqs. (B18) and (B19), one obtains

$$
\begin{aligned}
S_{j,j+1}=&\cot\theta_j|0\rangle\langle 0|\otimes|0\rangle\langle 1|\otimes XM_{j+1}XZ-\tan\theta_j|1\rangle\langle 1|\otimes|0\rangle\langle 1|\otimes XM_{j+1}XZ\\
&+\tan\theta_j|0\rangle\langle 0|\otimes|1\rangle\langle 0|\otimes M_{j+1}Z-\cot\theta_j|1\rangle\langle 1|\otimes|1\rangle\langle 0|\otimes M_{j+1}Z, \qquad \text{(B29)}
\end{aligned}
$$

where $j=1,2$ is strictly a label and does not denote qubit position. Multiplication yields the unenlightening expressions

$$
\begin{aligned}
(S_{1,2}\otimes I)(I\otimes S_{2,3})=&-\cot\theta_1\tan\theta_2|0\rangle\langle 0|\otimes|0\rangle\langle 1|\otimes XM_2XZ|0\rangle\langle 1|\otimes XM_3XZ\\
&-\cot\theta_1\cot\theta_2|0\rangle\langle 0|\otimes|0\rangle\langle 1|\otimes XM_2XZ|1\rangle\langle 0|\otimes M_3Z\\
&+\tan\theta_1\tan\theta_2|1\rangle\langle 1|\otimes|0\rangle\langle 1|\otimes XM_2XZ|0\rangle\langle 1|\otimes XM_3XZ\\
&+\tan\theta_1\cot\theta_2|1\rangle\langle 1|\otimes|0\rangle\langle 1|\otimes XM_2XZ|1\rangle\langle 0|\otimes M_3Z\\
&+\tan\theta_1\tan\theta_2|0\rangle\langle 0|\otimes|1\rangle\langle 0|\otimes M_2Z|1\rangle\langle 0|\otimes M_3Z\\
&+\tan\theta_1\cot\theta_2|0\rangle\langle 0|\otimes|1\rangle\langle 0|\otimes M_2Z|0\rangle\langle 1|\otimes XM_3XZ\\
&-\cot\theta_1\tan\theta_2|1\rangle\langle 1|\otimes|1\rangle\langle 0|\otimes M_2Z|1\rangle\langle 0|\otimes M_3Z\\
&-\cot\theta_1\cot\theta_2|1\rangle\langle 1|\otimes|1\rangle\langle 0|\otimes M_2Z|0\rangle\langle 1|\otimes XM_3XZ;\\
(I\otimes S_{2,3})(S_{1,2}\otimes I)=&\cot\theta_1\cot\theta_2|0\rangle\langle 0|\otimes|0\rangle\langle 1|\otimes|0\rangle\langle 1|XM_2XZ\otimes XM_3XZ\\
&+\cot\theta_1\tan\theta_2|0\rangle\langle 0|\otimes|0\rangle\langle 1|\otimes|1\rangle\langle 0|XM_2XZ\otimes M_3Z\\
&-\tan\theta_1\cot\theta_2|1\rangle\langle 1|\otimes|0\rangle\langle 1|\otimes|0\rangle\langle 1|XM_2XZ\otimes XM_3XZ\\
&-\tan\theta_1\tan\theta_2|1\rangle\langle 1|\otimes|0\rangle\langle 1|\otimes|1\rangle\langle 0|XM_2XZ\otimes M_3Z\\
&-\tan\theta_1\cot\theta_2|0\rangle\langle 0|\otimes|1\rangle\langle 0|\otimes|1\rangle\langle 0|M_2Z\otimes M_3Z\\
&-\tan\theta_1\tan\theta_2|0\rangle\langle 0|\otimes|1\rangle\langle 0|\otimes|0\rangle\langle 1|M_2Z\otimes XM_3XZ\\
&+\cot\theta_1\cot\theta_2|1\rangle\langle 1|\otimes|1\rangle\langle 0|\otimes|1\rangle\langle 0|M_2Z\otimes M_3Z\\
&+\cot\theta_1\tan\theta_2|1\rangle\langle 1|\otimes|1\rangle\langle 0|\otimes|0\rangle\langle 1|M_2Z\otimes XM_3XZ. \qquad \text{(B30)}
\end{aligned}
$$

As $XM_2XZ|0\rangle\langle 1|=\cot\theta_2|0\rangle\langle 1|$ while $|0\rangle\langle 1|XM_2XZ=-\tan\theta_2|0\rangle\langle 1|$, terms 1 and 3 in both expressions coincide; similarly, as $XM_2XZ|1\rangle\langle 0|=-\tan\theta_2|1\rangle\langle 0|$ while $|1\rangle\langle 0|XM_2XZ=\cot\theta_2|1\rangle\langle 0|$, terms 2 and 4 in both expressions coincide. Similar results apply to the remaining terms, and therefore $[S_{1,2}\otimes I, I\otimes S_{2,3}]=0$.

Third, one may likewise define generalizations of the surface stabilizers from the operations in Eq. (B24):

$$P_{\theta_1}^{1,2}X_1Z_2C_{\theta_1}^{1,2}C_{\theta_2}^{2,3} = S_{1,2}C_{\theta_1}^{1,2}C_{\theta_2}^{2,3} = C_{\theta_1}^{1,2}C_{\theta_2}^{2,3}X_1;$$
$$P_{\theta_n}^{n-1,n}Z_{n-1}X_nC_{\theta_{n-1}}^{n-2,n-1}C_{\theta_n}^{n-1,n} = S_{n-1,n}C_{\theta_{n-1}}^{n-2,n-1}C_{\theta_n}^{n-1,n} = C_{\theta_{n-1}}^{n-2,n-1}C_{\theta_n}^{n-1,n}X_n, \quad \text{(B31)}$$

so that $S_{1,2} = P_{\theta_1}^{1,2}X_1Z_2$ and $S_{n-1,n} = P_{\theta_n}^{n-1,n}Z_{n-1}X_n$. Following a similar analysis as above, it is straightforward to show that these operators also commute with the bulk generalized stabilizer generators, Eq. (B27). Likewise, $S_{1,2}^2 = I_{1,2}$ and $S_{n-1,n}^2 = I_{n-1,n}$.

Fourth, for X_{odd} and X_{even}, Eq. (B25), to represent a $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry, they should also square to the identity. We can make use of the results above for this purpose. Because $P_{\theta_j}^{j-1,j}$, Eq. (B23), is diagonal, one can write

$$S_{j,j+1} = P_{\theta_j}^{j-1,j}P_{\theta_{j+1}}^{j,j+1}Z_{j-1}X_jZ_{j+1} = Z_{j-1}Z_{j+1}P_{\theta_j}^{j-1,j}P_{\theta_{j+1}}^{j,j+1}X_j \quad \text{(B32)}$$

or

$$S_{2j+1,2j+2} = Z_{2j}Z_{2j+2}P_{\theta_{2j+1}}^{2j,2j+1}P_{\theta_{2j+2}}^{2j+1,2j+2}X_{2j+1} = P_{\theta_{2j+1}}^{2j,2j+1}P_{\theta_{2j+2}}^{2j+1,2j+2}X_{2j+1}Z_{2j}Z_{2j+2}, \quad \text{(B33)}$$

so that

$$P_{\theta_{2j+1}}^{2j,2j+1}P_{\theta_{2j+2}}^{2j+1,2j+2}X_{2j+1} = Z_{2j}Z_{2j+2}S_{2j+1,2j+2} = S_{2j+1,2j+2}Z_{2j}Z_{2j+2}. \quad \text{(B34)}$$

One can therefore rewrite X_{odd} as

$$X_{\text{odd}} = P_{\theta_1}^{1,2}X_1\left(\prod_{j=1}P_{\theta_{2j}}^{2j,2j+1}P_{\theta_{2j+1}}^{2j+1,2j+2}X_{2j+1}\right)Z_n = (Z_2S_{1,2})\left(\prod_{j=1}Z_{2j}Z_{2j+2}S_{2j+1,2j+2}\right)Z_n = (S_{1,2})\left(\prod_{j=1}S_{2j+1,2j+2}\right), \quad \text{(B35)}$$

where $\theta_{2j+1}\to\theta_{2j}$ is an unimportant shift; here we have used the fact that all the Z operators commute through the S operators, Eq. (B34). Then

$$X_{\text{odd}}^2 = (S_{1,2})\left(\prod_{j=1}S_{2j+1,2j+2}\right)(S_{1,2})\left(\prod_{j'=1}S_{2j'+1,2j'+2}\right) = I, \quad \text{(B36)}$$

because all generalized stabilizers commute with one another, as shown above, and then square to the identity, Eq. (B28). Thus the symmetry operator X_{odd} squares to the identity. A similar result follows for $X_{\text{even}} = (\prod_j S_{2j,2j+1})(S_{n-1,n})$. The symmetry operators written this way have an intuitive form, as products of generalized stabilizers on two-site blocks, counting either from the first or second site. It is important to keep in mind, however, that each generalized stabilizer operator acts on three sites.

4. SPT order for the state defined in Eq. (17)

Finally, one can apply these results to the analysis of SPT order. Again, we can treat the set of n $S_{j,j+1}$ operators as effective stabilizers that uniquely define the state, including the qubits at the boundaries. The X and Z gates on the boundary qubit are transformed by the C_θ operators into effective Pauli gates, and can be read directly from Eq. (B24):

$$\overline{X}_1 = P_{\theta_1}^{1,2}X_1Z_2;\ \overline{Z}_1 = Z_1;\ \overline{X}_n = P_{\theta_n}^{n-1,n}Z_{n-1}X_n;\ \overline{Z}_n = Z_n. \quad \text{(B37)}$$

Define the generators of the $\mathbb{Z}_2\times\mathbb{Z}_2$ as $O(g_1,g_2) = X_{\text{odd}}^{g_1}X_{\text{even}}^{g_2}$, where X_{even} and X_{odd} are given in Eq. (B25) and $g_1, g_2\in\{0,1\}$. We write $O(g_1,g_2)$ rather than $U(g_1,g_2)$ as the former is only unitary for the specific case of $\theta_i=\theta_c\ \forall i$. The goal is to determine the effective Pauli operators $\overline{X}_1'$ and $\overline{Z}_1'$ for the left side that satisfy $\overline{X}_1'O(g_1,g_2) = O(g_1,g_2)\overline{X}_1$ and $\overline{Z}_1'O(g_1,g_2) = O(g_1,g_2)\overline{Z}_1$; recall that $O(g_1,g_2)^2 = I$ so that $O(g_1,g_2)^{-1} = O(g_1,g_2)$. The latter is simpler:

$$\overline{Z}_1'O(g_1,g_2) = O(g_1,g_2)\overline{Z}_1 = \left(P_{\theta_1}^{1,2}X_1\right)^{g_1}\left(P_{\theta_1}^{1,2}P_{\theta_2}^{2,3}X_2\right)^{g_2}Z_1 = Z_1(-1)^{g_1}\left(P_{\theta_1}^{1,2}X_1\right)^{g_1}\left(P_{\theta_1}^{1,2}P_{\theta_2}^{2,3}X_2\right)^{g_2}, \quad \text{(B38)}$$

which gives $\overline{Z}_1' = (-1)^{g_1}\overline{Z}_1$. Consider the cases $O(1,0)\overline{X}_1$ and $O(0,1)\overline{X}_1$ separately:

$$\begin{aligned} O(1,0)\overline{X}_1 &= P_{\theta_1}^{1,2}X_1P_{\theta_1}^{1,2}X_1Z_2 = (|0\rangle_1\langle 0|X_2M_{\theta_1}^2X_2 + |1\rangle_1\langle 1|M_{\theta_1}^2)(|1\rangle_1\langle 1|X_2M_{\theta_1}^2X_2Z_2 + |0\rangle_1\langle 0|M_{\theta_1}^2Z_2)\\ &= (|0\rangle_1\langle 0|X_2M_{\theta_1}^2X_2M_{\theta_1}^2Z_2 + |1\rangle_1\langle 1|M_{\theta_1}^2X_2M_{\theta_1}^2X_2Z_2) = Z_2;\\ \overline{X}_1O(1,0) &= P_{\theta_1}^{1,2}X_1Z_2P_{\theta_1}^{1,2}X_1 = P_{\theta_1}^{1,2}X_1P_{\theta_1}^{1,2}X_1Z_2 = Z_2, \end{aligned} \quad \text{(B39)}$$

so that $[\overline{X}_1, O(1,0)] = 0$;

$$\begin{aligned} O(0,1)\overline{X}_1 &= P_{\theta_1}^{1,2}\left(P_{\theta_2}^{2,3}X_2P_{\theta_1}^{1,2}X_1\right)Z_2\\ \overline{X}_1O(0,1) &= P_{\theta_1}^{1,2}X_1Z_2P_{\theta_1}^{1,2}P_{\theta_2}^{2,3}X_2 = -P_{\theta_1}^{1,2}\left(X_1P_{\theta_1}^{1,2}P_{\theta_2}^{2,3}X_2\right)Z_2, \end{aligned} \quad \text{(B40)}$$

so that one need only compare the terms in parentheses:

$$\begin{aligned}P^{2,3}_{\theta_2}X_2P^{1,2}_{\theta_1}X_1 &= \big(|0\rangle_2\langle 1|X_3M^3_{\theta_2}X_3 + |1\rangle_2\langle 0|M^3_{\theta_2}\big)\big(|0\rangle_1\langle 1|X_2M^2_{\theta_1}X_2 + |1\rangle_1\langle 0|M^2_{\theta_1}\big)\\ &= (\tan\theta_1|0\rangle_1\langle 1| + \cot\theta_1|1\rangle_1\langle 0|)|0\rangle_2\langle 1|X_3M^3_{\theta_2}X_3 + (\cot\theta_1|0\rangle_1\langle 1| + \tan\theta_1|1\rangle_1\langle 0|)|1\rangle_2\langle 0|M^3_{\theta_2};\\ X_1P^{1,2}_{\theta_1}P^{2,3}_{\theta_2}X_2 &= \big(|1\rangle_1\langle 0|X_2M^2_{\theta_1}X_2 + |0\rangle_1\langle 1|M^2_{\theta_1}\big)\big(|0\rangle_2\langle 1|X_3M^3_{\theta_2}X_3 + |1\rangle_2\langle 0|M^3_{\theta_2}\big)\\ &= (\tan\theta_1|0\rangle_1\langle 1| + \cot\theta_1|1\rangle_1\langle 0|)|0\rangle_2\langle 1|X_3M^3_{\theta_2}X_3 + (\tan\theta_1|1\rangle_1\langle 0| + \cot\theta_1|0\rangle_1\langle 1|)|1\rangle_2\langle 0|M^3_{\theta_2},\end{aligned} \tag{B41}$$

which coincide. Therefore $\overline{X}_1' = (-1)^{g_2}\overline{X}_1$. The transformations on $\overline{Z}$ and $\overline{X}$ by the $\mathbb{Z}_2\times\mathbb{Z}_2$ operators are equivalent to conjugation under an effective operator $O_{\rm eff}(g_1, g_2) = \overline{X}^{g_1}\overline{Z}^{g_2}$, where $\overline{X}$ and $\overline{Z}$ for the left boundary are defined in Eq. ((B37). A similar result holds for the right boundary. Therefore $O_{\rm eff}(g_1, g_2)$ has the same form as for the regular cluster state, discussed above; this isn't surprising, as the cluster-state symmetry operators are included in the general form, Eq. (B25). To summarize, the states given by Eq. (17) possess $\mathbb{Z}_2\times\mathbb{Z}_2$ SPT order for all θ_i, albeit one that is generally neither unitary nor on-site.

APPENDIX C: ENTANGLEMENT SPECTRUM

Consider the entanglement spectrum for a bipartition of the state defined by the MPS matrices (12), where the boundary states are defined in Eq. (16), leading to the state given in Eq. (17). Applying Eq. (25) yields the rather intuitive-looking expressions

$$\Phi^R_{\ell,j} = \begin{cases} x_1|0\rangle & (\ell, j) = (n-1, 0);\\ x_2|1\rangle & (\ell, j) = (n-1, 1);\\ \frac{1}{\sqrt{2}}\big(\prod_{k=\ell+1}^{n-1} C^{k,k+1}_{\theta_k}\big)(x_1|0\rangle + x_2|1\rangle)\otimes|+\rangle^{\otimes(n-\ell-2)}\otimes|j\rangle & \ell < n-1,\end{cases} \tag{C1}$$

and

$$\Phi^L_{\ell,j} = \begin{cases} a_L\cos\theta_1|0\rangle + b_L\sin\theta_1|1\rangle & (\ell, j) = (1, 0);\\ a_L\sin\theta_1|0\rangle - b_L\cos\theta_1|1\rangle & (\ell, j) = (1, 1);\\ \frac{1}{\sqrt{2}}\big(\prod_{k=1}^{\ell-1} C^{k,k+1}_{\theta_k}\big)\begin{pmatrix}\cos\theta_\ell & \sin\theta_\ell\\ \sin\theta_\ell & -\cos\theta_\ell\end{pmatrix}|j\rangle\otimes|+\rangle^{\otimes(\ell-2)}\otimes(a_L|0\rangle + b_L|1\rangle) & \ell > 1,\end{cases} \tag{C2}$$

where x_1 and x_2 are defined below Eq. (17). The $\Phi^L_{\ell,j}$ and $\Phi^R_{\ell,j}$ correspond to 2^ℓ-dimensional and $2^{n-\ell}$-dimensional vectors respectively, whose index gets summed over in Eq. (26). A straightforward calculation employing Eqs. (C1) and (C2) yields

$$\begin{aligned}\langle 0|V^R_\ell|0\rangle &= \begin{cases}|x_1|^2 & \ell = n-1;\\ \frac{1}{2}\big[1 + \big(|x_1|^2 - |x_2|^2\big)\prod_{k=\ell+1}^{n-1}\cos(2\theta_k)\big] & \ell < n-1;\end{cases}\\ \langle 0|V^R_\ell|1\rangle &= \langle 1|V^R_\ell|0\rangle = 0;\\ \langle 1|V^R_\ell|1\rangle &= \begin{cases}|x_2|^2 & \ell = n-1;\\ \frac{1}{2}\big[1 - \big(|x_1|^2 - |x_2|^2\big)\prod_{k=\ell+1}^{n-1}\cos(2\theta_k)\big] & \ell < n-1.\end{cases}\end{aligned} \tag{C3}$$

and

$$\begin{aligned}\langle 0|V^L_\ell|0\rangle &= \begin{cases}|a_L|^2\cos^2\theta_1 + |b_L|^2\sin^2\theta_1 & \ell = 1;\\ \frac{1}{2}\big[1 + \big(|a_L|^2 - |b_L|^2\big)\prod_{k=1}^{\ell}\cos(2\theta_k)\big] & \ell > 1;\end{cases}\\ \langle 0|V^L_\ell|1\rangle &= \langle 1|V^L_\ell|0\rangle = \begin{cases}\big(|a_L|^2 - |b_L|^2\big)\cos\theta_1\sin\theta_1 & \ell = 1;\\ \frac{1}{2}\big(|a_L|^2 - |b_L|^2\big)\sin(2\theta_\ell)\prod_{k=1}^{\ell-1}\cos(2\theta_k) & \ell > 1;\end{cases}\\ \langle 1|V^L_\ell|1\rangle &= \begin{cases}|a_L|^2\sin^2\theta_1 + |b_L|^2\cos^2\theta_1 & \ell = 1;\\ \frac{1}{2}\big[1 - \big(|a_L|^2 - |b_L|^2\big)\prod_{k=1}^{\ell}\cos(2\theta_k)\big] & \ell > 1.\end{cases}\end{aligned} \tag{C4}$$

Consider the case of a bulk bipartition, $2 < \ell < n-1$. One obtains

$$V^R_\ell = \frac{1}{2}\begin{pmatrix}1+\alpha & 0\\ 0 & 1-\alpha\end{pmatrix};\quad V^L_\ell = \frac{1}{2}\begin{pmatrix}1+\beta\cos(2\theta_\ell) & \beta\sin(2\theta_\ell)\\ \beta\sin(2\theta_\ell) & 1-\beta\cos(2\theta_\ell)\end{pmatrix}, \tag{C5}$$

where $\alpha := (|x_1|^2 - |x_2|^2)\prod_{k=\ell+1}^{n-1}\cos(2\theta_k)$ and $\beta := (|a_L|^2 - |b_L|^2)\prod_{k=1}^{\ell-1}\cos(2\theta_k)$. The eigenvalues of $V_\ell^R V_\ell^L$ are readily obtained:

$$\lambda_\pm = \frac{1}{4}\left\{1 + \alpha\beta\cos(2\theta_\ell) \pm \sqrt{[1 + \alpha\beta\cos(2\theta_\ell)]^2 - (1-\alpha^2)(1-\beta^2)}\right\}. \tag{C6}$$

Equation (C6) corresponds to the entanglement spectrum of the (unnormalized) state defined by Eqs. (12) and (17).

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