

Hydrodynamics of quantum vortices on a closed surface

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We develop a neutral vortex fluid theory on closed surfaces with zero genus. The theory describes collective dynamics of many well-separated quantum vortices in a superfluid confined on a closed surface. Comparing to the case on a plane, the covariant vortex fluid equation on a curved surface contains an additional term proportional to Gaussian curvature multiplying the circulation quantum. This term describes the coupling between topological defects and curvature in the macroscopic level. For a sphere, the simplest nontrivial stationary vortex flow is obtained analytically and this flow is analogous to the celebrated zonal Rossby-Haurwitz wave in classical fluids on a nonrotating sphere. For this flow the difference between the coarse-grained vortex velocity field and the fluid velocity field generated by vortices is solely driven by curvature and vanishes in the corresponding vortex flow on a plane when the radius of the sphere goes to infinity.

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I. INTRODUCTION

Fluids on curved surfaces exhibit rich phenomena which are absent on a plane. The interplay between geometry, topology and fluid dynamics has been explored extensively in diverse platforms, including quantum Hall liquids [1–6], active matter [7–9], classical fluids [10–13], and soft materials [14–16].

The coupling between geometric potentials induced by curvature and quantum vortices plays an essential role in determining properties of superfluids on a curved surface [17,18]. For a superfluid film, a curved surface is realized by the underlying substrate [18]. Recent experimental advances in Bose-Einstein condensates (BECs) in International Space Station [19] now allow ultracold atomic bubbles [20], providing a promising possibility to investigate a bubble trapped superfluid experimentally. Motivated by the experimental progress, research interests on few body vortex dynamics on curved surfaces have been renewed [21–23], adding different perspectives on a more mathematical treatment of point vortex dynamics on curved surfaces [24–26]. However, the effects of curvature and topology on collective dynamics of quantum vortices remain unexplored, motivating us to consider vortex fluids on curved surfaces. Furthermore, static vortex distributions influenced by curvature remains a challenge [18], especially when the vortex number is large. Examining

stationary solutions of such vortex fluid equations would provide a feasible way to tackle this problem.

A vortex fluid is a coarse-grained model for a system consisting of a large number of point vortices and its dynamical equations describe collective dynamics of well-separated quantum vortices at large scales [27,28]. The theory reveals several emergent properties. For instance, a binary vortex fluid is compressible [28] while a chiral vortex fluid is incompressible [27]; there exists an odd viscous tensor and the circulation quantum plays the role of the nondissipative odd viscosity coefficient. The theory also predicts a universal long-time dynamics of the vorticity distribution in a dissipative superfluid and this prediction has been verified in experiments [29]. However, on a finite region with boundaries, boundary conditions are difficult to incorporate in general, hence a closed surface is a better venue for vortex fluids. Vortex fluids are also closely related to quantum Hall liquids [2] and fractons [30,31].

In this paper we develop a vortex fluid theory on orientable closed surfaces with zero genus. For a closed surface, the total vorticity must vanish and hence we consider binary vortex fluids containing equal number of vortices and antivortices. On a plane, the momentum flux tensor of the vortex fluid contains an emergent odd viscous tensor and a quantum pressure like stress tensor [28], preventing applying the minimal coupling principle directly to derive the covariant vortex fluid equation on a curved surface. We overcome this difficulty by introducing an auxiliary tensor which is mathematically equivalent to the original momentum flux tensor, however, is more readily amenable for applying the minimal coupling substitution. After the minimal coupling substitution and rewriting the equation in terms of the original momentum flux tensor, we obtain the vortex fluid equation on a closed surface in isothermal coordinates. The emergent curvature term plays the role of a source term in the vortex fluid equation and hence

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might be referred to as *curvature anomaly*. The generalized relation between the superfluid velocity field generated by the vortices and the coarse-grained vortex velocity field induces the equation of motion (EOM) of point vortices on closed surfaces, verifying the minimal coupling approach. A connection between the odd viscous tensor and Euler characteristic of the closed surface is obtained. For a sphere, an exact stationary vortex flow solution determined by Gaussian curvature is found, whose vorticity exhibits the profile of a vortex-dipole in spherical coordinates and its velocity distribution has the profile of a Kaufmann vortex in stereographic coordinates. It should be noted that the obtained vortex fluid equation holds also for infinitely large curved surfaces, where the vortex system does not have to be neutral.

II. QUANTUM VORTICES AND VORTEX FLUIDS ON A PLANE

In a superfluid, the circulation of a vortex is quantized in units of circulation quantum $\kappa \equiv 2\pi\hbar/m$ [32], and the vorticity has a singularity at the vortex core \mathbf{r}_i : $\omega(\mathbf{r}) = \nabla \times \mathbf{u} = \kappa\sigma_i\delta(\mathbf{r} - \mathbf{r}_i)$ with sign $\sigma_i = \pm 1$ for singly charged vortices. Here m is the atomic mass and \mathbf{u} is the fluid velocity generated by the vortex at $\mathbf{r} = \mathbf{r}_i$. This quantization arises from the single-valuedness of the macroscopic superfluid wave function. It ensures that the vorticity of a quantum vortex concentrates around the core region in the dynamics, which is not the general case for vortices in classical fluids [33]. Hence when the mean separation between quantum vortices is much larger than the vortex core size ℓ , the point vortex model governs the dynamics of quantum vortices [33–36], provided vortex annihilation can be neglected. In this regime, a superfluid at low temperatures is nearly incompressible.

Let us introduce complex coordinates $z = x^1 + ix^2$, $\partial_z \equiv (\partial_1 - i\partial_2)/2$, $\partial_{\bar{z}} \equiv (\partial_1 + i\partial_2)/2$ and complex velocity $u \equiv u^1 - iu^2$. For a system containing N_+ singly-charged quantum vortices and N_- antivortices, the superfluid velocity u generated by these vortices and the vortex velocity $v_i \equiv d\bar{z}_i/dt$ read

$$u = -\frac{1}{2\pi} \sum_{j=1}^N \frac{ik\sigma_j}{z - z_j}, \quad v_i = -\frac{1}{2\pi} \sum_{j, j \neq i}^N \frac{ik\sigma_j}{z_i(t) - z_j(t)}, \quad (1)$$

where $u = 2i\partial_z\psi$, the stream function $\psi(z) = -\kappa/2\pi \sum_i \sigma_i \log |(z - z_i)/\ell|$, and $N = N_+ + N_-$ is the total number of vortices. The vorticity is $\omega(\mathbf{r}) = \kappa \sum_i \sigma_i \delta(\mathbf{r} - \mathbf{r}_i)$. The above fluid velocity u appears to be a singular solution of incompressible two-dimensional (2D) Euler or Helmholtz equation [37]: $\partial_t\omega + \mathbf{u} \cdot \nabla\omega = 0$, which describes 2D nonviscous incompressible classical fluids.

In the point vortex regime, the slow motion of vortices is nearly decoupled from fast degree of freedom-acoustic modes. In this regime, a large number of well-separated quantum vortices are almost isolated and can be treated as a fluid [27,28]. On a plane, the corresponding hydrodynamical equation is [28]

$$\partial_t(\rho v^\alpha) + \partial_\beta T^{\alpha\beta} + \rho\partial^\alpha p = 0, \quad (2)$$

where $\rho(\mathbf{r}) \equiv \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$ is vortex number density, $\sigma(\mathbf{r}) \equiv \sum_i \sigma_i \delta(\mathbf{r} - \mathbf{r}_i) = \kappa^{-1}\omega$ is vortex charge density, v^α is vortex velocity field defined as $\rho v^\alpha \equiv \sum_i \delta(\mathbf{r} - \mathbf{r}_i)v_i^\alpha$, p is the fluid

pressure, $T^{\alpha\beta} = \rho v^\alpha v^\beta - \Pi^{\alpha\beta}$ is the momentum flux tensor, and

$$\Pi^{\alpha\beta} = -\eta^2\sigma\partial^\beta\left(\frac{1}{\rho}\partial^\alpha\sigma\right) - 8\pi\eta^2\sigma^2\delta^{\alpha\beta} - \sigma\tau^{\alpha\beta} \quad (3)$$

is the emergent Cauchy stress tensor with $\eta = \kappa/8\pi$. This emergent tensor $\Pi^{\alpha\beta}$ is absent in the Euler equation and describes emergent macroscopic effects of discrete quantum vortices. In particular,

$$\tau^{\alpha\beta} = -\eta(\epsilon_\gamma^\alpha\partial^\beta v^\gamma + \epsilon_\gamma^\beta\partial^\alpha v^\gamma) \quad (4)$$

is the nondissipative odd viscous tensor and η is identified as the odd viscosity coefficient. Here $\epsilon_2^1 = 1$, $\epsilon_1^2 = -1$, and $\epsilon_1^1 = \epsilon_2^2 = 0$. The presence of $\tau^{\alpha\beta}$ in Eq. (3) is due to that in a vortex system the parity symmetry is broken, namely $\eta \rightarrow -\eta$ under the parity transformation $(x^1, x^2) \rightarrow (-x^1, x^2)$ or $(x^1, -x^2)$. The odd viscosity effects in 2D fluids are very rich [38,39] and have been investigated in quantum Hall systems [40–43], chiral active matter [44–46], chiral superfluids [47], 2D vortex matter [27,28,48–50] and classical fluids [51,52].

The vortex fluid theory describes emergent collective dynamics of well-separated quantum vortices at large scales and is valid whenever the point vortex model (PVM) Eq. (1) is applicable. Vortex annihilations due to collisions, dissipative vortex dipole decay and boundary loss would modify the PVM. For a low temperature BEC containing many vortices, annihilations due to collisions limit dominantly the PVM approach. However, the collision rate depends on the vortex distribution [28] and for largely vorticity-polarized states, including the vortex shear flow [28], Onsager clustered states [34,54–58], and the enstrophy cascade [53], vortex number losses are negligible.

III. VORTEX FLUIDS ON CLOSED SURFACES

The widely used recipe on deriving laws of physics in curved spacetime from those in flat spacetime is the so-called minimal coupling (MC) principle. For our situation, it means the following substitution:

$$\delta_{\mu\nu} \rightarrow g_{\mu\nu}; \quad \partial_\mu \rightarrow \nabla_\mu, \quad (5)$$

where $g_{\mu\nu}$ is the metric on the surface, and ∇_μ is Levi-Civita covariant derivative. When acting a vector field V^ν ,

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda, \quad (6)$$

where

$$\Gamma_{\mu\lambda}^\nu = \frac{1}{2}g^{\nu\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho}\right) \quad (7)$$

is the connection coefficient-Christoffel symbol. The second covariant derivatives do not commute, namely

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)V^\mu = R_{\nu\alpha\beta}^\mu V^\nu, \quad (8)$$

where $R_{\nu\alpha\beta}^\mu$ is Riemann curvature tensor.

Unless specified, in the following we use isothermal coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = h(x^1, x^2)[(dx^1)^2 + (dx^2)^2], \quad (9)$$

namely, $g_{12} = g_{21} = 0$ and $g_{11} = g_{22} = h(x^1, x^2)$, where $h(x^1, x^2)$ is a positive function and exists locally for 2D surfaces [59]. In isothermal coordinates, calculations are

considerably simplified. For instance, $g^{\alpha\beta} = \delta^{\alpha\beta} h^{-1}$ and $v^\alpha = g^{\alpha\beta} v_\beta = h^{-1} v_\alpha$.

We define the vortex number density and vortex charge density on a curved surface as

$$\rho(x^\mu) = \frac{1}{\sqrt{\det g_{\mu\nu}}} \sum_i \delta(x^\mu - x_i^\mu), \quad (10)$$

$$\sigma(x^\mu) = \frac{1}{\sqrt{\det g_{\mu\nu}}} \sum_i \sigma_i \delta(x^\mu - x_i^\mu). \quad (11)$$

The assumption of absence of vortex annihilation ensures the the following continuity equations:

$$\partial_t \rho + \nabla_\mu J_n^\mu = 0, \quad \partial_t \sigma + \nabla_\mu J_c^\mu = 0, \quad (12)$$

where

$$J_n^\mu = \frac{1}{\sqrt{\det g_{\mu\nu}}} \sum_i \delta(\mathbf{r} - \mathbf{r}_i) v_i^\mu \equiv \rho v^\mu, \quad (13)$$

$$J_c^\mu = \frac{1}{\sqrt{\det g_{\mu\nu}}} \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \sigma_i v_i^\mu \equiv \rho w^\mu, \quad (14)$$

are the currents for number and charge, respectively.

We use Eq. (5) to obtain the relation between u and v on a curved surface from it on a plane [28] :

$$\rho v^\mu = \rho u^\mu - \eta \epsilon_v^\mu g^{\nu\alpha} \nabla_\alpha \sigma, \quad (15)$$

$$\rho w^\mu = \sigma u^\mu - \eta \epsilon_v^\mu g^{\nu\alpha} \nabla_\alpha \rho. \quad (16)$$

Consequently, $\omega_v - \omega = \eta \nabla_\mu (\frac{1}{\rho} \nabla^\mu \sigma)$, where $\omega_v = \epsilon_\alpha^\gamma \nabla_\gamma v^\alpha$ and $\omega = \epsilon_v^\mu \nabla_\mu u^\nu = 8\pi\eta\sigma$. The vortex fluid is compressible and $\nabla_\mu v^\mu = -\eta \epsilon_v^\mu \nabla_\mu (\frac{1}{\rho} \nabla^\nu \sigma) \neq 0$ [60]. Note that here $\epsilon_\alpha^\beta \equiv \epsilon^{\beta\gamma} \epsilon_{\gamma\alpha}$ and $\epsilon_{\gamma\alpha} = \sqrt{\det g_{\mu\nu}} \tilde{\epsilon}_{\gamma\alpha}$ is the Levi-Civita tensor and $\tilde{\epsilon}_{\gamma\alpha}$ is the Levi-Civita symbol. In isothermal coordinates, the tensor ϵ_α^β used here takes the same value as what is introduced previously [below Eq. (4)]. For a scalar f , $\nabla_\alpha f = \partial_\alpha f$, in complex coordinates, Eqs. (15) and (16) become

$$\rho v = \rho u - 2i\eta \frac{1}{h} \partial_z \sigma, \quad (17)$$

$$\rho w = \sigma u - 2\eta i \frac{1}{h} \partial_z \rho. \quad (18)$$

The above relations reveal that the velocity of a vortex at position \mathbf{r} is the fluid velocity excluding the flow generated by the vortex itself at \mathbf{r} . The superfluid velocity field u is irregular at a vortex core and subtracting the pole at the vortex core leads to a regular vortex velocity field v .

There is no solid reason why the MC principle must lead to correct results [61]. Hence justification is needed. To verify Eqs. (17) and (18), let us apply the relation (17), which is for coarse-grained variables, to discrete point vortices. The fluid velocity generated by these point vortices on a closed surface is $u = 2ih^{-1} \partial_z \psi$ with the stream function

$$\psi(z) = 8\pi\eta \sum_i \sigma_i G(z, z_i), \quad (19)$$

where $G(z, z_i)$ is the Green's function satisfying [26]

$$\Delta G(z, z_i) = -\delta_{z, z_i} + 1/\Omega, \quad (20)$$

$\Delta \equiv \nabla^\mu \nabla_\mu$, Ω is the area of the surface and

$$\delta_{z, z_k} \equiv h^{-1} \delta(z - z_k). \quad (21)$$

The fluid velocity at $z = z_k$ is

$$u_{z \rightarrow z_k} = \frac{16\pi\eta i}{h} \left[\sum_{i \neq k} \sigma_i \partial_z G(z, z_i)|_{z=z_k} + \sigma_k \lim_{z \rightarrow z_k} \partial_z G(z, z_k) \right], \quad (22)$$

where the last part is the contribution from the vortex at $z = z_k$ itself and contains a pole. To analyze the last term in Eq. (22), it is useful to isolate the logarithmic singularity of the Green's function [62]:

$$G(z, z_k) = \frac{1}{2\pi} [-\log |z - z_k| + H(z, z_k)], \quad (23)$$

where $H(z, z_k) = H(z_k, z)$ is a regular function. Expanding in a power series in z around z_k , we obtain

$$H(z, z_k) = h_0(z_k) + \frac{1}{2} h_1(z_k) (z - z_k) + \text{H.c.} + \mathcal{O}(|z - z_k|^2) \quad (24)$$

and

$$\begin{aligned} \partial_z H(z, z_k) &= \frac{1}{2} h_1(z_k) + \mathcal{O}(|z - z_k|) = \frac{1}{2} \partial_{z_k} h_0(z_k) \\ &+ \mathcal{O}(|z - z_k|). \end{aligned} \quad (25)$$

Here $h_0(z_k) = H(z_k, z_k)$ and $h_1(z_k) = \partial_z H_{z, z_k}|_{z=z_k}$. Let us now analyze the singular term in $\partial_z \sigma$. Noting that $2/(\pi h) \partial_z \partial_{\bar{z}} \log |z - z_k| = \delta_{z, z_k}$ and rearranging derivatives, we obtain

$$\lim_{z \rightarrow z_k} \partial_z \sigma = -\sigma_k \delta_{z, z_k} \partial_z \log h|_{z=z_k} - 2\sigma_k \frac{1}{z - z_k} \delta_{z, z_k}, \quad (26)$$

where we have used $\partial_{\bar{z}}(1/z) = \pi \delta(\mathbf{r})$. Hence the singular terms $\propto 1/(z - z_k)$ in Eq. (22) and Eq. (26) cancel and the remaining finite part in Eq. (17) gives rise precisely, by recognizing $v(z = z_k) = d\bar{z}_k(t)/dt$ and $\lim_{z \rightarrow z_k} \rho = \delta_{z, z_k}$, the EOM of point vortices on closed surfaces with zero genus [26]:

$$\sigma_k h \frac{d\bar{z}_k(t)}{dt} = 8\pi\eta i \left[2 \sum_{i \neq k} \sigma_k \sigma_i \partial_z G(z, z_i)|_{z=z_k} + \partial_{z_k} R_{\text{robin}}(z_k) \right], \quad (27)$$

where $R_{\text{robin}}(z_k) \equiv (1/2\pi)[h_0(z_k) + \log \sqrt{h(z_k)}]$ is the celebrated Robin function [62].

Note that Eq. (27) holds for infinitely large curved surfaces as well [24], and hence so do Eqs. (17) and (18). For an infinitely large surface, $R_{\text{robin}}(z_k) = (1/2\pi) \log \sqrt{h(z_k)}$. In contrast to the scenario on a plane, on a curved surface the self-energy of a vortex is position dependent and a single vortex may move driven by the geometrical potential (Robin function) [18]. It was not an easy task to obtain the EOM of point vortices on closed surfaces [26]. From the vortex fluid point of view, it is somewhat striking that relation (17) naturally generalized from it on a plane could lead to Eq. (27).

IV. DYNAMICAL EQUATIONS OF VORTEX FLUIDS ON CLOSED SURFACES

The Euler equation on a curved surface can be obtained from its form on a plane applying the MC principle [7,11]:

$$\partial_t u^\alpha + \nabla_\beta \mathcal{T}^{\alpha\beta} = 0, \quad (28)$$

where the momentum flux tensor $\mathcal{T}^{\alpha\beta} = u^\alpha u^\beta + p g^{\alpha\beta}$ (hereafter we set the fluid (mass) density $n = 1$). Unlike the case of Euler equation, we cannot apply the MC principle to Eq. (2) directly. The reason is that there are terms containing second derivatives of vectors in Eq. (2). On a plane, the order of derivatives of these terms are interchangeable, namely: $\partial_\beta \partial^\alpha \partial^\beta \sigma = \partial^\alpha \partial_\beta \partial^\beta \sigma$ and $\partial_\beta \partial^\gamma v^\alpha = \partial^\gamma \partial_\beta v^\alpha$. However on a curved surface, $\nabla_\beta \nabla^\alpha \nabla^\beta \sigma \neq \nabla^\alpha \nabla_\beta \nabla^\beta \sigma$, and $\nabla_\beta \nabla^\gamma v^\alpha \neq \nabla^\gamma \nabla_\beta v^\alpha$. At this stage, there is no preferred order for which the MC substitution should be applied.

Our strategy is to search for another tensor $Q^{\alpha\beta}$ such that

- (1) it does not contain derivatives of vectors;
- (2) $\partial_\beta T^{\alpha\beta} = \partial_\beta Q^{\alpha\beta}$.

To do so, it is convenient to use complex coordinates, in which, Eq. (2) becomes

$$\partial_t(\rho v) + \partial_z T_{z\bar{z}} + \partial_{\bar{z}} T + \rho \partial_z(2p) = 0, \quad (29)$$

where

$$T = \rho v v + 4\eta^2 \sigma \partial_z \left(\frac{1}{\rho} \partial_z \sigma \right) - 4i\eta \sigma \partial_z v, \quad (30)$$

$$\begin{aligned} T_{z\bar{z}} &= \rho v \bar{v} + 16\pi \eta^2 \sigma^2 + 4\eta^2 \sigma \partial_z \left(\frac{1}{\rho} \partial_z \sigma \right) \\ &= \rho v \bar{v} + 4i\eta \sigma \partial_z v - 4\eta^2 \sigma \partial_z \left(\frac{1}{\rho} \partial_z \sigma \right). \end{aligned} \quad (31)$$

In the last step we have used $\partial_z u = -4\pi i \eta \sigma$ and $u = v + 2i\eta \partial_z \sigma / \rho$ [28].

Let us now define

$$Q_{z\bar{z}} \equiv \rho v \bar{v} - 4i\eta v \partial_z \sigma + 4\eta^2 \frac{1}{\rho} \partial_z \sigma \partial_z \sigma, \quad (32)$$

$$Q \equiv \rho v v + 4i\eta v \partial_z \sigma - 4\eta^2 \frac{1}{\rho} \partial_z \sigma \partial_z \sigma. \quad (33)$$

Clearly condition (1) is satisfied. Since $T_{z\bar{z}} - Q_{z\bar{z}} = 4i\eta \partial_z(\sigma v) - 4\eta^2 \partial_z[(\sigma/\rho) \partial_z \sigma]$ and $T - Q = -4i\eta \partial_z(\sigma v) + 4\eta^2 \partial_z[(\sigma/\rho) \partial_z \sigma]$, it is easy to verify that $\partial_z Q_{z\bar{z}} + \partial_{\bar{z}} Q = \partial_z T_{z\bar{z}} + \partial_{\bar{z}} T$ which is the complex form of condition (2). Hence $Q^{\alpha\beta}$ defined in Eqs. (32) and (33) is the tensor we search for.

It is now ready to apply the MC principle to obtain the vortex fluid equation on a closed surface :

$$\partial_t(\rho v^\alpha) + \nabla_\beta Q^{\alpha\beta} + \rho \nabla^\alpha p = 0, \quad (34)$$

where

$$Q^{\alpha\beta} = \rho v^\alpha v^\beta + 2\eta v^\alpha \epsilon_\mu^\beta \nabla^\mu \sigma + \eta^2 \frac{1}{\rho} \epsilon_\mu^\alpha \epsilon_\nu^\beta \nabla^\mu \sigma \nabla^\nu \sigma \quad (35)$$

and the pressure p is determined by $\nabla_\mu(u^\nu \nabla_\nu u^\mu) = -\nabla_\mu \nabla^\mu p$.

It is crucial that the momentum flux tensor includes the odd viscous tensor $\tau^{\alpha\beta}(\partial_\mu \rightarrow \nabla_\mu)$. For this purpose, we need to write the dynamical equation in terms of $T^{\alpha\beta}$:

$$\partial_t(\rho v^\alpha) + \nabla_\beta T^{\alpha\beta} + \rho \nabla^\alpha p = \eta K \left(\eta \frac{\sigma}{\rho} \nabla^\alpha \sigma - 2\sigma \epsilon_\beta^\alpha v^\beta \right), \quad (36)$$

where

$$T^{\alpha\beta} = \rho v^\alpha v^\beta + \eta^2 \sigma \nabla^\beta \left(\frac{1}{\rho} \nabla^\alpha \sigma \right) + 8\pi \eta^2 \sigma^2 g^{\alpha\beta} + \sigma \tau^{\alpha\beta}, \quad (37)$$

$K = R_{1212} / \det g_{\mu\nu} = R_{1212} / h^2$ is Gaussian curvature. Here we have used $\epsilon_\nu^\mu \nabla_\mu u^\nu = 8\pi \eta \sigma$ and Eq. (15).

Comparing to Eq. (2), the conspicuous feature of Eq. (36) is that the combination of Gaussian curvature and the circulation quantum/odd viscosity plays the role of the coefficient of a source term. The presence of this additional term might be referred to as *curvature anomaly*. The momentum flux tensor $T^{\alpha\beta}$ is not symmetric for binary vortex fluids and it cannot be symmetrized in the usual way due to that its antisymmetric part

$$T^{12} - T^{21} = \eta^2 \sigma h^{-1} \nabla_\mu v^\mu \quad (38)$$

is not a total divergence. The hydrodynamics equation (37) is invariant under the following scaling transformation $x \rightarrow \lambda x$, $t \rightarrow \lambda^2 t$, $\rho \rightarrow \lambda^{-2} \rho$, $\sigma \rightarrow \lambda^{-2} \sigma$, $v^\alpha \rightarrow \lambda^{-1} v^\alpha$, $K \rightarrow \lambda^{-2} K$, $p \rightarrow \lambda^{-2} p$. The vortex core size ℓ plays the role of the ultraviolet cut-off of the hydrodynamics theory.

Since the odd viscous tensor $\tau^{\alpha\beta}$ is of fundamental importance and appears in a large class of fluids [39], it is worthwhile exploring its properties on a curved surface. From the definition of $\tau^{\alpha\beta}$, one obtains

$$\Delta v_\alpha \nabla_\beta \tau^{\alpha\beta} = -\eta K \epsilon_\beta^\alpha v^\beta \Delta v_\alpha. \quad (39)$$

For a closed orientable surface, due to Gauss-Bonnet theorem, we have

$$\int ds \frac{\Delta v_\alpha \nabla_\beta \tau^{\alpha\beta}}{\epsilon_\beta^\alpha v^\beta \Delta v_\alpha} = -\eta \int ds K = -2\pi \eta \chi(\mathcal{M}), \quad (40)$$

where $\chi(\mathcal{M}) = 2(2 - g)$ is Euler characteristic, and g is the genus of the surface. It should be noted that Eq. (40) holds for any value of g . Connecting Eq. (40) to physical observables deserves future investigations. The hydrodynamic equation (36) can be verified by substituting Eqs. (15) and (16) into Eq. (28). It should be mentioned that chiral vortex fluids have been studied on closed surfaces [49], where additional vorticity has to be introduced to ensure zero total vorticity.

V. VORTEX FLOW ON A SPHERE

We consider vortex fluids on a sphere embedded in \mathbb{R}^3 . We introduce the Cartesian coordinates

$$\xi = R \sin \theta \cos \phi, \quad \eta = R \sin \theta \sin \phi, \quad \zeta = R \cos \theta, \quad (41)$$

where R is the radius, θ is the polar angle and ϕ is the azimuthal angle. On a sphere, stereographic coordinates $z = x^1 + ix^2$ are isothermal coordinates and are related to the

spherical coordinates by $z = R \tan(\theta/2)e^{i\phi}$ (projection from the south pole). In terms of z , the Riemannian metric reads

$$h = \frac{4R^4}{(R^2 + |z|^2)^2} \quad (42)$$

and in spherical coordinates

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2. \quad (43)$$

A. Conserved quantities

It is known that for point vortices on a sphere, the quantities

$$L_{\xi} = \kappa R^2 \sum_j \sigma_j \sin \theta_j \cos \phi_j, \quad (44)$$

$$L_{\eta} = \kappa R^2 \sum_j \sigma_j \sin \theta_j \sin \phi_j, \quad (45)$$

$$L_{\zeta} = \kappa R^2 \sum_j \sigma_j \cos \theta_j, \quad (46)$$

are conserved [25]. In terms of collective variables,

$$L_{\xi} = \kappa R^2 \int ds \sigma \sin \theta \cos \phi, \quad (47)$$

$$L_{\eta} = \kappa R^2 \int ds \sigma \sin \theta \sin \phi, \quad (48)$$

$$L_{\zeta} = \kappa R^2 \int ds \sigma \cos \theta. \quad (49)$$

These conserved quantities are directly related to the corresponding fluid angular momentum $\int ds \mathbf{r} \times \mathbf{u}$ which is associated with the SO(3) symmetry. In stereographic coordinates, they become

$$L_{\xi} = \kappa R \int dx^1 dx^2 h^{3/2} \sigma x^1, \quad L_{\eta} = \kappa R \int dx^1 dx^2 h^{3/2} \sigma x^2, \quad (50)$$

$$L_{\zeta} = -\frac{\kappa}{2} \int dx^1 dx^2 h^{3/2} \sigma |z|^2 + \frac{\kappa R^2}{2} \int dx^1 dx^2 h^{3/2} \sigma. \quad (51)$$

Then it is easy to notice that, as $R \rightarrow \infty$,

$$L_{\xi}/R \propto P_{x^2} = -\kappa \sum_i \sigma_i x_i^1 = -\kappa \int dx^1 dx^2 \sigma x^1, \quad (52)$$

$$L_{\eta}/R \propto P_{x^1} = \kappa \sum_i \sigma_i x_i^2 = \kappa \int dx^1 dx^2 \sigma x^2, \quad (53)$$

where P_{x^2} and P_{x^1} are components of canonical momentum of vortices on a plane. Also, as $R \rightarrow \infty$,

$$L_{\zeta} \propto L = \kappa \sum_i \sigma_i |\mathbf{r}_i|^2 = \kappa \int dx^1 dx^2 \sigma |z|^2 \quad (54)$$

which is the canonical angular momentum of the point-vortex system on a plane. Hence there is a one-to-one correspondence between conserved quantities on a plane and on a sphere.

It is worthwhile to mention that the enstrophy

$$H \equiv \int ds \omega^2 \quad (55)$$

is conserved in any closed surface with zero genus, as

$$\frac{dH}{dt} = -2 \int ds \omega u^\mu \nabla_\mu \omega = - \int ds \nabla_\mu (u^\mu \omega^2) = 0. \quad (56)$$

However, the symmetry associated with this conservation law is not obvious [63].

B. Stationary vortex flows

For constant vortex density $\rho = \rho_0$ on a surface with constant Gaussian curvature $K = K_0$, the vortex fluid becomes incompressible $\nabla_\mu v^\mu = 0$ and Eq. (36) becomes

$$\partial_t \omega_v + \frac{1}{\rho_0} \epsilon_\alpha^\gamma \nabla_\gamma \nabla_\beta T^{\alpha\beta} = \frac{2\eta K_0}{\rho_0} v^\beta \nabla_\beta \sigma. \quad (57)$$

For a sphere, $K_0 = 1/R^2$, and we find a stationary solution of Eq. (57)

$$\sigma = \rho_0 \frac{K_0^{-1} - |z|^2}{K_0^{-1} + |z|^2}, \quad (58)$$

$$v^1 = -(4\pi\eta\rho_0 - K_0\eta)x^2, \quad v^2 = (4\pi\eta\rho_0 - K_0\eta)x^1. \quad (59)$$

Note that $\sigma(z=0) = \rho_0 = -\sigma(z=\infty)$. For this flow $\tau^{\alpha\beta} = 0$, $L_{\xi} = L_{\eta} = 0$ and $L_{\zeta} = 4/3\pi R^4 \kappa \rho_0$. The modulus of the vortex velocity field is

$$|v| = \sqrt{v_1 v^1 + v_2 v^2} = \frac{2R^2 |4\pi\eta\rho_0 - K_0\eta| |z|}{R^2 + |z|^2}, \quad (60)$$

having the profile of a Kaufmann vortex. For $|z| \ll R$, $|v| \propto |z|$, while $|v| \propto 1/|z|$ for $|z| \gg R$. The maximum value of $|v|$ is reached at $|z| = R$. The anomalous correction to the fluid velocity is

$$v^1 - u^1 = K_0\eta x^2, \quad v^2 - u^2 = -K_0\eta x^1 \quad (61)$$

and its modulus is $(v^1 - u^1)(v_1 - u_1) + (v^2 - u^2)(v_2 - u_2) = hK_0^2 \eta^2 |z|^2 = 4K_0\eta |z|^2 / (R^2 + |z|^2)^2$. The vorticity of the vortex velocity field also has an anomalous correction that is proportional to K_0

$$\omega_v - \omega = -2K_0\eta \frac{K_0^{-1} - |z|^2}{K_0^{-1} + |z|^2}. \quad (62)$$

When $R \rightarrow \infty$, $K_0 \rightarrow 0$, $\sigma \rightarrow \rho_0$ for $z \neq \infty$ (ρ_0 keeps a constant as $R \rightarrow \infty$), and this corresponds to rigid body rotation of a chiral vortex flow on a plane. The oppositely charged vortices accumulate at $z = \infty$. It is important to note that the anomalous corrections, i.e., the differences between v and u (or ω_v and ω), are proportional to curvature and vanish as $K_0 \rightarrow 0$.

It is helpful to express this stationary flow using spherical coordinates, for which $\mathbf{v} = v^\theta \partial_\theta + v^\phi \partial_\phi$ and

$$\sigma = \rho_0 \cos \theta, \quad v^\phi = R(4\pi\eta\rho_0 - K_0\eta), \quad v^\theta = 0. \quad (63)$$

The modulus of the vortex velocity field is

$$|v| = \sqrt{v^\theta v_\theta + v^\phi v_\phi} = R|(4\pi\eta\rho_0 - K_0\eta) \sin \theta|, \quad (64)$$

which vanishes at the poles and reaches the maximum at the equator (see Fig. 1). Since $u^\phi = 4\pi\eta R \rho_0$ and $u^\theta = 0$, we

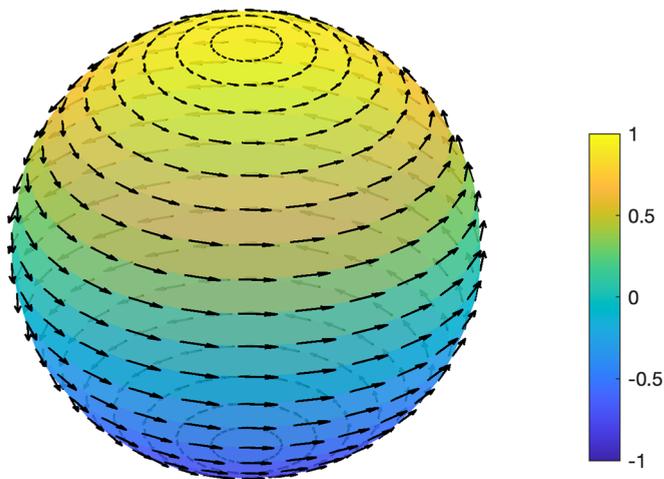


FIG. 1. Schematic of the stationary vortex flow on a sphere. The arrows represent the vortex velocity field \mathbf{v} and the background color shows the renormalized vorticity of the vortex fluid $\omega_v(\theta)/|\omega_v(0)|$.

have $|v - u|^2 = (v^\phi - u^\phi)(v_\phi - u_\phi) = K_0\eta^2 \sin^2 \theta$. The vorticity of the vortex fluid reads

$$\omega_v = 2(4\pi\rho_0 - K_0)\eta \cos \theta. \quad (65)$$

and the correction is $\omega_v - \omega = -2K_0\eta \cos \theta$. Due to compactness of the sphere, $\omega_v(\theta = 0) = -\omega_v(\theta = \pi) = 2(4\pi\rho_0 - K_0)$, the vorticity of this vortex flow has the profile of a vortex-dipole. It is worthwhile mentioning that the vortex flows we found here are analogous to zonal Rossby-Haurwitz flows in Euler fluids on a sphere [64,65], which play an important role in analyzing dynamics of atmosphere of Earth [66–68].

VI. CONCLUSION

We generalize the vortex fluid theory on a plane to closed surfaces with zero genus. The dynamical equation is derived using the minimal coupling principle from it on a flat surface. An additional curvature term emerges and describes the interaction between topological defects and curvature in the hydrodynamical level. Since the vortex fluid equation contains second derivatives of vectors, there is an ambiguity for applying the minimal coupling principle directly. Our method does get over this difficulty and provides a feasible recipe to investigate other complex fluids on curved surfaces. For a sphere, a nontrivial stationary vortex flow is found analytically. It poses a challenge to find analytic solutions for surfaces with non-constant Gaussian curvature. For surfaces with Gaussian curvature weakly depending on positions [$K(x^\mu) = K_0 + \delta K(x^\mu)$ with $\delta K(x^\mu) \ll K_0$], it might be possible to treat $\delta K(x^\mu)$ as perturbations and investigate the effects of nonconstant Gaussian curvature. These are interesting topics which are worthwhile exploring in the future.

The theory developed in this work leads to a broad understanding of the interaction between topological defects and curvature, and provides a theoretical framework for investigating rich phenomena involving a large number of quantum vortices [69–72] in bubble trapped Bose-Einstein condensates [73,74].

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