Penetration of arbitrary double potential barriers with probability unity: Implications for testing the existence of a minimum length

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Quantum tunneling across double potential barriers is studied. With the assumption that the real space is a continuum, it is rigorously proved that large barriers of arbitrary shapes can be penetrated by low-energy particles with a probability of unity, i.e., realization of resonant tunneling (RT), by simply tuning the interbarrier spacing. The results are demonstrated by tunneling of electrons and protons, in which resonant and sequential tunneling are distinguished. The critical dependence of tunneling probabilities on the barrier positions not only demonstrates the crucial role of phase factors but also points to the possibility of ultrahigh accuracy measurements near resonance. By contrast, the existence of a nonzero minimum length puts upper bounds on the barrier size and particle mass, beyond which effective RT ceases. A scheme is suggested for dealing with the practical difficulties arising from the delocalization of particle position due to the uncertainty principle. This work opens a possible avenue for experimental tests of the existence of a minimum length based on atomic systems.

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I. INTRODUCTION

The scenario of a minimum length ($L_{\min}$) plays an essential role in the quantum theory of gravity [1–11]. Breakdown of the Lorentz invariance may happen when the real space approaches such a minimum length scale [12–20], which is generally taken to be the Planck length ($l_P = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \times 10^{-35}$ m, where $\hbar$ is the reduced Planck's constant, G is the gravitational constant, and c is the speed of light) [7]. Despite numerous efforts [1–20], the question remains open regarding the existence of such a minimum length [7,21–23]. Since such a length scale is well below the lower bound of spatial resolution achieved by state-of-the-art instruments such as LIGO ($\sim 10^{-19}$ m) [24,25], it is a great challenge for experimental verification. Here, we show the possibility of tackling this problem by investigations of quantum tunneling across double potential barriers.

Quantum tunneling [26] is a classically forbidden phenomenon in which a particle passes through a potential barrier higher than the energy it possesses. In the early years of quantum mechanics, theories based on quantum tunneling have explained some puzzles of experimental observations like the thermionic and field-induced emission of electrons from metal surfaces [26], and the alpha decay of heavy nuclei [26]. In the ensuing decades, research on the quantum tunneling of electrons in condensed matter has led to fruitful discoveries [27–33] and enabled important inventions such as the scanning tunneling microscope (STM) [34] and tunneling diodes [35–37]. Since the pioneering works by Tsu and Esaki [31], Chang *et al.* [32], and Esaki and Chang [33], double barriers (DBs) have received a lot of attention while studying electron transport in heterostructures [37,38]. Resonant tunneling (RT) typically takes place in DB systems in which the incident electrons may pass through the barriers without being reflected, i.e., with a transmission probability of 100%. This behavior is due to the coherent interference of electron waves which cancel the reflected waves and enhance the transmitted ones, analogous to the resonant transmission through a Fabry-Perot etalon in optics. Typical interbarrier spacing of the devices based on RT is several tens of angstroms, matching the de Broglie wavelengths of electrons. In recent decades, the phenomena of RT in mesoscopic and nanoscale structures have continued to attract research interest [39–43].

Historically, RT of electrons was considered to gain experimental evidence from the negative differential resistance (NDR) found in the current-voltage (*I-V*) curves [32,35,37]. Later, an alternative mechanism was suggested for NDR, namely, sequential tunneling in which the phase memory of electron wave functions is lost due to inelastic scattering [37,44–51]. It was argued that resonant (coherent) tunneling is a prerequisite for sequential tunneling [51]. The effects of external electric field, inelastic scattering, and the repulsive interactions between electrons on RT were also studied [38,52,53]. Despite these efforts, consensus on the underlying physics is yet to be reached. There are still large discrepancies between theory and the measured *I-V* curves (e.g.,

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peak-to-valley ratio). The gap originates partly from the fact that, in calculations related to experiments, the simplest rectangular barriers (or their variants) are adopted, which usually differ significantly from the true barriers felt by electrons.

To resolve the puzzles, exact theoretical description of the conditions for RT across DBs is highly desired. For the simplest rectangular DBs, exact mathematical relation of energy and geometric conditions has been established [38,54,55]. For the more general and realistic situation where DBs are of arbitrary shapes, aside from the semiclassical approach [38], full quantum-level description of the RT conditions is still lacking. It is generally accepted that RT takes place when the energy of incident particle matches the energy levels of the quasibound states within the potential well between the two barriers [37,38,44,49,51,52]. In principle, this applies to electrons as well as the massive particles like protons, atoms, and molecules. Recent simulations have shown the RT of H and He atoms across small DBs, with the barrier height $E_b \sim 0.2$ eV [56,57] and $\sim$0.02 eV [58], respectively. However, when a particle tunnels across arbitrarily shaped DBs, it is unclear how the level-match condition can be reached, and rigorous theoretical descriptions remain elusive.

In this paper, we revisit this topic in DB systems consisting of equal barriers of arbitrary geometries. With the assumption of a continuously varied interbarrier spacing (equivalently, $L_{\min} = 0$), it is rigorously proved that quantum tunneling through the DB system with a probability of unity can always happen (i.e., RT) when the interbarrier spacing is appropriately chosen. Exact mathematical relation for RT is established. At the presence of a nonzero $L_{\min}$ ($L_{\min} > 0$), the interbarrier spacing varies discontinuously, which sets upper bounds for the barrier heights and particle mass, above which no RT may happen. The results are demonstrated by the tunneling of electrons, protons, and some typical bosons. A practically possible scheme is therefore provided for experimental tests of the existence of a nonzero minimum length.

The rest of this paper is organized as follows. Section II presents the analytic and numerical results on RT, with examples of typical particles like electrons, protons, and some bosons. The connection between RT and the continuity of real space is revealed. The constraints set by the existence of a minimum length, the practical obstacles due to the uncertainty principle, and plausible solutions are presented. We conclude in Sec. III with discussions on the impacts and future opportunities inspired by this paper.

II. RESULTS AND DISCUSSIONS

We begin in Part A of this section by performing general analysis on the transmission properties of quantum particles across DBs of arbitrary shapes and prove a theorem which establishes the mathematical condition for RT. Part B provides analyses on two typical models—rectangular and parabolic DBs. The quantum tunneling of electrons and protons is studied and compared, with emphasis on the differences between resonant and sequential tunneling. Based on the results of Parts A and B, we show in Part C the upper bounds of barrier heights for RT set by the Planck length. In Part D, the fundamental limits made by the uncertainty principle are studied, and a possible solution to position delocalization of the incident particles is suggested.

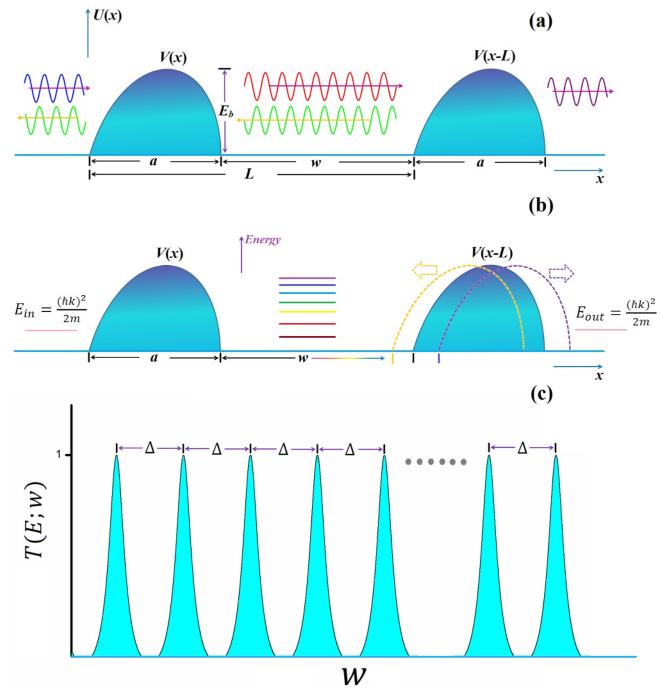

FIG. 1. Schematics of resonant tunneling (RT) across double barriers. (a) Quantum interference of the incident and reflected matter waves. (b) Modulation of the energy levels of the quasibound states between the two barriers, by varying the interbarrier spacing w. (c) RT spectrum as a function of w with a period of Δ ($= \frac{\pi}{k}$).

A. General analyses on arbitrary DBs

Generally, DBs consist of two identical or different single barriers, which are respectively referred to as homostructured and heterostructured hereafter. Unless otherwise stated, the DB considered here is homostructured in one-dimensional space, as shown in Fig. 1(a), with a barrier height E_b and barrier width a for each. Our analyses are based on the transfer matrix method, a powerful technique for studying the transmission properties in finite systems [31,59–62]. For the propagation of a quantum particle across a single barrier $V(x)$ with compact support (the intrinsic property of physical interactions), the transmitted and reflected amplitudes (A_L, B_L; A_R, B_R) of the wave functions (ψ_L, ψ_R) may be related by a transfer matrix (denoted by M) as follows [38,56,59–62]:

$$\begin{pmatrix} A_R \\ B_R \end{pmatrix} = M \begin{pmatrix} A_L \\ B_L \end{pmatrix} \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} A_L \\ B_L \end{pmatrix}. \quad (1)$$

The incoming wave function (with incident energy E) is $\psi_L = A_L e^{ikx} + B_L e^{-ikx}$, and the outgoing wave function is $\psi_R = A_R e^{ikx} + B_R e^{-ikx}$, where $k = \sqrt{2mE/\hbar^2}$, and m is the particle mass. The determinant $|M| = 1$, for systems where time-reversal symmetry preserves, and the transmission coefficient is given by [56] $T = \frac{1}{|m_{11}|^2} = \frac{1}{|m_{22}|^2}$. In general, the matrix elements m_{ij} ($i, j = 1, 2$) are complex numbers and obey the conjugate relations [56,59–62] of $m_{11}^* = m_{22}$ and

$m_{12}^* = m_{21}$. For a homostructured DB with an interbarrier spacing w, the following theorem holds:

Theorem. For any $E < E_b$, the transmission coefficient (tunneling probability) across a homostructured DB $T_{\rm DB}(E;w) = 1$ at $w = w_n = \frac{n\pi}{k} - \frac{\pi+\theta+2ka}{2k}$, where $\theta = \arg(m_{11}^2)$, n (referred to as resonance number) belongs to integers.

Proof. The updated transfer matrix for a single barrier $V(x)$ translated by a distance $L = a + w$, $V(x - L)$, is given by [57,63]

$$M(L) = \begin{pmatrix} m_{11} & m_{12}e^{-i2kL} \\ m_{21}e^{i2kL} & m_{22} \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12}e^{-i2k(a+w)} \\ m_{21}e^{i2k(a+w)} & m_{22} \end{bmatrix}. \tag{2}$$

The transfer matrix for the DB $[U(x) = V(x) + V(x - L)]$ is therefore [56]

$$M_{\rm DB} = M(L)M = \begin{bmatrix} m_{11} & m_{12}e^{-i2k(a+w)} \\ m_{21}e^{i2k(a+w)} & m_{22} \end{bmatrix} \times \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \tag{3}$$

The diagonal matrix element describing the transmission properties $(M_{\rm DB})_{11}$ is explicitly calculated to be $(M_{\rm DB})_{11} = m_{11}^2 + m_{12}m_{21}e^{-i2k(a+w)}$. Let $Z = m_{11}^2 \equiv |Z|e^{i\theta}$, $\phi = 2k(a + w)$, and $m_{12}m_{21} = |m_{12}|^2 = R$, with i the imaginary unit, and the angle $\theta = \arg(Z)$; the determinant $|M| = 1 = |m_{11}|^2 - |m_{12}|^2 = |Z| - R$ gives that $|Z| = 1 + R$, then $(M_{\rm DB})_{11} = (1 + R)e^{i\theta} + Re^{-i\phi} = e^{i\theta} + Re^{i\theta}[e^{-i(\phi+\theta)} + 1]$. When $e^{-i(\phi+\theta)} = -1$, i.e., $\phi + \theta = (2n-1)\pi$, with n being integers, one has $(M_{\rm DB})_{11} = e^{i\theta}$. It follows that the transmission coefficient $T_{\rm DB}(E;w) = \frac{1}{|(M_{\rm DB})_{11}|^2} = 1$, which corresponds to RT. Using the condition $\phi + \theta = (2n-1)\pi$, one has $2k(a + w) + \theta = (2n-1)\pi$, and consequently, $w = \frac{n\pi}{k} - \frac{\pi+\theta+2ka}{2k} \equiv w_n$ This completes the proof of the theorem.

It should be stressed here that the proof inherently includes the precondition that the interbarrier spacing w varies continuously ($L_{\min} = 0$) such that the angle ϕ can have any desired value to satisfy the RT condition. The theorem points to the possibility of penetration of arbitrarily large (but finite) potential barriers by low-energy particles with a probability of unity. For a quantum particle with incident energy E, it can completely tunnel across a homostructured DB of height E_b when the interbarrier spacing equals w_n described above, even in the case $E \ll E_b$. In addition, one sees that the barrier-barrier separations (w_n) for RT are solely determined by the parameters (θ, a) describing the transmission of single barriers. Physically, the onset of RT is due to the presence of quasibound states between the two barriers whose energy levels match that of the incident particles [37,38,44,49,51,52]. A direct consequence is that any quasibound energy levels ($E \leqslant E_b$) can be realized within the potential well set by the two barriers via simply tuning the interbarrier spacing, as illustrated in Fig. 1(b). Moreover, from its mathematical expression, one sees that $(M_{\rm DB})_{11}$ is the periodic function of w, with a period of $\tau = \frac{\pi}{k}$. For a fixed E, the tunneling probability $T[E;w]$ displays periodic variations with w, showing comblike structures with the resonance peaks positioned at $L_n = a + w_n$, and the distance between any two neighboring peaks is $\Delta = w_n - w_{n-1} = \frac{\pi}{k}$ [Fig. 1(c)]. The value of Δ is just half the de Broglie wavelength of the incident matter wave, indicating the key role of phase factor and quantum interference. Finally, the mathematical expression of w_n implies that there could be infinitely many resonance peaks in free space. The theorem applies rigorously to pointlike particles such as electrons [64,65]. We go on to show that the theorem holds valid for finite-sized particles. For a particle incident along the x direction, the effects due to finite size are determined by the distribution $\rho(x)$ of the physical quantity (e.g., charge) that the barriers (e.g., Coulomb interactions) arise from. Generally, a normalized $\rho(x)$ is subjected to the constraint $\int_{x_1}^{x_2} \rho(x)dx = 1$, with x_1, x_2 being the coordinates of two edge points. By defining the weight averaged center $x_c = \int_{x_1}^{x_2} x\rho(x)dx$, one can introduce the internal coordinate τ with reference to x_c, which is translational invariant: $\tau = x - x_c$, $\tau_1 = x_1 - x_c$, $\tau_2 = x_2 - x_c$, and $\int_{\tau_1}^{\tau_2} \rho(\tau)d\tau = 1$. Given that $V_0(x)$ is the potential barrier felt by a pointlike particle, the true barrier felt by a finite-sized incident particle is given by $V(x) = \int_{\tau_1}^{\tau_2} V_0(x)\rho(\tau)d\tau$. The DB felt by the particle is therefore changed from $U_0(x)[= V_0(x) + V_0(x - L)]$ to $U(x)[= V(x) + V(x - L)]$, with the variation $\delta U(x) = U(x) - U_0(x)$. As a result, the finite-sized effects are self-consistently incorporated into the new DB $U(x)$. It is straightforward that the theorem holds, and the incident particle behaves as a pointlike particle with the coordinate x_c during the process of tunneling across the new DB. The results may be readily extended to two- or three-dimensional systems in that the interaction potentials along the direction of propagation are equivalently described by some effective DBs, by considering the translational invariance in the plane perpendicular to the direction of propagation.

B. Tunneling across typical DBs: RT vs sequential tunneling

For homostructured rectangular DBs, analytic expressions of w_n are available, which enable in-depth understanding of the physics of RT. The matrix element m_{11} describing the transmission across single rectangular barrier (barrier height V_0) may be expressed as follows (Appendix A):

$$m_{11} = 2\gamma e^{-ika}[i(k^2 - \beta^2)\sinh(\beta a) + 2\beta k\cosh(\beta a)], \tag{4}$$

where $k = \sqrt{2mE/\hbar^2}$, $\beta = \sqrt{2m(V_0-E)/\hbar^2}$, and $\gamma = \frac{1}{4\beta k}$. Eq. (4) may be reduced to

$$m_{11} = 2\gamma e^{-ika} \times \sigma e^{i\alpha} = 2\gamma\sigma e^{i(\alpha-ka)}, \tag{5}$$

where $\sigma = \sqrt{A^2 + B^2}$, $A = (k^2 - \beta^2)\sinh(\beta a)$, $B = 2\beta k\cosh(\beta a)$, and the angle $\alpha = \arctan(\frac{A}{B})$. Therefore, $m_{11}^2 = \frac{(A^2+B^2)}{4\beta^2k^2}e^{i2(\alpha-ka)}$. Using the theorem stated above, the angle $\theta = 2(\alpha-ka)$, and then $\theta + 2ka = 2\alpha$. The interbarrier spacing is given by $w_n = \frac{n\pi}{k} - \frac{\pi+2\alpha}{2k}$. It follows that $2kw_n = (2n-1)\pi-2\alpha$, and one arrives at the equality $\tan(2kw_n) = \frac{\delta\tanh(\beta a)}{1-\frac{1}{4}\delta^2\tanh^2(\beta a)}$, where $\delta \equiv (\frac{\beta}{k} - \frac{k}{\beta})$. Alternatively, this equality can be obtained by direct calculation of the squared norm of the diagonal element $|(M_{\rm DB})_{11}|^2$, a function of interbarrier spacing w: The minimum of $|(M_{\rm DB})_{11}|^2$ leads to RT (Appendix B). The equality for $\tan(2kw_n)$ is in line with Ref. [55], which was

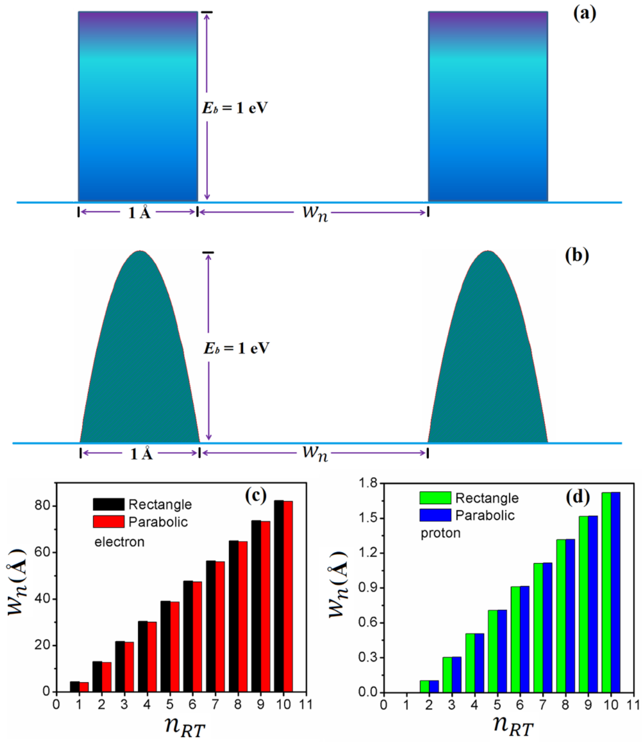

FIG. 2. Schematics of (a) rectangular and (b) parabolic double barriers (DBs). The interbarrier spacing of resonant tunneling (RT), w_n, as a function of resonance number (n_{RT}), for (c) electron and (d) proton across the DBs, at incident energy $E = 0.5$ eV.

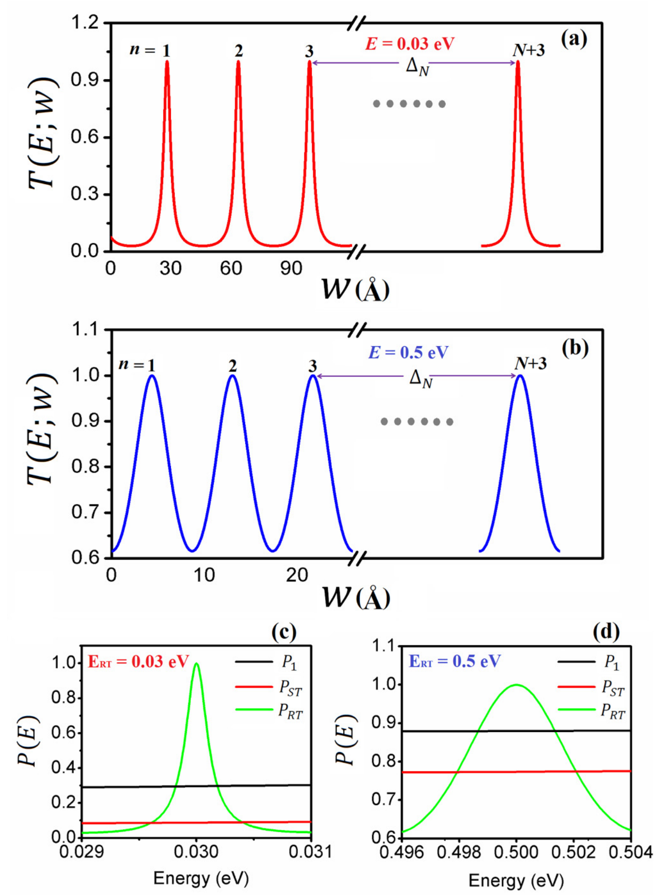

FIG. 3. Tunneling spectrum of electrons across the rectangular double barrier (DB) shown in Fig. 2(a). Resonant tunneling (RT) at resonance level (a) $E_{RT} = 0.03$ eV and (b) $E_{RT} = 0.5$ eV, as a function of interbarrier spacing w. Energy dependence of tunneling probability at fixed w, around (c) $E_{RT} = 0.03$ eV, $w = 100008.49$ Å and (d) $E_{RT} = 0.5$ eV, $w = 100991.45$ Å. The data lines labeled by P_1, P_{ST}, and P_{RT} correspond to tunneling through single barrier, sequential, and RT through the DB, respectively.

derived in a different way. In the special case when the incident energy is half the barrier height ($E = 0.5V_0$), $\beta = k$, and the angle $\alpha = 0$, one obtains a simplified relation that $2kw_n = (2n-1)\pi$, and $w_n = \frac{(n-1/2)\pi}{k} = (n-\frac{1}{2})(\frac{\lambda_d}{2})$, where $\lambda_d = \frac{2\pi}{k}$ is the de Broglie wavelength. In another special case when $k \ll \beta$ and $\beta a \gg 1$, i.e., the incident energy is far below the barrier height, one has $\alpha \cong -\frac{\pi}{2} + \frac{k}{2\beta}$ and $kw_n \cong n\pi - \frac{k}{2\beta}$, $w_n \cong \frac{n\pi}{k} - \frac{1}{2\beta}$. In both situations, the value of w_n is independent of the barrier width.

The results are applicable to electrons and other massive quantum particles. However, demonstration of the quantum interference effects leading to RT would be much more challenging due to the large difference in particle masses and the corresponding de Broglie wavelengths. Here, we perform systematic investigations on the RT characteristics of electrons and protons through two model systems: rectangular and parabolic DBs (Fig. 2). Compared with the analytic expressions for rectangular barriers, the transfer matrices for parabolic barriers are evaluated numerically [56,57]. Figures 2(c) and 2(d) show the calculated w_n for the RT of electrons and protons across the two types of DBs, as a function of the resonance numbers (n_{RT}). For the same n_{RT}, the w_n of electrons is much larger than that of proton owing to smaller mass. The different geometries of the potential barriers (rectangular vs parabolic) are reflected by the slight differences of w_n. Despite the differences, the overall comparable magnitudes of the two sets of w_n indicate that rectangular DBs may serve as approximations for qualitative description of some smoothly varying DBs with regular geometries.

At fixed energy E, the tunneling probability varies periodically with interbarrier spacing w. We have further studied such characteristics in the case of electrons and protons tunneling through rectangular DBs. Figures 3(a) and 3(b) show the transmission of electrons, at varying w for $E = 0.03$ and 0.5 eV. The effects of incident energy on the tunneling spectrum $T(E; w)$ are clearly seen. Higher energy not only results in a smaller period of oscillation ($\tau = \frac{\pi}{k}$) but also smaller peak-to-valley ratio. The resonance number can extend to very large integers if the perturbation from the environment is negligible and the coherence of wave functions is maintained. To show the role of coherence, we have studied the energy-dependent tunneling probability $P(E)$ of electrons at a fixed interbarrier spacing ($w \sim 10$μm). For resonant (coherent) tunneling, the quantity $P(E)$ [$= T(E; w)$] drops quickly with small deviations from the resonant energy level E_{RT}. For sequential tunneling, in which the phase coherence is destroyed in a two-step tunneling process, the quantity $P(E)$ is simply the product of the transmission coefficient across each single barrier: $P(E) = T_1(E)T_2(E) = T_1^2(E)$. Around the

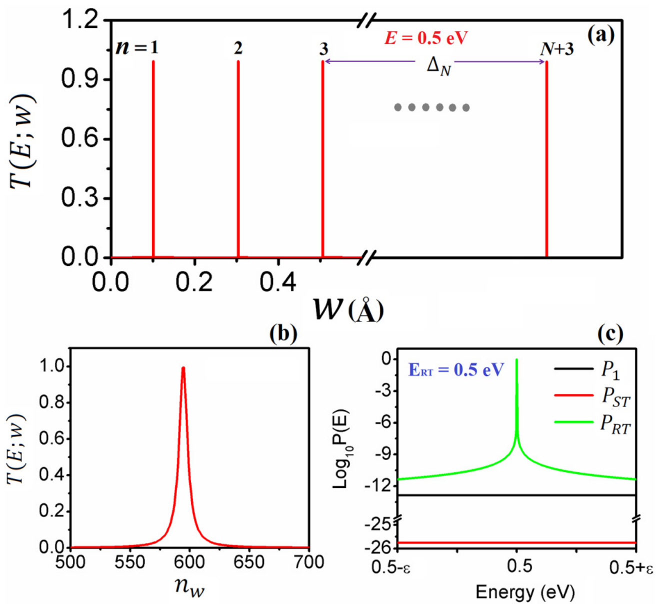

FIG. 4. Tunneling spectrum of protons across the rectangular double barrier shown in Fig. 2(a). Resonance at (a) $E_{\rm RT} = 0.5$ eV. Variations of tunneling probability with respect to small deviations from the resonant tunneling (RT) parameters: (b) Interbarrier spacing $w = (n_w - n_p) \times \Delta l + w_n$, $n_p = 596$, and $\Delta l = 10^{-15}$ Å; (c) Incident energy at the vicinity of $E_{\rm RT}$, for $w = 20.137016632763302$ Å, and the energy deviation $\varepsilon = 10^{-10}$ eV. All digits of w are meaningful.

resonant energy $E_{\rm RT}$ [Figs. 3(c) and 3(d)], the probability of sequential tunneling ($P_{\rm ST}$) changes smoothly with energy and is significantly smaller than unity at low incident energies.

For protons, more radical differences are encountered. Shown in Fig. 4(a) is the tunneling spectrum of protons at $E = E_b/2 = 0.5$ eV. The periodically repeated isolated lines imply much narrower resonant peaks with comparison with electrons. The enlarged structures of one resonant peak are shown in Fig. 4(b). Around the RT peaks, the squared norm of transfer matrix element may be expressed as follows (Appendix C):

$$|(M_{\rm DB})_{11}|^2 \cong 1 + \sinh^2(2ka)(k\Delta w)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta w}, \quad (6)$$

where Δw is the deviation from the peak position w_n. When $\Delta|M_{11}|^2_{\Delta w} = 1$, $T(E;\, w) = 0.5$, and one has

$$|\Delta w| = \frac{1}{k\sinh(2ka)}. \quad (7)$$

It follows that the term $2|\Delta w|$ is the full width at half maximum (FWHM) of the resonant peaks. For the DB [Fig. 2(a)] considered here, it turns out that $|\Delta w| \cong 4.235 \times 10^{-15}$ Å. When the deviation $\Delta w \sim 10^{-13}$ Å, the tunneling probability drops quickly to $T(E;\, w) \sim 10^{-3}$, in good agreement with the results presented in Fig. 4(b). In general, given that w is an approximate value to w_n, one can determine the significant digits of w by designating a deviation Δw when $|w - w_n| \leqslant \Delta w$ such that $T(E; w) \geqslant 1-\delta P$, where δP ($0 < \delta P < 1$) is the tolerance of decrease in tunneling probability at which significant tunneling (effective RT, events measurable in experiment) is maintained. Furthermore, using the proof of the theorem, we find that, for arbitrarily homostructured DBs, the deviation Δw at the tolerance δP may be given by (Appendix D)

$$\Delta w = \frac{1}{2k}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}}, \quad (8)$$

where $R = |m_{12}|^2$, k, and δP are defined as above. Here, we focus on the case that $\delta P = 0.5$, which yields the FWHM ($= 2\Delta w$). Indeed, the high sensitivity on barrier positions has been revealed by studies on RT transducers [66], which are based on electronic DB systems. Later, it was theoretically proposed that a tunneling electromechanical transducer may be employed to dynamically detect the Casimir forces between two conducting surfaces [67].

Such ultrahigh sensitivity on tunneling parameters is also found for the RT energies. Figure 4(c) compares the tunneling of protons across single and DBs at a fixed interbarrier spacing ($w \sim 20$ Å). Near resonance, $P(E)$ descends drastically from 1 to $\sim 10^{-11}$ by a tiny shift of $\varepsilon = 10^{-10}$ eV from $E_{\rm RT}$. At the vicinity of $E_{\rm RT}$, the dependence of $|(M_{\rm DB})_{11}|^2$ with deviation ΔE is given by (Appendix C) $|(M_{\rm DB})_{11}|^2 \cong 1 + \sinh^2(2ka)(\frac{kw}{2})^2(\frac{\Delta E}{E})^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta E}$. The FWHM at energy scale is therefore obtained when $\Delta|M_{11}|^2_{\Delta E} = 1$, and $|\frac{\Delta E}{E}| = \frac{2}{(kw)\sinh(2ka)}$. In our case, $|\frac{\Delta E}{E}| \approx 4.235 \times 10^{-16}$. When the energy broadening $\Delta E = \varepsilon = 10^{-10}$ eV, $|\frac{\Delta E}{E}| = 2 \times 10^{-10}$, $|(M_{\rm DB})_{11}|^2 \cong 2.23 \times 10^{11}$, and $P(E) = |(M_{\rm DB})_{11}|^{-2} \approx 10^{-11.35}$ compare well with the numerical results. Without resonance, the probability of a two-step tunneling, i.e., sequential tunneling decreases by >25 orders of magnitude [Fig. 4(c)]. The sharp contrast distinguishes RT from sequential tunneling. For the more generalized case, the allowed energy broadening ΔE may be calculated as follows (Appendix D):

$$\left|\frac{\Delta E}{E}\right| = \frac{1}{k(a+w)}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}}. \quad (9)$$

In the case of $R \gg 1$ (large reflection), for instance, tunneling through large barriers or tunneling by massive particles, Eq. (9) is reduced to $|\frac{\Delta E}{E}| \cong \frac{1}{k(a+w)R}\sqrt{\frac{\delta P}{1-\delta P}}$. For a single barrier $V(x)$, the reflection and tunneling probabilities (Appendix D) are related to R by $|r|^2 = R|t|^2 \equiv RT_1(E)$, subjected to the condition $|r|^2 + |t|^2 = 1$. It is straightforward that $R = \frac{1}{|t|^2} - 1 \cong \frac{1}{|t|^2} = \frac{1}{T_1(E)}$ when $R \gg 1$, where $T_1(E)$ is the tunneling probability across a single barrier.

As seen from Fig. 4(c), at the absence of phase coherence, the incident protons will be nearly completely reflected by a single barrier. On the contrary, when the interbarrier spacing equals w_n and phase coherence is maintained, the protons penetrate the two barriers with a probability of unity. Such effects are schematically illustrated in Fig. 5. The key role of quantum interference is demonstrated. Experimental verification may be carried out using atomically thin membranes, which have potential applications as proton sieve filters. Generally, the variation step (Δl) of the interbarrier spacing w_n required by RT should be the order of magnitude of Δw studied above and

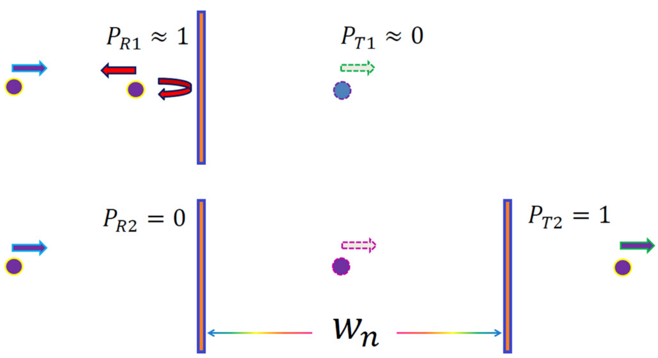

FIG. 5. Schematics of quantum tunneling of protons across single barrier (upper panel) and double barriers (lower panel) at the presence of resonant tunneling (RT). The probability of reflection is denoted by P_{Ri} and tunneling by P_{Ti}, $i = 1, 2$.

no less than the minimum length (i.e., $\Delta l \sim \Delta w \geqslant L_{\min}$) such that effective RT can be reached by tuning the interbarrier spacing. This is the topic of next subsection.

To reveal the effects of asymmetry, we have studied the transmission properties of heterostructured rectangular DBs, where the barrier width and height of the first and second barriers are a, V_1 and b, V_2, respectively. Figures 6(a) and 6(b) show, respectively, the calculated transmission coefficients of electron with an incident energy of $E = 0.1$ eV, at varying ratios of barrier widths (b/a) and barrier heights (V_2/V_1) between the two constituent barriers. In both cases, very similar dependence of tunneling with the barrier asymmetry is observed, and full transmission [$T(E) = 1$] takes places only for the homostructured case, where $a = b = 1$ Å and $V_1 = V_2 =$ 1eV. Generally, the total probability of coherent transmission across a rectangular DB may be approximately given by $T \approx 4T_1T_2/(T_1 + T_2)^2$ [68], where T_1 and T_2 are the transmission coefficients through each single barrier, respectively. We go further to study the effects of asymmetry on resonance levels, i.e., the quasibound states between the two barriers [Fig. 1(b)]. Figures 6(c) and 6(d) compare, respectively, the tunneling spectra for electrons and protons across a homostructured rectangular DB and the corresponding slightly distorted one. It is clearly seen that the distortions in both barrier width and height not only lead to significant changes in the tunneling probability but also induce small shifts on the resonance peak/level positions.

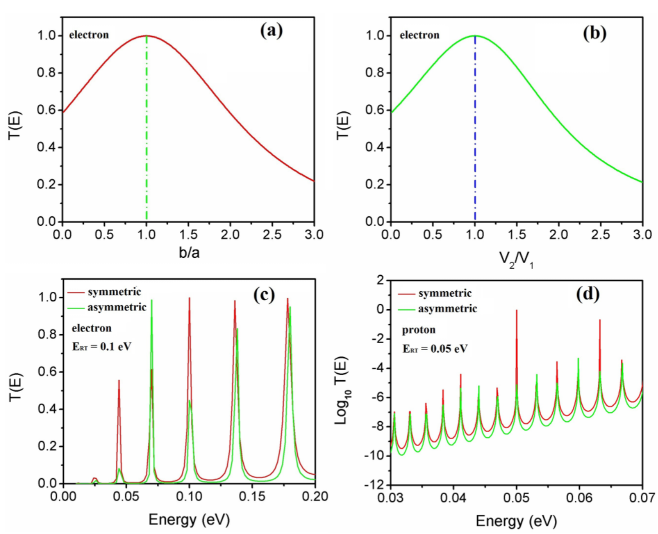

FIG. 6. Tunneling properties of (a)–(c) electrons and (d) protons across homostructured (symmetric) and heterostructured (asymmetric) rectangular double barriers (DBs). (a) Variation of transmission coefficient $T(E)$ with the barrier width ratio b/a, at barrier height $V_1 = V_2 = 1$ eV, and the interbarrier spacing $w = 109.995$ Å. (b) Variation of $T(E)$ with the barrier height ratio V_2/V_1, at barrier width $a = b = 1$ Å, and the interbarrier spacing $w = 109.995$ Å. (c) Tunneling spectrum of electron across symmetric and asymmetric DBs with given interbarrier spacing $w = 111.500$ Å, where the parameters $a = b = 2$ Å, $V_1 = V_2 = 1$ eV for the symmetric DB, and the changes of $\delta a = 0.2$ Å, $\delta b = 0.1$ Å, $\delta V_1 = 0.1$ eV, and $\delta V_2 = 0.2$ eV for the asymmetric one. (d) Like (c) but for proton tunneling, where the interbarrier spacing $w = 20.1596$ Å, $a = b = 1$ Å, and $V_1 = V_2 = 0.1$ eV for the symmetric one, and the changes $\delta a = 0.05$ Å and $\delta V_1 = 0.005$ eV for the asymmetric DB. For (a)–(c), T(0.1 eV) = 1, and for (d), T(0.05 eV) = 1.

For small changes of barrier structures induced by external forces or physical fields such as Casimir forces, finite temperatures, and the gravitational waves, the effects on the tunneling properties may be equivalently ascribed to perturbations on the constituent barriers and be treated within a unified framework. For the special case of resonance at half barrier height ($E_{\mathrm{RT}} = 0.5E_b$), like the derivation of Eqs. (6) and (C12), the dependence of $|(M_{DB})_{11}|^2$ on the more generalized perturbations to DB structure may be similarly deduced by using Taylor series to the second order as below:

$$|(M_{\mathrm{DB}})_{11}|^2 \cong 1 + \sinh^2(2ka)\left(\frac{kw}{2}\right)^2\left(\frac{\Delta E}{E}\right)^2 + \sinh^4(ka)\left[\left(\frac{\Delta V_1}{V_0}\right)^2 + \left(\frac{\Delta V_2}{V_0}\right)^2\right] + \sinh^2(2ka)[(k\Delta w)^2 + (k\Delta x_1)^2 + (k\Delta x_2)^2], \tag{10}$$

where the terms ΔE, ΔV_1, ΔV_2, Δw, Δx_1, and Δx_2 are the small magnitude of changes in the resonant energy E, in the barrier height (V_0) of the first and second barriers, in the interbarrier spacing (w), and in the barrier width (a) of the first and second barriers, respectively. In the following, we go further to evaluate separately the effects of Casimir forces and gravitational waves on the tunneling properties across rectangular DBs.

Casimir forces are attractive interactions which exist between two parallel neutral conducting plates due to the fluctuation of zero-point energy (ZPE) of the electromagnetic field in vacuum. The energy change per unit surface area may be calculated as follows [69]: $\Delta E_0(d) = -\frac{\pi^2}{720}\frac{\hbar c}{d^3}$, where d is the interplate spacing, $\hbar$ is the reduced Planck's constant, and c is the speed of light. Casimir forces typically operate in the range of submicron to micron [67,69]. For a DB with cross-section area A, the potential change at a distance of x due to Casimir effect is given by $\Delta V_0(x) = -A\frac{\pi^2}{720}\frac{\hbar c}{x^3}$, where $w_n \leqslant x \leqslant w_n + a + b$. In this paper, the interbarrier spacing $w_n \gg (a + b)$, and the correction to barrier height can therefore be taken as constant, which is $\Delta V_0(w_n) = -A\frac{\pi^2}{720}\frac{\hbar c}{w_n^3}$. Given that

TABLE I. Energetic and geometric parameters for the RT of electrons across homostructured rectangular DBs, and the decrease of transmission probability (δP) due to the strain ($h = 1 \times 10^{-21}$) induced by gravitational waves. For all the DBs, resonance takes place at $E_{\rm RT} = 0.5E_b$. The number of digits are set by $|\Delta w|$, $|\Delta E|$ when $T(E_{\rm RT}) = 0.5$.

n	χ_n	E_b (eV)	w_n (Å)	a (Å)	δP
1	45.5375	1	4.336	10	1.091×10^{-32}
		1	4.3360717180254546576	62.8515175683137030659	0.5
		39.5031326064004559839	0.6898913321086561634	10	0.5
10	45.2686	1	82.385	10	1.872×10^{-31}
		1	82.3853626424836420483	62.4803778905949016575	0.5
		39.0379762135153995928	13.1079353100644677709	10	0.5
100	43.2978	1	862.878	10	1.941×10^{-29}
		1	862.8782718870654662168	59.7602511637514837162	0.5
		35.7128761915465986476	144.3900008925093061407	10	0.5

$A \sim w_n^2$, then $\Delta V_0(w_n) = -\frac{\pi^2}{720}\frac{\hbar c}{w_n}$. It follows that $\Delta V_0 \cong 2.7$ and 27 meV, respectively, for $w_n = 1$ and 0.1μm, respectively. The effects of such a small magnitude of variations are usually negligible for electron tunneling through the DBs studied in this paper [Figs. 6(a)–6(c)], except for the situation when the resonance energy ($E_{\rm RT}$) is well below the barrier height [see, e.g., Fig. 3(c)] in which the effects of external perturbations cannot be neglected. Although practically challenging, the low-energy RT of electrons points to a possible experimental scheme of detecting the Casimir effect.

Finally, we evaluate the effects of gravitational waves, which are expected to induce expansion or contraction in the geometric size of a DB. Assuming that the strain induced by gravitational waves across the DB region is uniform, then the size changes of a homostructured rectangular DB as defined in Eq. (10) are given by $\Delta w = h \times w$, $\Delta x_1 = \Delta x_2 = h \times a$, where h is the gravitational-wave strain amplitude projected on the tunneling direction, and w and a are, respectively, the interbarrier spacing and barrier width. The peak value of h is adopted in our calculation, which is 1×10^{-21} [70]. A passing gravitational wave causes modification on the tunneling probability $T(E)$. In the case of $E_{\rm RT} = 0.5E_b$, the inverse of $T(E)$ may be deduced using Eq. (10) as follows:

$$|(M_{\rm DB})_{11}|^2 = 1 + \sinh^2(2ka)h^2[(kw_n)^2 + 2(ka)^2]. \quad (11)$$

The significant change of $T(E)$, in particular $T(E) = 0.5$, i.e., $\delta P = 0.5$ yields $|(M_{\rm DB})_{11}|^2 = 2$, which gives that

$$\sinh^2(2ka)h^2[(kw_n)^2 + 2(ka)^2] = 1. \quad (12)$$

The energetic and geometric parameters of a DB for the observation of significant effects induced by gravitational waves are therefore defined by Eq. (12). Let $\chi = 2ka$. Recalling that $2kw_n = (2n-1)\pi$ for resonance at half barrier height, Eq. (12) may be rewritten as

$$\sinh(\chi)\sqrt{[(2n-1)\pi]^2 + 2\chi^2} = 2h^{-1}, \quad (13)$$

where $k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{mE_b}}{\hbar}$. The nth root of Eq. (13) $\chi_n = 2ka$ puts constraint on the barrier height (E_b) and barrier width (a) at interbarrier spacing w_n. For instance, $\chi_1 \cong 45.5375$ at $w_1 \cong 4.33607172$ Å and $E_b = 1$eV require that $a \cong 62.85151757$ Å for a significant probability drop ($\delta P = 0.5$) of electron tunneling across a rectangular DB. By contrast, in a general homostructured rectangular DB, for instance, with $E_b = 1$eV, $a = 10$ Å, and $w_1 \cong 4.336$ Å at a resonant energy of 0.5 eV, only negligible probability drop ($\delta P = 1.091 \times 10^{-32}$) is expected. A similar situation is found for a given barrier width (e.g., $a = 10$ Å) where a barrier height ($E_b \cong 39.5031326$eV) with ultrahigh accuracy is required to get significant probability drop ($\delta P = 0.5$). The results are summarized in Table I, for $n = 1$, 10, and 100. The results indicate that the effects of gravitational waves on RT are usually negligible, except for the specially designed DBs in which the energetic and geometric parameters with ultrahigh accuracy present.

The analyses presented above are applicable to the tunneling of a single particle or the situation when the incident particles can be viewed as independent. Nontrivial modifications on the RT behavior are expected when the interactions among the incident particles are considered. For instance, it has been shown experimentally [71,72] and theoretically [73] that the electrostatic feedback of the space charge dynamically stored in the well of a DB heterostructure produces intrinsic bistability displayed in the I-V curve. Generally, the interactions cause splitting of the energy levels of the incident particles from a δ-function-like distribution to a broadened energy spectrum, which is physically equivalent to the picture that the tunneling particle induces a (self-consistent) modification of the experienced potential barriers.

Let Γ_n be the energy broadening of the nth resonance level (E_n) between the two barriers [Fig. 1(b)]. Near resonance, the transmission probability may be expressed using the Breit-Wigner formula $T(E) \cong \frac{\Gamma_n^2}{(E-E_n)^2+\Gamma_n^2}$. When $\Gamma_n > \Delta E$, the intrinsic FWHM at the energy scale as defined above, the tunneling probability would be enhanced by energy broadening. Equivalently, the inverse of $T(E)$, $|(M_{\rm DB})_{11}|^2$, is reduced. Such an effect due to resonance level broadening may be phenomenologically described as the reduction of the k value in Eq. (6), by replacing k with effective $k' = k-\Delta k$, with $\Delta k = \frac{k}{2}|\frac{\Delta E}{E}| = \frac{k}{2}\frac{\Gamma_n}{E}$ (Appendix C). For resonance at half barrier height $2E = E_b$, one has $\Delta k = k\frac{\Gamma_n}{E_b}$, and $k' = k(1-\frac{\Gamma_n}{E_b})$. Then Eq. (6) is rewritten as follows:

$$|(M_{\rm DB})_{11}|^2 \cong 1 + \sinh^2\left[2k\left(1 - \frac{\Gamma_n}{E_b}\right)a\right]\left[k\left(1 - \frac{\Gamma_n}{E_b}\right)\Delta w\right]^2. \quad (14)$$

Consequently, the deviation Δw which defines the FWHM at length scale is

$$|\Delta w| = \frac{1}{k\left(1-\frac{\Gamma_n}{E_b}\right)\sinh\left[2k\left(1-\frac{\Gamma_n}{E_b}\right)a\right]}. \tag{15}$$

For $\Gamma_n = 0$, the particle-particle interactions are negligible, and there is no level broadening; Eq. (15) is simply reduced to Eq. (7), which defines the intrinsic FWHM of resonance at given energy. For any $\Gamma_n > 0$, it is clearly seen that $|\Delta w|$ is enlarged with comparison with the intrinsic value. Depending on the value of Γ_n, the actual accuracy of the position measurement based on the RT of DBs could be significantly lower than the ideal case where $\Gamma_n = 0$. For the special case when $\Gamma_n \cong E_b$, $|\Delta w|$ would be infinitely large, and there is no restriction on the deviation of interbarrier spacing. This is understandable since $\Gamma_n \cong E_b$ implies that the resonance levels form a continuous spectrum, and RT takes place for any incident energy $E\,(\leqslant E_b)$. At stationary states, the energy spectrum can be described by the density of states $D(E)$, which may be numerically calculated using *ab initio* methods like density functional theory (DFT) calculations or quantum Monte Carlo simulations. The tunneling probability P_t is therefore obtained by integration on the spectrum: $P_t = \int T(E)D(E)dE$.

C. Upper bounds of RT barriers set by the Planck length

The critical dependence of the tunneling probabilities on the barrier positions not only demonstrates the crucial role of phase factors but also points to the possibility of ultrahigh accuracy measurements near resonance. As shown above, the deviation of $|\Delta w| \cong 4.235 \times 10^{-15}$ Å leads to a 50% drop of $P(E)$ of protons across a rectangular DB. Such a deviation is several orders of magnitude below the smallest length scale sensed by LIGO [24,25,70]. Even smaller $|\Delta w|$ is expected for heavier particles or larger barriers. As mentioned above, to have measurable RT within some tolerance δP, an upper bound of deviation from the exact peak position w_n is given by $\Delta w = |w - w_n|$. Suppose that the elementary variation step of distance is Δl; if the real space is a continuum ($L_{\min} = 0$), then Δl can be arbitrarily small, and in principle, w_n can always be reached by a finite number of operations [i.e., $N = [\frac{\Delta w}{\Delta l}]$, for a variation step of Δl]. In this case, the theorem stated above always holds. On the contrary, if a nonzero $L_{\min}$ exits ($L_{\min} > 0$), to have significant RT, the variation step should satisfy $\Delta w \geqslant \Delta l \geqslant L_{\min}$. Let $n = \lfloor \log_{10}(\Delta w) \rfloor$. Then it is a feasible choice to set $\Delta l = 10^n$, such that $\Delta w = N\Delta l$, with N being an integer and the modulo $\Delta w \bmod \Delta l = 0$. Consequently, the existence of a minimum length leads to the inequality of $|\Delta w| \geqslant L_{\min}$ and therefore puts some upper bounds for the particle mass, barrier height, and barrier width above which RT will cease. In the special case of tunneling across rectangular DBs at $E = 0.5E_b$, realization of RT requires that $|\Delta w| = \frac{1}{k\sinh(2ka)} \geqslant L_{\min}$, which may be rewritten as

$$\chi \sinh(\chi) \leqslant \frac{2a}{L_{\min}}, \tag{16}$$

where $\chi = 2ka$, $k = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{\frac{mV_0}{\hbar^2}}$. The upper bound of the term mV_0 is therefore determined for a given barrier with a.

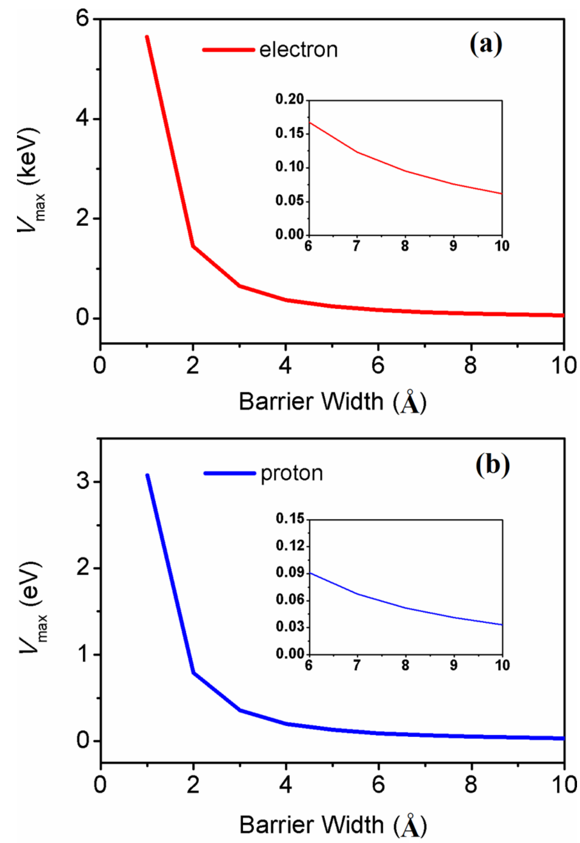

FIG. 7. Calculated upper bounds ($V_{\max}$) of the barrier height of rectangular double barriers set by the Planck length for (a) electrons and (b) protons, as a function of barrier width. The values of $V_{\max}$ at barrier width of 6–10 Å are highlighted in the insets.

Assuming that the minimum length is identical to the Planck length ($L_{\min} = l_P$), the upper bounds of barrier height ($V_{\max}$) for electrons and protons are calculated and shown in Fig. 7. At barrier widths of $a = 1$, 5, and 10 Å, the $V_{\max}$ is ~5652.72, 239.42, and 61.32 eV for electrons and ~3.08, 0.13, and 0.03 eV for protons, respectively. It is seen that $V_{\max}$ of RT decreases fast with increasing barrier width. For instance, when the barrier width increases to $a = 20$ and 30 Å, the value of $V_{\max}$ is respectively ~15.70 and 7.08 eV for electrons, which may be feasible for experimental tests using metal-insulator-metal DBs. Meanwhile, the same variation trend is found for electrons and protons, with the magnitude of $V_{\max}$ being scaled by a factor of $\eta = \frac{m_e}{m_p} \cong \frac{1}{1836}$, where m_e and m_p are, respectively, the mass of electrons and protons. This is due to the conjugate relation that the particle mass times barrier height (mV_0) is a constant at fixed barrier width. For the general case of tunneling through arbitrary DBs, the constraint imposed on the particle mass and barrier size due to a nonzero minimum length is given by (Appendix D)

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}} \geqslant L_{\min}, \tag{17}$$

where $R = |m_{12}|^2$, and δP ($0 < \delta P < 1$) has the same meaning as above. Provided that $L_{\min} = l_P$ and $\delta P = 0.5$, the

TABLE II. Parameters describing the RT of protons across rectangular DBs at $E = 0.5E_b$. With the deviation of ΔE or $|\Delta w|$ from the parameters for resonance, the tunneling probability drops from 1 to 0.5. The corresponding momentum broadening Δp, and the minimum standard deviation of particle positions Δx_m are calculated using the relation $\Delta p \Delta x_m \geqslant \hbar/2$. In all cases the barrier width $a = 1$ Å.

E_b (eV)	w (Å)	ΔE (eV)	$|\Delta w|$ (Å)	Δp (kg m/s)	Δx_m (m)
1	20.137016632763302	2.103×10^{-16}	4.235×10^{-15}	3.443×10^{-39}	15314.7
0.5	20.17790917547	1.320×10^{-12}	5.328×10^{-11}	3.056×10^{-35}	1.725
0.2	20.1380336	2.671×10^{-9}	2.690×10^{-7}	9.778×10^{-32}	5.392×10^{-4}
0.1	20.15963	1.101×10^{-7}	2.220×10^{-5}	5.700×10^{-30}	9.251×10^{-6}

inequality reduces to

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)}} \geqslant l_P. \tag{18}$$

Since the parameter R is generally an increasing function of barrier size (Appendix D), the upper bounds on the barrier size of RT are therefore determined by the Planck length. When the energy broadening due to particle-particle interactions is considered, the quantity k is reduced by a factor of $(1-\frac{\Gamma}{V_0})$, where Γ is the level broadening. Based on the analysis above, the product mV_0 will be enlarged by a factor of $(1-\frac{\Gamma}{V_0})^{-2}$.

D. Fundamental limits made by the uncertainty principle and possible solution

For a group of incident particles, given that the standard derivation of energy distribution σ_E is approximately the term ΔE for $\Delta|M_{11}|^2_{\Delta E} = 1$ and $P(E) = 0.5$, the narrow window of energy dispersion implies that the momenta of the particles distribute dominantly within a narrow interval with a small standard derivation (Δp). Because of the uncertainty principle, the standard derivation of position Δx is expected to be large. Table II lists the energy and momentum broadening and estimated standard position derivations of protons when $P(E) = 0.5$ at $E = 0.5E_b$, for several rectangular DBs with $E_b = 1$, 0.5, 0.2, and 0.1 eV, and $w \sim 20$ Å. It is clearly seen that the energy broadening increases significantly with decreasing barrier height, resulting in reduced standard derivations of position. For $E_b = 1$eV, the requirement of ultrahigh monochromaticity of incident energies leads to a very small Δp and consequently a quite large Δx ($\sim 1.53 \times 10^4$ m), which is practically very challenging, if not impossible for experimental tests. Much smaller Δx (9.25×10^{-6} m) is found when E_b decreases to 0.1 eV.

It should be stressed here that the constraint on the standard deviations of particle momentum and position imposed by the uncertainty principle does not exclude the possibility that a subgroup of particles with ultrahigh monochromaticity coexists with another subgroup of particles with large energy broadening. The reason is that, by definition, the standard derivation of some physical quantity of a single particle is the statistical average over many events, which may be equivalently evaluated by the statistical results of many identical particles within a small interval of time. Indeed, this is in line with the impossibility of measuring the quantum state of a single system [74]. Therefore, a possible recipe to the practical difficulty of position delocalization is to have much larger standard derivation of kinetic energy distribution than that required for the half drop of $P(E)$, i.e., $\sigma_E \gg \Delta E$, such that the standard derivation of momentum σ_p is much larger than the Δp corresponding to ΔE. Consider two microcanonical ensembles containing N weakly interacting identical bosons which follow the kinetic energy distributions $g_1(E)$ and $g_2(E)$, respectively: $N = \int_0^\infty g_1(E)dE = \int_0^\infty g_2(E)dE$. In addition, they have the same averaged kinetic energies: $\langle E\rangle = \int_0^\infty E g_1(E)dE = \int_0^\infty E g_2(E)dE$. The key difference is the standard deviation of kinetic energies: $\sigma_{E2} \gg \sigma_{E1} \sim \Delta E$, i.e., the energy broadening of the first group of particles [distribution described by $g_1(E)$] is approximately the energy deviation for the half drop of $P(E)$, while it is much smaller than that of the second group. The distribution function of the mixed $2N$-particle ensemble is $g(E) = g_1(E) + g_2(E)$, with the standard deviation $\sigma_E = \sqrt{\frac{\sigma_{E1}^2+\sigma_{E2}^2}{2}} \approx \frac{\sigma_{E2}}{\sqrt{2}} \gg \Delta E$. Therefore, mixing of the two groups of identical particles has drastically increased the energy broadening and reduced the position uncertainty. Meanwhile, enough particles for resonant transmission are maintained. The modifications introduced by the procedure are illustrated in Fig. 8. For the general case of $P(E) = 1 - \delta P$, the energy broadening ΔE is given by Eq. (9) and can be similarly analyzed. In weakly interacting dilute atomic gases, two-body collisions dominate the interactions which simply exchange particle momenta and therefore keep the kinetic energy distributions unchanged.

The critical dependence of the tunneling behavior with small energy deviations requires that the energies of the incident particles to distribute within a very narrow range or,

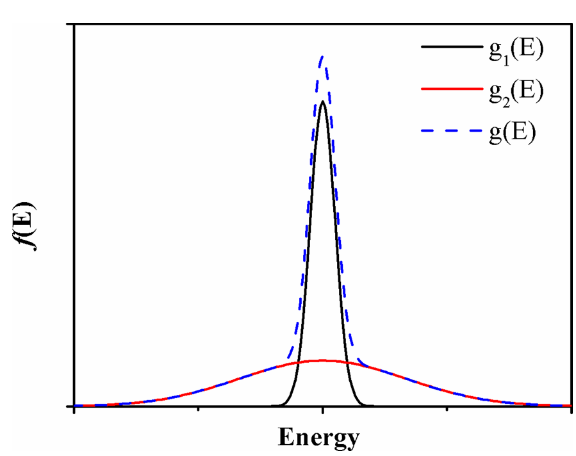

FIG. 8. Schematic diagram for the kinetic energy distribution $[f(E)]$ of identical particles in microcanonical ensembles: Particle groups of high monochromaticity $[f(E) = g_1(E)]$, low monochromaticity $[f(E) = g_2(E)]$, and their superposition $[f(E) = g(E) = g_1(E) + g_2(E)]$.

TABLE III. Like Table II but for the RT of some typical bosons with incident energy $E = 0.5E_b$. In all cases the barrier width $a = 1$ Å, and barrier height $E_b = 0.01V_{\max}$, with $V_{\max}$ being the upper bound set by the Planck length. The Cooper pairs of electrons are represented by $e^- \dots e^-$. The energy broadening and resulted uncertainties of momenta and positions of mixed particle groups are displayed in the same lines to make a comparison. The broadening parameter of energy is chosen such that $\sigma_E \gtrsim k_B T_c$, with k_B the Boltzmann constant and T_c the phase transition temperatures.

Boson	E_b (eV)	w (Å)	$\lvert\Delta w\rvert$ (Å)	ΔE (eV)	σ_E (eV)	Δp (kg m/s)	σ_p (kg m/s)	Δx_m (m)	σ_x (m)
$e^- \dots e^-$ (in Nb)	28.26	6.3439	3.16×10^{-3}	1.41×10^{-12}	1×10^{-3}	1.43×10^{-37}	1.02×10^{-28}	368.15	5.19×10^{-7}
^{4}He	7.69×10^{-3}	6.3439	3.16×10^{-3}	3.83×10^{-16}	5×10^{-4}	1.43×10^{-37}	1.87×10^{-25}	368.15	2.82×10^{-10}
^{7}Li	4.39×10^{-3}	6.3439	3.16×10^{-3}	2.19×10^{-16}	1×10^{-10}	1.43×10^{-37}	6.54×10^{-32}	368.15	8.07×10^{-4}
^{23}Na	1.34×10^{-3}	6.3439	3.16×10^{-3}	6.67×10^{-17}	1×10^{-10}	1.43×10^{-37}	2.15×10^{-31}	368.15	2.45×10^{-4}
^{87}Rb	3.53×10^{-4}	6.3439	3.16×10^{-3}	1.76×10^{-17}	1×10^{-10}	1.43×10^{-37}	8.13×10^{-31}	368.15	6.49×10^{-5}

ideally, with a δ-function-like kinetic energy distribution. In practice, the first group of particles may be prepared using the Bose-Einstein condensates [75–77], in which a large fraction of atoms from a Bose gas occupy the same quantum state, and the momenta of all involved bosons are expected to have approximately the same value: condensation in the momentum space. The second group of particles may be prepared at temperatures slightly above the critical temperature T_c of phase transition from normal states to the new quantum states like superconductivity, superfluidity, or Bose-Einstein condensation. The common feature is that the particles in the first group are of high monochromaticity in the momentum space as well as of high coherence in their wave functions: Both are key factors for the realization of RT in DB systems. At temperatures above T_c, the quantum motions of the incident gas atoms may be described in terms of wave packets [78]. As an example, the RT of some typical bosons (Cooper pairs of superconducting Nb, ^{4}He, ^{7}Li, ^{23}Na, and ^{87}Rb) across rectangular DBs is studied, and the related parameters are presented in Table III. The effects of energy broadening through mixing identical bosons of different ensembles are evidenced by the significantly reduced standard position deviations. As mentioned above, phase coherence of the incident particles plays a key role in the RT process. The quantity characterizing the strength of phase coherence is the coherence length ξ, which sets the upper limit for the interbarrier spacing: $w \leqslant \xi$. Within the BCS theory, the coherence length of Cooper pairs is given by $\xi = \hbar v_F/\pi\Delta_0$, where v_F is the Fermi velocity, and Δ_0 is the order parameter. For a BEC condensate, the coherence length is [79] $\xi \sim 1/\sqrt{na_s}$, where n is the particle density, and a_s is the scattering length. In the BCS-BEC crossover region ($\Delta_0 \sim E_F$, the Fermi energy), $\xi \sim 1/k_F$, inverse of the Fermi wave vector, scales as $1/\sqrt[3]{n}$. In this case, the coherence length of both Cooper pairs and BEC condensates decreases monotonically with particle density. Typical coherence length of Cooper pairs can span from hundreds of angstroms to microns. Nevertheless, preparation of the first group of particles with ultrahigh monochromaticity remains challenging even with a state-of-the-art technique. Another challenge to experimental tests may be the acceleration of the condensates to desired incident velocities while maintaining the states of condensation [80,81]. Alternatively, the experimentally observed intrinsic bistability in the electron-based asymmetric DB systems [71,72] may have enlightenment to design similar experimental architectures for testing the existence of $L_{\min}$ based on the RT of massive quantum particles (e.g., protons, atoms, molecules). Indeed, atom-based interferometers have demonstrated their feasibility in ultrahigh precision measurements [82–89].

III. CONCLUSIONS

To summarize, we have studied quantum tunneling across DBs and arrived at a theorem which leads to several physical consequences. First, by tuning the interbarrier spacing, it is possible that low-energy particles penetrate arbitrary finite-sized potential barriers completely via RT. This result points to the possibility of significant tunneling of massive quantum particles across large barriers at mild conditions. Secondly, it is possible to construct any desired quasibound energy levels within the quantum well formed by the two barriers via adjustment of the interbarrier spacing. Thirdly, for the RT of quantum particles, it is possible to detect the tiny variations of energy levels and positions of the involved potential barriers with unprecedented accuracy. Finally, the critical dependence on interbarrier spacing (consequently, the phase difference) demonstrates again the vital role of phase factor of wave function, which has manifested itself in some remarkable phenomena such as the Aharonov-Bohm effect [90].

Demonstration of the abovementioned results involves two key factors: (i) continuity of the real space and (ii) energy monochromaticity of the incident particles. The first is determined by whether a nonzero minimum length ($L_{\min}$) exists, and the second is affected by the uncertainty principle. Provided that $L_{\min} = 0$, the distances in real space change continuously, and RT can always be realized at given incident energies. On the contrary, the existence of a nonzero $L_{\min}$ will set constraints (upper bounds) for the particle mass, barrier height, and barrier width beyond which no RT is expected. In realistic applications, the energy broadening due to the interactions between the incident particles also puts a limit on the accessible accuracy of position determination, and the phase coherence length of the incident particles sets an upper bound for the size of DBs. Meanwhile, to surmount the practical difficulty (position delocalization of incident particles) owing to the uncertainty principle, we suggest a plausible scheme in which the high- and low-monochromatic beams of identical particle groups are mixed. Potential applications of Bose-Einstein condensates in the scheme are discussed. This work reveals the deep connection between two seemingly different branches of quantum physics—quantum tunneling and quantum gravity—and opens a possible avenue for testing the existence of a minimum length.

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APPENDIX A: MATRIX ELEMENT m_{11} FOR TUNNELING ACROSS SINGLE RECTANGULAR BARRIER

Within the transfer matrix method, we derive the diagonal matrix element m_{11} that describes the transmission across a single rectangular barrier. For a quantum particle with incident energy E tunneling through a rectangular barrier with the height of V_0 and width a, the transfer matrix may be given by [56]

$$M_1 = \frac{1}{2ik}\begin{bmatrix}(ik+\beta)e^{-(ik-\beta)a} & (ik-\beta)e^{-(ik+\beta)a}\\ (ik-\beta)e^{(ik+\beta)a} & (ik+\beta)e^{(ik-\beta)a}\end{bmatrix}\frac{1}{2\beta}\begin{pmatrix}\beta+ik & \beta-ik\\ \beta-ik & \beta+ik\end{pmatrix}$$
$$= \frac{\gamma}{i}\begin{Bmatrix}(ik+\beta)^2 e^{-(ik-\beta)a}-(ik-\beta)^2 e^{-(ik+\beta)a} & (\beta^2+k^2)[e^{-(ik-\beta)a}-e^{-(ik+\beta)a}]\\ (\beta^2+k^2)[e^{(ik-\beta)a}-e^{(ik+\beta)a}] & (ik+\beta)^2 e^{(ik-\beta)a}-(ik-\beta)^2 e^{(ik+\beta)a}\end{Bmatrix},$$

where $k=\sqrt{2mE/\hbar^2}$, $\beta=\sqrt{2m(V_0-E)/\hbar^2}$, and $\gamma=\frac{1}{4\beta k}$.

The first diagonal term is $m_{11}=\frac{\gamma}{i}[(ik+\beta)^2 e^{-(ik-\beta)a}-(ik-\beta)^2 e^{-(ik+\beta)a}]=-i\gamma[(ik+\beta)^2 e^{-(ik-\beta)a}-(ik-\beta)^2 e^{-(ik+\beta)a}]$, which can be reduced to

$$m_{11}=-2i\gamma e^{-ika}[(\beta^2-k^2)\sinh(\beta a)+2i\beta k\cosh(\beta a)],$$

and finally, one has

$$m_{11}=2\gamma e^{-ika}[i(k^2-\beta^2)\sinh(\beta a)+2\beta k\cosh(\beta a)]. \tag{A1}$$

APPENDIX B: DEDUCTION OF ALTERNATIVE RT CONDITION

In this appendix, we deduce the RT condition for homostructured rectangular DBs.

For a DB consisting of single rectangular barriers with the height V_0 and barrier widths a and b, the diagonal element M_{11} of the transfer matrix M may be expressed as follows [57]:

$$|M_{11}|^2 = 1+\frac{(\beta^2+k^2)^2}{4\beta^2k^2}[\sinh^2(\beta b)+\sinh^2(\beta a)]+2\left[\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\right]^2\sinh^2(\beta b)\sinh^2(\beta a)$$
$$-\frac{1}{16k^4\beta^4}(\beta^2+k^2)^2\sinh(\beta b)\sinh(\beta a)(\{[(\beta^2+k^2)^2-8k^2\beta^2]\cosh\beta(a+b)-(\beta^2+k^2)^2$$
$$\times\cosh\beta(a-b)\}\cos2[k(b+w)-ka]-4k\beta(\beta^2-k^2)\sinh\beta(a+b)\sin2[k(b+w)-ka]), \tag{B1}$$

where $k=\sqrt{2mE/\hbar^2}$, $\beta=\sqrt{2m(V_0-E)/\hbar^2}$, and E is energy of the incident particle.

In the case of a homostructured DB, $a=b$, then

$$|M_{11}|^2 = 1+\frac{(\beta^2+k^2)^2}{4\beta^2k^2}[2\sinh^2(\beta a)]+2\left[\frac{(\beta^2+k^2)^2}{4\beta^2k^2}\right]^2\sinh^4(\beta a)-\frac{1}{16k^4\beta^4}(\beta^2+k^2)^2\sinh^2(\beta a)$$
$$\times(\{[(\beta^2+k^2)^2-8k^2\beta^2]\cosh(2\beta a)-(\beta^2+k^2)^2\}\cos(2kw)-4k\beta(\beta^2-k^2)\sinh(2\beta a)\sin(2kw)). \tag{B2}$$

For a given E, $|M_{11}|^2=\frac{1}{T(E;\,w)}$, which is the function of interbarrier spacing w, the minimum of $|M_{11}|^2$ gives the maximum of transmission coefficient $T(E;\,w)$, i.e., RT. The condition of RT can be established by $\frac{\partial}{\partial w}|M_{11}|^2=0$. It follows that

$$\{[(\beta^2+k^2)^2-8k^2\beta^2]\cosh(2\beta a)-(\beta^2+k^2)^2\}(-2k)\sin(2kw)-4k\beta(\beta^2-k^2)\sinh(2\beta a)(2k)\cos(2kw)=0,$$

and consequently,

$$\tan(2kw)=\frac{4k\beta(\beta^2-k^2)\sinh(2\beta a)}{(\beta^2+k^2)^2-[(\beta^2+k^2)^2-8k^2\beta^2]\cosh(2\beta a)}. \tag{B3}$$

By dividing the term $\beta^2 k^2$ in both numerator and denominator, Eq. (B3) changes to

$$\tan(2kw) = \frac{4\left(\frac{\beta}{k} - \frac{k}{\beta}\right)\sinh(2\beta a)}{\left(\frac{\beta}{k} + \frac{k}{\beta}\right)^2 - \left[\left(\frac{\beta}{k} + \frac{k}{\beta}\right)^2 - 8\right]\cosh(2\beta a)} \equiv \frac{4\delta\sinh(2\beta a)}{(\delta^2+4) - (\delta^2-4)\cosh(2\beta a)}$$
$$= \frac{\delta\sinh(2\beta a)}{\left(1+\frac{1}{4}\delta^2\right) + \left(1-\frac{1}{4}\delta^2\right)\cosh(2\beta a)},$$

where $\delta \equiv (\frac{\beta}{k} - \frac{k}{\beta})$.

Recalling that $\sinh(2\beta a) = 2\sinh(\beta a)\cosh(\beta a)$ and $\cosh(2\beta a) = 2\cosh^2(\beta a) - 1$, one has

$$\tan(2kw) = \frac{2\delta\sinh(\beta a)\cosh(\beta a)}{\left(1+\frac{1}{4}\delta^2\right) + \left(1-\frac{1}{4}\delta^2\right)[2\cosh^2(\beta a) - 1]} = \frac{2\delta\sinh(\beta a)\cosh(\beta a)}{\frac{1}{2}\delta^2 + \left(1-\frac{1}{4}\delta^2\right)\cosh^2(\beta a)},$$

which can be reduced to

$$\tan(2kw) = \frac{\delta\tanh(\beta a)}{\left(1-\frac{1}{4}\delta^2\right) + \frac{1}{4}\frac{\delta^2}{\cosh^2(\beta a)}} = \frac{\delta\tanh(\beta a)}{\left(1-\frac{1}{4}\delta^2\right) + \frac{\delta^2}{4}\mathrm{sech}^2(\beta a)}. \tag{B4}$$

Using the equality $\mathrm{sech}^2(\beta a) = 1 - \tanh^2(\beta a)$, Eq. (B4) is finally reduced to

$$\tan(2kw) = \frac{\delta\tanh(\beta a)}{1 - \frac{\delta^2}{4}\tanh^2(\beta a)}. \tag{B5}$$

APPENDIX C: DEPENDENCE OF TUNNELING ON SMALL POSITION AND ENERGY CHANGES

In this appendix, we deduce the mathematical expressions describing the dependence of squared norm of the diagonal transfer matrix element $|(M_{\mathrm{DB}})_{11}|^2$, with respect to slight deviations from the peak positions and incident energies at RT, for the special case when the incident energy is half the barrier height (V_0) of a homostructured rectangular DB (width of single barrier: a). The inverse of $|(M_{\mathrm{DB}})_{11}|^2$ then describes the dependence of tunneling behavior on small position and energy changes.

In general, $|(M_{\mathrm{DB}})_{11}|^2 \equiv f(E; w)$ is the function of incident energy E and interbarrier spacing w. At the vicinity of RT, the function $f(E; w)$ can be expressed as functions of small deviations from RT parameters using the Taylor series, by considering the fact that $|(M_{\mathrm{DB}})_{11}|^2 = 1$ and $(\frac{\partial f}{\partial w}) = 0$, $(\frac{\partial f}{\partial E}) = 0$ at the RT point.

(I) For constant E, the dependence on deviation (Δw) from the RT positions (w_n) is

$$|(M_{\mathrm{DB}})_{11}|^2 \equiv f(E; w) \cong 1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial w^2}\right)(\Delta w)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta w}. \tag{C1}$$

Using the expressions for rectangular DBs (Appendix B), one has

$$\frac{\partial f}{\partial w} = -\left(\frac{1}{16\beta^4 k^4}\right)(\beta^2+k^2)^2\sinh^2(\beta a)[g(\beta,k)(-2k)\sin(2kw) + h(\beta,k)(2k)\cos(2kw)], \tag{C2}$$

$$\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2\sinh^2(\beta a)}{4\beta^4 k^2}[g(\beta,k)\cos(2kw) + h(\beta,k)\sin(2kw)], \tag{C3}$$

where $k = \sqrt{2mE/\hbar^2}$, $\beta = \sqrt{2m(V_0-E)/\hbar^2}$, and

$$g(\beta,k) \equiv [(\beta^2+k^2)^2 - 8\beta^2 k^2]\cosh(2\beta a) - (\beta^2+k^2)^2, \quad h(\beta,k) = -4\beta k(\beta^2-k^2)\sinh(2\beta a).$$

The condition $(\frac{\partial f}{\partial w}) = 0$ gives that $g(\beta,k)\sin(2kw) = h(\beta,k)\cos(2kw)$, and then

$$\tan(2kw) = \frac{h(\beta,k)}{g(\beta,k)}. \tag{C4}$$

Using Eq. (C4), $\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2\sinh^2(\beta a)}{4\beta^4 k^2} h(\beta,k)\sin(2kw)[\cot^2(2kw) + 1]$, and then

$$\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2\sinh^2(\beta a)}{4\beta^4 k^2}\frac{h(\beta,k)}{\sin(2kw)}. \tag{C5}$$

For rectangular DBs, we have the general relation $2kw = (2n-1)\pi - 2\alpha$, and $\alpha = \arctan[\frac{(k^2-\beta^2)}{2\beta k}\tanh(\beta a)]$.

Consequently,

$$\sin(2kw) = \sin(2\alpha) = \frac{2\tan\alpha}{1+\tan^2\alpha} = \frac{2\frac{(k^2-\beta^2)}{2\beta k}\tanh(\beta a)}{1+\tan^2\alpha}.$$

Finally,

$$\frac{\partial^2 f}{\partial w^2} = \frac{(\beta^2+k^2)^2\sinh^2(\beta a)}{4\beta^4 k^2}\frac{h(\beta,k)}{\sin(2kw)} = \frac{(\beta^2+k^2)^2\sinh^2(2\beta a)}{2\beta^2}(1+\tan^2\alpha). \tag{C6}$$

It is clear that $\frac{\partial^2 f}{\partial w^2} > 0$ holds for all allowed incident energies E, which proves that the term $|(M_{\rm DB})_{11}|^2$ arrives at its minimum, and its reciprocal gives the maximum of transmission probability, i.e., 1.

When the incident energy is half the barrier height $\beta = k$, we have $\alpha = 0$, and $\frac{\partial^2 f}{\partial w^2} = 2k^2\sinh^2(2ka)$, and therefore,

$$|(M_{\rm DB})_{11}|^2 \cong 1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial w^2}\right)\times(\Delta w)^2 = 1+\sinh^2(2ka)(k\Delta w)^2. \tag{C7}$$

(II) For constant w, the dependence on deviation (ΔE) from the RT energies ($E_{\rm RT}$) is

$$|(M_{\rm DB})_{11}|^2 \equiv f(E;w) \cong 1 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial E^2}\right)(\Delta E)^2 \equiv 1 + \Delta|M_{11}|^2_{\Delta E}. \tag{C8}$$

Compared with $(\frac{\partial^2 f}{\partial w^2})$, computation of $(\frac{\partial^2 f}{\partial E^2})$ is much more complicated. Alternatively, we directly consider the dependence of $|(M_{\rm DB})_{11}|^2$ with energy deviation (ΔE) to the second order. For the special situation $\beta = k$, the mathematical expression of $|(M_{\rm DB})_{11}|^2$ is reduced to

$$|(M_{\rm DB})_{11}|^2 = 1 + 2\sinh^2(ka) + 2\sinh^4(ka) + \sinh^2(ka)[\cosh(2ka)+1]\cos(2kw). \tag{C9}$$

Recalling that $2kw = (2n-1)\pi$ for $\beta = k$, the term $\cos(2kw)$ may be expressed by Taylor series around the RT point with respect to Δk to the second order:

$$\cos(2kw) \cong -1 + \tfrac{1}{2}(2w)^2(\Delta k)^2 = -1 + 2(w\Delta k)^2. \tag{C10}$$

Substitution of $\cos(2kw)$ into Eq. (C10) leads to the following:

$$|(M_{\rm DB})_{11}|^2 \cong 1 + \sinh^2(2ka)(w\Delta k)^2. \tag{C11}$$

Using $k = \sqrt{2mE/\hbar^2}$, and then $\Delta k = \frac{\Delta E}{2\sqrt{E}}\sqrt{2m/\hbar^2} = \frac{k}{2}\frac{\Delta E}{E}$, one finally arrives at

$$|(M_{\rm DB})_{11}|^2 \cong 1 + \sinh^2(2ka)\left(\frac{kw}{2}\right)^2\left(\frac{\Delta E}{E}\right)^2. \tag{C12}$$

APPENDIX D: GENERALIZED CONSTRAINTS ON BARRIER SIZE DUE TO A MINIMUM LENGTH

In this appendix, we deduce the constraint on the barrier size (barrier height, barrier width) for effective RT at the presence of a nonzero minimum length ($L_{\rm min}$). As defined above, effective RT implies that, given the deviation Δw when $|w - w_n| \leqslant \Delta w$, the inequality $T(E;w) \geqslant 1-\delta P$ holds, where δP ($0 < \delta P < 1$) is the tolerance of decrease in tunneling probability at which significant tunneling is measurable. Based on the proof of the theorem, we have

$$T_{\rm DB}(E;w) = \frac{1}{|(M_{\rm DB})_{11}|^2} = \frac{1}{|e^{i\theta}\{1+R[e^{-i(\phi+\theta)}+1]\}|^2} = \frac{1}{|1+R[e^{-i(\phi+\theta)}+1]|^2}. \tag{D1}$$

Then Δw is determined by the equality as follows:

$$\frac{1}{|1+R[e^{-i(\phi+\theta)}+1]|^2} = 1-\delta P. \tag{D2}$$

Equivalently,

$$|1+R[e^{-i(\phi+\theta)}+1]|^2 = \frac{1}{1-\delta P}. \tag{D3}$$

It follows that

$$|1+R+R[\cos(\phi+\theta) - i\sin(\phi+\theta)]|^2 = \frac{1}{1-\delta P}, \tag{D4}$$

$$[1+R+R\cos(\phi+\theta)]^2 + R^2\sin^2(\phi+\theta) = \frac{1}{1-\delta P}, \tag{D5}$$

$$(1+R)^2 + R^2 + 2(1+R)R\cos(\phi+\theta) = \frac{1}{1-\delta P}, \tag{D6}$$

and then reduces to

$$2R(1+R)[1+\cos(\phi+\theta)] = \frac{\delta P}{1-\delta P}. \tag{D7}$$

Consequently, we arrive at

$$\cos(\phi+\theta) = -1 + \frac{1}{2R(1+R)}\frac{\delta P}{1-\delta P}. \tag{D8}$$

For a homostructured DB system, the two parameters $\theta = \arg(m_{11}^2)$, and $R = |m_{12}|^2$ are solely determined by a single barrier $V(x)$. The tunable parameter is $\phi = 2k(a+w)$, via variation of the interbarrier spacing w by a small magnitude of Δw. At the vicinity of RT, $|\Delta w| << w_n$. Around $\phi+\theta = (2n-1)\pi$, i.e., the RT points, expansion of $\cos(\phi+\theta)$ using Taylor series to the second order, we have

$$\cos(\phi+\theta) \cong -1 + \tfrac{1}{2}(\Delta\phi)^2, \tag{D9}$$

where $\Delta\phi = \pm 2k\Delta w$. Comparison of Eq. (D8) and (D9) gives that

$$2k\Delta w = \sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}}. \tag{D10}$$

Finally, we get

$$\Delta w = \frac{1}{2k}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}}. \tag{D11}$$

To achieve effective RT, the existence of $L_{\min}$ requires that

$$\Delta w = \frac{1}{2k}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}} \geqslant L_{\min}. \tag{D12}$$

For a single barrier $V(x)$, the reflection coefficient is given by [59–62]

$$|r|^2 = \frac{|m_{12}|^2}{|m_{11}|^2} = RT_1(E) = R|t|^2, \tag{D13}$$

where $R = |m_{12}|^2$, and $T_1(E) = |t|^2$ is the transmission coefficient across $V(x)$ at energy E. Conservation of probability current gives that $|r|^2 + |t|^2 = 1$. Qualitatively, $|r|^2$ increases with barrier width a and barrier height E_b, which indicates that R is the increasing function of barrier size parameters a and E_b: $R = R\,(a, E_b)$. Larger barrier size results in larger value of R. With substitution of k by $\frac{\sqrt{2mE}}{\hbar}$, the inequality in Eq. (D12) is therefore

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}} \geqslant L_{\min}. \tag{D14}$$

This is the constraint imposed on the particle mass, barrier height, and barrier width due to the minimum length.

In the case $\delta P = 0.5$, FWHM $(= 2\Delta w)$ is obtained. Given that $L_{\min} = l_P$, we have

$$\frac{\hbar}{2\sqrt{2mE}}\sqrt{\frac{1}{R(1+R)}} \geqslant l_P. \tag{D15}$$

For a fixed particle mass m and incident energy E, the inequality in Eq. (D15) sets upper bounds on R and, consequently, the upper bounds for barrier size of $V(x)$: the barrier width a and barrier height E_b.

Furthermore, we can derive the constraint on the broadening of incident energy by using Eq. (D9). In this case, w $(= w_n)$ and a are fixed, $\Delta\phi = 2\Delta k(a+w)$. Using $k = \frac{\sqrt{2mE}}{\hbar}$, we have $\Delta k = \frac{k}{2}\frac{\Delta E}{E}$, then $2\Delta k = k\frac{\Delta E}{E}$, and $\Delta\phi = k(a+w)\frac{\Delta E}{E}$. It follows that

$$(\Delta\phi)^2 = \frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}. \tag{D16}$$

Then

$$\Delta\phi = k(a+w)\left|\frac{\Delta E}{E}\right| = \sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}}. \tag{D17}$$

Finally, we have

$$\left|\frac{\Delta E}{E}\right| = \frac{1}{k(a+w)}\sqrt{\frac{1}{R(1+R)}\frac{\delta P}{1-\delta P}}. \tag{D18}$$

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