

Remnants of the nonrelativistic Casimir effect on the lattice

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The Casimir effect is a fundamental quantum phenomenon induced by the zero-point energy for a quantum field. It is well known for relativistic fields with a linear dispersion relation, while its existence or absence for nonrelativistic fields with a quadratic dispersion is an unsettled question. Here, we investigate the Casimir effects for various dispersion relations on the lattice. We find that Casimir effects for dispersions proportional to an even power of momentum are absent in a long distance but a remnant of the Casimir effect survives in a short distance. Such a remnant Casimir effect will be experimentally observed in materials with quantum fields on the lattice, such as thin films, narrow nanoribbons, and short nanowires. In terms of this effect, we also give a reinterpretation of the Casimir effect for massive fields.

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I. INTRODUCTION

Zero-point energy is a fundamental concept predicted in quantum mechanics. In fact, the Casimir effect [1] is a quantum phenomenon induced by the zero-point energy for photons confined by a spatial boundary condition and was experimentally established [2] (see Refs. [3–7] for reviews). The photonic Casimir effect will be useful in the field of nanophotonics [8], while analogous effects for quantum fields on the lattice in solid states, such as electrons, phonons, and magnons, will open engineering fields which may be named *Casimir electronics* and *Casimir spintronics*.

For relativistic degrees of freedom with the linear dispersion relations as shown in Fig. 1(a), such as the photon and the massless Dirac fermion, the occurrence of the Casimir effect is well known. On the other hand, an unsettled question is whether the Casimir effect for nonrelativistic fields exists. In general, the existence or absence of the Casimir effect for a quantum field depends on the following three points:

- (i) the existence/absence of zero-point energy,
- (ii) the form of the dispersion relation, and
- (iii) the spatial boundary condition.

Absence of the Casimir effect can be realized by any of these conditions. For example, when we consider a free Schrödinger field with a quadratic dispersion as shown in Fig. 1(b), as usual in solid state physics, we can prove that the zero-point energy is exactly zero. In this sense, by reason (i), we may say that the Casimir effect for nonrelativistic

fields described as the Schrödinger field does not occur. For reason (ii), there are some examples. As shown in Fig. 1(c), a flat band has no momentum-dependent dispersion. For such a constant dispersion, the discretization of momenta in a finite volume never produces any difference from the zero-point energy in the infinite volume, so that the Casimir effect does not occur. As another example, as shown in Fig. 1(d), for a field with two quadratic dispersions consisting of positive and negative eigenvalues, one can prove that the Casimir effect disappears under a boundary condition (see, e.g., Refs. [9–12] and a textbook [6]) [13]. Also, in this sense, the Casimir effect for nonrelativistic fields seems to be prohibited. However, the understanding of such a tendency is still not clear.

In this letter, we theoretically study the Casimir effect for degrees of freedom with some types of dispersion relations on the lattice, by using a lattice regularization approach [14–16]. We show that the Casimir effect for quantum fields on the lattice with an even-order dispersion tends to be absent (which is consistent with the case in continuous space), but a remnant of the Casimir effect survives in a short distance (which is distinct from the continuous case). We call this phenomenon the *remnant Casimir effect*. This effect gives us a deeper understanding of the Casimir effect, and moreover, it will be experimentally observed in finite-size condensed matter systems, such as thin films, narrow nanoribbons, and short nanowires of materials. Throughout this letter, we use the natural units $\hbar = v = 1$ for the reduced Planck constant \hbar and the velocity of a particle v .

II. CASIMIR EFFECT ON THE LATTICE

First, we define the Casimir effect on the lattice. For simplicity, we consider a real scalar field in the three-dimensional (3D) lattice space (the cases in other spatial dimensions are straightforward). We set the periodic boundary condition (other boundaries will be discussed later) for the z axis in momentum space as $k_z \rightarrow 2l\pi/L_z$ or $ak_z \rightarrow 2l\pi/N_z$, where

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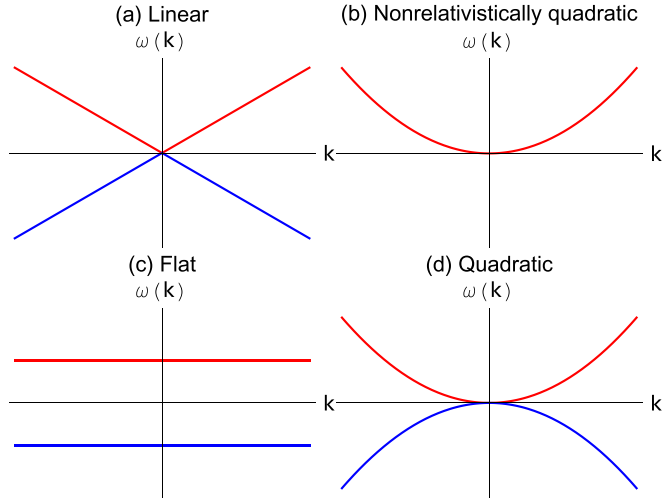


FIG. 1. Examples of typical dispersion relations. (a) A relativistic linear dispersion. (b) A nonrelativistically quadratic dispersion. (c) A flat dispersion. (d) A quadratic dispersion.

$L_z \equiv aN_z$ is the distance with the lattice constant a , and $l = 0, 1, \dots, N_z - 1$ within the first Brillouin zone (BZ), $0 \leq ak_i < 2\pi$ ($i = x, y, z$). The Casimir energy E_{Cas} for the lattice number N_z on the z axis (per the lattice number $N_x N_y$ on the surface area) is defined as the difference between the zero-point energies E_0^{sum} with discrete energy levels $\omega_{\mathbf{k}_\perp, l}$ and E_0^{int} with a continuous energy level $\omega_{\mathbf{k}}$ [14,15] (also see the Supplemental Material [17]):

$$aE_{\text{Cas}} \equiv aE_0^{\text{sum}}(N_z) - aE_0^{\text{int}}(N_z), \quad (1)$$

$$aE_0^{\text{sum}}(N_z) = \int_{\text{BZ}} \frac{d^2(ak_\perp)}{(2\pi)^2} \left[\frac{1}{2} \sum_{l=0}^{N_z-1} a|\omega_{\mathbf{k}_\perp, l}| \right], \quad (2)$$

$$aE_0^{\text{int}}(N_z) = \int_{\text{BZ}} \frac{d^2(ak_\perp)}{(2\pi)^2} \left[\frac{N_z}{2} \int_{\text{BZ}} \frac{d(ak_z)}{2\pi} a|\omega_{\mathbf{k}}| \right], \quad (3)$$

where the momentum integral is over the first BZ, $k_\perp^2 \equiv k_x^2 + k_y^2$, and $d^2(ak_\perp) \equiv d(ak_x)d(ak_y)$. The factor of $+\frac{1}{2}$ comes from the zero-point energy for the real scalar field.

III. DISPERSION RELATION DEPENDENCE OF CASIMIR EFFECT

By using the definition in Eq. (1), we can investigate the Casimir energies for various dispersion relations. Massless dispersion relations with a 3D momentum $\mathbf{k} = (k_x, k_y, k_z)$ in continuous space are defined as

$$a\omega_{\mathbf{k}} = \pm |a\mathbf{k}|^s = \pm (a^2 k_x^2 + a^2 k_y^2 + a^2 k_z^2)^{s/2}, \quad (4)$$

where s is the order of dispersion. For example, $s = 1$ corresponds to a field with the linear dispersion $\omega_{\mathbf{k}} = \pm |\mathbf{k}|$, such as the massless Klein-Gordon field. Here, $s = 2$ is a field with the quadratic dispersion $\omega_{\mathbf{k}} = \pm \mathbf{k}^2$. A dispersion relation in lattice space is obtained by replacing $a\mathbf{k}$ with $a\tilde{\mathbf{k}} = a(\tilde{k}_x, \tilde{k}_y, \tilde{k}_z)$

TABLE I. Classification of Casimir effects for fields with various dispersion relations in the continuous spacetime or the lattice space defined by Eq. (5).

Order s	Continuous	Lattice (periodic boundary)
1 ($m = 0$)	Lasting	Lasting
1 ($m \neq 0, m \neq \infty$)	Damping	Damping
2	No	Remnant ($N_z = 1$)
4	No	Remnant ($N_z = 1, 2$)
\vdots		
$2n$	No	Remnant ($N_z = 1, 2, \dots, n$)
$2n - 1$ ($m = 0$)	Lasting	Lasting

using $a^2 \tilde{k}_i^2 = 2 - 2 \cos ak_i$:

$$a\omega_{\mathbf{k}} = \pm |a\tilde{\mathbf{k}}|^s = \pm \left[\sum_{i=x,y,z} (2 - 2 \cos ak_i) \right]^{s/2}. \quad (5)$$

By substituting this form into Eqs. (2) and (3), we can calculate the Casimir energy.

In Fig. 2, we show the results for the dimensionless Casimir energy aE_{Cas} and the Casimir coefficient (which is a convenient quantity to check the N_z^3 dependence of aE_{Cas}) $C_{\text{Cas}}^{[3]} \equiv N_z^3 aE_{\text{Cas}}$ for the linear ($s = 1$), quadratic ($s = 2$), quartic ($s = 4$), or sextic ($s = 6$) dispersion relation. The result at $s = 1$ is evaluated by numerical integrals, and the others can be analytically obtained. For $s = 1$, we see the well-known behaviors of the Casimir effect: aE_{Cas} decreases as N_z increases. In the large N_z region, $C_{\text{Cas}}^{[3]}$ approximately approaches the analytic solution known in continuous space $C_{\text{Cas}}^{[3]} = -\pi^2/90$. On the other hand, the result for $s = 2$ is quite distinct, where we find that a nonzero E_{Cas} survives only at $N_z = 1$, while E_{Cas} is exactly zero at $N_z > 1$. It is easy to analytically derive this behavior: using $aE_0^{\text{sum}} = 2$ and $aE_0^{\text{int}} = 3$ at $N_z = 1$, we obtain $aE_{\text{Cas}} \equiv aE_0^{\text{sum}} - aE_0^{\text{int}} = -1$, while using $aE_0^{\text{sum}} = aE_0^{\text{int}}$ at $N_z > 1$, $E_{\text{Cas}} = 0$. Furthermore, we can easily derive that this behavior does not depend on the spatial dimension. Similarly, also for $s = 2n$ ($n = 1, 2, \dots$), we find a nonzero E_{Cas} only at $N_z \leq n$. We call this behavior the remnant Casimir effect. Note that, for odd-order dispersions $s = 2n - 1$, E_{Cas} is nonzero at any N_z , and $C_{\text{Cas}}^{[2+s]} \equiv N_z^{2+s} aE_{\text{Cas}}$ approximately approaches a constant value.

We comment on the other types of boundary conditions. For the antiperiodic boundary [$ak_z \rightarrow (2l + 1)\pi/N_z$ with $l = 0, \dots, N_z - 1$], the remnant Casimir effect does not change qualitatively, except for its magnitude and sign. For a phenomenological boundary [$ak_z \rightarrow l\pi/N_z$ with $l = 1, \dots, 2N_z$ (or $l = 0, \dots, 2N_z - 1$)], there are no remnants at any N_z for $s = 2$, while we find a remnant appears at $N_z = 1$ for $s = 4, 6$ and remnants at $N_z < n + 1$ for $s = 4n, 4n + 2$. Thus, various boundary conditions can induce the remnant Casimir effect.

IV. CLASSIFICATION OF CASIMIR EFFECTS

Based on the above investigation, we classify typical behaviors of Casimir effects into four types (see Table I for the periodic boundary):

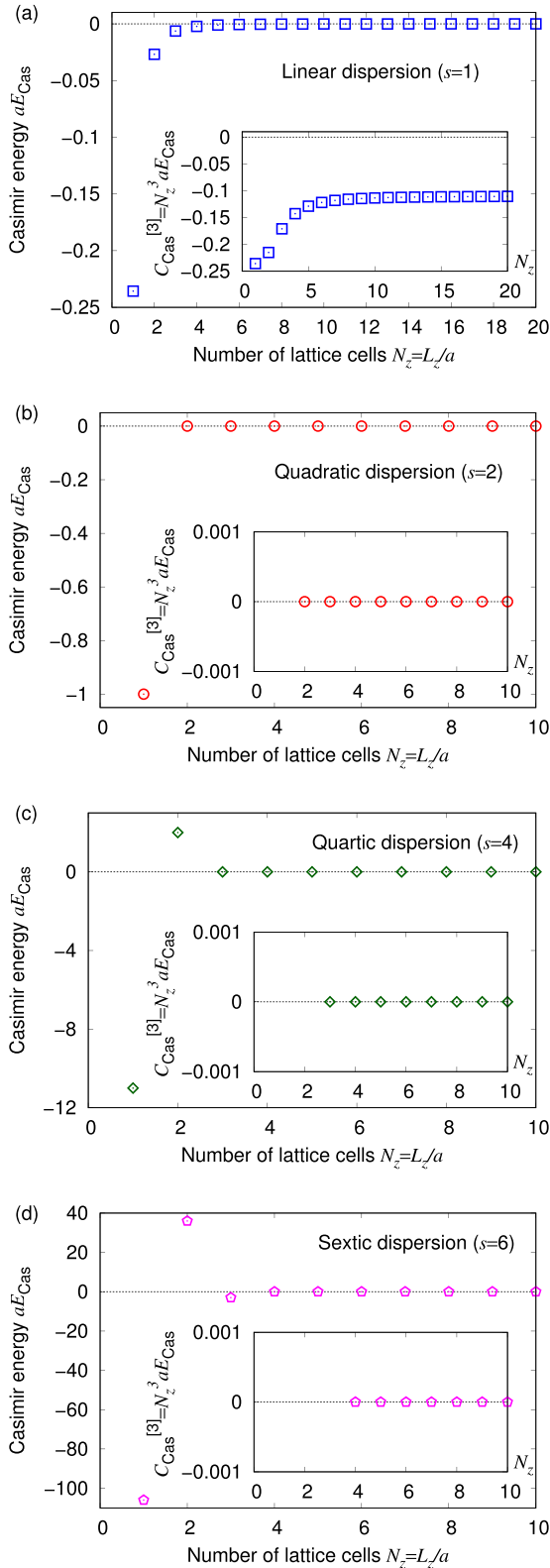


FIG. 2. Casimir energies aE_{Cas} for massless real scalar fields on the lattice, where the dispersion relation is (a) linear ($s = 1$), (b) quadratic ($s = 2$), (c) quartic ($s = 4$), or (d) sextic ($s = 6$) order. Insets: Casimir coefficients $C_{Cas}^{[3]}$. The behaviors for $s = 2, 4$, and 6 are the remnant Casimir effect.

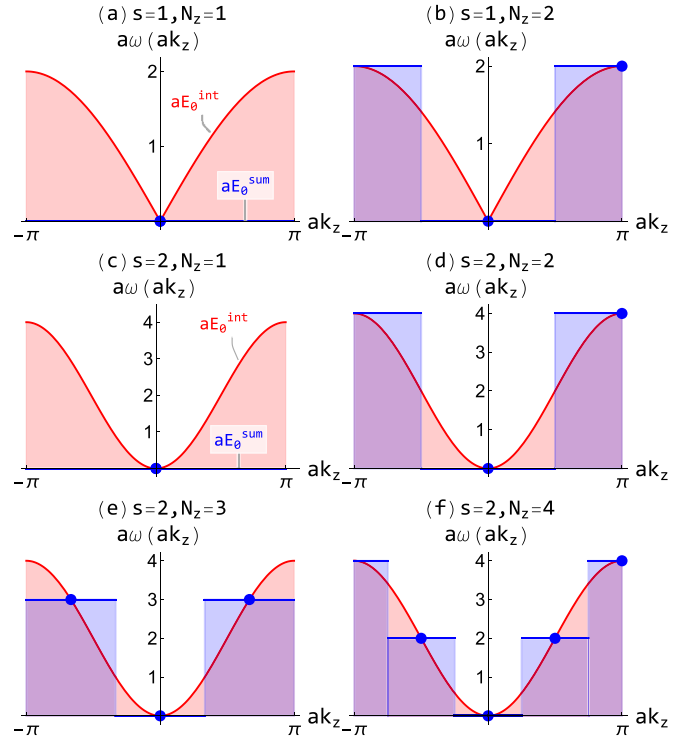


FIG. 3. Graphical interpretation for Casimir effects on the lattice. Blue points: discrete energy levels. Red curves: continuous energy levels. Blue and red colored regions correspond to aE_0^{sum} and aE_0^{int} , respectively. (a) and (b) Linear dispersion at $N_z = 1, 2$. (c)–(f) Quadratic dispersion at $N_z = 1, 2, 3, 4$.

- (1) No Casimir effect—In any distance, the Casimir energy is exactly zero. A representative example is a field with the even-order dispersion in continuous space.
- (2) Lasting Casimir effect—The Casimir energy is nonzero in a long distance, and the Casimir coefficient approaches asymptotically to a nonzero value. Such a behavior is known for fields with a linear dispersion in both the continuous and lattice space.
- (3) Damping Casimir effect—The Casimir energy is nonzero in a long distance, but the Casimir coefficient asymptotically approaches zero. Such a behavior is well known for massive fields [3,18,19] because a mass parameter usually tends to suppress the Casimir effect in a long distance.
- (4) Remnant Casimir effect—The Casimir energy in a long distance is exactly zero, but nonzero values (remnants) survive only in a short distance. Such a behavior is realized in fields with an even-order dispersion in lattice space. This is the main finding in this letter.

V. GRAPHICAL INTERPRETATION

Here, we give a graphical interpretation for Casimir effects on the lattice. By the definition in Eq. (1), the Casimir energy is defined as the difference between E_0^{sum} and E_0^{int} , where E_0^{sum} is the sum over discretized k_z , and E_0^{int} is the integral with respect to continuous k_z . As shown in Fig. 3, such a difference can be graphically understood by comparing

(i) the sum of the areas of rectangles with a width $2\pi/N_z$ and a height $[2 - 2\cos(2l\pi/N_z)]^{s/2}$ (blue region) and (ii) the integral of the dispersion relation with respect to ak_z (red region), within a BZ. For example, Figs. 3(a) and 3(b) for $s = 1$ show that E_0^{sum} and E_0^{int} are different from each other. Such a difference survives at any N_z , so that the resultant Casimir energies are nonzero at any N_z . Similarly, in Fig. 3(c) for $s = 2$ at $N_z = 1$, this situation does not change, which is the origin of the only remnant for $s = 2$. On the other hand, in Figs. 3(d)–3(f) for $s = 2$ at $N_z > 1$, we find an exact cancellation between E_0^{sum} and E_0^{int} , which leads to the exactly zero Casimir energy. Mathematically, this is nothing but the property of $2 - 2\cos ak_z$. Thus, the Casimir effect on the lattice can be visually understood as a difference or cancellation of the zero-point energies.

VI. REINTERPRETATION OF MASSIVE DISPERSION

Here, we discuss a connection between the the Casimir effect for the $s = 1$ massive dispersion and the remnant Casimir effect for even-order dispersions. The $s = 1$ dispersion relation with a mass (or a gap) m is expanded around $|\mathbf{k}|/m = 0$ as

$$\begin{aligned}\omega_{\mathbf{k}} &= \pm\sqrt{\mathbf{k}^2 + m^2} \\ &= \pm\left(m + \frac{\mathbf{k}^2}{2m} - \frac{\mathbf{k}^4}{8m^3} + \frac{\mathbf{k}^6}{16m^5} + \dots\right).\end{aligned}\quad (6)$$

The first term is a flat band without any momentum dependence, as in Fig. 1(c), and never contributes to the Casimir effect. The other terms are regarded as the sum of various even-order dispersion relations.

Using this form, the zero-point energy is rewritten as

$$\begin{aligned}E_0 &= \int \frac{d\mathbf{k}}{(2\pi)^3} \omega_{\mathbf{k}} = \int \frac{d\mathbf{k}}{(2\pi)^3} \sqrt{\mathbf{k}^2 + m^2} \\ &= \int \frac{d\mathbf{k}}{(2\pi)^3} \left(m + \frac{\mathbf{k}^2}{2m} - \frac{\mathbf{k}^4}{8m^3} + \frac{\mathbf{k}^6}{16m^5} + \dots\right).\end{aligned}\quad (7)$$

We reexamine the Casimir effect from the last form of the zero-point energy by using lattice regularization [20]. In Fig. 4, we show the comparison between the Casimir energies for the massive dispersion calculated from the third form of Eq. (7) and the sum of remnant Casimir energies for lower even-order dispersion ($s = 2, 4, 6, \dots$) calculated from the last form of Eq. (7), where we fix the dimensionless mass as $am = 5$. Even at $N_z = 1$, the sum of the remnants up to $s = 4$ is almost consistent with that for the massive dispersion. We find that, even at $N_z = 3$, the remnants up to $s = 6$ become a good approximation. Thus, we have derived the picture that the Casimir energy for $s = 1$ massive fields on the lattice is regarded as the sum of the remnant Casimir energies for massless fields of lower even orders.

It should be noted that, in continuous space, this picture of finite sum is not correct since the Casimir energies for even-order dispersion relations are exactly zero. The Casimir energy in continuous space is obtained only by infinite summation of all orders. This inconsistency is because the dispersion relation on the lattice is slightly different from that in continuous space.

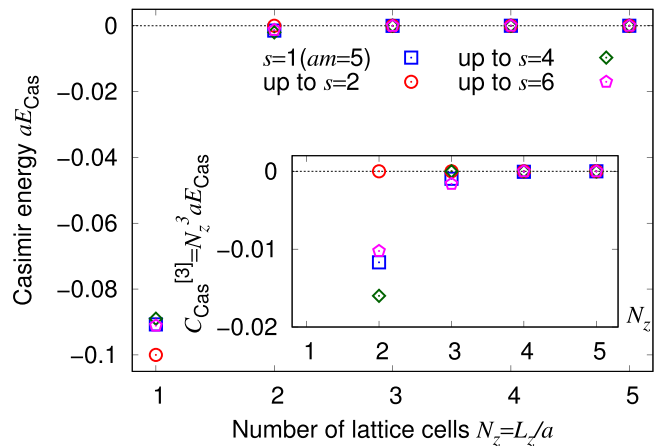


FIG. 4. Comparison between the Casimir energy for a massive real scalar field and the sum of remnant Casimir energies for lower-order dispersions. The mass parameter is fixed as $am = 5$.

VII. EXPERIMENTAL REALIZATIONS

We emphasize that not only the lattice Casimir effect but also the remnant Casimir effect intrinsically arise in relativistic condensed matter systems when its energy levels are discretized by the finite size of a system. For example, a quadratic dispersion relation as shown in Fig. 1(d) appears in AB-stacked bilayer graphene [21,22], multi-Weyl semimetals [23–25], and AB-stacked honeycomb bilayer magnets [26]. In such materials, the Casimir effect is intrinsically realized on nanoribbons of two-dimensional materials and in thin films of 3D materials, and the Casimir energy is defined as an energy difference relative to the infinite bulk. In this sense, the Casimir energy is a physical quantity well defined in principle, and thus, the remnant Casimir effect in even-order dispersion systems is also well defined. We stress that such an energy difference can influence thermodynamic properties of materials, such as magnetization. As a result, the Casimir effect can be measured: One can observe it as a remnant behavior of a physical quantity in a short distance and an unchanged behavior in a long distance.

We remark on two points. First, the magnitude of the remnant Casimir energy in a short distance is generally comparable with the free energy of the original (infinite-bulk) system (which is proportional to aE_0^{int}) because this remnant is generated from a large difference between aE_0^{sum} and aE_0^{int} . Second, the unchanged behavior in a long distance, the disappearance of the Casimir effect, can be regarded as a no-go theorem of the Casimir effect, which is always helpful for the correct understanding for finite-size effects. Note that the band structures in realistic materials may be slightly different from the exact $2 - 2\cos ak$ band due to the existence of interactions or impurities, but even in such a case, one can observe an approximate effect: approximate remnants in a short distance and approximate zeros in a long distance. In addition, if one knows the form of $2 - 2\cos ak$ in the infrared region but does not know its ultraviolet structure, the nonzero Casimir effect in a long distance implies a difference from $2 - 2\cos ak$ in the ultraviolet region.

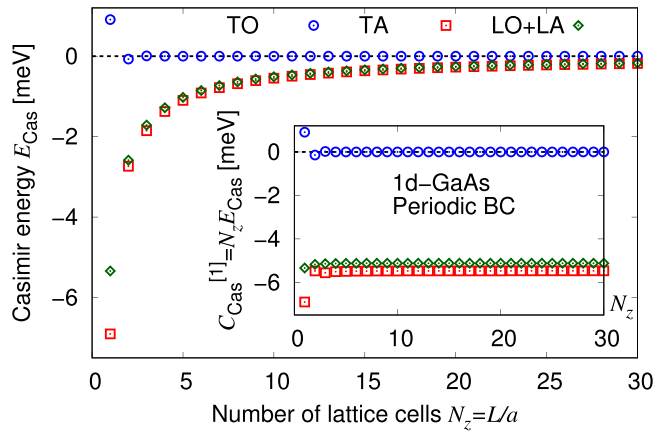


FIG. 5. Casimir energy and Casimir coefficient for phonons in one-dimensional-like GaAs nanowires with the periodic boundary condition.

As a demonstration, we consider phonon fields in GaAs nanowires along the z axis, where there are four types of phonons: the transverse optical (TO), longitudinal optical (LO), transverse acoustic (TA), and longitudinal acoustic (LA) modes. For dispersion relations, we use a model with parameters estimated by fitting the experimental data of the four modes in the whole BZ [27] (see the Supplemental Ma-

terial [17]). In Fig. 5, we show the numerical results for the Casimir energy with the periodic boundary. We find that E_{Cas} for the TO modes is strongly suppressed compared with the other modes, except for $N_z = 1$. This is because the dispersion relations for the TO modes are quadraticlike around both $ak_z = 0$ and π , and other modes are not. This is regarded as an approximate realization of the remnant Casimir effect.

VIII. CONCLUSIONS

In this letter, we have proposed the remnant Casimir effect which is a unique property of even-order dispersion on the lattice. This is a well-defined concept in principle and should be experimentally realized. Our findings will become common knowledge on Casimir effects on the lattice and will play a key role in the Casimir engineering, which is the application of quantum fluctuations on the lattice to engineering fields, such as Casimir photonics [8], Casimir electronics, and Casimir spintronics.

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$$E_{\text{Cas}}^{[s]} = 2 \frac{1}{(4\pi)^{(d-1)/2}} \frac{\pi^{-1/2} \Gamma\left(\frac{d+s}{2}\right) \zeta(d+s)}{\Gamma\left(-\frac{s}{2}\right)} \frac{2^{d-1+s}}{L^{d-1+s}}.$$
- When s is an even number, $E_{\text{Cas}}^{[s]} = 0$ because of $\Gamma\left(-\frac{s}{2}\right)$. Also, an alternative interpretation, it is well known that the Casimir energy for relativistic fields with a nonzero and finite mass is characterized by the modified Bessel function, and its infinite-mass limit goes to zero [3,18,19].
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- [17] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/PhysRevResearch.5.L022054> for (1) In S1, we compare the definitions of Casimir energies in continuous space and in lattice space and (2) In S2, we show the detailed analyses of the phononic Casimir effect in a one-dimensional GaAs nanowire.
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