Universal bounds on the performance of information-thermodynamic engine

Tomohiro Tanogami,[1] Tan Van Vu,[2] and Keiji Saito[3]
[1]Department of Earth and Space Science, Osaka University, Osaka 560-0043, Japan
[2]Analytical Quantum Complexity RIKEN Hakubi Research Team, RIKEN Center for Quantum Computing (RQC), 2-1 Hirosawa, Wako, Saitama 351-0198, Japan
[3]Department of Physics, Kyoto University, Kyoto 606-8502, Japan

(Received 4 June 2023; accepted 19 October 2023; published 22 December 2023)

We investigate fundamental limits on the performance of information processing systems from the perspective of information thermodynamics. We first extend the thermodynamic uncertainty relation (TUR) to a subsystem. Specifically, for a bipartite composite system consisting of a system of interest X and an auxiliary system Y, we show that the relative fluctuation of an arbitrary current for X is lower bounded not only by the entropy production associated with X but also by the information flow between X and Y. As a direct consequence of this *bipartite* TUR, we prove universal tradeoff relations between the output power and efficiency of an information-thermodynamic engine in the fast relaxation limit of the auxiliary system. In this limit, we further show that the Gallavotti-Cohen symmetry is satisfied even in the presence of information flow. This symmetry leads to universal relations between the fluctuations of information flow and entropy production in the linear response regime. We illustrate these results with simple examples: coupled quantum dots and coupled linear overdamped Langevin equations. Interestingly, in the latter case, the equality of the bipartite TUR is achieved even far from equilibrium, which is a very different property from the standard TUR. Our results will be applicable to a wide range of systems, including biological systems, and thus provide insight into the design principles of biological systems.

DOI: 10.1103/PhysRevResearch.5.043280

I. INTRODUCTION

Biological systems maintain their functions by acquiring or using information about fluctuating environments. For example, *E. coli* regulates its flagellar motors by processing information about external ligand concentrations to adapt to the environment [1–6]. A gene network senses a sudden increase in protein concentration and then suppresses mRNA transcription to maintain protein levels [7–10]. While these systems rely on a negative feedback mechanism that suppresses intrinsic noise by using information about fluctuating environments, some molecular machines can even convert information into output work. Such examples include F_oF_1-ATP synthase, where F_1 motor converts energy and information provided by F_o motor into the synthesis of ATP molecules [11–13]. To elucidate the general design principles underlying biological systems, it is necessary to investigate the fundamental limits on the performance of such information processing systems.

Stochastic thermodynamics has revealed various fundamental limits to the thermodynamic aspects of such fluctuating mesoscale systems [14–17]. For example, the thermodynamic uncertainty relation (TUR) states that suppressing the relative fluctuations of an arbitrary time-integrated current $\hat{\mathcal{J}}_\mathcal{T}$ necessarily involves a thermodynamic cost [18–20]:

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}_\mathcal{T}]}{\langle\hat{\mathcal{J}}_\mathcal{T}\rangle^2} \geqslant \frac{2}{\Delta\sigma}, \tag{1}$$

where $\langle\hat{\mathcal{J}}_\mathcal{T}\rangle$ and $\mathrm{Var}[\hat{\mathcal{J}}_\mathcal{T}]$ denote the mean and variance of $\hat{\mathcal{J}}_\mathcal{T}$, and $\Delta\sigma$ denotes the total entropy production up to time $\mathcal{T}$. While the validity of TUR in its original form (1) is limited to steady-state currents in Markov jump processes and overdamped Langevin dynamics, TUR-type inequalities even revealed that there is a fundamental limit to the performance of a thermodynamic heat engine. Specifically, a heat engine with a finite output power cannot achieve the Carnot efficiency as long as the fluctuation of the output power is finite [21–24]. Furthermore, for a stationary cross-transport system with input and output currents, which can be regarded as fuel (positive entropy) and load (negative entropy), respectively, the input-output fluctuation inequalities hold in the linear response regime [25,26]. These inequalities state that the fluctuation of the output current is smaller than that of the input current, while the relative fluctuation of the output current is larger than that of the input current.

In this paper, we aim to find similar fundamental limits for information processing systems, in particular for an information-thermodynamic engine that converts information into output work. Information thermodynamics, which is essentially stochastic thermodynamics for subsystems, is a thermodynamic framework for information flow between two interacting subsystems, either autonomous or nonautonomous [27]. This theory reveals that the information flow between subsystems can significantly affect the thermodynamic constraints of each subsystem. While information thermodynamics has its origins in the thought experiment of Maxwell's demon, it has recently been applied to information processing at the cellular level in biological systems [5,28–31] and even to fully developed fluid turbulence [32].

Here, we consider a composite system consisting of a system of interest X and an auxiliary system Y, described by continuous-time Markov jump processes or diffusion processes with only even variables and parameters under time reversal. Our main results can be summarized as follows.

(i) *Bipartite TUR*. We first extend the standard TUR (1) to a subsystem. For arbitrary time-integrated current $\hat{\mathcal{J}}_{\mathcal{T}}$ for X with arbitrary observation time $\mathcal{T}$, we prove that [cf. Eq. (26)]

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}}]}{\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle^2} \geqslant \frac{2(1+\delta_{\mathcal{J}})^2}{\Delta S^X_{\mathrm{tot}} - \Delta I^X}. \tag{2}$$

Here, $\Delta S^X_{\mathrm{tot}}$ denotes the entropy production associated with X, and ΔI^X denotes the time-integrated information flow, which is the amount of information exchanged with the auxiliary system Y. The additional term $\delta_{\mathcal{J}}$ reflects the contribution of the interaction with Y. This *bipartite* TUR states that the relative fluctuation of the current for the subsystem X is lower bounded not only by the entropy production associated with X, but also by the information transfer between X and Y. In particular, if Y evolves much faster than X, we can further show that $\delta_{\mathcal{J}} \to 0$ in the steady state. In this case, the bipartite TUR gives a tighter bound than the standard TUR (1). While here we derive the bipartite TUR in the steady state, this relation is valid even for systems under arbitrary time-dependent driving from arbitrary initial states (see Appendix A).

(ii) *Tradeoff relations*. As a consequence of the bipartite TUR, we show that there are fundamental limits on the performance of an information-thermodynamic engine. When the system of interest X acts as a steady-state information-thermodynamic engine, its performance can be quantified, e.g., by the negative entropy production rate $|\dot{S}^X_{\mathrm{env}}|$ in the environment and the information-thermodynamic efficiency $\eta^X_S := |\dot{S}^X_{\mathrm{env}}|/|\dot{I}^X|$, which quantifies how efficiently the engine converts information into the negative entropy production. In the typical case where the auxiliary system Y evolves much faster than the engine X, we prove universal tradeoff relations between $|\dot{S}^X_{\mathrm{env}}|$ and η^X_S [cf. Eqs. (74) and (76)]:

$$\left|\dot{S}^X_{\mathrm{env}}\right| \leqslant D_S\frac{1-\eta^X_S}{\eta^X_S} \tag{3}$$

and

$$\left|\dot{S}^X_{\mathrm{env}}\right| \leqslant D_I\eta^X_S(1-\eta^X_S), \tag{4}$$

where D_S and D_I denote the fluctuation of the stochastic medium entropy production and the time-integrated stochastic information flow, respectively. These inequalities state that an information engine with a finite negative entropy production rate cannot achieve $\eta^X_S = 1$ as long as the fluctuations D_S and D_I are finite. In order to achieve a finite negative entropy production rate with $\eta^X_S = 1$, the fluctuations D_S and D_I must diverge.

(iii) *Gallavotti-Cohen symmetry*. In addition to the TURs, the Gallavotti-Cohen symmetry [33–35] also provides important information about fluctuations of currents. For the scaled cumulant generating function $\mu(\lambda_S, \lambda_I)$ with the counting fields λ_S and λ_I for the stochastic medium entropy production and the time-integrated stochastic information flow, we prove that the following Gallavotti-Cohen symmetry holds in the fast relaxation limit of Y [cf. Eq. (116)]:

$$\mu(\lambda_S, \lambda_I) = \mu(-\lambda_S - 1, -\lambda_I - 1). \tag{5}$$

(iv) *Input-output fluctuation inequalities*. As a direct consequence of the Gallavotti-Cohen symmetry, we show that the input-output fluctuation inequalities hold even in the case where information flow is regarded as an input or output current. That is, in the linear response regime where X acts as a steady-state information-thermodynamic engine, we prove that [cf. Eqs. (122) and (123)]

$$D_S \leqslant D_I, \tag{6}$$

$$\frac{D_I}{(\dot{I}^X)^2} \leqslant \frac{D_S}{\left(\dot{S}^X_{\mathrm{env}}\right)^2}. \tag{7}$$

These inequalities state that the fluctuation of the output current (negative entropy production) is smaller than that of the input current (information flow), while the relative fluctuation of the output current is larger than that of the input current.

We illustrate these results with two simple examples: coupled quantum dots and coupled linear overdamped Langevin equations. Interestingly, the latter provides an example where the equality of the bipartite TUR is achieved even far from equilibrium. This is in contrast to the standard TUR (1), where the equality is guaranteed only in the near-equilibrium limit [20,36]. While the bipartite TUR is generally not valid for systems with broken time-reversal symmetry, such as underdamped Langevin dynamics [37–43], many relevant biological systems are often described by continuous-time Markov jump processes or diffusion processes with only even variables and parameters under time reversal. Therefore, these results will be applicable to a wide range of systems, including biological systems, and thus shed new light on our understanding of the design principles of biological systems.

This paper is organized as follows. In Sec. II, we introduce important information-theoretic quantities and briefly review the framework of information thermodynamics in a general setup. In Sec. III A, we describe the bipartite TUR, which is the first main result of this paper. The detailed derivation of the bipartite TUR is presented in Sec. III B. In Sec. III C, we show that the bipartite TUR reduces to the form of the standard TUR if the auxiliary system evolves much faster than the system of interest. We discuss the equality condition of the bipartite TUR in Sec. III D. In Sec. IV, we show that the bipartite TUR gives universal bounds on the performance of an information-thermodynamic engine, which is the second main result of this paper. In Sec. V A, as the third main result of this paper, we prove that the Gallavotti-Cohen symmetry holds even in the presence of information flow in the fast relaxation limit of the auxiliary system. As a corollary to this symmetry, we show that the input-output fluctuation inequalities are valid even in the case where information flow is regarded as an input or output current in Sec. V B. In Sec. VI, we illustrate our results with two examples. In Sec. VII, we conclude this paper with some remarks.

II. SETUP

We consider a composite system that consists of two subsystems, X (system of interest) and Y (auxiliary system), whose time evolution is described by Markov jump processes or overdamped Langevin equations. Let x_t and y_t be the states of X and Y at time t, respectively. We assume that the system satisfies the bipartite property: the transition probability $p(x_{t+dt}, y_{t+dt}|x_t, y_t)$ satisfies

$$p(x_{t+dt}, y_{t+dt}|x_t, y_t) = p(x_{t+dt}|x_t, y_t)p(y_{t+dt}|x_t, y_t) \tag{8}$$

for $dt \to 0^+$. This property means that X and Y do not jump simultaneously in the case of Markov jump processes and that the noises acting on X and Y are uncorrelated in the case of diffusion processes. In this paper, we focus mainly on Markov jump processes, while the extension to the overdamped Langevin case is straightforward. Let $p_t(x, y)$ be the probability of state (x, y) at time t. The time evolution of $p_t(x, y)$ is described by the master equation

$$\partial_t p_t(x, y) = \sum_{x'} w^y_{xx'} p_t(x', y) + \sum_{y'} w^{yy'}_x p_t(x, y'), \tag{9}$$

where $w^y_{xx'}$ ($w^{yy'}_x$) is the time-independent transition rate from state (x', y) to (x, y) [(x, y') to (x, y)] with $w^y_{xx} = -\sum_{x'(\neq x)} w^y_{x'x}$ ($w^{yy}_x = -\sum_{y'(\neq y)} w^{y'y}_x$). The rate matrix is assumed to be irreducible to ensure the uniqueness of the stationary distribution $p_{ss}(x, y)$. Note that X and Y can affect each other's transition rates, although they cannot jump simultaneously.

A. Information-theoretic quantities

We introduce important information-theoretic quantities. The strength of the correlation between X and Y can be quantified by the mutual information [44]:

$$I[X : Y] := \sum_x \sum_y p_t(x, y) \ln \frac{p_t(x, y)}{p^X_t(x) p^Y_t(y)}, \tag{10}$$

where $p^X_t(x) = \sum_y p_t(x, y)$ and $p^Y_t(y) = \sum_x p_t(x, y)$ denote the marginal distributions for X and Y, respectively. The mutual information is non-negative and is equal to zero if and only if X and Y are independent.

The directional information from one variable to the other can be quantified by *information flow* [45], which is defined as

$$\begin{aligned} \dot{I}^X &:= \lim_{dt\to 0^+} \frac{I[X_{t+dt} : Y_t] - I[X_t : Y_t]}{dt} \\ &= \sum_x \sum_{x'} \sum_y w^y_{xx'} p_t(x', y) \ln \frac{p_t(y|x)}{p_t(y|x')}, \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{I}^Y &:= \lim_{dt\to 0^+} \frac{I[X_t : Y_{t+dt}] - I[X_t : Y_t]}{dt} \\ &= \sum_x \sum_y \sum_{y'} w^{yy'}_x p_t(x, y') \ln \frac{p_t(x|y)}{p_t(x|y')}, \end{aligned} \tag{12}$$

where $p_t(y|x) = p_t(x, y)/p^X_t(x)$ and $p_t(x|y) = p_t(x, y)/p^Y_t(y)$ denote the conditional probabilities. From the bipartite property, the sum of $\dot{I}^X$ and $\dot{I}^Y$ gives the time derivative of the mutual information [46]:

$$d_t I[X : Y] = \dot{I}^X + \dot{I}^Y. \tag{13}$$

In the steady-state condition, $\dot{I}^X$ and $\dot{I}^Y$ have opposite signs because $d_t I[X : Y] = 0$. If $\dot{I}^X > 0$, the correlation between X and Y increases due to transitions in X. In other words, X gains information about Y. If $\dot{I}^X < 0$, in contrast, X_{t+dt} is less correlated with Y_t than X_t. In this case, the information is destroyed or exploited by X.

B. Second law of information thermodynamics

Here, we formulate the second law of information thermodynamics. To this end, we impose the local detailed balance condition to ensure that the system is thermodynamically consistent [15,17,47]. Then, the entropy change in the environment due to transitions in X and Y is identified as

$$\begin{aligned} \dot{S}_{\text{env}} &= \sum_x \sum_{x'} \sum_y w^y_{xx'} p_t(x', y) \ln \frac{w^y_{xx'}}{w^y_{x'x}} \\ &\quad + \sum_x \sum_y \sum_{y'} w^{yy'}_x p_t(x, y') \ln \frac{w^{yy'}_x}{w^{y'y}_x} \\ &=: \dot{S}^X_{\text{env}} + \dot{S}^Y_{\text{env}}. \end{aligned} \tag{14}$$

The average rate of the system entropy is identified as the time derivative of the system's Shannon entropy $S[X, Y] := -\sum_{x,y} p_t(x, y) \ln p_t(x, y)$:

$$\begin{aligned} d_t S[X, Y] &= \sum_x \sum_{x'} \sum_y w^y_{xx'} p_t(x', y) \ln \frac{p_t(x', y)}{p_t(x, y)} \\ &\quad + \sum_x \sum_y \sum_{y'} w^{yy'}_x p_t(x, y') \ln \frac{p_t(x, y')}{p_t(x, y)}. \end{aligned} \tag{15}$$

Then, the total entropy production rate $\dot{\sigma}$ is given by

$$\dot{\sigma} = d_t S[X, Y] + \dot{S}_{\text{env}} \geqslant 0, \tag{16}$$

where the non-negativity is proved by using $\ln a \leqslant a - 1$ ($a \geqslant 0$). The non-negativity of the total entropy production rate is a manifestation of the second law of thermodynamics and is sometimes called the second law of stochastic thermodynamics [17].

From the bipartite property, $\dot{\sigma}$ can be decomposed into two parts:

$$\dot{\sigma} = \dot{\sigma}^X + \dot{\sigma}^Y. \tag{17}$$

Here, $\dot{\sigma}^X$ and $\dot{\sigma}^Y$ denote the partial entropy production rate due to transitions in X and Y, respectively [48]:

$$\begin{aligned} \dot{\sigma}^X &:= \sum_x \sum_{x'} \sum_y w^y_{xx'} p_t(x', y) \ln \frac{w^y_{xx'} p_t(x', y)}{w^y_{x'x} p_t(x, y)} \\ &= \dot{S}^X_{\text{tot}} - \dot{I}^X, \end{aligned} \tag{18}$$

$$\begin{aligned} \dot{\sigma}^Y &:= \sum_x \sum_y \sum_{y'} w^{yy'}_x p_t(x, y') \ln \frac{w^{yy'}_x p_t(x, y')}{w^{y'y}_x p_t(x, y)} \\ &= \dot{S}^Y_{\text{tot}} - \dot{I}^Y, \end{aligned} \tag{19}$$

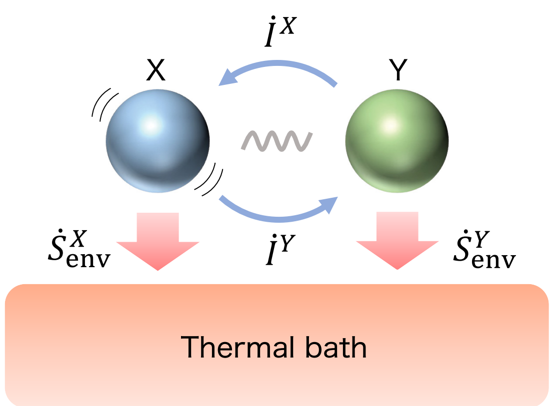

FIG. 1. Schematic of the second law of information thermodynamics. In the case where X (blue) acts as a steady-state information-thermodynamic engine, X converts information ($\dot{I}^X < 0$) into negative entropy production ($\dot{S}^X_{\mathrm{env}} < 0$), while Y (green) gains information about X ($\dot{I}^Y = -\dot{I}^X > 0$) with the thermodynamic cost $\dot{S}^Y_{\mathrm{env}} > 0$. Although only a single thermal bath is depicted here, our results hold even in the presence of multiple thermal baths.

where $\dot{S}^Z_{\mathrm{tot}}$ $(Z = X, Y)$ can be interpreted as the entropy production rate associated with Z, which consists of the time derivative of Z's Shannon entropy $S[Z] := -\sum_z p^Z_t(z)\ln p^Z_t(z)$ and the entropy change in the environment due to transitions in Z:

$$\dot{S}^X_{\mathrm{tot}} := \sum_x \sum_{x'} \sum_y w^y_{xx'} p_t(x', y) \ln \frac{w^y_{xx'} p^X_t(x')}{w^y_{x'x} p^X_t(x)}$$
$$= d_t S[X] + \dot{S}^X_{\mathrm{env}}, \tag{20}$$

$$\dot{S}^Y_{\mathrm{tot}} := \sum_x \sum_y \sum_{y'} w^{yy'}_x p_t(x, y') \ln \frac{w^{yy'}_x p^Y_t(y')}{w^{y'y}_x p^Y_t(y)}$$
$$= d_t S[Y] + \dot{S}^Y_{\mathrm{env}}. \tag{21}$$

From the definition of the partial entropy production rates (18) and (19), it immediately follows that $\dot{\sigma}^X$ and $\dot{\sigma}^Y$ are individually non-negative:

$$\dot{\sigma}^X = \dot{S}^X_{\mathrm{tot}} - \dot{I}^X \geqslant 0, \tag{22}$$

$$\dot{\sigma}^Y = \dot{S}^Y_{\mathrm{tot}} - \dot{I}^Y \geqslant 0. \tag{23}$$

This is the so-called second law of information thermodynamics (see also Fig. 1). The important point here is that $\dot{S}^X_{\mathrm{tot}}$ ($\dot{S}^Y_{\mathrm{tot}}$) can be negative if $\dot{I}^X$ ($\dot{I}^Y$) is negative. This apparent violation of the second law of thermodynamics caused by information flow lies at the heart of the mechanism of Maxwell's demon [27]. In this case, X acts as an information-thermodynamic engine that converts information into output work or negative entropy production.

III. THERMODYNAMIC UNCERTAINTY RELATION FOR BIPARTITE SYSTEMS

In this section, we explain our first main result, which can be regarded as an extension of the standard TUR (1) to bipartite systems. Hereafter, we assume that the whole system is in the steady state. See Appendixes A and B for time-dependent cases.

A. Bipartite TUR

Let $\hat{\mathcal{J}}_{\mathcal{T}}$ be a generalized time-integrated current for the subsystem X with an arbitrary antisymmetric weight $d^y_{xx'} = -d^y_{x'x}$:

$$\hat{\mathcal{J}}_{\mathcal{T}} := \sum_x \sum_{x'(\neq x)} \sum_y \hat{n}^y_{xx'} d^y_{xx'}, \tag{24}$$

where $\hat{n}^y_{xx'}$ denotes the number of transitions from the state (x', y) to (x, y) during the time interval $[0, \mathcal{T}]$. For example, the choice of $d^y_{xx'} = \ln w^y_{xx'}/w^y_{x'x}$ yields the stochastic entropy change in the environment due to transitions in X during $[0, \mathcal{T}]$. In the steady state, the time-integrated stochastic information flow can also be expressed in this form [see (81)]. We remark that when there are multiple environments with the label $\nu = 1, 2, \dots$, the weight $d^y_{xx'}$ can depend on ν. The ensemble average of $\hat{\mathcal{J}}_{\mathcal{T}}$ reads as

$$\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle = \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} \sum_y w^y_{xx'} p_{\mathrm{ss}}(x', y) d^y_{xx'}. \tag{25}$$

Our first main result is the following inequality: for arbitrary observation time $\mathcal{T}$,

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}}]}{\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle^2} \geqslant \frac{2(1+\delta_{\mathcal{J}})^2}{\Delta S^X_{\mathrm{tot}} - \Delta I^X}, \tag{26}$$

where $\Delta S^X_{\mathrm{tot}} := \int_0^{\mathcal{T}} dt\, \dot{S}^X_{\mathrm{tot}}$ and $\Delta I^X := \int_0^{\mathcal{T}} dt\, \dot{I}^X$ denote the entropy production and time-integrated information flow associated with X, respectively. Here, $\delta_{\mathcal{J}} := \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q / \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle$, and $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q$ is defined as

$$\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q := \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} \sum_y w^y_{xx'} q_t(x', y) d^y_{xx'}, \tag{27}$$

where q_t satisfies the following equation with $q_0 = 0$:

$$\partial_t q_t(x, y) = \left[\sum_{x'} w^y_{xx'} q_t(x', y) + \sum_{y'} w^{yy'}_x q_t(x, y') \right]$$
$$+ \sum_{x'} w^y_{xx'} p_{\mathrm{ss}}(x', y). \tag{28}$$

This additional current term $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q$ reflects the contribution of the interaction with Y. Indeed, if the transition rate for X and the weight are independent of Y, i.e., $w^y_{xx'} = w_{xx'}$ and $d^y_{xx'} = d_{xx'}$, we can prove that $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q = 0$. For the derivation, see Appendix B, where we consider the bipartite TUR in a more general case, applicable to a transient state. If, in addition, X and Y are independent and thus $\Delta I^X = 0$, then the bipartite TUR (26) reduces to the standard TUR (1), where the relative fluctuation of the current for X is lower bounded by the entropy production associated with X. In the general case where X and Y are correlated, the bipartite TUR (26) states that the relative fluctuation of the current for the subsystem X is lower bounded not only by the entropy production associated with X, but also by the information transfer between X and Y.

There are two important cases where the additional current term $\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_q$ can be ignored and thus $\delta_{\mathcal{J}} \to 0$. The first case occurs in the short-time limit $\mathcal{T} \to 0$. Since $q_0 = 0$, it immediately follows that $\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_q \to 0$ as $\mathcal{T} \to 0$, while the information flow ΔI^X remains finite in general. The second case occurs in the long-time limit $\mathcal{T} \to \infty$ where there is a separation of timescales between X and Y. That is, the case where the observation time is long and Y evolves much faster than X. The proof of $\delta_{\mathcal{J}} \to 0$ in this case will be described in detail in Sec. III C. Since this case is typically realized when X acts as an information-thermodynamic engine, we will mainly focus on this case in the following sections.

B. Derivation of the bipartite TUR

Here, we prove the bipartite TUR (26) by using the generalized Cramér-Rao inequality [20,36,44,49]. We remark that the bipartite TUR can also be proved more directly from the master equation or Langevin equation [50] (see also Appendix A for the direct derivation for overdamped Langevin equations). We consider the following auxiliary dynamics parametrized by θ with $p_0^\theta = p_{\rm ss}$:

$$\partial_t p_t^\theta(x,y) = \sum_{x'} w_{xx'}^y(\theta) p_t^\theta(x',y) + \sum_{y'} w_x^{yy'} p_t^\theta(x,y'). \tag{29}$$

Here, $w_{xx'}^y(\theta)$ denotes the parametrized transition rate

$$\begin{aligned} w_{xx'}^y(\theta) &:= w_{xx'}^y e^{\theta Z_{xx'}^y} \qquad \text{for} \quad x \neq x', \\ w_{xx}^y(\theta) &:= -\sum_{x'(\neq x)} w_{x'x}^y e^{\theta Z_{x'x}^y}, \end{aligned} \tag{30}$$

where

$$Z_{xx'}^y := \frac{w_{xx'}^y p_{\rm ss}(x',y) - w_{x'x}^y p_{\rm ss}(x,y)}{w_{xx'}^y p_{\rm ss}(x',y) + w_{x'x}^y p_{\rm ss}(x,y)}. \tag{31}$$

Let $\mathbb{P}_\theta(\Gamma)$ be the parametrized path probability for the trajectory $\Gamma = \{x_t, y_t\}_{t=0}^{\mathcal{T}}$,

$$\mathbb{P}_\theta(\Gamma) = p_{\rm ss}(x_0,y_0) \exp\left[-\sum_x\sum_y \left(\sum_{x'(\neq x)} w_{x'x}^y(\theta) + \sum_{y'(\neq y)} w_x^{y'y}\right)\hat{\tau}_x^y + \sum_x\sum_y\left(\sum_{x'(\neq x)} \hat{n}_{xx'}^y \ln w_{xx'}^y(\theta) + \sum_{y'(\neq y)} \hat{n}_x^{yy'} \ln w_x^{yy'}\right)\right], \tag{32}$$

where $\hat{n}_x^{yy'}$ denotes the number of transitions from the state (x,y') to (x,y) during the time interval $[0,\mathcal{T}]$, and $\hat{\tau}_x^y$ denotes the empirical dwell time in state (x,y), defined as the total amount of time spent in state (x,y) along the trajectory Γ:

$$\hat{\tau}_x^y := \int_0^{\mathcal{T}} dt\, \delta_{xx_t}\delta^{yy_t}, \tag{33}$$

where δ_{xx_t} (δ^{yy_t}) denotes the Kronecker delta, which is 1 if $x = x_t$ $(y = y_t)$, and zero otherwise. We denote by $\mathbb{I}(\theta) := -\langle\partial_\theta^2 \ln \mathbb{P}_\theta(\Gamma)\rangle_\theta$ the corresponding Fisher information [44], where $\langle\cdot\rangle_\theta$ denotes the average with respect to $\mathbb{P}_\theta$. The generalized Cramér-Rao inequality then yields [20,36,44,49]

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}}]}{\left(\partial_\theta\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_\theta|_{\theta=0}\right)^2} \geqslant \frac{1}{\mathbb{I}(0)}. \tag{34}$$

Here, $\mathbb{I}(0)$ can be expressed as

$$\begin{aligned} \mathbb{I}(0) &= \left\langle \sum_x\sum_y\sum_{x'(\neq x)} \frac{\partial^2}{\partial\theta^2} w_{x'x}^y(\theta)\hat{\tau}_x^y - \sum_x\sum_y\sum_{x'(\neq x)}\frac{\partial^2}{\partial\theta^2}\ln w_{xx'}^y(\theta)\hat{n}_{xx'}^y\right\rangle_\theta\Bigg|_{\theta=0} \\ &= \int_0^{\mathcal{T}} dt \sum_x\sum_{x'(\neq x)}\sum_y w_{xx'}^y p_{\rm ss}(x',y)\left(Z_{xx'}^y\right)^2 \\ &= \frac{1}{2}\int_0^{\mathcal{T}} dt \sum_x\sum_{x'(\neq x)}\sum_y \frac{\left[w_{xx'}^y p_{\rm ss}(x',y) - w_{x'x}^y p_{\rm ss}(x,y)\right]^2}{w_{xx'}^y p_{\rm ss}(x',y) + w_{x'x}^y p_{\rm ss}(x,y)} \\ &\leqslant \frac{1}{2}\int_0^{\mathcal{T}} dt \sum_x\sum_{x'(\neq x)}\sum_y w_{xx'}^y p_{\rm ss}(x',y)\ln\frac{w_{xx'}^y p_{\rm ss}(x',y)}{w_{x'x}^y p_{\rm ss}(x,y)} = \frac{\Delta S_{\rm tot}^X - \Delta I^X}{2}, \end{aligned} \tag{35}$$

where we have used the inequality $2(a-b)^2/(a+b) \leqslant (a-b)\ln a/b$. To calculate $\partial_\theta\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_\theta|_{\theta=0}$, we expand $p_t^\theta(x,y)$ in terms of θ as $p_t^\theta(x,y) = p_{\rm ss}(x,y) + \theta q_t(x,y) + O(\theta^2)$. By substituting this expression into (29), we find that

$$\partial_t q_t(x,y) = \left[\sum_{x'} w_{xx'}^y q_t(x',y) + \sum_{y'} w_x^{yy'} q_t(x,y')\right] + \sum_{x'} w_{xx'}^y p_{\rm ss}(x',y). \tag{36}$$

Then, $\partial_\theta \langle \hat{\mathcal{J}}_\mathcal{T} \rangle_\theta |_{\theta=0}$ can be calculated as

$$\begin{aligned}
&\partial_\theta \langle \hat{\mathcal{J}}_\mathcal{T} \rangle_\theta |_{\theta=0} \\
&= \partial_\theta \int_0^\mathcal{T} dt \sum_x \sum_{x'(\neq x)} \sum_y w^y_{xx'}(\theta) p^\theta_t(x', y) d^y_{xx'} \Bigg|_{\theta=0} \\
&= \langle \hat{\mathcal{J}}_\mathcal{T} \rangle + \langle \hat{\mathcal{J}}_\mathcal{T} \rangle_q .
\end{aligned} \tag{37}$$

We thus arrive at the inequality (26).

C. Fast relaxation limit of Y

Here, we show that $\langle \hat{\mathcal{J}}_\mathcal{T} \rangle_q \ll \langle \hat{\mathcal{J}}_\mathcal{T} \rangle$ in the long-time limit $\mathcal{T} \to \infty$ if Y relaxes much faster than X. Let τ_X and τ_Y be the timescale of X and Y, respectively. We assume a separation of timescales: $\tau_Y \ll \tau_X$, i.e., the auxiliary system Y evolves much faster than the system of interest X. This situation is typically realized when Y acts as a Maxwell's demon, i.e., when Y measures the state of X and performs feedback control [45]. We introduce a dimensionless slow time $\tau := t/\tau_X$ and a small parameter $\epsilon := \tau_Y/\tau_X \ll 1$. Correspondingly, we nondimensionalize the transition rates as $\widetilde{w}^y_{xx'} := \tau_X w^y_{xx'}$ and $\widetilde{w}^{yy'}_x := \tau_Y w^{yy'}_x$.

We first take the long-time limit $\mathcal{T} \to \infty$, i.e., $\mathcal{T} \gg \tau_X$, and assume that $q_t(x, y)$ reaches a stationary solution $q_{ss}(x, y)$. Then, p_{ss} and q_{ss} satisfy the following equations:

$$\sum_{x'} \widetilde{w}^y_{xx'} p_{ss}(x', y) + \frac{1}{\epsilon} \sum_{y'} \widetilde{w}^{yy'}_x p_{ss}(x, y') = 0, \tag{38}$$

$$\left[\sum_{x'} \widetilde{w}^y_{xx'} q_{ss}(x', y) + \frac{1}{\epsilon} \sum_{y'} \widetilde{w}^{yy'}_x q_{ss}(x, y') \right] + \sum_{x'} \widetilde{w}^y_{xx'} p_{ss}(x', y) = 0. \tag{39}$$

We now assume that p_{ss} and q_{ss} have asymptotic expansions in terms of the asymptotic sequences $\{\epsilon^n\}_{n=0}^\infty$ as $\epsilon \to 0$:

$$p_{ss} = p^{(0)}_{ss} + \epsilon p^{(1)}_{ss} + \cdots , \tag{40}$$

$$q_{ss} = q^{(0)}_{ss} + \epsilon q^{(1)}_{ss} + \cdots . \tag{41}$$

Here, we impose the normalization condition

$$\sum_y p^{(0)}_{ss}(x, y) = p^X_{ss}(x), \tag{42}$$

$$\sum_y q^{(0)}_{ss}(x, y) = q^X_{ss}(x), \tag{43}$$

where we have introduced $q^X_{ss}(x) := \sum_y q_{ss}(x, y)$. Note that q^X_{ss} satisfies the normalization condition $\sum_x q^X_{ss}(x) = 0$.

By substituting these expansions into (38) and (39), we find that the leading order yields

$$\sum_{y'} \widetilde{w}^{yy'}_x p^{(0)}_{ss}(x, y') = 0, \tag{44}$$

$$\sum_{y'} \widetilde{w}^{yy'}_x q^{(0)}_{ss}(x, y') = 0. \tag{45}$$

Let $\pi_{ss}(y|x)$ be the normalized zero eigenvector that satisfies $\sum_{y'} \widetilde{w}^{yy'}_x \pi_{ss}(y'|x) = 0$. Due to the irreducibility of the rate matrix, this normalized zero eigenvector is unique for each x. Then, from the normalization condition, $p^{(0)}_{ss}$ and $q^{(0)}_{ss}$ should have the form

$$p^{(0)}_{ss}(x, y) = p^X_{ss}(x) \pi_{ss}(y|x), \tag{46}$$

$$q^{(0)}_{ss}(x, y) = q^X_{ss}(x) \pi_{ss}(y|x). \tag{47}$$

The subleading order of (38) and (39) yields

$$\sum_{x'} \widetilde{w}^y_{xx'} p^{(0)}_{ss}(x', y) + \sum_{y'} \widetilde{w}^{yy'}_x p^{(1)}_{ss}(x, y') = 0, \tag{48}$$

$$\left[\sum_{x'} \widetilde{w}^y_{xx'} q^{(0)}_{ss}(x', y) + \sum_{y'} \widetilde{w}^{yy'}_x q^{(1)}_{ss}(x, y') \right] + \sum_{x'} \widetilde{w}^y_{xx'} p^{(0)}_{ss}(x', y) = 0. \tag{49}$$

Note that (48) and (49) are linear equations for $p^{(1)}_{ss}$ and $q^{(1)}_{ss}$ with the matrix $\widetilde{w}^{yy'}_x$, which has the left zero eigenvector 1 because $\sum_y \widetilde{w}^{yy'}_x = 0$. This property guarantees that the solutions $p^{(1)}_{ss}$ and $q^{(1)}_{ss}$ exist only under the solvability conditions

$$\sum_{x'} \overline{w}_{xx'} p^X_{ss}(x') = 0, \tag{50}$$

$$\sum_{x'} \overline{w}_{xx'} q^X_{ss}(x') = 0, \tag{51}$$

which correspond to (48) and (49) summed over y, respectively. Here, we have introduced the effective transition rate $\overline{w}_{xx'} := \sum_y \widetilde{w}^y_{xx'} \pi_{ss}(y|x')$. Then, from the Perron-Frobenius theorem, q^X_{ss} can be expressed as $q^X_{ss} = \mathcal{N} p^X_{ss}$, where $\mathcal{N}$ denotes the normalization constant. Because q^X_{ss} satisfies the normalization condition $\sum_x q^X_{ss}(x) = 0$, we obtain $\mathcal{N} = 0$. Thus, in the fast relaxation limit of Y, we have

$$p_{ss}(x, y) = p^X_{ss}(x) \pi_{ss}(y|x) + O(\epsilon), \tag{52}$$

$$q_{ss}(x, y) = O(\epsilon). \tag{53}$$

Therefore, the additional current term $\langle \hat{\mathcal{J}}_\mathcal{T} \rangle_q$ appearing in the bipartite TUR (26) is much smaller than $\langle \hat{\mathcal{J}}_\mathcal{T} \rangle$ in the long-time limit $\mathcal{T} \to \infty$:

$$\delta_\mathcal{J} := \frac{\langle \hat{\mathcal{J}}_\mathcal{T} \rangle_q}{\langle \hat{\mathcal{J}}_\mathcal{T} \rangle} \to 0. \tag{54}$$

To summarize, in the fast relaxation limit of Y, the bipartite TUR (26) reduces to the form similar to the standard TUR (1) in the long-time limit:

$$\frac{D_\mathcal{J}}{J^2} \geqslant \frac{1}{\dot{S}^X_{tot} - \dot{I}^X}, \tag{55}$$

where $D_\mathcal{J} := \lim_{\mathcal{T}\to\infty} \mathrm{Var}[\hat{\mathcal{J}}_\mathcal{T}]/2\mathcal{T}$ denotes the fluctuation of $\hat{\mathcal{J}}_\mathcal{T}$, and $J := \lim_{\mathcal{T}\to\infty} \langle \hat{\mathcal{J}}_\mathcal{T} \rangle / \mathcal{T}$ denotes the mean current. Note that (55) gives a tighter lower bound on the fluctuation of a current than the standard TUR because $\dot{S}^X_{tot} - \dot{I}^X$ is smaller than or equal to the total entropy production $\dot{\sigma}$. If the partial entropy production of Y is zero, then (55) can also be obtained from the standard TUR.

D. Equality condition

The equality of the bipartite TUR in the fast relaxation limit of Y (55) can be achieved even far from equilibrium. This nontrivial fact will be shown later with a simple example in Sec. VI B. This property is a stark difference from the standard TUR, where the equality is guaranteed only in the near-equilibrium limit [20,36]. Here, before showing the example in Sec. VI B, we discuss a possible scenario to achieve the equality of the bipartite TUR (26) in a somewhat abstract manner.

We first consider the equality condition of the generalized Cramér-Rao inequality at $\theta = 0$ (34). Because the generalized Cramér-Rao inequality is based on the Cauchy-Schwarz inequality, the equality condition is satisfied if and only if the following relation holds [36]:

$$\hat{\mathcal{J}}_{\mathcal{T}} - \langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle = C\frac{\partial}{\partial\theta}\ln\mathbb{P}_\theta(\Gamma)\bigg|_{\theta=0}, \tag{56}$$

where C is a constant. The right-hand side of (56) is given by

$$C\frac{\partial}{\partial\theta}\ln\mathbb{P}_\theta(\Gamma)\bigg|_{\theta=0} = C\sum_x\sum_{x'(\neq x)}\sum_y Z^y_{xx'}\left(\hat{n}^y_{xx'} - w^y_{xx'}\hat{\tau}^y_{x'}\right). \tag{57}$$

The left-hand side of (56) reads as

$$\hat{\mathcal{J}}_{\mathcal{T}} - \langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle = \hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}} + \hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}} - \langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle, \tag{58}$$

where we have decomposed the current $\hat{\mathcal{J}}_{\mathcal{T}}$ into two parts [50]:

$$\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}} := \sum_x\sum_{x'(\neq x)}\sum_y d^y_{xx'}\left(\hat{n}^y_{xx'} - w^y_{xx'}\hat{\tau}^y_{x'}\right), \tag{59}$$

$$\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}} := \sum_x\sum_{x'(\neq x)}\sum_y d^y_{xx'}w^y_{xx'}\hat{\tau}^y_{x'}. \tag{60}$$

Note that $\langle\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}\rangle = 0$ and $\langle\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}\rangle = \langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle$. By comparing (57) and (58), we expect that $d^y_{xx'} = CZ^y_{xx'}$ to be the optimal choice. However, due to the presence of $\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}} - \langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle$, the equality condition is generally not satisfied.

In the standard TUR, it is known that the equality can be achieved by including the generalized time-integrated static observable in addition to the current $\hat{\mathcal{J}}_{\mathcal{T}}$ [20,50,51]. Here, we consider the following generalized time-integrated static observable:

$$\hat{\mathcal{O}}_{\mathcal{T}} := \sum_x\sum_y \rho^y_x\hat{\tau}^y_x, \tag{61}$$

where ρ^y_x is an arbitrary weight that depends on a state (x, y). Even for the observable $\hat{\mathcal{J}}_{\mathcal{T}} + \hat{\mathcal{O}}_{\mathcal{T}}$ instead of $\hat{\mathcal{J}}_{\mathcal{T}}$, by following the same argument described in Sec. III B, we can derive the following *bipartite-correlation* TUR:

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}} + \hat{\mathcal{O}}_{\mathcal{T}}]}{\left[\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle + \langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_q + \langle\hat{\mathcal{O}}_{\mathcal{T}}\rangle_q\right]^2} \geqslant \mathbb{I}(0) \tag{62}$$

$$\geqslant \frac{2}{\Delta S^X_{\mathrm{tot}} - \Delta I^X}, \tag{63}$$

where $\langle\hat{\mathcal{O}}_{\mathcal{T}}\rangle_q$ is defined as

$$\langle\hat{\mathcal{O}}_{\mathcal{T}}\rangle_q := \int_0^{\mathcal{T}} dt\sum_x\sum_y q_t(x, y)\rho^y_x. \tag{64}$$

In this case, the equality condition of the inequality (62) is given by

$$\hat{\mathcal{J}}_{\mathcal{T}} + \hat{\mathcal{O}}_{\mathcal{T}} - \langle\hat{\mathcal{J}}_{\mathcal{T}} + \hat{\mathcal{O}}_{\mathcal{T}}\rangle = C\frac{\partial}{\partial\theta}\ln\mathbb{P}_\theta(\Gamma)\bigg|_{\theta=0}. \tag{65}$$

Then, we find that the choice

$$d^y_{xx'} = CZ^y_{xx'}, \tag{66}$$

$$\rho^y_x = -\sum_{x'(\neq x)} d^y_{x'x}w^y_{x'x} \tag{67}$$

yields

$$\hat{\mathcal{J}}_{\mathcal{T}} + \hat{\mathcal{O}}_{\mathcal{T}} - \langle\hat{\mathcal{J}}_{\mathcal{T}} + \hat{\mathcal{O}}_{\mathcal{T}}\rangle = \hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}} = C\frac{\partial}{\partial\theta}\ln\mathbb{P}_\theta(\Gamma)\bigg|_{\theta=0}, \tag{68}$$

where we have used the fact that $\hat{\mathcal{O}}_{\mathcal{T}} = -\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}$ for this choice. Thus, the equality of (62) is achieved for this choice of $\hat{\mathcal{J}}_{\mathcal{T}}$ and $\hat{\mathcal{O}}_{\mathcal{T}}$. However, even if we can achieve the equality of (62), the equality condition of the second inequality (63) is not satisfied in general. Still, in the overdamped Langevin case, the inequality $\mathbb{I}(0) \geqslant 2/(\Delta S^X_{\mathrm{tot}} - \Delta I^X)$ becomes an equality (see [52] for the detailed discussion). Therefore, the equality of the bipartite-correlation TUR is achieved in the overdamped Langevin case even far from equilibrium for the choice (66) and (67):

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}]}{\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle^2} = \frac{2}{\Delta S^X_{\mathrm{tot}} - \Delta I^X}, \tag{69}$$

where we have used $\langle\hat{\mathcal{O}}_{\mathcal{T}}\rangle_q = -\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_q$. In the long-time limit $\mathcal{T}\to\infty$, (69) can be rewritten as

$$\frac{D^{\mathrm{I}}_{\mathcal{J}}}{J^2} = \frac{1}{\dot{S}^X_{\mathrm{tot}} - \dot{I}^X}, \tag{70}$$

where $D^{\mathrm{I}}_{\mathcal{J}} := \lim_{\mathcal{T}\to\infty}\mathrm{Var}[\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}]/2\mathcal{T}$ denotes the fluctuation of $\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}$. Note that $D^{\mathrm{I}}_{\mathcal{J}}$ is generally different from $D_{\mathcal{J}}$, and thus the equality (70) does generally not correspond to the equality of the bipartite TUR in the fast relaxation limit of Y (55). To put it another way, if $D^{\mathrm{I}}_{\mathcal{J}} = D_{\mathcal{J}}$ in the fast relaxation limit, then the equality of (55) is achieved. Since $\langle\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}\rangle = 0$, the covariance between $\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}$ and $\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}$ reads as

$$\begin{aligned}\mathrm{Cov}[\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}, \hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}] &= \langle\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}\rangle \\ &= C\left\langle\frac{\partial}{\partial\theta}\ln\mathbb{P}_\theta(\Gamma)\bigg|_{\theta=0}\hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}\right\rangle \\ &= C\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_q.\end{aligned} \tag{71}$$

Therefore, the difference between $D^{\mathrm{I}}_{\mathcal{J}}$ and $D_{\mathcal{J}}$ is given by

$$\begin{aligned}D_{\mathcal{J}} - D^{\mathrm{I}}_{\mathcal{J}} &= \lim_{\mathcal{T}\to\infty}\frac{1}{2\mathcal{T}}\mathrm{Var}[\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}} + \hat{\mathcal{J}}^{\mathrm{II}}_{\mathcal{T}}] - \lim_{\mathcal{T}\to\infty}\frac{1}{2\mathcal{T}}\mathrm{Var}[\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}] \\ &= D^{\mathrm{II}}_{\mathcal{J}} + \lim_{\mathcal{T}\to\infty}\frac{C}{\mathcal{T}}\langle\hat{\mathcal{J}}_{\mathcal{T}}\rangle_q \\ &= D^{\mathrm{II}}_{\mathcal{J}},\end{aligned} \tag{72}$$

in the fast relaxation limit of Y. Thus, $D_{\mathcal{J}}^{\rm II}=0$ is a sufficient condition for (55) to hold with equality. In Sec. VI B, we give an example that satisfies this sufficient condition.

IV. TRADEOFF RELATIONS

In this section, we focus on the regime where Y evolves much faster than X and show that the bipartite TUR in this regime (55) provides tradeoff relations for the performance of information processing systems. In Sec. IV A, we consider the situation where the subsystem X can be regarded as a steady-state information-thermodynamic engine, while the external system Y plays the role of a memory of Maxwell's demon. From the second law of information thermodynamics (22), this situation corresponds to the case where $0<-\dot{S}_{\rm env}^X\leqslant -\dot{I}^X$. While this situation may be typical in the regime of the fast relaxation limit of Y, we can also consider the case where the slow system X plays the role of a memory and measures the state of the fast system Y, i.e., $0<\dot{I}^X\leqslant \dot{S}_{\rm env}^X$. Even for this case, we can show that the bipartite TUR (55) provides tradeoff relations on the performance of the memory, which will be described in Sec. IV B.

A. Information-thermodynamic engine: $0<-\dot{S}_{\rm env}^X\leqslant -\dot{I}^X$

Here, we show that the bipartite TUR gives several universal bounds on the performance of information-thermodynamic engines. In this case, both the entropy production and the information flow associated with X are negative and satisfy the relation $0<-\dot{S}_{\rm env}^X\leqslant -\dot{I}^X$. Then, the performance of an information-thermodynamic engine can be quantified by, e.g., the information-thermodynamic efficiency [45]:

$$\eta_S^X := \frac{\left|\dot{S}_{\rm env}^X\right|}{|\dot{I}^X|}, \tag{73}$$

which satisfies $0\leqslant \eta_S^X\leqslant 1$ as a direct consequence of the second law of information thermodynamics. This efficiency quantifies how efficiently the engine X converts information into negative entropy production. In addition to this information-thermodynamic efficiency, the negative entropy production rate itself is an important indicator characterizing the performance of an information-thermodynamic engine. Here, we show that there is the following tradeoff relation between η_S^X and $|\dot{S}_{\rm env}^X|$:

$$\left|\dot{S}_{\rm env}^X\right| \leqslant D_S\frac{1-\eta_S^X}{\eta_S^X}, \tag{74}$$

where D_S denotes the fluctuation of the stochastic medium entropy production $\Delta\hat{S}_{\rm env}^X$,

$$D_S := \lim_{\mathcal{T}\to\infty}\frac{1}{2\mathcal{T}}\mathrm{Var}\left[\Delta\hat{S}_{\rm env}^X\right]. \tag{75}$$

This inequality states that an information engine with a finite negative entropy production rate cannot achieve $\eta_S^X=1$ as long as the fluctuation D_S is finite. In order to achieve a finite negative entropy production rate with $\eta_S^X=1$, the fluctuation D_S must diverge. We can also prove a similar tradeoff relation where the negative entropy production rate is bounded by the fluctuation of the time-integrated stochastic information flow $\Delta\hat{I}^X$ instead of D_S:

$$\left|\dot{S}_{\rm env}^X\right| \leqslant D_I\eta_S^X\left(1-\eta_S^X\right), \tag{76}$$

where

$$D_I := \lim_{\mathcal{T}\to\infty}\frac{1}{2\mathcal{T}}\mathrm{Var}[\Delta\hat{I}^X]. \tag{77}$$

The inequalities (74) and (76) are the second main results of this paper.

In Sec. IV A 1, we provide detailed proof of these inequalities. In Sec. IV A 2, we briefly discuss which of the two inequalities (74) and (76) gives a tighter bound on the negative entropy production rate. In Sec. IV A 3, we derive a tradeoff relation in terms of power, i.e., output work produced per unit of time, instead of the negative entropy production. This relation can be regarded as a direct extension of the tradeoffs for heat engines [21,22] to information-thermodynamic engines.

1. Derivation of (74) and (76)

Here, we derive the tradeoff relations (74) and (76) by using the bipartite TUR in the fast relaxation limit of Y (55), which can be rewritten as follows:

$$|J| \leqslant D_{\mathcal{J}}\frac{\dot{S}_{\rm env}^X-\dot{I}^X}{|J|}. \tag{78}$$

Let us choose stochastic medium entropy production associated with X as current $\hat{\mathcal{J}}_{\mathcal{T}}$ in (78):

$$\begin{aligned}\hat{\mathcal{J}}_{\mathcal{T}} &= \Delta\hat{S}_{\rm env}^X\\ &:= \sum_x\sum_{x'(\neq x)}\sum_y \hat{n}_{xx'}^y\ln\frac{w_{xx'}^y}{w_{x'x}^y},\end{aligned} \tag{79}$$

which satisfies $\langle\Delta\hat{S}_{\rm env}^X\rangle=\Delta S_{\rm env}^X$. Then, we immediately obtain the tradeoff between entropy production and efficiency (74).

Another type of the tradeoff relation (76) can be derived by choosing the time-integrated stochastic information flow as current $\hat{\mathcal{J}}_{\mathcal{T}}$ in (78). Here, the instantaneous stochastic information flow is defined as the partial rate of change of the stochastic mutual information $\hat{I}(x_t:y_t) := \ln[p_t(x_t,y_t)/p_t^X(x_t)p_t^Y(y_t)]$:

$$\begin{aligned}\hat{\dot{I}}^X &:= \sum_n\delta(t-t_n)\ln\frac{p_{t_n}(y_{t_n}|x_{t_n^+})}{p_{t_n}(y_{t_n}|x_{t_n^-})}\\ &\quad+\left.\frac{1}{p_t(x,y)}\sum_{x'}w_{xx'}^y p_t(x',y)\right|_{(x,y)=(x_t,y_t)}\\ &\quad-\left.\frac{1}{p_t^X(x)}\sum_{x'}\overline{w}_{xx'}p_t^X(x')\right|_{x=x_t},\end{aligned} \tag{80}$$

where t_n denotes the time at which X jumps from $x_{t_n^-}$ to $x_{t_n^+}$, and $\overline{w}_{xx'}:=\sum_y w_{xx'}^y p_t(y|x')$ denotes the effective transition rate. In the steady state, the last two terms vanish so that the time-integrated stochastic information flow reads as

$$\Delta\hat{I}^X = \sum_x\sum_{x'(\neq x)}\sum_y\hat{n}_{xx'}^y\ln\frac{\pi_{\rm ss}(y|x)}{\pi_{\rm ss}(y|x')}, \tag{81}$$

which satisfies $\langle\Delta\hat{I}^X\rangle=\Delta I^X$. Substituting $\hat{\mathcal{J}}_{\mathcal{T}}=\Delta\hat{I}^X$ in the bipartite TUR (78), we obtain the inequality (76). Note that we can also obtain a tradeoff relation between the information flow and efficiency:

$$|\dot{I}^X| \leqslant D_I\big(1-\eta_S^X\big). \tag{82}$$

2. Tightness of the bounds

Here, we consider which of the two inequalities (74) and (76) gives a tighter bound on the negative entropy production rate. The difference between the two upper bounds reads as

$$D_S\frac{1-\eta_S^X}{\eta_S^X} - D_I\eta_S^X\big(1-\eta_S^X\big) = \frac{1-\eta_S^X}{\eta_S^X}D_I\bigg[\frac{D_S}{D_I} - \big(\eta_S^X\big)^2\bigg]. \tag{83}$$

Therefore, the bound (74) is tighter than (76) when $\sqrt{D_S/D_I} < \eta_S^X \leqslant 1$, while (76) becomes tighter than (74) when $0 \leqslant \eta_S^X < \sqrt{D_S/D_I}$. Note that D_S/D_I may depend on η_S^X.

In the linear response regime with $\dot{S}_{\rm env}^X \leqslant 0$ and $\dot{I}^X \leqslant 0$, we can prove the input-output fluctuation inequality $D_S \leqslant D_I$ (for the derivation, see Sec. V B). Beyond the linear response regime, however, the input-output fluctuation inequality can be violated, i.e., D_S can become larger than D_I [26].

3. Tradeoff between power and efficiency

While we have focused on the negative entropy production rate to characterize the performance of an information-thermodynamic engine, we can also derive a tradeoff relation in terms of power, i.e., output work produced per unit of time. To define power, we assume that the transition rates satisfy the local detailed balance condition of the following form [17]:

$$\ln\frac{w_{xx'}^y}{w_{x'x}^y} = \beta\big(\epsilon_{x'y} - \epsilon_{xy} + \Delta_{xx'}^y\big), \tag{84}$$

where $\beta = (k_{\rm B}T)^{-1}$ denotes the inverse temperature, ϵ_{xy} denotes the energy of the state (x, y), and $\Delta_{xx'}^y$ denotes the energy provided by an external agent during the transition $(x', y) \to (x, y)$. Then, the average rate of heat absorbed by X from the environment is identified as

$$\dot{Q}^X = -k_{\rm B}T\sum_x\sum_{x'(\neq x)}\sum_y w_{xx'}^y p_t(x', y)\ln\frac{w_{xx'}^y}{w_{x'x}^y}. \tag{85}$$

Similarly, the average rate of work done by the external agent to X is identified as

$$\dot{W}^X := \sum_x\sum_{x'(\neq x)}\sum_y w_{xx'}^y p_t(x', y)\Delta_{xx'}^y. \tag{86}$$

Finally, the average rate of change of internal energy reads as

$$\dot{E}^X := \sum_x\sum_{x'(\neq x)}\sum_y w_{xx'}^y p_t(x', y)(\epsilon_{xy} - \epsilon_{x'y}). \tag{87}$$

If we regard x as an externally manipulated control parameter driving Y, then $\dot{E}^X$ can also be identified as the power delivered from X to Y [12,53]:

$$\dot{E}^X = \dot{W}^{X\to Y}. \tag{88}$$

Similarly, we can define $\dot{W}^Y$, $\dot{Q}^Y$, and $\dot{W}^{Y\to X}$. Then, the first law of stochastic thermodynamics for each subsystem can be expressed as follows (in an averaged form):

$$\dot{W}^{X\to Y} = \dot{W}^X + \dot{Q}^X, \tag{89}$$

$$\dot{W}^{Y\to X} = \dot{W}^Y + \dot{Q}^Y. \tag{90}$$

By using these relations, we can rewrite the second law of information thermodynamics in the steady state as

$$\beta\dot{W}^X - \beta\dot{W}^{X\to Y} - \dot{I}^X \geqslant 0, \tag{91}$$

$$\beta\dot{W}^Y - \beta\dot{W}^{Y\to X} - \dot{I}^Y \geqslant 0. \tag{92}$$

Here, we have assumed that both X and Y are each in contact with a thermal bath at temperature T, while the extension to the case of different temperatures is straightforward [13]. Note that $\dot{W}^{X\to Y} = -\dot{W}^{Y\to X}$ and $\dot{I}^X = -\dot{I}^Y$ in the steady state. Therefore, $\dot{W}^X$ and $\dot{W}^Y$ cannot both be negative.

Now, suppose that X operates as an information-thermodynamic engine, i.e., $\dot{W}^Y > 0$ and $\dot{W}^X < 0$. In this case, we can introduce the following efficiency:

$$\eta_W^X := \frac{|\beta\dot{W}^X|}{\beta\dot{W}^{Y\to X} + \dot{I}^Y}, \tag{93}$$

which satisfies $0 \leqslant \eta_W^X \leqslant 1$, as can be seen from the second law of information thermodynamics. The denominator $\beta\dot{W}^{Y\to X} + \dot{I}^Y = -\beta\dot{W}^{X\to Y} - \dot{I}^X \geqslant 0$ is called the *transduced capacity* [11,53] because it constraints the conversion of the input power $\dot{W}^Y$ into the output power $|\dot{W}^X|$ as $\beta\dot{W}^Y \geqslant \beta\dot{W}^{Y\to X} + \dot{I}^Y \geqslant |\beta\dot{W}^X|$. The efficiency η_W^X quantifies how efficiently X converts the transduced capacity into the output power $|\dot{W}^X|$.

Now we derive a tradeoff relation between the output power and the efficiency η_W^X by using the bipartite TUR (78). Let us choose the stochastic work as current $\hat{\mathcal{J}}_\mathcal{T}$ in (78):

$$\begin{aligned}\hat{\mathcal{J}}_\mathcal{T} &= \Delta\hat{W}^X \\ &:= \sum_x\sum_{x'(\neq x)}\sum_y \hat{n}_{xx'}^y\Delta_{xx'}^y.\end{aligned} \tag{94}$$

Then, the bipartite TUR gives

$$|\dot{W}^X| \leqslant \frac{D_W}{k_{\rm B}T}\frac{1-\eta_W^X}{\eta_W^X}, \tag{95}$$

where D_W denotes the fluctuation of the output work defined by

$$D_W := \lim_{\mathcal{T}\to\infty}\frac{1}{2\mathcal{T}}{\rm Var}[\Delta\hat{W}^X]. \tag{96}$$

The inequality (95) states that an information engine with a finite output power cannot achieve $\eta_W^X = 1$ as long as the fluctuation D_W is finite.

B. Memory: $0 < \dot{I}^X \leqslant \dot{S}_{\rm env}^X$

Although we have assumed that X evolves much slower than Y, there may be a situation where X measures the state of Y, i.e., $\dot{I}^X > 0$. Even in this case, we can also prove similar tradeoff relations concerning the performance of the memory

X. We first note that both the entropy production and the information flow associated with X are positive and satisfy the relation $0 < \dot{I}^X \leqslant \dot{S}^X_{\rm env}$. Then, we can introduce the following information-thermodynamic efficiency:

$$\eta_I^X := \frac{\dot{I}^X}{\dot{S}^X_{\rm env}}, \tag{97}$$

which satisfies $0 \leqslant \eta_I^X \leqslant 1$. In contrast to η_S^X, this efficiency quantifies how efficiently X gains information about Y relative to the energy dissipation or thermodynamic cost. Now we choose the time-integrated stochastic information flow as current $\hat{\mathcal{J}}_{\mathcal{T}}$ in the bipartite TUR (78). By noting the positivity of $\dot{S}^X_{\rm env}$ and $\dot{I}^X$, the bipartite TUR gives the following inequality:

$$\dot{I}^X \leqslant D_I \frac{1-\eta_I^X}{\eta_I^X}. \tag{98}$$

This inequality states that a memory with a finite information flow can never attain $\eta_I^X = 1$ as long as D_I is finite.

If we choose the stochastic entropy production as current, $\hat{\mathcal{J}}_{\mathcal{T}} = \Delta\hat{S}^X_{\rm env}$, then we can obtain a similar tradeoff relation where the information flow is bounded by the fluctuation of the stochastic entropy production D_S instead of D_I:

$$\dot{I}^X \leqslant D_S \eta_I^X \left(1-\eta_I^X\right). \tag{99}$$

V. GALLAVOTTI-COHEN SYMMETRY AND INPUT-OUTPUT FLUCTUATION INEQUALITIES

In this section, we prove that the Gallavotti-Cohen symmetry [33–35] is satisfied in the fast relaxation limit of Y. As a consequence of this symmetry, we can further show that the input-output fluctuation inequalities hold in the linear response regime even in the presence of an information flow.

A. Gallavotti-Cohen symmetry

Let $\mu(\lambda_S, \lambda_I)$ be the scaled cumulant generating function of the time-integrated currents $\Delta\hat{S}^X_{\rm env}$ and $\Delta\hat{I}^X$ defined by

$$\mu(\lambda_S, \lambda_I) := \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln\left\langle e^{\lambda_S \Delta\hat{S}^X_{\rm env} - \lambda_I \Delta\hat{I}^X}\right\rangle, \tag{100}$$

where λ_S and λ_I are the counting fields for $\Delta\hat{S}^X_{\rm env}$ and $\Delta\hat{I}^X$, respectively. In this section, we prove that $\mu(\lambda_S, \lambda_I)$ satisfies the following Gallavotti-Cohen symmetry in the fast relaxation limit of Y:

$$\mu(\lambda_S, \lambda_I) = \mu(-\lambda_S - 1, -\lambda_I - 1). \tag{101}$$

To prove this, we first note that $\mu(\lambda_S, \lambda_I)$ can be rewritten as

$$\mu(\lambda_S, \lambda_I) = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln \sum_x \sum_y G_{\mathcal{T}}(x, y), \tag{102}$$

where $G_{\mathcal{T}}(x, y)$ denotes the generating function conditioned to a final state (x, y):

$$G_{\mathcal{T}}(x, y) := \int d\Delta S^X_{\rm env} d\Delta I^X p_{\mathcal{T}}\left(x, y, \Delta S^X_{\rm env}, \Delta I^X\right) \times e^{\lambda_S \Delta S^X_{\rm env} - \lambda_I \Delta I^X}, \tag{103}$$

where $p_{\mathcal{T}}(x, y, \Delta S^X_{\rm env}, \Delta I^X)$ denotes the joint probability density such that the state of the system at time $\mathcal{T}$ is (x, y) and the entropy production and information flow generated up to that time are $\Delta S^X_{\rm env}$ and ΔI^X, respectively. Therefore, the property of the scaled cumulant generating function $\mu(\lambda_S, \lambda_I)$ is encoded in the property of the time-evolution equation of the generating function $G_{\mathcal{T}}(x, y)$. The time-evolution equation of $G_{\mathcal{T}}(x, y)$ can be obtained by noting that the time-evolution equation of $p_{\mathcal{T}}(x, y, \Delta S^X_{\rm env}, \Delta I^X)$ reads as

$$\partial_\tau p_\tau\left(x, y, \Delta S^X_{\rm env}, \Delta I^X\right) = \sum_{x'(\neq x)} \left[\widetilde{w}^y_{xx'} p_\tau\left(x', y, \Delta S^X_{\rm env} - \ln\frac{\widetilde{w}^y_{xx'}}{\widetilde{w}^y_{x'x}}, \Delta I^X - \ln\frac{\pi_{\rm ss}(y|x)}{\pi_{\rm ss}(y|x')}\right) - \widetilde{w}^y_{x'x} p_\tau\left(x, y, \Delta S^X_{\rm env}, \Delta I^X\right)\right]$$
$$+ \frac{1}{\epsilon} \sum_{y'(\neq y)} \left[\widetilde{w}^{yy'}_x p_\tau\left(x, y', \Delta S^X_{\rm env}, \Delta I^X\right) - \widetilde{w}^{y'y}_x p_\tau\left(x, y, \Delta S^X_{\rm env}, \Delta I^X\right)\right], \tag{104}$$

where we have used the dimensionless slow time $\tau := \mathcal{T}/\tau_X$ and dimensionless transition rates $\widetilde{w}^y_{xx'} := \tau_X w^y_{xx'}$ and $\widetilde{w}^{yy'}_x := \tau_Y w^{yy'}_x$. Then, we find that the time evolution of $G_\tau(x, y)$ is described by the following tilted dynamics:

$$\partial_\tau G_\tau(x, y) = \sum_{x'} \left[\mathcal{L}^X_{\lambda_S,\lambda_I}\right]^y_{xx'} G_\tau(x', y) + \frac{1}{\epsilon} \sum_{y'} \left[\mathcal{L}^Y_{\lambda_S,\lambda_I}\right]^{yy'}_x G_\tau(x, y'), \tag{105}$$

where $\mathcal{L}^X_{\lambda_S,\lambda_I}$ and $\mathcal{L}^Y_{\lambda_S,\lambda_I}$ denote the tilted generators given by

$$\left[\mathcal{L}^X_{\lambda_S,\lambda_I}\right]^y_{xx'} := \begin{cases} \widetilde{w}^y_{xx'} \exp\left(\lambda_S \ln\frac{\widetilde{w}^y_{xx'}}{\widetilde{w}^y_{x'x}} - \lambda_I \ln\frac{\pi_{\rm ss}(y|x)}{\pi_{\rm ss}(y|x')}\right) & (x \neq x'), \\ -\sum_{x'(\neq x)} \widetilde{w}^y_{x'x} & (x = x'), \end{cases} \tag{106}$$

$$\left[\mathcal{L}^Y_{\lambda_S,\lambda_I}\right]^{yy'}_x := \begin{cases} \widetilde{w}^{yy'}_x & (y \neq y'), \\ -\sum_{y'(\neq y)} \widetilde{w}^{y'y}_x & (y = y'). \end{cases} \tag{107}$$

We now assume that G_τ has asymptotic expansions in terms of the asymptotic sequences $\{\epsilon^n\}_{n=0}^{\infty}$ as $\epsilon \to 0$:

$$G_\tau = G_\tau^{(0)} + \epsilon G_\tau^{(1)} + \cdots . \tag{108}$$

Here, we impose the normalization condition

$$\sum_y G_\tau^{(0)}(x, y) = \sum_y G_\tau(x, y) =: G_\tau^X(x). \tag{109}$$

By substituting this expansion into (105), we find that the leading order gives

$$\sum_{y'} [\mathcal{L}^Y_{\lambda_S,\lambda_I}]_x^{yy'} G_\tau^{(0)}(x, y') = 0. \tag{110}$$

From the Perron-Frobenius theorem and the normalization condition, we find that $G_\tau^{(0)}$ has the form

$$G_\tau^{(0)}(x, y) = G_\tau^X(x)\pi_{ss}(y|x). \tag{111}$$

The subleading order of (105) yields

$$\partial_\tau G_\tau^{(0)}(x, y) = \sum_{x'} [\mathcal{L}^X_{\lambda_S,\lambda_I}]^y_{xx'} G_\tau^{(0)}(x', y) + \sum_{y'} [\mathcal{L}^Y_{\lambda_S,\lambda_I}]_x^{yy'} G_\tau^{(1)}(x, y'). \tag{112}$$

From the solvability condition for $G_\tau^{(1)}$, we obtain the effective dynamics for $G_\tau^X(x)$:

$$\partial_\tau G_\tau^X(x) = \sum_{x'} \left[\overline{\mathcal{L}}^X_{\lambda_S,\lambda_I}\right]_{xx'} G_\tau^X(x'), \tag{113}$$

where $\overline{\mathcal{L}}^X_{\lambda_S,\lambda_I}$ denotes the effective tilted generator given by

$$\begin{aligned}\left[\overline{\mathcal{L}}^X_{\lambda_S,\lambda_I}\right]_{xx'} &:= \sum_y [\mathcal{L}^X_{\lambda_S,\lambda_I}]^y_{xx'}\pi_{ss}(y|x') \\ &= \begin{cases} \sum_y \left[\widetilde{w}^y_{xx'} \exp\left(\lambda_S \ln \frac{\widetilde{w}^y_{xx'}}{\widetilde{w}^y_{x'x}} - \lambda_I \ln \frac{\pi_{ss}(y|x)}{\pi_{ss}(y|x')}\right)\pi_{ss}(y|x')\right] & (x \neq x'), \\ -\sum_{x'(\neq x)} \left[\sum_y \widetilde{w}^y_{x'x}\pi_{ss}(y|x)\right] & (x = x'). \end{cases}\end{aligned} \tag{114}$$

Importantly, this effective tilted generator satisfies the following property:

$$\left(\overline{\mathcal{L}}^X_{\lambda_S,\lambda_I}\right)^\top = \overline{\mathcal{L}}^X_{-\lambda_S-1,-\lambda_I-1}, \tag{115}$$

where $\top$ denotes the matrix transpose. Because the scaled cumulant generating function is equal to the largest eigenvalue of this effective tilted generator, the Gallavotti-Cohen symmetry follows from this property:

$$\mu(\lambda_S, \lambda_I) = \mu(-\lambda_S - 1, -\lambda_I - 1). \tag{116}$$

B. Input-output fluctuation inequalities

In the linear response regime, where the scaled cumulant generating function can be approximated by a quadratic form [17], the Gallavotti-Cohen symmetry (116) constrains its form as

$$\mu(\lambda_S, \lambda_I) = a\left(\lambda_S + \frac{1}{2}\right)^2 + b\left(\lambda_S + \frac{1}{2}\right)\left(\lambda_I + \frac{1}{2}\right) + c\left(\lambda_I + \frac{1}{2}\right)^2 - \frac{1}{4}(a + b + c), \tag{117}$$

where a, b, c are constants. From the convexity of $\mu(\lambda_S, \lambda_I)$, these coefficients satisfy $a \geqslant 0, c \geqslant 0$, and $ac - b^2/4 \geqslant 0$. By noting that

$$\dot{S}^X_{\text{env}} = \frac{\partial}{\partial\lambda_S}\mu(\lambda_S, \lambda_I)\Big|_{\lambda_S,\lambda_I=0} = a + \frac{b}{2}, \tag{118}$$

$$\dot{I}^X = -\frac{\partial}{\partial\lambda_I}\mu(\lambda_S, \lambda_I)\Big|_{\lambda_S,\lambda_I=0} = -c - \frac{b}{2}, \tag{119}$$

these coefficients are further constrained by the second law of information thermodynamics to satisfy $a + b + c \geqslant 0$.

1. Information-thermodynamic engine: $0 < -\dot{S}^X_{\text{env}} \leqslant -\dot{I}^X$

We consider the case of $-\dot{I}^X \geqslant \dot{S}^X_{\text{env}}$, i.e., $c \geqslant a$, which includes the case where X acts as an information-thermodynamic engine with $0 < -\dot{S}^X_{\text{env}} \leqslant -\dot{I}^X$. Since we have

$$D_S = \frac{1}{2}\frac{\partial^2}{\partial\lambda_S^2}\mu(\lambda_S, \lambda_I)\Big|_{\lambda_S,\lambda_I=0} = a, \tag{120}$$

$$D_I = \frac{1}{2}\frac{\partial^2}{\partial\lambda_I^2}\mu(\lambda_S, \lambda_I)\Big|_{\lambda_S,\lambda_I=0} = c, \tag{121}$$

these relations between the coefficients a, b, c lead to the following input-output fluctuation inequalities:

$$D_S \leqslant D_I, \tag{122}$$

$$\frac{D_I}{(\dot{I}^X)^2} \leqslant \frac{D_S}{\left(\dot{S}^X_{\text{env}}\right)^2}. \tag{123}$$

These inequalities state that the fluctuation of the output current (negative entropy production) is smaller than that of the input current (information flow), while the relative fluctuation of the output current is larger than that of the input current.

2. Memory: $0 < \dot{I}^X \leqslant \dot{S}^X_{\text{env}}$

We can also derive input-output fluctuation inequalities when $-\dot{I}^X \leqslant \dot{S}^X_{\text{env}}$, i.e., $c \leqslant a$, which includes the case where X plays the role of a memory with $0 < \dot{I}^X \leqslant \dot{S}^X_{\text{env}}$. In this case, the information flow $\dot{I}^X$ corresponds to the output current while the entropy production rate $\dot{S}^X_{\rm env}$ corresponds to the input current. Obviously, we have the following relations:

$$D_S \geqslant D_I, \tag{124}$$

$$\frac{D_I}{(\dot{I}^X)^2} \geqslant \frac{D_S}{(\dot{S}^X_{\rm env})^2}. \tag{125}$$

VI. EXAMPLES

In this section, we illustrate our results, the tradeoffs for information-thermodynamic engines and the input-output fluctuation inequalities, using two simple examples. The first example is coupled quantum dots, which is one of the simplest models of autonomous Maxwell's demon [45,54]. As a second example, we consider coupled linear overdamped Langevin equations, which ubiquitously appear in biological contexts with the linear noise approximation [5,55–57]. Interestingly, the equality condition of the tradeoffs (74) and (76) is satisfied even far from equilibrium in this case.

A. Coupled quantum dots

1. Model

We consider the system composed of two single-level quantum dots X and Y. Let $x \in \{0, 1\}$ and $y \in \{0, 1\}$ be occupation variables on each particle site, where $x = 1$ and $y = 1$ ($x = 0$ and $y = 0$) represent that the site of X and Y is filled (empty), respectively. The energy of X is ϵ_X when it is filled with a particle and zero when it is empty. A single-particle site of X exchanges particles with two particle reservoirs $\nu = L, R$ at temperature T and chemical potential μ_ν. We assume that $\Delta\mu := \mu_L - \mu_R > 0$. Let $p_t(x, y)$ be the probability of state (x, y) at time t. The time evolution of $p_t(x, y)$ is described by the master equation

$$\begin{aligned}\partial_t p_t(x, y) = \sum_\nu \left[w^{(\nu)y}_{xx'} p_t(x', y) - w^{(\nu)y}_{x'x} p_t(x, y)\right] \\ + w^{yy'}_x p_t(x, y') - w^{y'y}_x p_t(x, y),\end{aligned} \tag{126}$$

where $x' := 1 - x$ and $y' := 1 - y$. Here, $w^{(\nu)y}_{xx'}$ denotes the time-independent transition rate from x' to x induced by the reservoir ν, which satisfies the local detailed balance condition

$$\frac{w^{(\nu)y}_{10}}{w^{(\nu)y}_{01}} = \exp[-\beta(\epsilon_X - \mu_\nu)]. \tag{127}$$

We suppose that the transition rates have the form

$$w^{(L)0}_{10} = \tilde{\Gamma}_X f_L, \qquad w^{(R)0}_{10} = \Gamma_X f_R, \tag{128}$$

$$w^{(L)1}_{10} = \Gamma_X f_L, \qquad w^{(R)1}_{10} = \tilde{\Gamma}_X f_R, \tag{129}$$

$$w^{(L)0}_{01} = \tilde{\Gamma}_X(1 - f_L), \qquad w^{(R)0}_{01} = \Gamma_X(1 - f_R), \tag{130}$$

$$w^{(L)1}_{01} = \Gamma_X(1 - f_L), \qquad w^{(R)1}_{01} = \tilde{\Gamma}_X(1 - f_R), \tag{131}$$

where $f_\nu := \{\exp[\beta(\epsilon_X - \mu_\nu)] + 1\}^{-1}$ is the Fermi distribution function, and Γ_X ($\tilde{\Gamma}_X$) denotes a positive coupling strength. Below, we focus on the case where $\tilde{\Gamma}_X \ll \Gamma_X$. The above form of transition rates implies that the coupling strength of the R (L) reservoir changes from Γ_X to $\tilde{\Gamma}_X$ when Y is filled (empty) with a particle. The transition rates associated with Y are given as follows:

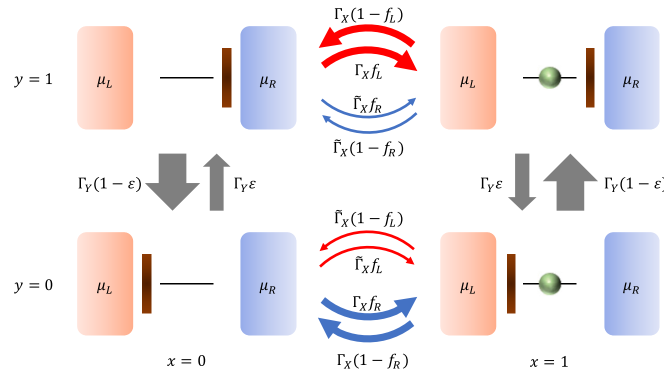

FIG. 2. Schematic of the coupled quantum dots. The single-particle site of X exchanges particles (green dot) with the two particle reservoirs at a chemical potential μ_L and μ_R. The position of the wall represents the state of Y: the wall is inserted on the left side when $y = 0$, while it is inserted on the right side when $y = 1$. The red and blue arrows represent the transition rates $w^{(L)y}_{xx'}$ and $w^{(R)y}_{xx'}$, respectively. The gray arrows represent the transition rates associated with Y, $w^{yy'}_x$. The thickness of these arrows indicates the magnitude of each transition rate.

$$w^{yy'}_x = \begin{cases}\Gamma_Y\varepsilon & (y, y') = (1 - x, x), \\ \Gamma_Y(1 - \varepsilon) & (y, y') = (x, 1 - x),\end{cases} \tag{132}$$

where Γ_Y is a coupling strength, and ε can be interpreted as an error probability with $0 \leqslant \varepsilon \leqslant 1$.

In this model, the subsystem Y acts as Maxwell's demon when ε is sufficiently small. To understand this point intuitively, let us consider the state of Y as representing the position of the wall, which is inserted between the single site of X and the reservoir. In other words, when $y = 0$ ($y = 1$), the wall is inserted between the site of X and the L (R) reservoir and prohibits the transition due to the L (R) reservoir by changing the coupling strength from Γ_X to $\tilde{\Gamma}_X$ (see Fig. 2). As a result, particles are transferred from the R to L reservoirs against the chemical potential difference.

2. Fast relaxation limit of Y

Hereafter, we focus on the case where Y is faster than X, i.e., $\Gamma_Y \gg \Gamma_X \gg \tilde{\Gamma}_X$. By performing a perturbation expansion following Sec. III C, we can show that $p_t(x, y) \simeq p^X_t(x)\pi_{\rm ss}(y|x)$ with

$$\pi_{\rm ss}(y = x|x) = 1 - \varepsilon, \tag{133}$$

$$\pi_{\rm ss}(y = 1 - x|x) = \varepsilon. \tag{134}$$

The effective dynamics for X is then given by

$$\partial_t p^X_t(x) \simeq \sum_\nu \left[\overline{w}^{(\nu)}_{xx'} p^X_t(x') - \overline{w}^{(\nu)}_{x'x} p^X_t(x)\right], \tag{135}$$

where $\overline{w}_{xx'}^{(\nu)} := \sum_y w_{xx'}^{(\nu)y}\pi_{ss}(y|x')$ denotes the effective transition rates:

$$\overline{w}_{10}^{(L)} = [\varepsilon\Gamma_X + (1-\varepsilon)\tilde{\Gamma}_X]f_L, \tag{136}$$

$$\overline{w}_{01}^{(L)} = [(1-\varepsilon)\Gamma_X + \varepsilon\tilde{\Gamma}_X](1-f_L), \tag{137}$$

$$\overline{w}_{10}^{(R)} = [(1-\varepsilon)\Gamma_X + \varepsilon\tilde{\Gamma}_X]f_R, \tag{138}$$

$$\overline{w}_{01}^{(R)} = [\varepsilon\Gamma_X + (1-\varepsilon)\tilde{\Gamma}_X](1-f_R). \tag{139}$$

Thus, in the fast relaxation limit of Y, the system X can be considered as an autonomous system where the coupling strength of the reservoirs changes autonomously. More specifically, when $x = 1$, the coupling strength of the R reservoir changes from the original strength Γ_X to a smaller value $\tilde{\Gamma}_X$, while that of the L reservoir remains unchanged. In contrast, when $x = 0$, the coupling strength of the L reservoir becomes small while that of the R reservoir remains at the original strength Γ_X. This autonomous control is probabilistic and has the error probability ε.

3. Tradeoff between power and efficiency

We first consider the tradeoff between the negative entropy production rate and information-thermodynamic efficiency (74). Note that (74) corresponds to the tradeoff between power and efficiency (95) because $\dot{S}_{\rm env}^X = \beta\dot{W}^X$ in this model.

We first calculate the average rate of chemical work $\dot{W}^X$. By defining $b_{xx'}$ ($x' = 1-x$) as

$$b_{xx'} := \begin{cases} 1 & (x=1, x'=0), \\ -1 & (x=0, x'=1), \end{cases} \tag{140}$$

we note that $b_{xx'}\mu_\nu$ corresponds to the energy provided by the particle reservoir ν during the transition $(x', y) \to (x, y)$. Then, the average rate of chemical work reads as

$$\begin{aligned} \dot{W}^X &= \sum_\nu\sum_x\sum_y w_{xx'}^{(\nu)y} p_{ss}(x', y) b_{xx'}\mu_\nu \\ &\simeq \sum_\nu\sum_x\sum_y w_{xx'}^{(\nu)y}\pi_{ss}(y|x')p_{ss}^X(x')b_{xx'}\mu_\nu \\ &= \sum_\nu\sum_x \overline{w}_{xx'}^{(\nu)}p_{ss}^X(x')b_{xx'}\mu_\nu \\ &= J_X\Delta\mu, \end{aligned} \tag{141}$$

where, in the second line, we have used $p_{ss}(x', y) \simeq \pi_{ss}(y|x')p_{ss}^X(x')$ in the fast relaxation limit of Y. In the last line, J_X denotes the net particle current from L to R, which is conjugate with the chemical potential difference $\Delta\mu$:

$$\begin{aligned} J_X &= \overline{w}_{10}^{(L)}p_{ss}^X(0) - \overline{w}_{01}^{(L)}p_{ss}^X(1) \\ &= \Gamma_X\frac{\varepsilon^2 f_L(1-f_R) - (1-\varepsilon)^2(1-f_L)f_R}{1+(2\varepsilon-1)(f_L-f_R)} + O(\tilde{\Gamma}_X). \end{aligned} \tag{142}$$

The net particle current J_X becomes negative when ε is smaller than the critical value ε_*, which can be evaluated as

$$\varepsilon_* = \frac{(1-f_L)f_R}{f_L-f_R}\left[-1+\sqrt{1+\frac{f_L-f_R}{(1-f_L)f_R}}\right] + O\left(\frac{\tilde{\Gamma}_X}{\Gamma_X}\right). \tag{143}$$

Note that $\varepsilon_* < \frac{1}{2}$ because $f_L(1-f_R) > f_R(1-f_L)$, which follows from the condition $\Delta\mu = \mu_L - \mu_R > 0$.

Similarly, the information flow can be expressed as

$$\begin{aligned} \dot{I}^X &= \sum_\nu\sum_x\sum_y w_{xx'}^{(\nu)y}p_{ss}(x', y)\ln\frac{p_{ss}(y|x)}{p_{ss}(y|x')} \\ &\simeq \sum_\nu\sum_x\sum_y w_{xx'}^{(\nu)y}\pi_{ss}(y|x')p_{ss}^X(x')\ln\frac{\pi_{ss}(y|x)}{\pi_{ss}(y|x')} \\ &= J_I F_I, \end{aligned} \tag{144}$$

where in the second line we have used $p_{ss}(x', y) \simeq \pi_{ss}(y|x')p_{ss}^X(x')$ in the fast relaxation limit of Y. In the last line, F_I denotes the information affinity defined as

$$\begin{aligned} F_I &:= \ln\frac{\pi_{ss}(0|0)\pi_{ss}(1|1)}{\pi_{ss}(0|1)\pi_{ss}(1|0)} \\ &= 2\ln\frac{1-\varepsilon}{\varepsilon}, \end{aligned} \tag{145}$$

and J_I denotes the probability current that is conjugate with F_I:

$$\begin{aligned} J_I &= \sum_\nu\left[w_{01}^{(\nu)0}\pi_{ss}(0|1)p_{ss}^X(1) - w_{10}^{(\nu)0}\pi_{ss}(0|0)p_{ss}^X(0)\right] \\ &= J_X + O(\tilde{\Gamma}_X). \end{aligned} \tag{146}$$

Thus, the tight-coupling condition is satisfied in the limit $\tilde{\Gamma}_X/\Gamma_X \ll 1$. Since $\varepsilon_* < \frac{1}{2}$, the information flow $\dot{I}^X$ also becomes negative when $\varepsilon < \varepsilon_*$.

The fluctuation of the chemical work can be calculated by considering the tilted dynamics (see Appendix C). The result reads as

$$D_W = D_n\Delta\mu^2, \tag{147}$$

where D_n denotes the fluctuation of the net particle current:

$$\begin{aligned} D_n = \frac{\Gamma_X}{2}\Bigg\{&\frac{(1-\varepsilon)^2(1-f_L)f_R + \varepsilon^2 f_L(1-f_R)}{1+(2\varepsilon-1)(f_L-f_R)} \\ &-\frac{2\left[(1-\varepsilon)^2(1-f_L)f_R - \varepsilon^2 f_L(1-f_R)\right]^2}{[1+(2\varepsilon-1)(f_L-f_R)]^3}\Bigg\} + O(\tilde{\Gamma}_X). \end{aligned} \tag{148}$$

We now focus on the case of $\varepsilon < \varepsilon_*$, where the system X acts as an information-thermodynamic engine with $\dot{W}^X < 0$. The corresponding information-thermodynamic efficiency reads as

$$\eta_W^X = \eta_S^X = \frac{|J_X|F_X}{|J_I|F_I} \simeq \frac{F_X}{F_I} \leqslant 1, \tag{149}$$

where $F_X := \beta\Delta\mu$ denotes the thermodynamic affinity conjugate with J_X. The ε dependence of the efficiency η_W^X and the output power $|\dot{W}^X|$ is shown in Figs. 3(a) and 3(b), respectively. From this figure, we can see that the output power does

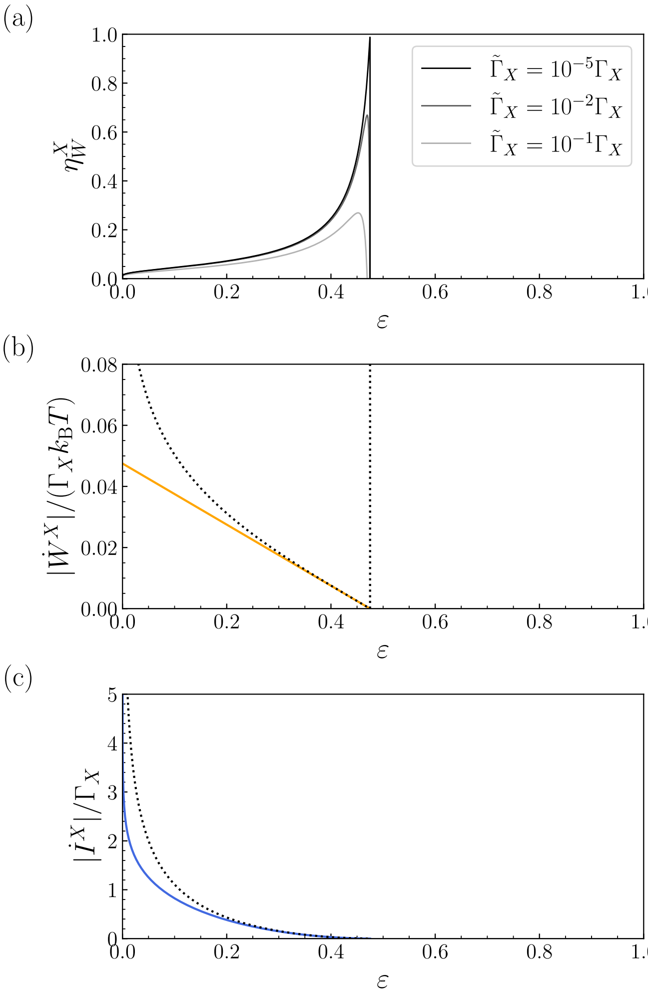

FIG. 3. (a) ε dependence of information-thermodynamic efficiency η_W^X. (b) ε dependence of the power $|\dot{W}^X|$ with $\tilde{\Gamma}_X = 10^{-5}\Gamma_X$. The dotted line denotes the upper bound of (95). (c) ε dependence of the information flow $|\dot{I}^X|$ with $\tilde{\Gamma}_X = 10^{-5}\Gamma_X$. The dotted line denotes the upper bound of (82). In all panels, the parameter values are $\varepsilon_X = 1$, $\mu_L = 1.1$, $\mu_R = 0.9$, and $k_B T = 1$.

not remain finite as $\eta_W^X \to 1$. This result is consistent with the tradeoff between power and information-thermodynamic efficiency (95) as illustrated in Fig. 3(b).

We next consider the tradeoff relation where the negative entropy production is bounded by the fluctuation of the time-integrated stochastic information flow D_I (76). In terms of the power $\dot{W}^X$, it can be expressed as

$$|\dot{W}^X| \leqslant D_I k_B T \eta_W^X \left(1 - \eta_W^X\right). \tag{150}$$

The fluctuation of the information flow D_I can also be calculated by using the tilted dynamics as

$$D_I = D_n F_I^2, \tag{151}$$

which satisfies $\beta\sqrt{D_W/D_I} = F_X/F_I \simeq \eta_W^X$. Therefore, from (83), it follows that the upper bound of (150) is exactly the same as that of (95) for $\varepsilon < \varepsilon_*$.

For comparison, we also plot the information flow and its upper bound (82) in Fig. 3(c). As in the case of the output power, the information flow also vanishes as the efficiency η_S^X $(= \eta_W^X)$ approaches 1. We note that $|\dot{I}^X| \to \infty$ as $\varepsilon \to 0$ because the information affinity F_I diverges.

4. Input-output fluctuation inequalities

We now consider the input-output fluctuation inequalities for $\varepsilon < \varepsilon_*$, where the entropy production $(\dot{S}_{\rm env}^X = \beta\dot{W}^X)$ and the information flow correspond to the output and input currents, respectively. Since $F_X/F_I \leqslant 1$, we can easily confirm that $D_S \leqslant D_I$ and $D_I/(\dot{I}^X)^2 = D_S/(\dot{S}_{\rm env}^X)^2$. Thus, the input-output fluctuation inequalities are satisfied even beyond the linear response regime in this model. Furthermore, the equality is achieved for the inequality regarding the relative fluctuations.

B. Coupled linear overdamped Langevin equations

1. Model

We consider the following coupled linear overdamped Langevin equations:

$$\dot{x}_t = -\omega^X x_t + \omega^{XY} y_t + \sqrt{2D^X}\xi_t^X, \tag{152}$$

$$\dot{y}_t = \omega^{YX} x_t - \omega^Y y_t + \sqrt{2D^Y}\xi_t^Y, \tag{153}$$

where ξ_t^Z $(Z = X, Y)$ is a zero-mean white Gaussian noise that satisfies $\langle \xi_t^Z \xi_{t'}^{Z'}\rangle = \delta_{ZZ'}\delta(t - t')$, and $D^Z > 0$ denotes the noise intensity. Here, x_t relaxes exponentially with decay rate $\omega^X > 0$ and is affected by y_t with rate ω^{XY}, while y_t also relaxes exponentially with decay rate $\omega^Y > 0$ and detects x_t with the differential gain ω^{YX}. We assume that

$$\omega^X\omega^Y - \omega^{XY}\omega^{YX} > 0 \tag{154}$$

to ensure that the system reaches a steady state [58]. The corresponding Fokker-Planck equation reads as

$$\partial_t p_t(x, y) = -\partial_x J_t^X(x, y) - \partial_y J_t^Y(x, y), \tag{155}$$

where $J_t^X(x, y)$ and $J_t^Y(x, y)$ denote the probability currents:

$$J_t^X(x, y) := (-\omega^X x + \omega^{XY} y)p_t(x, y) - D^X\partial_x p_t(x, y), \tag{156}$$
$$J_t^Y(x, y) := (\omega^{YX} x - \omega^Y y)p_t(x, y) - D^Y\partial_y p_t(x, y). \tag{157}$$

While this model is exactly solvable, it is widely used to describe biological systems such as signal transduction networks and gene regulatory networks [5,55–57]. In the context of heat engines, this model includes a Brownian gyrator [59] and can be experimentally realized in, e.g., electronic and colloidal systems [60,61]. Note that the system can be far from equilibrium due to the nonreciprocal interactions (when $\omega^{XY} \neq \omega^{YX}$) or the heat flow (when $D^X \neq D^Y$).

2. Fast relaxation limit of Y

Hereafter, we focus on the case where Y relaxes much faster than X. We introduce a dimensionless slow time $\tau := \omega^X t$ and a small parameter $\epsilon := \omega^X/\omega^Y \ll 1$. Correspondingly, we introduce dimensionless rates $\bar{\omega}^{XY} := \omega^{XY}/\omega^X$ and $\bar{\omega}^{YX} := \omega^{YX}/\omega^Y$ and dimensionless noise intensities $\bar{D}^X := D^X/\omega^X$ and $\bar{D}^Y := D^Y/\omega^Y$. From the condition (154) and the positivity of ω^X and ω^Y, we note that $\bar{\omega}^{XY}\bar{\omega}^{YX} < 1$. Then, the time-evolution equations (152) and (153) can be rewritten as

$$\dot{x}_\tau = [-x_\tau + \bar{\omega}^{XY} y_\tau] + \sqrt{2\bar{D}^X}\xi^X_\tau, \tag{158}$$

$$\dot{y}_\tau = \frac{1}{\epsilon}[\bar{\omega}^{YX} x_\tau - y_\tau] + \sqrt{\frac{2\bar{D}^Y}{\epsilon}}\xi^Y_\tau. \tag{159}$$

In the fast relaxation limit $\epsilon \to 0$, the joint probability density $p_\tau(x, y)$ can be approximated as $p_\tau(x, y) \simeq p^X_\tau(x)\pi_{\rm ss}(y|x)$, where

$$\pi_{\rm ss}(y|x) = \frac{1}{\sqrt{2\pi\bar{D}^Y}}\exp\left[-\frac{1}{2\bar{D}^Y}(y - \bar{\omega}^{YX}x)^2\right]. \tag{160}$$

The resulting effective dynamics for X reads as

$$\dot{x}_\tau = -(1 - \bar{\omega}^{XY}\bar{\omega}^{YX})x_\tau + \sqrt{2\bar{D}^X}\xi^X_\tau. \tag{161}$$

3. Tradeoff between negative entropy production and efficiency

We first consider the tradeoff between the negative entropy production and efficiency (74). Note that, unlike the previous example, this tradeoff is not the same as the tradeoff between power and efficiency (95) because there is no externally applied work in this system. In the steady state with the fast relaxation limit of Y, the entropy production rate associated with X reads as

$$\begin{aligned}\dot{S}^X_{\rm env} &= \frac{1}{D^X}\langle(-\omega^X x_t + \omega^{XY} y_t)\circ \dot{x}_t\rangle \\ &= \omega^X\bar{\omega}^{XY}\bar{\omega}^{YX}\left(\frac{\bar{\omega}^{XY}\bar{D}^Y}{\bar{\omega}^{YX}\bar{D}^X} - 1\right),\end{aligned} \tag{162}$$

where the symbol $\circ$ denotes the Stratonovich product. We note that $\dot{S}^X_{\rm env}$ is induced by the fast variable y_t, which does not appear in the effective dynamics for X (161). In other words, $\dot{S}^X_{\rm env}$ is an entropy production invisible from the effective dynamics, which is called *hidden entropy* [62,63]. Similarly, the information flow can be calculated as

$$\begin{aligned}\dot{I}^X &= \int dx\,dy\,J^X_{\rm ss}(x, y)\partial_x \ln\frac{p_{\rm ss}(x, y)}{p^X_{\rm ss}(x)p^Y_{\rm ss}(y)} \\ &= \frac{\bar{\omega}^{YX}\bar{D}^X}{\bar{\omega}^{XY}\bar{D}^Y}\dot{S}^X_{\rm env}.\end{aligned} \tag{163}$$

In the context of the Brownian gyrator, we can show that there is a torque, which remains finite even in the fast relaxation limit of Y. Both the medium entropy production rate $\dot{S}^X_{\rm env}$ and the information flow $\dot{I}^X$ are proportional to this "hidden" torque. The fluctuation of the entropy production can be calculated by considering the tilted dynamics. The result reads as (see Appendix D for the derivation)

$$D_S = \omega^X\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2. \tag{164}$$

We now focus on the case where $\omega^{XY}\omega^{YX} > 0$ and $\omega^{XY}D^Y < \omega^{YX}D^X$. In this case, both the entropy production rate and information flow become negative, i.e., X acts as an information-thermodynamic engine. Then, the corresponding information-thermodynamic efficiency is given by

$$\eta^X_S = \frac{|\dot{S}^X_{\rm env}|}{|\dot{I}^X|} = \frac{\omega^{XY}D^Y}{\omega^{YX}D^X} \leqslant 1. \tag{165}$$

Combining (165) and (164), we find that the upper bound on the negative entropy production rate (74) is

$$\begin{aligned}D_S\frac{1-\eta^X_S}{\eta^X_S} &= \omega^X\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2\frac{1 - \frac{\bar{\omega}^{XY}\bar{D}^Y}{\bar{\omega}^{YX}\bar{D}^X}}{\frac{\bar{\omega}^{XY}\bar{D}^Y}{\bar{\omega}^{YX}\bar{D}^X}} \\ &= |\dot{S}^X_{\rm env}|.\end{aligned} \tag{166}$$

Thus, the equality condition is satisfied even far from equilibrium in this case. This is in contrast to the standard long-time TUR, where the equality is guaranteed only in the near-equilibrium limit.

We next consider the tradeoff relation where the negative entropy production is bounded by the fluctuation of the time-integrated stochastic information flow D_I (76). The fluctuation of the information flow D_I can also be calculated by using the tilted dynamics as

$$D_I = \omega^X\frac{\bar{D}^X}{\bar{D}^Y}(\bar{\omega}^{YX})^2, \tag{167}$$

which satisfies $\sqrt{D_S/D_I} = \eta^X_S$. Therefore, from (83), it follows that the upper bound of (76) is exactly the same as that of (74). This implies that the tradeoff between the information flow and efficiency (82) also achieves the equality in this case:

$$\begin{aligned}D_I(1-\eta^X_S) &= \omega^X\frac{\bar{D}^X}{\bar{D}^Y}(\bar{\omega}^{YX})^2\left(1 - \frac{\bar{\omega}^{XY}\bar{D}^Y}{\bar{\omega}^{YX}\bar{D}^X}\right) \\ &= |\dot{I}^X|.\end{aligned} \tag{168}$$

We now consider the possibility of achieving finite negative entropy production even when $\eta^X_S \to 1$. We first note that the negative entropy production can be expressed in terms of η^X_S as

$$|\dot{S}^X_{\rm env}| = \omega^X\bar{\omega}^{XY}\bar{\omega}^{YX}(1 - \eta^X_S). \tag{169}$$

Since $0 < \bar{\omega}^{XY}\bar{\omega}^{YX} < 1$, we find that $|\dot{S}^X_{\rm env}| \to 0$ as $\eta^X_S \to 1$ as long as ω^X is finite. In contrast, if ω^X is scaled as $\omega^X = \omega_0/(1-\eta^X_S)$, the negative entropy production can remain finite even in the limit $\eta^X_S \to 1$:

$$|\dot{S}^X_{\rm env}| = \omega_0\bar{\omega}^{XY}\bar{\omega}^{YX}. \tag{170}$$

As can be seen from the tradeoff relation (166), the fluctuation of entropy production blows up as $\eta^X_S \to 1$ in this case [see Fig. 4(a)]:

$$\begin{aligned}D_S &= \omega^X\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 \\ &= \omega_0\frac{\eta^X_S}{1-\eta^X_S}\bar{\omega}^{XY}\bar{\omega}^{YX}.\end{aligned} \tag{171}$$

Similarly, the information flow can also remain finite in the limit $\eta^X_S \to 1$:

$$|\dot{I}^X| = \omega_0\bar{\omega}^{XY}\bar{\omega}^{YX}\frac{1}{\eta^X_S}, \tag{172}$$

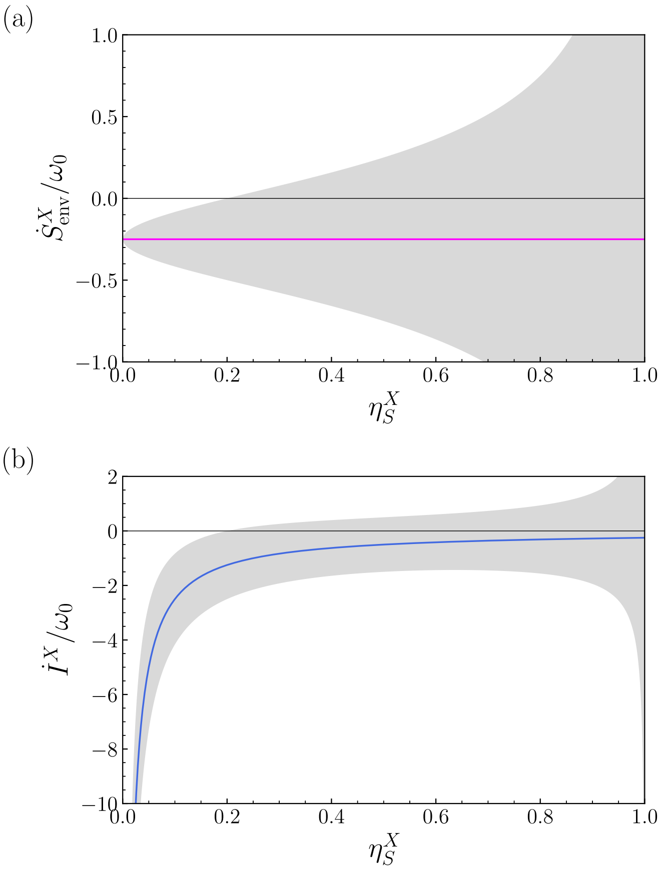

FIG. 4. η^X_S dependence of the entropy production rate (a) and information flow (b) with the scaling $\omega^X = \omega_0/(1-\eta^X_S)$. Here, we assume that ω_0 is constant. The gray shaded region represents the fluctuations of the entropy production and information flow quantified by $\sqrt{D_S/\omega_0}$ and $\sqrt{D_I/\omega_0}$, respectively. The parameter values are $\bar{\omega}^{XY} = \bar{\omega}^{XY} = 0.5$.

at the expense of the blowup of the fluctuation of information flow as $\eta^X_S \to 1$ [see Fig. 4(b)]:

$$D_I = \omega^X \frac{\bar{D}^X}{\bar{D}^Y}(\bar{\omega}^{YX})^2 = \omega_0 \frac{1}{\eta^X_S(1-\eta^X_S)}\bar{\omega}^{XY}\bar{\omega}^{YX}. \tag{173}$$

4. Equality condition of bipartite TUR for this model

Here, we discuss the reason why the equality of the tradeoffs (166) and (168) is achieved in this model. We first recall that these tradeoffs are special cases of the bipartite TUR in the fast relaxation limit of Y (55). In this model, the time-integrated generalized current $\hat{\mathcal{J}}_\mathcal{T}$ for the subsystem X can be expressed as

$$\hat{\mathcal{J}}_\mathcal{T} = \int_{t=0}^{t=\mathcal{T}} g(x_t, y_t) \circ dx_t \tag{174}$$

with an arbitrary weight function $g(x, y)$. As in Sec. III D, the current can be decomposed as $\hat{\mathcal{J}}_\mathcal{T} = \hat{\mathcal{J}}^{\rm I}_\mathcal{T} + \hat{\mathcal{J}}^{\rm II}_\mathcal{T}$ with

$$\hat{\mathcal{J}}^{\rm I}_\mathcal{T} := \int_{t=0}^{t=\mathcal{T}} g(x_t, y_t) \cdot \sqrt{2D^X} dW^X_t, \tag{175}$$

$$\hat{\mathcal{J}}^{\rm II}_\mathcal{T} := \int_0^\mathcal{T} f(x_t, y_t) dt, \tag{176}$$

where the center dot denotes the Ito product, W^X_t denotes the Wiener process, and

$$f(x, y) := g(x, y)(-\omega^X x + \omega^{XY} y) + D^X \partial_x g(x, y). \tag{177}$$

We now show that this model satisfies the sufficient condition for the bipartite TUR in the fast relaxation limit of Y (55) to hold with equality, described in Sec. III D. First, the weight of the current should be proportional to that of the partial entropy production:

$$\begin{aligned} g(x, y) &= C \frac{1}{2D^X} \frac{J^X_{\rm ss}(x_t, y_t)}{p_{\rm ss}(x_t, y_t)} \\ &= C \frac{1}{2D^X} \omega^X \bar{\omega}^{XY} \left(1 - \frac{\bar{\omega}^{YX} \bar{D}^X}{\bar{\omega}^{XY} \bar{D}^Y}\right)(y - \bar{\omega}^{YX} x). \end{aligned} \tag{178}$$

The time-integrated stochastic information flow $\Delta \hat{I}^X$ in the steady state with the fast relaxation limit of Y is an example that satisfies this condition:

$$\begin{aligned} \Delta \hat{I}^X &= \int_{t=0}^{t=\mathcal{T}} \partial_x \ln \frac{p_{\rm ss}(x_t, y_t)}{p^X_{\rm ss}(x_t) p^Y_{\rm ss}(y_t)} \circ dx_t \\ &\simeq \int_{t=0}^{t=\mathcal{T}} \partial_x \ln \pi_{\rm ss}(y_t | x_t) \circ dx_t \\ &= \int_{t=0}^{t=\mathcal{T}} \frac{\bar{\omega}^{YX}}{\bar{D}^Y}(y_t - \bar{\omega}^{YX} x_t) \circ dx_t. \end{aligned} \tag{179}$$

Second, for this choice of the current, the fluctuation of $\hat{\mathcal{J}}^{\rm II}_\mathcal{T}$ must go to zero in the fast relaxation limit of Y:

$$D^{\rm II}_\mathcal{J} := \lim_{\mathcal{T}\to\infty} \frac{1}{2\mathcal{T}} {\rm Var}\big[\hat{\mathcal{J}}^{\rm II}_\mathcal{T}\big] = 0. \tag{180}$$

In this model, we can confirm that this condition is indeed satisfied by explicitly calculating $D^{\rm II}_\mathcal{J}$ (see Appendix D 2). As a result, the equality of (55) is achieved for a current that satisfies the condition (178) in the fast relaxation limit of Y:

$$\frac{D_\mathcal{J}}{J^2} = \frac{1}{\dot{S}^X_{\rm tot} - \dot{I}^X}. \tag{181}$$

Note that the condition described above is only a sufficient condition. In fact, the equality (181) holds for more diverse types of currents that do not even satisfy the condition (178) in this model. To see this, note that the current $\hat{\mathcal{J}}_\mathcal{T}$ that satisfies the condition (178) can be expressed as

$$\begin{aligned} \hat{\mathcal{J}}_\mathcal{T} &= C \int_{t=0}^{t=\mathcal{T}} (y_t - \bar{\omega}^{YX} x_t) \circ dx_t \\ &= C \left[\int_{t=0}^{t=\mathcal{T}} y_t \circ dx_t - \frac{1}{2} \bar{\omega}^{YX} x_t^2 \Big|_{t=0}^{t=\mathcal{T}} \right]. \end{aligned} \tag{182}$$

The important point here is that the second term is a boundary term and can be ignored when considering the long-time statistical properties of $\hat{\mathcal{J}}_\mathcal{T}$. (For the effect of such a boundary term on the large deviation, see [64].) Therefore, any current $\hat{\mathcal{J}}_\mathcal{T}$ that has the same long-time statistical properties as (182) satisfies the equality (181). The example includes the stochastic medium entropy production $\Delta\hat{S}^X_{\mathrm{env}}$:

$$\begin{aligned}\Delta\hat{S}^X_{\mathrm{env}} &= \int_{t=0}^{t=\mathcal{T}} \frac{1}{D^X}(-\omega^X x_t + \omega^{XY} y_t)\circ dx_t \\ &= \frac{\bar{\omega}^{XY}}{\bar{D}^X}\int_{t=0}^{t=\mathcal{T}} y_t \circ dx_t - \frac{1}{2\bar{D}^X} x_t^2\Big|_{t=0}^{t=\mathcal{T}}. \end{aligned} \tag{183}$$

Hence, the choice $\hat{\mathcal{J}}_{\mathcal{T}} = \Delta\hat{S}^X_{\mathrm{env}}$ also satisfies the equality (181), although it does not satisfy the condition (178). Indeed, we can show that the following relation holds:

$$\frac{D_S^{\mathrm{I}}}{\left(\dot{S}^X_{\mathrm{env}}\right)^2} > \frac{D_S}{\left(\dot{S}^X_{\mathrm{env}}\right)^2} = \frac{1}{\dot{S}^X_{\mathrm{tot}} - \dot{I}^X}. \tag{184}$$

Here, $D_S^{\mathrm{I}} := \lim_{\mathcal{T}\to\infty} \mathrm{Var}[\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}]/2\mathcal{T}$ denotes the fluctuation of $\hat{\mathcal{J}}^{\mathrm{I}}_{\mathcal{T}}$ with $\hat{\mathcal{J}}_{\mathcal{T}} = \Delta\hat{S}^X_{\mathrm{env}}$, which is given by

$$D_S^{\mathrm{I}} = \omega^X\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 + (1 - \bar{\omega}^{XY}\bar{\omega}^{YX})\right]. \tag{185}$$

5. Input-output fluctuation inequalities

We finally consider the input-output fluctuation inequalities for the case of $\omega^{XY}\omega^{YX} > 0$ and $\omega^{XY}D^Y < \omega^{YX}D^X$, where the entropy production and the information flow correspond to the output and input currents, respectively. From the relation $\sqrt{D_S/D_I} = \eta_S^X$, it immediately follows that $D_S \leqslant D_I$ and $D_I/(\dot{I}^X)^2 = D_S/(\dot{S}^X_{\mathrm{env}})^2$. Thus, as in the previous example, the input-output fluctuation inequalities are satisfied even beyond the linear response regime in this model, and the equality is achieved for the inequality regarding the relative fluctuations.

VII. CONCLUDING REMARKS

In this paper, we have obtained several fundamental limits for information processing systems. Specifically, we have derived a TUR-type inequality for bipartite systems that provides a universal lower bound on the relative fluctuation of an arbitrary current for a system of interest by the associated partial entropy production, which includes the information flow. This bipartite TUR includes the standard TUR as a special case and incorporates the effect of the interaction with external auxiliary systems. As a corollary to this inequality, we have derived universal tradeoff relations between the negative entropy production rate and the information-thermodynamic efficiency, which can be regarded as an extension of the tradeoffs for heat engines [21,22] to information-thermodynamic engines. Furthermore, in the fast relaxation limit of the auxiliary system, we have shown that the Gallavotti-Cohen symmetry holds even in the presence of information flow. From this symmetry, we can show that the input-output fluctuation inequalities are also valid for information processing systems. We have illustrated our results with two simple examples: coupled quantum dots and coupled linear overdamped Langevin equations. In particular, we have seen that the latter provides an example where the equality of the bipartite TUR is achieved even far from equilibrium.

Here, we provide some remarks on previous studies related to our results. We first note that the bipartite TUR in the short-time limit $\mathcal{T} \to 0$ is already proved in [52] using the Cauchy-Schwarz inequality. Our first main result (26) can be regarded as an extension of the short-time bipartite TUR to an arbitrary observation time $\mathcal{T}$. TUR-type inequalities including measurement and feedback are also derived from fluctuation theorems in [65,66]. While these relations include a contribution of information induced by measurement and feedback processes, this contribution appears in the form of total entropy production rather than partial entropy production. Therefore, our bipartite TUR can provide more stringent bounds on the precision of currents under measurement and feedback control. The standard TUR has also been discussed as a tool for inferring entropy production [52,67–71]. In this context, the bipartite TUR proved here may provide a promising approach to estimating a partial entropy production, especially an information flow.

Next, we remark on the range of validity of the bipartite TUR. While here we have presented the bipartite TUR in the steady state, this relation is valid even for systems under arbitrary time-dependent driving from arbitrary initial states. In Appendix A, we provide a proof of the bipartite TUR in a general form for the case of overdamped Langevin equations. It should also be noted that the bipartite TUR is generally not valid for systems with broken time-reversal symmetry, such as underdamped Langevin dynamics [37–43], as in the standard TUR. However, many relevant biological systems are often described by continuous-time Markov jump processes or diffusion processes with only even variables and parameters under time reversal. Therefore, the results described in this paper will be applicable to a wide range of systems, including biological systems.

In this study, we have focused mainly on the case where an auxiliary system evolves much faster than the system of interest. Such a separation of timescales allows the dynamics of a composite system to be reduced to the effective dynamics of the system of interest, and thus various universal relations similar to those found for a single system hold. While we expect such a separation of timescales to be ubiquitous in biological systems, extending our results to cases where there is no clear timescale separation would be important for elucidating the design principles of biological systems.

ACKNOWLEDGMENTS

We are grateful to T. Ariga for fruitful discussions. T.T. is supported by JSPS KAKENHI Grant No. JP23K19035 and JST, PRESTO Grant No. JPMJPR23O6, Japan. T.V.V. is supported by JSPS KAKENHI Grant No. JP23K13032. K.S. is supported by JSPS KAKENHI Grants No. JP23H01099, No. JP19H05603, and No. JP19H05791.

APPENDIX A: BIPARTITE TUR FOR OVERDAMPED LANGEVIN EQUATIONS

In this Appendix, we derive the bipartite TUR for overdamped Langevin equations. While the derivation based on the generalized Cramér-Rao inequality described in Sec. III B is also valid for this case, here we prove the bipartite TUR more directly from the Langevin equations, following Ref. [50]. We provide a proof of the bipartite TUR in a general form that is valid not only for a steady state, but also for systems under arbitrary time-dependent driving from arbitrary initial states. We note that this direct approach is valid even for Markov jump processes [50].

We consider the following coupled overdamped Langevin equations:

$$\dot{x}_t = F_t^X(x_t, y_t) + \sqrt{2D^X}\xi_t^X, \quad \text{(A1)}$$

$$\dot{y}_t = F_t^Y(x_t, y_t) + \sqrt{2D^Y}\xi_t^Y, \quad \text{(A2)}$$

where $F_t^Z(x, y)$ $(Z = X, Y)$ denotes the time-dependent drift term, D^Z denotes the noise intensity, and ξ_t^Z is a zero-mean white Gaussian noise that satisfies $\langle \xi_t^Z \xi_{t'}^{Z'} \rangle = \delta_{ZZ'}\delta(t - t')$. The independence of the noises ξ_t^X and ξ_t^Y ensures that the system satisfies the bipartite property. The corresponding Fokker-Planck equation reads as

$$\partial_t p_t(x, y) = -\partial_x J_t^X(x, y) - \partial_y J_t^Y(x, y), \quad \text{(A3)}$$

where J_t^Z denotes the probability current:

$$J_t^X(x, y) := F_t^X(x, y)p_t(x, y) - D^X\partial_x p_t(x, y), \quad \text{(A4)}$$

$$J_t^Y(x, y) := F_t^Y(x, y)p_t(x, y) - D^Y\partial_y p_t(x, y). \quad \text{(A5)}$$

Let $\hat{\mathcal{J}}_\mathcal{T}$ be the time-integrated generalized current for the subsystem X with an arbitrary time-dependent weight function $g_t(x, y)$:

$$\hat{\mathcal{J}}_\mathcal{T} := \int_{t=0}^{t=\mathcal{T}} g_t(x_t, y_t) \circ dx_t. \quad \text{(A6)}$$

Converting from the Stratonovich to the Ito product, the current can be decomposed into two parts $\hat{\mathcal{J}}_\mathcal{T} = \hat{\mathcal{J}}_\mathcal{T}^{\mathrm{I}} + \hat{\mathcal{J}}_\mathcal{T}^{\mathrm{II}}$ with

$$\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{I}} := \int_{t=0}^{t=\mathcal{T}} g_t(x_t, y_t) \cdot \sqrt{2D^X}dW_t^X, \quad \text{(A7)}$$

$$\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{II}} := \int_0^\mathcal{T} f_t(x_t, y_t)dt, \quad \text{(A8)}$$

where W_t^X denotes the Wiener process, and

$$f_t(x, y) := g_t(x, y)F_t^X(x, y) + D^X\partial_x g_t(x, y). \quad \text{(A9)}$$

We introduce the following quantity:

$$\hat{A}_\mathcal{T} := \int_{t=0}^{t=\mathcal{T}} \frac{1}{2D^X}\frac{J_t^X(x_t, y_t)}{p_t(x_t, y_t)} \cdot \sqrt{2D^X}dW_t^X. \quad \text{(A10)}$$

The second moment of this quantity gives the partial entropy production for X:

$$\begin{aligned}\langle \hat{A}_\mathcal{T}^2 \rangle &= \int_0^\mathcal{T} dt \int dx\, dy \frac{1}{2D^X}\frac{[J_t^X(x, y)]^2}{p_t(x, y)} \\ &= \frac{1}{2}\Delta\sigma^X = \frac{1}{2}\big(\Delta S_{\mathrm{tot}}^X - \Delta I^X\big).\end{aligned} \quad \text{(A11)}$$

Furthermore, we can easily confirm that $\langle \hat{A}_\mathcal{T} \rangle = 0$ and $\langle \hat{A}_\mathcal{T}\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{I}} \rangle = \langle \hat{\mathcal{J}}_\mathcal{T} \rangle$. Therefore, we find that

$$\langle \hat{A}_\mathcal{T}(\hat{\mathcal{J}}_\mathcal{T} - \langle \hat{\mathcal{J}}_\mathcal{T} \rangle) \rangle = \langle \hat{\mathcal{J}}_\mathcal{T} \rangle + \langle \hat{A}_\mathcal{T}\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{II}} \rangle. \quad \text{(A12)}$$

By using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}[\langle \hat{\mathcal{J}}_\mathcal{T} \rangle + \langle \hat{A}_\mathcal{T}\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{II}} \rangle]^2 &= \langle \hat{A}_\mathcal{T}(\hat{\mathcal{J}}_\mathcal{T} - \langle \hat{\mathcal{J}}_\mathcal{T} \rangle) \rangle^2 \\ &\leqslant \langle \hat{A}_\mathcal{T}^2 \rangle \mathrm{Var}[\hat{\mathcal{J}}_\mathcal{T}] \\ &= \frac{1}{2}\big(\Delta S_{\mathrm{tot}}^X - \Delta I^X\big)\mathrm{Var}[\hat{\mathcal{J}}_\mathcal{T}],\end{aligned} \quad \text{(A13)}$$

which has a form similar to that of the bipartite TUR (26). In fact, we can show that the additional current term $\langle \hat{A}_\mathcal{T}\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{II}} \rangle$ in (A13) exactly equal to $\langle \hat{\mathcal{J}}_\mathcal{T} \rangle_q$ in the bipartite TUR (26). To see this, we first rewrite $\langle \hat{A}_\mathcal{T}\hat{\mathcal{J}}_\mathcal{T}^{\mathrm{II}} \rangle$ as follows [50]:

$$\begin{aligned}\langle A_\mathcal{T} J_\mathcal{T}^{\mathrm{II}} \rangle &= \left\langle \int_{t'=0}^{t'=\mathcal{T}} \frac{1}{2D^X}\frac{J_{t'}^X(x_{t'}, y_{t'})}{p_{t'}(x_{t'}, y_{t'})} \cdot \sqrt{2D^X}dW_{t'}^X \int_0^\mathcal{T} f_t(x_t, y_t)dt \right\rangle \\ &= -\int_0^\mathcal{T} dt \int dx\, dy\, f_t(x, y) \int_0^t dt' \int dx'dy'\, p(x, y, t|x', y', t')\partial_{x'}J_{t'}^X(x', y') \\ &= \int_0^\mathcal{T} dt \int dx\, dy\, g_t(x, y)\big[F_t^X(x, y) - D^X\partial_x\big]\tilde{q}_t(x, y),\end{aligned} \quad \text{(A14)}$$

where

$$\tilde{q}_t(x, y) := -\int_0^t dt' \int dx'dy'\, p(x, y, t|x', y', t')\partial_{x'}J_{t'}^X(x', y'). \quad \text{(A15)}$$

In the second line of (A14), we have used the Doob transform [72–76], which maps a stochastic process conditioned on a future event [in this case, $(x_t, y_t) = (x, y)$] to an unconditioned stochastic process with an additional drift term. By differentiating (A15) with respect to t, we obtain the time-evolution equation of $\tilde{q}_t(x, y)$:

$$\begin{aligned}\partial_t\tilde{q}_t(x, y) &= -\int dx'dy'\,\delta(x - x')\delta(y - y')\partial_{x'}J_t^X(x', y') - \int_0^t dt' \int dx'dy'\,\partial_t p(x, y, t|x', y', t')\partial_{x'}J_{t'}^X(x', y') \\ &= -\partial_x J_t^X(x, y) + \mathcal{L}_t[\tilde{q}](x, y),\end{aligned} \quad \text{(A16)}$$

with $\tilde{q}_0 = 0$, where $\mathcal{L}_t$ denotes the Fokker-Planck operator. Thus, $\langle A_{\mathcal{T}} J^{\mathrm{II}}_{\mathcal{T}} \rangle$ is exactly equal to $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q$. We remark that this conclusion is also confirmed by noting that $\hat{A}_{\mathcal{T}}$ corresponds to the θ derivative of the path probability $\partial_\theta \ln \mathbb{P}_\theta(\Gamma)|_{\theta=0}$ used in the generalized Cramér-Rao inequality.

Note that $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q$ generally reflects not only the contribution of interaction with Y, but also the effect of nonstationarity. This point will be clarified in the next Appendix.

APPENDIX B: RELATION TO THE CONVENTIONAL TRANSIENT TUR

In this Appendix, we consider the bipartite TUR in a general form that is applicable to a transient state, derived in the previous section. Here, we prove that $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q = \mathcal{T}\partial_{\mathcal{T}} \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle - \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle$, if the system is time homogeneous and the transition rate for X and the weight are independent of Y as $w^y_{xx'} = w_{xx'}$ and $d^y_{xx'} = d_{xx'}$. In this case, the bipartite TUR becomes

$$\frac{\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}}]}{[\mathcal{T}\partial_{\mathcal{T}} \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle]^2} \geqslant \frac{2}{\Delta S^X_{\mathrm{tot}} - \Delta I^X}, \tag{B1}$$

which has a form similar to the conventional transient TUR [50,77]. From this result, we can also confirm that $\langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q = 0$ when the system is in the steady state. While we focus on the Markov jump processes in the following, the same result can be obtained for diffusion processes.

By noting that the transition rate $w^y_{xx'} = w_{xx'}$ and the weight of the current $d^y_{xx'} = d_{xx'}$ do not depend on Y, we obtain

$$\begin{aligned} \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q &= \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} \sum_y w_{xx'} q_t(x', y) d_{xx'} \\ &= \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} w_{xx'} q^X_t(x') d_{xx'}. \end{aligned} \tag{B2}$$

As in (A15), we can easily show that $q^X_t(x') = \sum_y q_t(x, y)$ has the form

$$\begin{aligned} q^X_t(x) &= \int_0^t dt' \sum_{x'} p(x, t|x', t') \sum_{x''} w_{x'x''} p^X_{t'}(x'') \\ &= \int_0^t dt' \sum_{x'} p(x, t|x', t') \partial_{t'} p^X_{t'}(x'). \end{aligned} \tag{B3}$$

By substituting (B3) into (B2) and integrating by parts, we obtain

$$\begin{aligned} \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q &= \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} w_{xx'} d_{xx'} \\ &\quad \times \int_0^t dt' \sum_{x'} p(x, t|x', t') \partial_{t'} p^X_{t'}(x') \\ &= - \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} w_{xx'} d_{xx'} \\ &\quad \times \int_0^t dt' \sum_{x'} \partial_{t'} p(x, t|x', t') p^X_{t'}(x'). \end{aligned} \tag{B4}$$

Since the system is time homogeneous, we have $\partial_{t'} p(x, t|x', t') = -\partial_t p(x, t|x', t')$, and thus

$$\begin{aligned} &\int_0^t dt' \sum_{x'} \partial_{t'} p(x, t|x', t') p^X_{t'}(x') \\ &= - \int_0^t dt' \partial_t \sum_{x'} p(x, t|x', t') p^X_{t'}(x') \\ &= -t \partial_t p^X_t(x). \end{aligned} \tag{B5}$$

Hence, by integrating by parts, we obtain

$$\begin{aligned} \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle_q &= \int_0^{\mathcal{T}} dt \sum_x \sum_{x'(\neq x)} w_{xx'} d_{xx'} t \partial_t p^X_t(x) \\ &= \mathcal{T}\partial_{\mathcal{T}} \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle - \langle \hat{\mathcal{J}}_{\mathcal{T}} \rangle. \end{aligned} \tag{B6}$$

APPENDIX C: COUPLED QUANTUM DOTS

In this Appendix, we provide a detailed calculation of the fluctuation of the chemical work D_W in the fast relaxation limit of Y for the coupled quantum dots introduced in Sec. VI A. The fluctuation of the information flow D_I can be calculated in a similar way. The stochastic chemical work is defined as

$$\Delta \hat{W}^X := \sum_\nu \sum_x \sum_{x'(\neq x)} \sum_y \hat{n}^{(\nu)y}_{xx'} b_{xx'} \mu_\nu, \tag{C1}$$

where

$$b_{xx'} := \begin{cases} 1 & (x = 1, x' = 0), \\ -1 & (x = 0, x' = 1). \end{cases} \tag{C2}$$

Then, the fluctuation of the stochastic chemical work is defined as

$$D_W := \lim_{\mathcal{T}\to\infty} \frac{1}{2\mathcal{T}} \mathrm{Var}[\Delta \hat{W}^X]. \tag{C3}$$

The fluctuation D_W can be obtained from the scaled cumulant generating function defined by

$$\mu(\lambda) = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln \langle e^{\lambda \Delta \hat{W}^X} \rangle. \tag{C4}$$

As described in Sec. V A, the scaled cumulant generating function can be calculated by considering the generating function conditioned to a final state (x, y):

$$G_{\mathcal{T}}(x, y) := \int d\Delta W^X p_{\mathcal{T}}(x, y, \Delta W^X) e^{\lambda \Delta W^X}. \tag{C5}$$

The time evolution of $G_{\mathcal{T}}(x, y)$ reads as

$$\begin{aligned} \partial_\tau G_\tau(x, y) &= \sum_\nu \sum_{x'} [\mathcal{L}^X_\lambda]^{(\nu)y}_{xx'} G_\tau(x', y) \\ &\quad + \frac{1}{\epsilon} \sum_{y'} [\mathcal{L}^Y_\lambda]^{yy'}_x G_\tau(x, y'), \end{aligned} \tag{C6}$$

where $\mathcal{L}^X_\lambda$ and $\mathcal{L}^Y_\lambda$ denote the tilted generators given by

$$[\mathcal{L}^X_\lambda]^{(\nu)y}_{xx'} := \begin{cases} \widetilde{w}^{(\nu)y}_{xx'} e^{\lambda b_{xx'} \mu_\nu} & (x \neq x'), \\ -\sum_{x'(\neq x)} \widetilde{w}^{(\nu)y}_{x'x} & (x = x'), \end{cases} \tag{C7}$$

$$\left[\mathcal{L}_\lambda^Y\right]_x^{yy'} := \begin{cases} \widetilde{w}_x^{yy'} & (y \neq y'), \\ -\sum_{y'(\neq y)} \widetilde{w}_x^{y'y} & (y = y'), \end{cases} \tag{C8}$$

where we have used the dimensionless slow time $\tau := \Gamma_X \mathcal{T}$ and dimensionless transition rates $\widetilde{w}_{xx'}^{(\nu)y} := w_{xx'}^{(\nu)y}/\Gamma_X$ and $\widetilde{w}_x^{yy'} := w_x^{yy'}/\Gamma_Y$ with a small parameter $\epsilon := \Gamma_X/\Gamma_Y \ll 1$ (do not confuse ϵ with the error probability ε).

Since we are interested in the fast relaxation limit of Y, we can consider the effective tilted dynamics for $G_\tau^X := \sum_y G_\tau$. By performing a perturbation expansion as in Sec. V A, we obtain

$$\partial_\tau G_\tau^X(x) = \sum_{x'} \left[\overline{\mathcal{L}}_\lambda^X\right]_{xx'} G_\tau^X(x'), \tag{C9}$$

where $\overline{\mathcal{L}}_\lambda^X$ denotes the effective tilted generator given by

$$\left[\overline{\mathcal{L}}_\lambda^X\right]_{xx'} := \sum_\nu \sum_y \left[\mathcal{L}_\lambda^X\right]_{xx'}^{(\nu)y} \pi_{\rm ss}(y|x'), \tag{C10}$$

which can be expressed as

$$\overline{\mathcal{L}}_\lambda^X = \begin{pmatrix} -\sum_\nu \overline{w}_{10}^{(\nu)} & \sum_\nu \overline{w}_{01}^{(\nu)} e^{-\lambda\mu_\nu} \\ \sum_\nu \overline{w}_{10}^{(\nu)} e^{\lambda\mu_\nu} & -\sum_\nu \overline{w}_{01}^{(\nu)} \end{pmatrix}, \tag{C11}$$

where we have introduced the effective transition rate $\overline{w}_{xx'}^{(\nu)} := \sum_y \widetilde{w}_{xx'}^{(\nu)y} \pi_{\rm ss}(y|x')$. The largest eigenvalue $\theta_{\max}(\lambda)$ of this matrix is

$$\theta_{\max}(\lambda) = \frac{1}{2}\sum_\nu \left[\overline{w}_{10}^{(\nu)} + \overline{w}_{01}^{(\nu)}\right]\left\{-1 + \sqrt{1 - \frac{4\left[\overline{w}_{01}^{(L)}\overline{w}_{10}^{(R)}(1 - e^{-\lambda\Delta\mu}) + \overline{w}_{10}^{(L)}\overline{w}_{01}^{(R)}(1 - e^{\lambda\Delta\mu})\right]}{\left(\sum_\nu \left[\overline{w}_{10}^{(\nu)} + \overline{w}_{01}^{(\nu)}\right]\right)^2}}\right\}. \tag{C12}$$

Then, the second derivative of $\theta_{\max}$ gives the fluctuation D_W:

$$\begin{aligned} D_W &= \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\mu(\lambda)\bigg|_{\lambda=0} \\ &= \frac{1}{2}\Gamma_X \frac{\partial^2}{\partial\lambda^2}\theta_{\max}(\lambda)\bigg|_{\lambda=0} \\ &= D_n \Delta\mu^2, \end{aligned} \tag{C13}$$

where D_n denotes the fluctuation of the net particle current:

$$\begin{aligned} D_n &= \frac{\Gamma_X}{2}\left\{\frac{1}{\sum_\nu \left[\overline{w}_{10}^{(\nu)} + \overline{w}_{01}^{(\nu)}\right]}\left[\overline{w}_{01}^{(L)}\overline{w}_{10}^{(R)} + \overline{w}_{10}^{(L)}\overline{w}_{01}^{(R)}\right] - \frac{2}{\left(\sum_\nu \left[\overline{w}_{10}^{(\nu)} + \overline{w}_{01}^{(\nu)}\right]\right)^3}\left[\overline{w}_{01}^{(L)}\overline{w}_{10}^{(R)} - \overline{w}_{10}^{(L)}\overline{w}_{01}^{(R)}\right]^2\right\} \\ &= \frac{\Gamma_X}{2}\left\{\frac{(1-\varepsilon)^2(1-f_L)f_R + \varepsilon^2 f_L(1-f_R)}{1 + (2\varepsilon - 1)(f_L - f_R)} - \frac{2\left[(1-\varepsilon)^2(1-f_L)f_R - \varepsilon^2 f_L(1-f_R)\right]^2}{[1 + (2\varepsilon-1)(f_L - f_R)]^3}\right\} + O(\tilde{\Gamma}_X). \end{aligned} \tag{C14}$$

APPENDIX D: COUPLED LINEAR OVERDAMPED LANGEVIN EQUATIONS

In this Appendix, we provide a detailed calculation of the fluctuation of the entropy production D_S in the fast relaxation limit of Y for the coupled linear overdamped Langevin equations introduced in Sec. VI B. The fluctuation of the information flow D_I can be calculated in a similar way. We also prove that $D_{\mathcal{J}}^{\rm II} := \lim_{\mathcal{T}\to\infty} {\rm Var}[\hat{\mathcal{J}}_{\mathcal{T}}^{\rm II}]/2\mathcal{T} \to 0$ in the fast relaxation limit of Y for the generalized current $\hat{\mathcal{J}}_{\mathcal{T}}$ whose weight function satisfies the condition (178).

1. Calculation of D_S

The fluctuation of the stochastic medium entropy production is defined as

$$D_S := \lim_{\mathcal{T}\to\infty} \frac{1}{2\mathcal{T}} {\rm Var}\left[\Delta\hat{S}_{\rm env}^X\right], \tag{D1}$$

where $\Delta\hat{S}_{\rm env}^X$ denotes the stochastic medium entropy production:

$$\Delta\hat{S}_{\rm env}^X = \int_{\tau=0}^{\tau=\omega^X\mathcal{T}} g(x_\tau, y_\tau) \circ dx_\tau, \tag{D2}$$

where we have used the dimensionless slow time $\tau = \omega^X t$, and the weight function is defined as

$$g(x, y) := \frac{1}{\bar{D}^X}(-x + \bar{\omega}^{XY} y). \tag{D3}$$

The fluctuation D_S can be obtained from the scaled cumulant generating function defined by

$$\mu(\lambda) = \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \ln\left\langle e^{\lambda\Delta\hat{S}_{\rm env}^X}\right\rangle. \tag{D4}$$

To compute the scaled cumulant generating function, we introduce the generating function conditioned to an initial state $(x_0, y_0) = (x, y)$, defined by

$$G_{\mathcal{T}}(x, y) := \left\langle e^{\lambda\Delta\hat{S}_{\rm env}^X}\middle| x, y\right\rangle. \tag{D5}$$

The time evolution of $G_{\mathcal{T}}$ is described by the Feynman-Kac formula [78]

$$\partial_\tau G_\tau(x, y) = \mathcal{L}_\lambda^\dagger[G_\tau](x, y), \tag{D6}$$

where $\mathcal{L}_\lambda^\dagger$ denotes the tilted generator defined by

$$\mathcal{L}_\lambda^\dagger = \mathcal{L}_\lambda^{X\dagger} + \frac{1}{\epsilon}\mathcal{L}_\lambda^{Y\dagger} \tag{D7}$$

with

$$\mathcal{L}_\lambda^{X\dagger} := \bar{F}^X(x,y)[\partial_x + \lambda g(x,y)] + \bar{D}^X[\partial_x + \lambda g(x,y)]^2, \tag{D8}$$

$$\mathcal{L}_\lambda^{Y\dagger} := \bar{F}^Y(x,y)\partial_y + \bar{D}^Y\partial_y^2, \tag{D9}$$

where $\bar{F}^X(x,y) := -x + \bar{\omega}^{XY}y$ and $\bar{F}^Y(x,y) := \bar{\omega}^{YX}x - y$ denote the dimensionless drift terms. The largest eigenvalue of this tilted generator gives the scaled cumulant generating function.

Since we are interested in the fast relaxation limit of Y, we can further simplify the problem by considering the effective tilted generator for X, as follows. We first assume that G_τ have asymptotic expansions in terms of the asymptotic sequences $\{\epsilon^n\}_{n=0}^\infty$ as $\epsilon \to 0$:

$$G_\tau = G_\tau^{(0)} + \epsilon G_\tau^{(1)} + \cdots. \tag{D10}$$

Here, we impose the normalization condition

$$\begin{aligned}\int dy\, \pi_{\rm ss}(y|x)G_\tau^{(0)}(x,y) &= \int dy\, \pi_{\rm ss}(y|x)G_\tau(x,y)\\ &=: G_\tau^X(x),\end{aligned} \tag{D11}$$

where $\pi_{\rm ss}$ denotes the zero eigenfunction for $\mathcal{L}_0^Y$. By substituting this expansion into (D6), we find that the leading order gives

$$\mathcal{L}_\lambda^{Y\dagger}\big[G_\tau^{(0)}\big](x,y) = 0. \tag{D12}$$

Since $\mathcal{L}_\lambda^{Y\dagger} = \mathcal{L}_0^{Y\dagger}$, the zero eigenfunction for $\mathcal{L}_\lambda^{Y\dagger}$ is 1. From the Perron-Frobenius theorem and the normalization condition, we find that $G_\tau^{(0)}$ has the form

$$G_\tau^{(0)}(x,y) = G_\tau^X(x). \tag{D13}$$

The subleading order of (D6) gives

$$\partial_\tau G_\tau^{(0)}(x,y) = \mathcal{L}_\lambda^{X\dagger}\big[G_\tau^{(0)}\big](x,y) + \mathcal{L}_\lambda^{Y\dagger}\big[G_\tau^{(1)}\big](x,y). \tag{D14}$$

From the solvability condition for $G_\tau^{(1)}(x,y)$, we obtain the effective dynamics for $G_\tau^X(x)$:

$$\partial_\tau G_\tau^X(x) = \overline{\mathcal{L}}_\lambda^{X\dagger}\big[G_\tau^X\big](x). \tag{D15}$$

Here, $\overline{\mathcal{L}}_\lambda^{X\dagger}$ denotes the effective tilted generator defined by

$$\begin{aligned}\overline{\mathcal{L}}_\lambda^{X\dagger} &:= \int dy\, \pi_{\rm ss}(y|x)\mathcal{L}_\lambda^{X\dagger}\\ &= -(1-\bar{\omega}^{XY}\bar{\omega}^{YX})x\partial_x + \bar{D}^X\partial_x^2\\ &\quad + \lambda\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 + \frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2x^2 - 1 - 2(1-\bar{\omega}^{XY}\bar{\omega}^{YX})x\partial_x\right]\\ &\quad + \lambda^2\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 + \frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2x^2\right].\end{aligned} \tag{D16}$$

We first note that when $\lambda = 0$, the largest eigenvalue is 0 with the corresponding eigenfunction $\phi_{\lambda=0}(x) = 1$. To find the largest eigenvalue for general λ, we impose the Gaussian ansatz $\phi_\lambda(x) = \exp[-K(\lambda)x^2/2]$. Then, the largest eigenvalue $\theta_{\max}(\lambda)$ should satisfy

$$\begin{aligned}\theta_{\max}(\lambda) &= \frac{\mathcal{L}_\lambda^\dagger\phi_\lambda(x)}{\phi_\lambda(x)}\\ &= (1-\bar{\omega}^{XY}\bar{\omega}^{YX})x^2K(\lambda) + \bar{D}^X[-K(\lambda) + K^2(\lambda)x^2]\\ &\quad + \lambda\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 + \frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2x^2 - 1 + 2(1-\bar{\omega}^{XY}\bar{\omega}^{YX})x^2K(\lambda)\right]\\ &\quad + \lambda^2\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 + \frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2x^2\right].\end{aligned} \tag{D17}$$

Because this relation holds for arbitrary x, comparing the coefficients of the quadratic form yields

$$\theta_{\max}(\lambda) = -\bar{D}^XK(\lambda) + \lambda\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 - 1\right] + \lambda^2\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2 \tag{D18}$$

and

$$(1-\bar{\omega}^{XY}\bar{\omega}^{YX})K(\lambda) + \bar{D}^XK^2(\lambda) + \lambda\left[\frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2 + 2(1-\bar{\omega}^{XY}\bar{\omega}^{YX})K(\lambda)\right] + \lambda^2\frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2 = 0. \tag{D19}$$

We now expand K in terms of λ as

$$K(\lambda) = \lambda K^{(1)} + \lambda^2K^{(2)} + \cdots. \tag{D20}$$

Here, note that $K(\lambda)$ should go to zero as $\lambda \to 0$, because $\theta_{\max}(\lambda)$ is the largest eigenvalue and $\theta_{\max}(\lambda) \to 0$ as $\lambda \to 0$. Then, by substituting this expansion into (D19), we find that the leading order yields

$$(1-\bar{\omega}^{XY}\bar{\omega}^{YX})K^{(1)}+\frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2=0. \tag{D21}$$

From this equation, we obtain

$$K^{(1)}=-\frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX}). \tag{D22}$$

The subleading order of (D19) gives

$$(1-\bar{\omega}^{XY}\bar{\omega}^{YX})K^{(2)}+\bar{D}^X(K^{(1)})^2+2(1-\bar{\omega}^{XY}\bar{\omega}^{YX})K^{(1)} + \frac{1}{\bar{D}^X}(1-\bar{\omega}^{XY}\bar{\omega}^{YX})^2=0. \tag{D23}$$

Since $1-\bar{\omega}^{XY}\bar{\omega}^{YX}>0$, we find that $K^{(2)}=0$. Therefore, the largest eigenvalue is

$$\theta_{\max}(\lambda)=\lambda\left[\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2-\bar{\omega}^{XY}\bar{\omega}^{YX}\right]+\lambda^2\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2. \tag{D24}$$

From this result, we can calculate D_S as

$$\begin{aligned} D_S &= \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\mu(\lambda)\bigg|_{\lambda=0} \\ &= \frac{1}{2}\omega^X\frac{\partial^2}{\partial\lambda^2}\theta_{\max}(\lambda)\bigg|_{\lambda=0} \\ &= \omega^X\frac{\bar{D}^Y}{\bar{D}^X}(\bar{\omega}^{XY})^2. \end{aligned} \tag{D25}$$

2. Proof of $D_{\mathcal{J}}^{\rm II}\to 0$

Here, we prove that $D_{\mathcal{J}}^{\rm II}:=\lim_{\mathcal{T}\to\infty}\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}}^{\rm II}]/2\mathcal{T}=0$ in the fast relaxation limit of Y for the generalized current $\hat{\mathcal{J}}_{\mathcal{T}}$ whose weight function is given by $g(x,y)=C(y-\bar{\omega}^{YX}x)$. For this weight function, $\hat{\mathcal{J}}_{\mathcal{T}}^{\rm II}$ can be expressed as

$$\hat{\mathcal{J}}_{\mathcal{T}}^{\rm II}:=\int_0^{\omega^X\mathcal{T}}f(x_\tau,y_\tau)d\tau, \tag{D26}$$

where

$$\begin{aligned} f(x,y) &:= C(y-\bar{\omega}^{YX}x)(-x+\bar{\omega}^{XY}y) \\ &\quad +\bar{D}^X\partial_x C(y-\bar{\omega}^{YX}x). \end{aligned} \tag{D27}$$

The fluctuation $D_{\mathcal{J}}^{\rm II}$ can be obtained from the following scaled cumulant generating function

$$\mu(\lambda)=\lim_{\mathcal{T}\to\infty}\frac{1}{\mathcal{T}}\ln\left\langle e^{\lambda\hat{\mathcal{J}}_{\mathcal{T}}^{\rm II}}\right\rangle, \tag{D28}$$

which corresponds to the largest eigenvalue of the following tilted generator [78]:

$$\mathcal{L}_\lambda^\dagger=\mathcal{L}_\lambda^{X\dagger}+\frac{1}{\epsilon}\mathcal{L}_\lambda^{Y\dagger} \tag{D29}$$

with

$$\mathcal{L}_\lambda^{X\dagger}:=\bar{F}^X(x,y)\partial_x+\bar{D}^X\partial_x^2+\lambda f(x,y), \tag{D30}$$

$$\mathcal{L}_\lambda^{Y\dagger}:=\bar{F}^Y(x,y)\partial_y+\bar{D}^Y\partial_y^2. \tag{D31}$$

The effective tilted generator is then given by

$$\begin{aligned} \overline{\mathcal{L}}_\lambda^{X\dagger} &:= \int dy\,\pi_{\rm ss}(y|x)\mathcal{L}_\lambda^{X\dagger} \\ &= -(1-\bar{\omega}^{XY}\bar{\omega}^{YX})x\partial_x+\bar{D}^X\partial_x^2 \\ &\quad +\lambda C(-\bar{\omega}^{YX}\bar{D}^X+\bar{\omega}^{XY}\bar{D}^Y). \end{aligned} \tag{D32}$$

By performing a similar calculation as in the previous section, we finally obtain

$$\theta_{\max}(\lambda)=\lambda C\big(-\bar{\omega}^{YX}\bar{D}^X+\bar{\omega}^{XY}\bar{D}^Y\big). \tag{D33}$$

Thus, we find that $D_{\mathcal{J}}^{\rm II}:=\lim_{\mathcal{T}\to\infty}\mathrm{Var}[\hat{\mathcal{J}}_{\mathcal{T}}^{\rm II}]/2\mathcal{T}=0$.

[1] N. Barkai and S. Leibler, Robustness in simple biochemical networks, Nature (London) **387**, 913 (1997).

[2] U. Alon, M. G. Surette, N. Barkai, and S. Leibler, Robustness in bacterial chemotaxis, Nature (London) **397**, 168 (1999).

[3] Y. Tu, T. S. Shimizu, and H. C. Berg, Modeling the chemotactic response of Escherichia coli to time-varying stimuli, Proc. Natl. Acad. Sci. USA **105**, 14855 (2008).

[4] G. Lan, P. Sartori, S. Neumann, V. Sourjik, and Y. Tu, The energy–speed–accuracy trade-off in sensory adaptation, Nat. Phys. **8**, 422 (2012).

[5] S. Ito and T. Sagawa, Maxwell's demon in biochemical signal transduction with feedback loop, Nat. Commun. **6**, 7498 (2015).

[6] H. H. Mattingly, K. Kamino, B. B. Machta, and T. Emonet, Escherichia coli chemotaxis is information limited, Nat. Phys. **17**, 1426 (2021).

[7] S. S. Shen-Orr, R. Milo, S. Mangan, and U. Alon, Network motifs in the transcriptional regulation network of Escherichia coli, Nat. Genet. **31**, 64 (2002).

[8] T. I. Lee, N. J. Rinaldi, F. Robert, D. T. Odom, Z. Bar-Joseph, G. K. Gerber, N. M. Hannett, C. T. Harbison, C. M. Thompson, I. Simon *et al.*, Transcriptional regulatory networks in Saccharomyces cerevisiae, Science **298**, 799 (2002).

[9] G. Karlebach and R. Shamir, Modelling and analysis of gene regulatory networks, Nat. Rev. Mol. Cell Biol. **9**, 770 (2008).

[10] S. Otsubo and T. Sagawa, Information-thermodynamic characterization of stochastic Boolean networks, arXiv:1803.04217.

[11] E. Lathouwers and D. A. Sivak, Internal energy and information flows mediate input and output power in bipartite molecular machines, Phys. Rev. E **105**, 024136 (2022).

[12] M. P. Leighton and D. A. Sivak, Inferring subsystem efficiencies in bipartite molecular machines, Phys. Rev. Lett. **130**, 178401 (2023).

[13] M. Grelier, D. A. Sivak, and J. Ehrich, Unlocking the potential of information flow: Maximizing free-energy transduction in a model of an autonomous rotary molecular motor, arXiv:2304.06690.

[14] K. Sekimoto, *Stochastic Energetics* (Springer, New York, 2010).

[15] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, Rep. Prog. Phys. **75**, 126001 (2012).

[16] C. Van den Broeck and M. Esposito, Ensemble and trajectory thermodynamics: A brief introduction, Phys. A (Amsterdam) **418**, 6 (2015).
[17] L. Peliti and S. Pigolotti, *Stochastic Thermodynamics: An Introduction* (Princeton University Press, Princeton, NJ, 2021).
[18] A. C. Barato and U. Seifert, Thermodynamic uncertainty relation for biomolecular processes, Phys. Rev. Lett. **114**, 158101 (2015).
[19] J. M. Horowitz and T. R. Gingrich, Thermodynamic uncertainty relations constrain non-equilibrium fluctuations, Nat. Phys. **16**, 15 (2020).
[20] N. Shiraishi, Optimal thermodynamic uncertainty relation in Markov jump processes, J. Stat. Phys. **185**, 19 (2021).
[21] N. Shiraishi, K. Saito, and H. Tasaki, Universal trade-off relation between power and efficiency for heat engines, Phys. Rev. Lett. **117**, 190601 (2016).
[22] P. Pietzonka and U. Seifert, Universal trade-off between power, efficiency, and constancy in steady-state heat engines, Phys. Rev. Lett. **120**, 190602 (2018).
[23] V. Holubec and A. Ryabov, Cycling tames power fluctuations near optimum efficiency, Phys. Rev. Lett. **121**, 120601 (2018).
[24] V. Holubec and A. Ryabov, Fluctuations in heat engines, J. Phys. A: Math. Theor. **55**, 013001 (2022).
[25] S. Saryal, M. Gerry, I. Khait, D. Segal, and B. K. Agarwalla, Universal bounds on fluctuations in continuous thermal machines, Phys. Rev. Lett. **127**, 190603 (2021).
[26] N. Shiraishi, T. Van Vu, and K. Saito, Is there any upper bound on current fluctuation in cross-transport beyond linear response regime? arXiv:2203.15466.
[27] J. M. Parrondo, J. M. Horowitz, and T. Sagawa, Thermodynamics of information, Nat. Phys. **11**, 131 (2015).
[28] A. C. Barato, D. Hartich, and U. Seifert, Efficiency of cellular information processing, New J. Phys. **16**, 103024 (2014).
[29] P. Sartori, L. Granger, C. F. Lee, and J. M. Horowitz, Thermodynamic costs of information processing in sensory adaptation, PLoS Comput. Biol. **10**, e1003974 (2014).
[30] D. Hartich, A. C. Barato, and U. Seifert, Sensory capacity: An information theoretical measure of the performance of a sensor, Phys. Rev. E **93**, 022116 (2016).
[31] S. Amano, M. Esposito, E. Kreidt, D. A. Leigh, E. Penocchio, and B. M. W. Roberts, Insights from an information thermodynamics analysis of a synthetic molecular motor, Nat. Chem. **14**, 530 (2022).
[32] T. Tanogami and R. Araki, Information-thermodynamic bound on information flow in turbulent cascade, arXiv:2206.11163.
[33] G. Gallavotti and E. G. D. Cohen, Dynamical ensembles in nonequilibrium statistical mechanics, Phys. Rev. Lett. **74**, 2694 (1995).
[34] J. Kurchan, Fluctuation theorem for stochastic dynamics, J. Phys. A: Math. Gen. **31**, 3719 (1998).
[35] J. L. Lebowitz and H. Spohn, A Gallavotti–Cohen-type symmetry in the large deviation functional for stochastic dynamics, J. Stat. Phys. **95**, 333 (1999).
[36] Y. Hasegawa and T. Van Vu, Uncertainty relations in stochastic processes: An information inequality approach, Phys. Rev. E **99**, 062126 (2019).
[37] T. Van Vu and Y. Hasegawa, Uncertainty relations for underdamped Langevin dynamics, Phys. Rev. E **100**, 032130 (2019).
[38] J. S. Lee, J.-M. Park, and H. Park, Thermodynamic uncertainty relation for underdamped Langevin systems driven by a velocity-dependent force, Phys. Rev. E **100**, 062132 (2019).
[39] L. P. Fischer, H.-M. Chun, and U. Seifert, Free diffusion bounds the precision of currents in underdamped dynamics, Phys. Rev. E **102**, 012120 (2020).
[40] J. S. Lee, J.-M. Park, and H. Park, Universal form of thermodynamic uncertainty relation for Langevin dynamics, Phys. Rev. E **104**, L052102 (2021).
[41] R.-S. Fu and T. R. Gingrich, Thermodynamic uncertainty relation for Langevin dynamics by scaling time, Phys. Rev. E **106**, 024128 (2022).
[42] A. Dechant, Bounds on the precision of currents in underdamped Langevin dynamics, arXiv:2202.10696.
[43] P. Pietzonka, Classical pendulum clocks break the thermodynamic uncertainty relation, Phys. Rev. Lett. **128**, 130606 (2022).
[44] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. (Wiley, Hoboken, NJ, 2006).
[45] J. M. Horowitz and M. Esposito, Thermodynamics with continuous information flow, Phys. Rev. X **4**, 031015 (2014).
[46] R. Chetrite, M. L. Rosinberg, T. Sagawa, and G. Tarjus, Information thermodynamics for interacting stochastic systems without bipartite structure, J. Stat. Mech. (2019) 114002.
[47] C. Maes, Local detailed balance, SciPost Phys. Lect. Notes **32**, (2021).
[48] N. Shiraishi and T. Sagawa, Fluctuation theorem for partially masked nonequilibrium dynamics, Phys. Rev. E **91**, 012130 (2015).
[49] A. Dechant, Multidimensional thermodynamic uncertainty relations, J. Phys. A: Math. Theor. **52**, 035001 (2019).
[50] C. Dieball and A. Godec, Direct route to thermodynamic uncertainty relations and their saturation, Phys. Rev. Lett. **130**, 087101 (2023).
[51] A. Dechant and S.-i. Sasa, Improving thermodynamic bounds using correlations, Phys. Rev. X **11**, 041061 (2021).
[52] S. Otsubo, S. Ito, A. Dechant, and T. Sagawa, Estimating entropy production by machine learning of short-time fluctuating currents, Phys. Rev. E **101**, 062106 (2020).
[53] J. Ehrich and D. A. Sivak, Energy and information flows in autonomous systems, Front. Phys. **11**, 1108357 (2023).
[54] P. Strasberg, G. Schaller, T. Brandes, and M. Esposito, Thermodynamics of a physical model implementing a Maxwell demon, Phys. Rev. Lett. **110**, 040601 (2013).
[55] S. Tănase-Nicola, P. B. Warren, and P. R. ten Wolde, Signal detection, modularity, and the correlation between extrinsic and intrinsic noise in biochemical networks, Phys. Rev. Lett. **97**, 068102 (2006).
[56] P. Thomas, A. V. Straube, and R. Grima, The slow-scale linear noise approximation: An accurate, reduced stochastic description of biochemical networks under timescale separation conditions, BMC Syst. Biol. **6**, 39 (2012).
[57] A. K. Maity, P. Chaudhury, and S. K. Banik, Role of relaxation time scale in noisy signal transduction, PLoS One **10**, e0123242 (2015).
[58] C. W. Gardiner, *Handbook of Stochastic Methods*, 4th ed. (Springer, Berlin, 2009).
[59] R. Filliger and P. Reimann, Brownian gyrator: A minimal heat engine on the nanoscale, Phys. Rev. Lett. **99**, 230602 (2007).

[60] K.-H. Chiang, C.-L. Lee, P.-Y. Lai, and Y.-F. Chen, Electrical autonomous brownian gyrator, Phys. Rev. E **96**, 032123 (2017).

[61] A. Argun, J. Soni, L. Dabelow, S. Bo, G. Pesce, R. Eichhorn, and G. Volpe, Experimental realization of a minimal microscopic heat engine, Phys. Rev. E **96**, 052106 (2017).

[62] A. Celani, S. Bo, R. Eichhorn, and E. Aurell, Anomalous thermodynamics at the microscale, Phys. Rev. Lett. **109**, 260603 (2012).

[63] K. Kawaguchi and Y. Nakayama, Fluctuation theorem for hidden entropy production, Phys. Rev. E **88**, 022147 (2013).

[64] J. du Buisson and H. Touchette, Dynamical large deviations of linear diffusions, Phys. Rev. E **107**, 054111 (2023).

[65] P. P. Potts and P. Samuelsson, Thermodynamic uncertainty relations including measurement and feedback, Phys. Rev. E **100**, 052137 (2019).

[66] T. Van Vu and Y. Hasegawa, Uncertainty relation under information measurement and feedback control, J. Phys. A: Math. Theor. **53**, 075001 (2020).

[67] J. Li, J. M. Horowitz, T. R. Gingrich, and N. Fakhri, Quantifying dissipation using fluctuating currents, Nat. Commun. **10**, 1666 (2019).

[68] T. Van Vu, V. T. Vo, and Y. Hasegawa, Entropy production estimation with optimal current, Phys. Rev. E **101**, 042138 (2020).

[69] S. K. Manikandan, D. Gupta, and S. Krishnamurthy, Inferring entropy production from short experiments, Phys. Rev. Lett. **124**, 120603 (2020).

[70] S. Otsubo, S. K. Manikandan, T. Sagawa, and S. Krishnamurthy, Estimating time-dependent entropy production from non-equilibrium trajectories, Commun. Phys. **5**, 11 (2022).

[71] B. Das, S. K. Manikandan, and A. Banerjee, Inferring entropy production in anharmonic brownian gyrators, Phys. Rev. Res. **4**, 043080 (2022).

[72] J. L. Doob, Conditional Brownian motion and the boundary limits of harmonic functions, Bull. Soc. Math. France **85**, 431 (1957).

[73] R. Chetrite and H. Touchette, Nonequilibrium Markov processes conditioned on large deviations, in *Ann. H. Poincaré* Vol. 16 (Springer, Berlin, 2015), pp. 2005–2057.

[74] S. N. Majumdar and H. Orland, Effective Langevin equations for constrained stochastic processes, J. Stat. Mech. (**2015**) P06039.

[75] S. Pigolotti, I. Neri, É. Roldán, and F. Jülicher, Generic properties of stochastic entropy production, Phys. Rev. Lett. **119**, 140604 (2017).

[76] A. Dechant and S.-i. Sasa, Continuous time reversal and equality in the thermodynamic uncertainty relation, Phys. Rev. Res. **3**, L042012 (2021).

[77] K. Liu, Z. Gong, and M. Ueda, Thermodynamic uncertainty relation for arbitrary initial states, Phys. Rev. Lett. **125**, 140602 (2020).

[78] H. Touchette, Introduction to dynamical large deviations of Markov processes, Physica A (Amsterdam) **504**, 5 (2018).