Modeling particle loss in open systems using Keldysh path integral and second order cumulant expansion

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For open quantum systems, integration of the bath degrees of freedom using the second order cumulant expansion in the Keldysh path integral provides an alternative derivation of the effective action for systems coupled to general baths. The baths can be interacting and not necessarily Markovian. Using this method in the Markovian limit, we compute the particle-loss dynamics in various models of ultracold atomic gases, including a one-dimensional Bose-Hubbard model with two-particle losses and a multicomponent Fermi gas with interactions tuned by an optical Feshbach resonance. We explicitly demonstrate that the limit of strong two-body losses can be treated by formulating an indirect loss scheme to describe the bath-system coupling. The particle-loss dynamics thus obtained is valid at all temperatures. For the one-dimensional Bose-Hubbard model, we compare it to solutions of the phenomenological rate equations. The latter are shown to be accurate at high temperatures.

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I. INTRODUCTION

Open quantum systems constitute one of the most interesting challenges of the field, both from a fundamental point of view and of course in connection with applications or experimental realizations. This class of problems offers an interesting interplay between coherent Hamiltonian dynamics and incoherent, dissipative dynamics emerging from coupling to the environment [1–3]. From a fundamental point of view, after being considered a nuisance due to the destruction of coherence, dissipative processes are now regarded as a resource allowing us to engineer quantum states of matter [4–7] or to assist for quantum transport properties [8–10]. Among the various types of dissipative processes, particle losses have played a special role, and recent experiments in cold atomic systems have allowed us to control them in an exquisite manner. This is, for example, the case of losses in weakly interacting Bose gases [11,12] or fermionic systems where local losses are realized [13,14]. More recently, cavities have also provided an interesting playground for this kind of physics [15–17].

Theoretically studying dissipative phenomena is a considerable challenge, and several approaches have been used to deal with the coupling to the environment, which is often modeled as a bath. One common approach resorts to non-Hermitian Hamiltonians [3,18–20]. The resulting (non-Hermitian) models can be analyzed using, e.g., the powerful tools of integrability [21,22] as well as various field-theoretic and numerical techniques [18,19], which include bosonization and the renormalization group [20,22,23]. However, because such treatments often neglect or at most treat in an approximate way the so-called quantum-jump term of the Lindblad master equation, it is difficult to assess how reliably they can describe the dynamics of the system observables. Other approaches deal with the full Lindblad master equation [24–27], assuming the bath is Markovian [1,28]. In this paper, we are concerned with another generic method to deal with such out-of-equilibrium systems, namely, the Keldysh formalism and its path integral formulation [29,30].

Approaches based on the path integral have a long history beginning with the pioneering work of Feynman and Vernon [31]. They have often been used in the context of quantum dissipation, where the coupling to ohmic, subohmic, or superohmic baths generates an effective long-range interaction in imaginary time [32–35]. In a more general context, they have also been used to describe nonequilibrium dynamics in the presence of coupling to baths [29,36], time-dependent environments [10,37], and single-particle losses [30,38–40].

Despite these important developments, several phenomena linked to the coupling to the environment still remain elusive. In particular, in ultracold atomic systems the dynamics in the presence particle losses involve *a priori* processes beyond single-particle losses and include two- or three-particle losses as well. Quite generally, the description of particle losses has assumed the validity of phenomenological rate equations of the form

$$\frac{dn}{dt} = -\gamma_1 n - \gamma_2 n^2, \tag{1}$$

where n is the particle density and γ_1 and γ_2 are the one- and two-particle loss rates, respectively. However, the correct

functional form and range of validity of these phenomenological equations and how to account for many-particle effects are still not fully understood. For ultracold atoms near Feshbach resonances, a theoretical description of the two-body loss rate, γ_2, was developed based on the S-matrix calculation in Ref. [41] and some properties of three-body losses could be determined from Bethe ansatz solutions [42]. Braaten *et al.* [43] combined effective field theory with the Lindblad master equation to obtain a universal relation for the two-atom inelastic loss rate for two-species Fermi gases.

In this paper, to go beyond the phenomenological description of particle losses provided by Eq. (1) and be able to account for many-particle effects, we develop a path integral formalism that relies on the second order cumulant expansion. The formalism can, in principle, also deal with interacting as well as non-Markovian baths. However, here we demonstrate it by studying the Markovian dynamics of many-particle systems in the presence of one- and two-particle losses and briefly discuss how to go beyond the Markovian limit in Appendix C. In the absence of interactions in the bath, the resulting approach is exact for one-particle losses and approximately valid for two-particle loss problems at weak coupling. To deal with the strong coupling regime, we also introduce an indirect two-body loss scheme. We discuss two experimentally relevant examples of this indirect-loss scheme [a lossy one-dimensional Bose-Hubbard model and a multicomponent Fermi gas near an optical Feshbach resonance (OFR)] and explicitly derive the loss rate equations. In both cases, we find their functional form deviates from Eq. (1). In the case of the lossy Bose-Hubbard model, we explicitly show that, at low-temperatures where quantum coherence is important, predictions of the microscopic theory deviate substantially from those obtained using the phenomenological Eq. (1).

The rest of this paper is organized as follows. In Sec. II, we describe the derivation of the Keldysh path integral using the second order cumulant expansion. We first illustrate the method with a model consisting of a single-mode (fermonic or bosonic) oscillator coupled to a bath to which particles can be lost (cf. Sec. II A). The general formalism is discussed in Sec. II B. However, since only the results of Sec. II A will be used in the rest of the paper, Sec. II B can be skipped on a first reading. Section III briefly describes how the results of Sec. II A are applied to describe a quantum gas with one-body losses. Sections IV and V deal with the applications of the formalism to two models for relevant for the physics of ultracold atomic gases with two-body losses: Sec. IV is concerned with a lossy Bose-Hubbard model in one dimension, whereas Sec. V deals with the model of a multicomponent Fermi gas near OFR. Some technical details and useful results are described to the Appendices.

II. FORMALISM

A. Single-mode case

To begin, let us illustrate the method by considering a single bosonic or fermionic mode (system A) coupled to a bath (B) by means of a quadratic Hamiltonian:

$$H_A = \epsilon_0 a^\dagger a + H_{A,\mathrm{int}}, \tag{2}$$

$$H_B = \sum_\alpha \omega_\alpha b_\alpha^\dagger b_\alpha + H_{B,\mathrm{int}}, \tag{3}$$

$$H_{AB}(t) = f(t)\sum_\alpha [g_\alpha a^\dagger b_\alpha + g_\alpha^* b_\alpha^\dagger a]. \tag{4}$$

Here $a, a^\dagger$ describe a bosonic (fermionic) mode in the A and $b_\alpha, b_\alpha^\dagger$ set of bosonic (fermionic) modes in B labeled by a continuum index α. We shall assume that A and bath B are in equilibrium (not necessarily with each other) at $t = -\infty$ and the interaction between them, $H_{AB}(t)$, is switched according to a protocol described by the function $f(t)$. Both A and B can be interacting systems with, e.g., $H_{B,\mathrm{int}} = g_{\mathrm{int}} \sum_{\alpha\beta\gamma\delta} b_\alpha^\dagger b_\beta^\dagger b_\gamma b_\delta$.

The Keldysh generating functional [29] for the system introduced above can be written as follows:

$$Z[\bar V, V] = \int D[\bar a a] D[\bar b_\alpha b_\alpha]\, e^{iS}. \tag{5}$$

The Keldysh action is $S = S_A + S_B + S_V + S_{AB}$, where

$$S_A = \int_C dt\, [i\bar a \partial_t a - H_A], \tag{6}$$

$$S_B = \int_C dt \left[i\sum_\alpha \bar b_\alpha \partial_t b_\alpha - H_{\mathrm{int},B} \right], \tag{7}$$

$$S_V = \int_C dt\, [\bar a(t) V(t) + \bar V(t) a(t)], \tag{8}$$

$$S_{AB} = -\sum_\alpha \int_C dt\; f(t)[g_\alpha \bar a(t) b_\alpha(t) + g_\alpha^* \bar b_\alpha(t) a(t)]. \tag{9}$$

In the above expressions, C is the Keldysh contour, which runs from $t = -\infty$ to $t = +\infty$ and back to $t = -\infty$ [29]; $\bar V, V$ are sources that couple the system degrees of freedom: Any n-point correlation of the system A can be obtained by conveniently taking functional derivatives of $Z[\bar V, V]$ with respect to $\bar V$ and V. The generating functional is normalized so $Z[\bar V = 0, V = 0] = 1$, since in the absence of external sources it merely describes the unitary evolution of the initial state from $t = -\infty$ to $+\infty$ and back to $t = -\infty$.

Starting from the above functional integral representation, we can define the Feynman-Vernon influence functional [31] $\mathcal{F}[\bar a, a]$ as the result of formally integrating out the bath degrees of freedom, i.e.,

$$\mathcal{F}[\bar a, a] = \int D[\bar b_\alpha b_\alpha]\, e^{i(S_B + S_{AB})}. \tag{10}$$

It is often not possible to obtain $\mathcal{F}$ in a closed form and, therefore, we have to resort to approximations. For a general (e.g., interacting) bath that is weakly coupled to the system, the second order cumulant expansion provides a good starting point to capture the dissipative dynamics induced by the bath. Furthermore, it is exact if the bath Hamiltonian H_B and the coupling H_{AB} are quadratic and linear in the fields $\bar b_\alpha, b_\alpha$, respectively. Using the cumulant expansion to second order (see Appendix B), we approximate

$$\mathcal{F}[\bar a, a] = \langle e^{iS_{AB}}\rangle_B \simeq \exp\left[i\langle S_{AB}\rangle_B - \tfrac{1}{2}\left(\langle S_{AB}^2\rangle_B - \langle S_{AB}\rangle_B^2\right)\right] = e^{i\mathcal{L}[\bar a, a]}, \tag{11}$$

where $\langle\cdots\rangle_B$ stands for the average over the bath degrees of freedom. Taking $\langle S_{AB}\rangle_B = 0$ after assuming that the bath conserves the number of bath excitations in its initial state and dynamics, we obtain the following dissipative effective action:

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}\sum_{\alpha}\int_C dt_1 dt_2\, |g_\alpha|^2 f(t_1) f(t_2)\\ &\quad\times [\langle b_\alpha(t_1)\bar{b}_\alpha(t_2)\rangle_B\, \bar{a}(t_1)a(t_2)\\ &\quad+ \langle \bar{b}_\alpha(t_1) b_\alpha(t_2)\rangle_B\, a(t_1)\bar{a}(t_2)]\\ &= \frac{i}{2}\sum_{m,n=\pm}\int dt_1 dt_2 v^{mn}(t_1,t_2)\bar{a}_m(t_1)a_n(t_2).\end{aligned} \tag{12}$$

In the last expression, we have split the integrals over the Keldysh contour C. After rewriting them as a single integral where t_1, t_2 run from $-\infty$ to $+\infty$, we have introduced the subindices $m, n = \pm$ to denote on which branch of C the time argument of the fields a, b, etc. lies. We have also introduced the functions $v^{mn}(t_1,t_2) = f(t_1)f(t_2)g^{mn}(t_1-t_2)$, where

$$\begin{aligned}g^{mn}(t_1-t_2) &= s_n s_m \sum_\alpha |g_\alpha|^2 [\langle b_{\alpha m}(t_1)\bar{b}_{\alpha n}(t_2)\rangle_B\\ &\quad + z\langle \bar{b}_{\alpha n}(t_2) b_{\alpha m}(t_1)\rangle_B].\end{aligned} \tag{13}$$

In the above expression, $s_{m=\pm} = \pm 1$ and $z = +1$ ($z = -1$) for bosons (fermions). In the Appendices, within Markovian limit, we show that $g^{mn}(t_1-t_2) \sim \delta(t_1-t_2)$. Thus, we arrive at the following expression for the effective couplings (see Appendix C for more detail):

$$v^{++}(t_1,t_2) \simeq \nu_0 |\langle g\rangle f(t_1)|^2 \delta(t_1-t_2), \tag{14}$$

$$v^{--}(t_1,t_2) \simeq \nu_0 |\langle g\rangle f(t_1)|^2 \delta(t_1-t_2), \tag{15}$$

$$v^{-+}(t_1,t_2) \simeq -2\nu_0 |\langle g\rangle f(t_1)|^2 \delta(t_1-t_2) \tag{16}$$

$$v^{+-}(t_1,t_2) \simeq 0, \tag{17}$$

where $\langle g\rangle$ is an average system-bath coupling constant. Upon denoting $\gamma(t) = \nu_0 |\langle g\rangle f(t)|^2$ for the loss rate of particles to the bath, we obtain the following result:

$$\begin{aligned}\mathcal{L} &= -i\int dt\, \gamma(t)\Big[\bar{a}_-(t)a_+(t)\\ &\quad - \frac{1}{2}(\bar{a}_+(t)a_+(t) + \bar{a}_-(t)a_-(t))\Big].\end{aligned} \tag{18}$$

This is the dissipative part of the action characteristic of a Markovian bath. It can also be obtained from the evolution of the density matrix according to the Lindblad master equation (see Appendix A, which is based on Ref. [30] and references therein).

Finally, let us compute the particle loss in system A caused by switching on the coupling to bath B at $t = 0$ for an infinitesimal time δt. This calculation can be carried out by the path integral version of time-dependent perturbation theory, i.e., by perturbatively expanding the effective dissipative action to leading order in $\mathcal{L}$, which yields

$$\begin{aligned}\langle a^\dagger(t)a(t)\rangle &= \int D[\bar{a}a]\, \bar{a}_-(t)a_+(t)\, e^{i(S_A+\mathcal{L})},\\ &= \int D[\bar{a}a]\, \bar{a}_-(t)a_+(t)[1+\mathcal{L}+O(\mathcal{L}^2)]\, e^{iS_A},\\ &\simeq \langle \bar{a}_-(t)a_+(t)\rangle_A + i\langle \bar{a}_-(t)a_+(t)\mathcal{L}\rangle_A,\\ &= n_a + \gamma\int_0^{+\infty} dt' \langle \bar{a}_-(t)a_+(t)\bar{a}_-(t')a_+(t')\rangle_A\\ &\quad - \frac{1}{2}\gamma\int_0^{+\infty} dt' \langle \bar{a}_-(t)a_+(t)\bar{a}_+(t')a_+(t')\rangle_A\\ &\quad - \frac{1}{2}\gamma\int_0^{+\infty} dt' \langle \bar{a}_-(t)a_+(t)\bar{a}_-(t')a_-(t')\rangle_A,\end{aligned} \tag{19, 20}$$

where we have set $\gamma(t) = \gamma\theta(t)$ and denoted $n_a = \langle a^\dagger a\rangle$, which is the occupation in the initial state (i.e., for $t < 0$). Assuming system A is noninteracting, we have

$$\langle \bar{a}_-(t)a_+(t)\bar{a}_-(t')a_+(t')\rangle_A = z n_a^2, \tag{21}$$

$$\begin{aligned}\langle \bar{a}_-(t)a_+(t)\bar{a}_+(t')a_+(t')\rangle_A &= \tilde{\theta}(t-t')n_a(1+zn_a)\\ &\quad + \theta(t'-t)zn_a^2,\end{aligned} \tag{22}$$

$$\begin{aligned}\langle \bar{a}_-(t)a_+(t)\bar{a}_-(t')a_-(t')\rangle_A &= \theta(t-t')n_a(1+zn_a)\\ &\quad + \tilde{\theta}(t'-t)zn_a^2.\end{aligned} \tag{23}$$

Hence,

$$n_a(t) = \langle a^\dagger(t)a(t)\rangle = n_a - \gamma n_a\int_0^t dt'\,\theta(t-t'). \tag{24}$$

Setting $t = \delta t \gg D^{-1}$ (where D^{-1} is the characteristic response time of the bath, see Appendix C), we arrive at the following one-body loss-rate equation:

$$\frac{dn_a(t)}{dt} = -\gamma n_a(t), \tag{25}$$

On the right-hand side, with accuracy $O(\gamma\delta t)$, we have replaced n_a by $n_a(\delta t)$.

B. General case

In this section, we generalize the above results. Since only the results obtained in the previous section will be necessary, the applications discussed in this section can be entirely skipped on a first reading. The starting point is again the Hamiltonian describing the unitary dynamics of a system (A) coupled to a bath (B).

$$H = H_A(t) + H_B + H_{AB}(t), \tag{26}$$

where $H_A(t)$, H_B, and $H_{AB}(t)$ denote the Hamiltonians of the system, bath, and their coupling, respectively. We have assumed that, in general, the system Hamiltonian and its coupling to the bath can be explicitly time dependent. We shall consider a rather general form of the coupling between system and bath:

$$H_{AB}(t) = \sum_{\mathbf{q},\mathbf{p}}[g_{\mathbf{qp}}(t)A^\dagger_{\mathbf{q}}B_{\mathbf{p}} + \text{H.c.}], \tag{27}$$

where

$$A_{\mathbf{q}} = a^\dagger_{\bar{q}_1}\cdots a^\dagger_{\bar{q}_N} a_{q_1}a_{q_2}\cdots a_{q_{N'}}, \tag{28}$$

$$B_{\mathbf{p}} = b^\dagger_{\bar{p}_1}\cdots b^\dagger_{\bar{p}_M} b_{p_1}b_{p_2}\cdots b_{p_{M'}} \tag{29}$$

are products of arbitrary number operators (with $N \neq N'$ and $M \neq M'$, in general) acting either on the system or the bath; $\mathbf{p} = \{p_1, \cdots, p_{M'}; \bar{p}_1, \cdots, \bar{p}_M\}$ and $\mathbf{q} = \{q_1, \cdots, q_N; \bar{q}_1, \cdots, \bar{q}_{N'}\}$ are the quantum numbers carried by those operators; $g_{\mathbf{qp}}(t)$ are the set of system-bath couplings. The system-bath coupling is switched on according to a certain protocol that determines the explicit time dependence of the $g_{\mathbf{qp}}(t)$.

In general, we are not able to obtain the Feynman-Vernon influence functional $\mathcal{F}$ exactly and here we will resort to a second order cumulant expansion:

$$\mathcal{F}[\bar{a}, a] = \exp\left[i\langle S_{\rm AB}\rangle_B - \tfrac{1}{2}(\langle S_{\rm AB}^2\rangle_B - \langle S_{\rm AB}\rangle_B^2) + \cdots\right] = e^{i\mathcal{L}[\bar{a},a]}. \tag{30}$$

Explicitly, for the system-bath coupling introduced in Eq. (27), the first-order correction takes the form

$$\langle S_{AB}\rangle_B = \sum_{\mathbf{pq}} \int_C dt \big[g_{\mathbf{qp}}(t)\bar{A}_{\mathbf{q}}(t)\langle B_{\mathbf{p}}(t)\rangle_B + g^*_{\mathbf{qp}}(t)\langle \bar{B}_{\mathbf{p}}(t)\rangle_B A_{\mathbf{q}}(t)\big]. \tag{31}$$

Note that this term does not describe dissipation and only modifies the unitary evolution of system A. Thus, it can be conveniently absorbed into H_S by defining the operators $B_{\mathbf{p}}$ and $B^\dagger_{\mathbf{p}}$ in Eq. (27) to have zero averages, i.e., $\langle B_{\mathbf{p}}\rangle_B = \langle B^\dagger_{\mathbf{p}}\rangle_B = 0$. In addition, this is automatically fulfilled if B and $B^\dagger$ change the number of particles in the bath but the bath Hamiltonian H_B and its initial state conserve this number. Therefore, in what follows, we set $\langle S_{\rm AB}\rangle_B = 0$ and do not discuss it any further.

The second-order correction can be brought to the following form in terms of bath correlators:

$$\begin{aligned}\mathcal{L} &= \frac{i}{2!}\langle S_{AB}^2\rangle_B \\ &= \frac{i}{2}\sum_{\mathbf{q}_2\mathbf{q}_1}\int_C dt_1 dt_2 [u_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)\bar{A}_{\mathbf{q}_1}(t_1)\bar{A}_{\mathbf{q}_2}(t_2) \\ &\quad + \bar{u}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)A_{\mathbf{q}_1}(t_1)A_{\mathbf{q}_2}(t_2) \\ &\quad + v_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)\bar{A}_{\mathbf{q}_1}(t_1)A_{\mathbf{q}_2}(t_2) \\ &\quad + \bar{v}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)A_{\mathbf{q}_1}(t_1)\bar{A}_{\mathbf{q}_2}(t_2)],\end{aligned} \tag{32}$$

where

$$u_{\mathbf{q}_1,\mathbf{q}_2}(t_1,t_2) = \sum_{\mathbf{p}_1\mathbf{p}_2} g^*_{\mathbf{p}_1\mathbf{q}_1}(t_1)g^*_{\mathbf{p}_2\mathbf{q}_2}(t_2)C_{BB}(\mathbf{p}_1t_1,\mathbf{p}_2t_2), \tag{33}$$

$$\bar{u}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2) = \sum_{\mathbf{p}_1\mathbf{p}_2} g_{\mathbf{p}_1\mathbf{q}_1}(t_1)g_{\mathbf{p}_2\mathbf{q}_2}(t_2)C_{\bar{B}\bar{B}}(\mathbf{p}_1t_1,\mathbf{p}_2t_2), \tag{34}$$

$$v_{\mathbf{q}_1,\mathbf{q}_2}(t_1,t_2) = \sum_{\mathbf{p}_1\mathbf{p}_2} g^*_{\mathbf{p}_1\mathbf{q}_1}(t_1)g_{\mathbf{p}_2\mathbf{q}_2}(t_2)C_{B\bar{B}}(\mathbf{p}_1t_1,\mathbf{p}_2t_2), \tag{35}$$

$$\bar{v}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2) = \sum_{\mathbf{p}_1\mathbf{p}_2} g_{\mathbf{p}_1\mathbf{q}_1}(t_1)g^*_{\mathbf{p}_2\mathbf{q}_2}(t_2)C_{\bar{B}B}(\mathbf{p}_1t_1,\mathbf{p}_2t_2). \tag{36}$$

In the above equation, the following notation has been introduced ($X, Y = B, \bar{B}$):

$$C_{XY}(\mathbf{p}_1t_1,\mathbf{p}_2t_2) = \langle T_C[X(\mathbf{p}_1,t_1)Y(\mathbf{p}_2,t_2)]\rangle_B \tag{37}$$

for the bath two-point correlation functions of operators $B, B^\dagger$. In the context of the Keldysh path integral, each one of the above correlation functions becomes a 2×2 matrix in the superindices $m, n = \pm$ after expanding the integrals over C so t_1, t_2 run from $-\infty$ to $+\infty$. The superindices are inherited by the couplings $u, v, \bar{u}, \bar{v}$ introduced in Eq. (36), which also become the 2×2 matrices $u^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)$, $v^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)$, etc.

Explicit consideration of the time dependence of the coupling matrices $u^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)$, $v^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2)$, etc., for a given bath may allow us to identify regimes where the Markovian approximation applies. This is typically the case when the response of the bath is much faster than the characteristic timescale of the system dynamics. Thus, in the Markovian regime we can assume that $u^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t_1,t_2) \propto \delta(t_1 - t_2)$, etc. (although some of them may also vanish as was the case in the previous example). This provides an additional simplification of the second-order term of the cumulant expansion and leads to the following result:

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_A, \tag{38}$$

where S_N is the normal part,

$$\mathcal{L}_N = \frac{i}{2}\sum_{\mathbf{q}_2\mathbf{q}_1,m,n=\pm}\int dt\ \tilde{v}^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t)\bar{A}_{\mathbf{q}_1,m}(t)A_{\mathbf{q}_2,n}(t), \tag{39}$$

and $\mathcal{L}_A$ is the anomalous part:

$$\mathcal{L}_A = \frac{i}{2}\sum_{\mathbf{q}_2\mathbf{q}_1,m,n=\pm}\int dt\ \big[\bar{u}^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t)A_{\mathbf{q}_1m}(t)A_{\mathbf{q}_2n}(t), + u^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t)\bar{A}_{\mathbf{q}_1m}(t)\bar{A}_{\mathbf{q}_2n}(t)\big]. \tag{40}$$

In the above expressions, we have introduced the following system-bath coupling matrices:

$$\tilde{v}^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t) = s_m s_n\big[v^{mn}_{\mathbf{q}_1\mathbf{q}_2}(t) + z\bar{v}^{nm}_{\mathbf{q}_2\mathbf{q}_1}(t)\big], \tag{41}$$

where $z = -1$ ($z = +1$) if the operator products $A_{\mathbf{q}}, A^\dagger_{\mathbf{q}}$ in $H_{\rm SB}$ have fermionic (bosonic) statistics and $s_+ = +1$ ($s_- = -1$). We can classify the different terms according to the type of time arguments of the system degrees of freedom $A_{\mathbf{q}}(t), \bar{A}_{\mathbf{q}}(t)$. The first two terms on the right-hand side contain $A_{\mathbf{q}m}(t), \bar{A}_{\mathbf{q}n}(t)$, whose time arguments lie on different branches of the Keldysh contour C and therefore $m \neq n$. These terms correspond to the so-called quantum jump terms of the Lindblad master equation (see Sec. III and Appendix A). The remaining terms contain $A_{\mathbf{q}m}(t), \bar{A}_{\mathbf{q}n}(t)$ with time arguments lying on the same branch of C, i.e., with $m = n$. Such terms contribute to the anticommutator terms of the Lindblad master equation in the operator language (see Appendix A) and give rise to the anti-Hermitian part of the effective non-Hermitian Hamiltonian in the operator language [1].

Finally, although the last few expressions above have been derived under the assumptions of Markovianity of the bath, we want to emphasize that the approach used here is not limited to the Markovian regime and can be used as a starting point to include effects beyond Markovianity. The latter are outside the range of applicability of the Lindblad master equation or its path integral formulation as introduced in Ref. [30].

In the following sections, we shall consider a number of applications to particle loss and show that, although derived

using the cumulant expansion up to second order, which may appear to be only valid for a weak system-bath coupling, it is possible to reformulate the models to describe the limit of very strong two-body losses exactly.

III. ONE-BODY LOSS

Before considering systems with two-body losses, it is interesting to generalize the results of Sec. II A to describe an ultracold gas coupled to a bath to which it can lose one particle at a time. This section largely relies on the results obtained in Sec. II A. We generalize the Hamiltonian to a uniform gas and therefore the fields carry a momentum index $\mathbf{k}$. The Hamiltonian reads

$$H_A = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}, \tag{42}$$

$$H_B = \sum_{\mathbf{k},\alpha} \epsilon_{\mathbf{k},\alpha} b^\dagger_{\mathbf{k}\alpha} b_{\mathbf{k}\alpha}, \tag{43}$$

$$H_{AB} = f(t) \sum_{\mathbf{k},\alpha} [g_\alpha a^\dagger_{\mathbf{k}} b_{\mathbf{k}\alpha} + g^*_\alpha\, b^\dagger_{\mathbf{k}\alpha} a_{\mathbf{k}}], \tag{44}$$

where the fields $a_{\mathbf{k}}$ and $b_{\mathbf{k},\alpha}$ are either fermionic or bosonic. In ultracold atomic gases, the couplings g_α, g_α are often not known from first principles. Instead, what is measured is the loss rate of particles. Following the same steps as in Sec. II A while keeping track of the momentum index $\mathbf{k}$, we arrive at the following effective action in the Markovian limit ($m, n = \pm$):

$$\mathcal{S}_{A,\mathrm{eff}} = S_A + \mathcal{L}, \tag{45}$$

$$S_A = \sum_{\mathbf{k},mn} \int dt \left[\sigma^3_{mn} \bar{a}_{\mathbf{k}m} (i\partial_t - \epsilon_{\mathbf{k}}) a_{\mathbf{k}n}\right], \tag{46}$$

$$\mathcal{L} = -i \sum_{\mathbf{k},mn} \int dt\, \gamma(t) \left[\left(\sigma^-_{mn} - \frac{1}{2}\sigma^0_{mn}\right) \bar{a}_{\mathbf{k}m} a_{\mathbf{k}n}\right], \tag{47}$$

where we have introduced the following short-hand notations: $\sigma^3_{++} = -\sigma^3_{--} = 1$, $\sigma^-_{-+} = 1$, $\sigma^0_{++} = \sigma^0_{--} = 1$, and zero otherwise.

Like in Sec. II A, we can obtain the rate of particle loss by assuming the coupling to the bath is switched on at $t = 0$ [i.e., $f(t) = \theta(t)$] and computing the change in the total particle number density using perturbation theory in $\mathcal{L}$:

$$n_A(t) = \frac{1}{\Omega} \sum_{\mathbf{k}} \langle a^\dagger_{\mathbf{k}}(t) a_{\mathbf{k}}(t)\rangle \tag{48}$$

$$= \frac{1}{\Omega} \sum_{\mathbf{k}} \int D[\bar{a}a]\, \bar{a}_{\mathbf{k}-}(t) a_{\mathbf{k}+}(t)\, e^{i(S_A+\mathcal{L})}. \tag{49}$$

Following the same steps as in Sec. II A, we compute the leading-order change of $n_{\mathbf{k}}(t) = \langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle$. Thus, we obtain rate equations for the momentum distribution $dn_{\mathbf{k}}(t)/dt = -\gamma n_{\mathbf{k}}(t)$, and hence the rate of change of the particle density follows:

$$\frac{dn_A(t)}{dt} = -\gamma n_A(t). \tag{50}$$

Note that the rate of change is proportional to the density, which is characteristic of the one-body loss process.

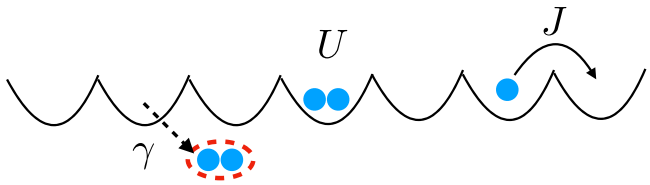

FIG. 1. Scheme of the lossy 1D Bose-Hubbard model. J is the nearest-neighbor hopping and U is the on-site interaction. The loss parametrized by γ is the one-body loss rate of doublons, i.e., doubly occupied sites

IV. LOSSY 1D BOSE HUBBARD MODEL

Next, we discuss how to describe two-body losses by coupling a system to a bath. We first consider an ultracold gas in a deep optical lattice which can be described by the one-dimensional Bose-Hubbard model [44]. A laser is applied that photoassociates atoms in doubly occupied lattice sites (doublons) into molecules. The molecules are quickly lost from the trap, resulting in a loss of two bosons (see Fig. 1). In the limit where the loss of the photoassociated molecule is very fast, the double occupancy is strongly suppressed. In other words, in the presence of this coupling, the states containing doublons are rapidly projected out. However, virtual transitions to states containing doublons can still have an effect on the system dynamics. This kind of loss dynamics was experimentally studied in Refs. [6,45] and theoretically described using an approach based on an effective master equation derived in Ref. [46], which yields a phenomenological rate equation like Eq. (1). In the following, we provide a microscopic theory for the loss dynamics and compare it to the phenomenological loss equation.

A convenient way to describe the dynamics of the Bose-Hubbard model in a subspace containing no doublons relies on the Jordan-Wigner transformation [44] that relates hard-core bosons to fermions:

$$c_j = K_j a_j, \quad K_j = \prod_{l<j} (1 - 2n_l), \tag{51}$$

where $n_j = a^\dagger_j a_j = c^\dagger_j c_j = 0, 1$ measures the occupation of site j. The transformation holds true provided $(a^\dagger_j)^2 = a^2_j = 0$ (i.e., no doublons) in the relevant Fock subspace. In this subspace, the kinetic energy of the hardcore bosons can be written in terms of the Jordan-Wigner fermions (see Fig. 1):

$$H_c = -J \sum_j [c^\dagger_j c_{j+1} + \mathrm{H.c.}]. \tag{52}$$

In addition, we note that the original hopping operator of bosons $-J\sum_i [a^\dagger_j a_{j+1} + \mathrm{H.c.}]$ also allows for transitions that create virtual doublons. To allow for such processes, we introduce a doublon field on each site $d^\dagger_i$ that is coupled to the (hardcore) bosons by means of $-J\sum_j (d_j + d_{j+1}) a^\dagger_j a^\dagger_{j+1} +$ H.c. Following the Jordan-Wigner transformation, this coupling becomes

$$H_{cd} = -J \sum_j [(c^\dagger_j c^\dagger_{j+1} + c^\dagger_{j-1} c^\dagger_j) d_j + \mathrm{H.c.}]. \tag{53}$$

Finally, we note that the doublon has an excitation energy equal to U, and therefore

$$H_d = U \sum_j d_j^\dagger d_j. \tag{54}$$

This rather heuristic derivation of the Hamiltonian in the limit where the doublons are suppressed is confirmed below by showing that it is a convenient Hubbard-Stranovich decoupling of the effective interaction generated in the strongly interacting limit of the one-dimensional Bose-Hubbard model, i.e., for $U \gg J$ [47].

The generating functional for the above model is

$$Z[\bar{V}, V] = \int D[\bar{d}d] D[\bar{c}c]\, e^{iS}, \tag{55}$$

where $S = S_c + S_d + S_{dc} + S_V$:

$$S_c = \sum_j \int_C dt\, [\bar{c}_j (i\partial_t - \mu) c_j + J(\bar{c}_j c_{j+1} + \bar{c}_{j+1} c_j)], \tag{56}$$

$$S_{dc} = J \sum_j \int_C dt [(\bar{c}_j \bar{c}_{j+1} + \bar{c}_{j-1}\bar{c}_j) d_j + \bar{d}_j (c_{j+1} c_j + c_j c_{j-1})], \tag{57}$$

$$S_V = \sum_j \int_C dt\, [\bar{V}_j c_j + \bar{c}_j V_j]. \tag{58}$$

As mentioned above, the doublon field will be treated as Hubbard-Stranotovich field and therefore its action does not contain a time-derivative term $i\bar{d}\partial_t d$:

$$S_d = -U \sum_j \int_C dt\, \bar{d}_j d_j. \tag{59}$$

The calculations to be described below can also be carried out including such derivative terms, which is not important in the limit where $U \gg J$ (see next section for a discussion where a similar situation is encountered). We further assume that the system is coupled to a bath that removes the doublons. This coupling is described by the following term in the Keldysh action:

$$S_{dB} = -\sum_{j,\alpha} \int_C dt f(t) [g_\alpha \bar{d}_j b_{j\alpha} + g_\alpha^* \bar{b}_{j\alpha} d_j], \tag{60}$$

where the bath modes have the following quadratic action:

$$S_B = \sum_{j,\alpha} \int_C dt\, \bar{b}_{j\alpha} (i\partial_t - \omega_\alpha) b_{j\alpha}. \tag{61}$$

We integrate out the bath following the same steps as in Sec. II A, which in the Markovian limit yields

$$\mathcal{L}_d = -i \sum_{j,mn} \int dt\, \gamma(t) \left(\sigma^-_{mn} - \frac{1}{2}\sigma^0_{mn}\right) \bar{d}_{jm} d_{jn}. \tag{62}$$

Combining this term with the action for the doublon, the following effective action is obtained:

$$S_{d,\text{eff}} = \sum_{j,mn} \int dt\, \bar{d}_{jm} G^{-1}_{mn}(t) d_{jn}, \tag{63}$$

$$G^{-1}_{mn}(t) = -U\sigma^3_{mn} - i\gamma(t)\left(\sigma^-_{mn} - \tfrac{1}{2}\sigma^0_{mn}\right). \tag{64}$$

Finally, we integrate out the doublon field by making the following change of integration variables in the functional integral:

$$d_{jm}(t) = d'_{jm}(t) - \sum_{m',n'} G_{m,m'}(t)\sigma^3_{m',n'} h_{j,n'}(t), \tag{65}$$

$$\bar{d}_{jm}(t) = \bar{d}'_{jm}(t) - \sum_{m',n'} \bar{h}_{j,m'}(t)\sigma^3_{m',n'} G_{n',m}(t), \tag{66}$$

where we have denoted $h_{j,m} = -J(c_{j+1,m}c_{j,m} + c_{j,m}c_{j-1,m})$, $\bar{h}_{j,m} = -J(\bar{c}_{j,m}\bar{c}_{j+1,m} + \bar{c}_{j-1,m}\bar{c}_{j,m})$, and

$$G_{mn}(t) = \frac{-U\sigma^3_{mn} - i\gamma(t)\left(\sigma^-_{mn} + \frac{1}{2}\sigma^0_{mn}\right)}{U^2 + (\gamma(t)/2)^2}. \tag{67}$$

The resulting integral over $\bar{d}'_j, d'_j$ is Gaussian and yields a constant prefactor to the generating functional in Eq. (55). In addition, there is an exponential factor with the following effective action in the exponent:

$$S'_{\text{eff}} = -\sum_{j,mn} \int dt\, \bar{h}_{j,m}(t)(\sigma^3 G(t) \sigma^3)_{mn} h_{j,n}(t). \tag{68}$$

Note that in the limit where the coupling to the bath vanishes and $\gamma(t) = 0$, $G_{mn}(t) = -\sigma^3_{mn}/U$, we obtain

$$S'_{\text{eff}} = \frac{1}{U} \sum_{j,mn} \int dt\, \bar{h}_{j,m}(t)\sigma^3_{mn} h_{j,n}(t) \tag{69}$$

$$= \frac{J^2}{U} \sum_{j,mn} \int dt\, \sigma^3_{mn} [2\bar{c}_{j+1,m}\bar{c}_{j,m}c_{j,n}c_{j+1,n} - \bar{c}_{j+1,m}\bar{c}_{j,m}c_{j,n}c_{j-1,n} - \bar{c}_{j-1,m}\bar{c}_{j,m}c_{j,n}c_{j+1,n}], \tag{70}$$

which is the Keldysh action for the effective Hamiltonian obtained using strong coupling perturbation theory in Ref. [47] for the 1D Bose-Hubbard model in the limit where $U \gg J$ and in the subspace with no doublons.

Next, we switch on the coupling to the bath so $\gamma(t) = \gamma\theta(t)$ and compute the particle-loss rate. It is convenient to work in the Bloch wave basis, where $c_k = \sum_j e^{-ikx_j} c_j/\sqrt{M}$, M is the number of lattice sites, $x_j = j$, $k = 2\pi l/M$, $l = -M/2 + 1, \ldots, M/2$ (assuming periodic boundary conditions and M to be even). Thus, the full effective action $S_{\text{eff}} = S_c + S'_{\text{eff}}$ reads

$$S_{\text{eff}} = \sum_{k,mn} \int dt\, \sigma^3_{mn} \bar{c}_{k,m}(i\partial_t - \epsilon_k) c_{k,n} - \frac{1}{2M} \sum_{pkq,mn} \int dt\, \sigma^3_{mn} \tilde{U}_{pkq}(t) \bar{c}_{p,m}\bar{c}_{k,m} c_{k+q,n} c_{p-q,n} - \frac{i}{M} \sum_{pkq} \int dt\, \Gamma_{pkq}(t) \bar{c}_{p,-}\bar{c}_{k,-} c_{k+q,+} c_{p-q,+}, \tag{71}$$

where $\epsilon_k = -J\cos k$ is the single-particle dispersion, $\tilde{U}(p,k,q) = U_{pkq}(t) - i\sigma^3_{mn}\Gamma_{pkq}(t)$ and $\Gamma_{pkq}(t)$ are given by

$$U_{pkq}(t) = \frac{-8J^2 U F_{pkq}}{U^2 + (\gamma(t)/2)^2}, \tag{72}$$

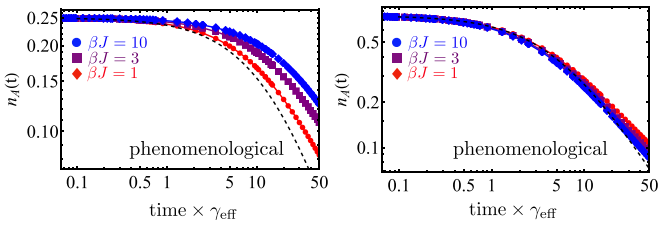

FIG. 2. Left: Evolution of the lattice filling starting from an initial filling of $n_A(t=0) = 0.25$ for different initial temperatures on a log-log scale. The initial temperature is determined by the initial state, which is assumed thermal. The phenomenological result shows large deviations from the microscopic theory, especially at low temperatures. Right: Evolution of the lattice filling starting from initial filling of $n_A(t=0) = 0.75$ for different initial temperatures on a log-log scale. At larger initial fillings, the deviation from the phenomenological rate equation becomes smaller.

$$\Gamma_{pkq}(t) = \frac{4J^2\gamma(t)F_{pkq}}{U^2 + (\gamma(t)/2)^2}, \tag{73}$$

$$F_{pkq} = \cos(q)\cos^2\left(\frac{p+k}{2}\right). \tag{74}$$

In the limit where $J \ll \max\{U, \gamma(t)\}$, both $U_{pkq}(t)$ and $\Gamma_{pkq}(t)$ are perturbatively small. Using perturbation theory to leading order in $U_{pkq}(t)$ and $\Gamma_{pkq}(t)$, we obtain the following loss rate equation for the distribution function of Jordan-Wigner fermions (see Appendix D for further details),

$$\frac{dn_p(t)}{dt} = -\gamma_{\rm eff}\int \frac{dk}{2\pi} C_{kp}^2\, n_p(t) n_k(t), \tag{75}$$

where $C_{kp} = \sin(\frac{p-k}{2})\cos(\frac{p+k}{2})$ and the effective loss rate is

$$\gamma_{\rm eff} = \frac{16J^2\gamma}{U^2 + \gamma^2/4}. \tag{76}$$

Integrating over p yields the following rate equation for the lattice filling $n_A(t) = N_A(t)/M$:

$$\frac{dn_A(t)}{dt} = -\gamma_{\rm eff}\int \frac{dpdk}{(2\pi)^2} C_{kp}^2\, n_p(t) n_k(t). \tag{77}$$

The form factor C_{kp}^2 was not present in the approach used in Ref. [46], which neglected inter-site correlations. The expression containing the form factor was later obtained by solving the Lindblad master equation using an approximation termed as a time-dependent generalized Gibbs ensemble [48,49]. As illustrated below, the form factor turns out to be very important when the filling of the lattice is low and at low initial temperatures. The phenomenological loss rate equation and the two-body loss coefficient γ_2 [cf. Eq. (1)] [6,45,46] can be obtained in the high-temperature limit where the Fermi-Dirac distribution of the Jordan-Wigner fermions approaches $n_A(t)$ and is independent of the momentum. Therefore,

$$\frac{dn_A(t)}{dt} = -\gamma_T\, n_A^2(t), \tag{78}$$

with two-body loss coefficient $\gamma_2 = \gamma_T = \gamma_{\rm eff}/4$. Figure 2 compares the solution of the phenomenological rate equation with the numerical solution of Eq. (75) obtained from

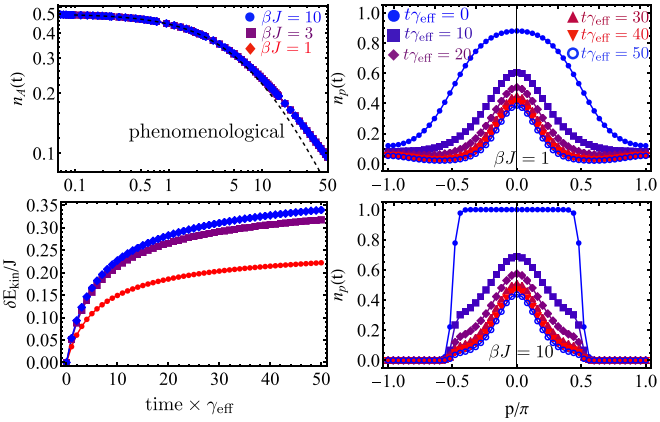

FIG. 3. Top left panel: Decay of the lattice filling for an initially half-filled system and different initial temperatures. The dependence of the initial temperature is very weak (note the log-log scale). Bottom left panel: Change of kinetic energies with different initial temperatures. Systems with a lower initial temperature undergo a larger change in the kinetic energy. Upper right panel: Time evolution of the distribution of Jordan-Wigner fermions for an inverse temperature $k_BT = J$ (i.e., $\beta J = 1$). Note a large depletion in the distribution near $k = \pm\pi/2$. Bottom right panel: Evolution of the distribution of Jordan-Wigner fermions for $k_BT = J/10$ (i.e., $\beta J = 10$).

the microscopic Keldysh action at different initial temperatures determined by the initial momentum distribution of the Jordan-Wigner fermions. Indeed, since after $t = 0$ the system is out of equilibrium, a global temperature can no longer be defined. At small values of the initial lattice filling [e.g., $n_A(t=0) = 0.25$], the results from the phenomenological rate equation strongly deviate from the predictions of the microscopic theory, Eq. (75). The difference between the two rate equations becomes smaller as temperature increases because, as mentioned above, the phenomenological rate equation is recovered from Eq. (75) in the high-temperature limit, where $T \gg J$. However, at a higher initial lattice filling [$n_A(t=0) = 0.75$], the error incurred by using the phenomenological rate equation becomes smaller at all the studied temperatures. Thus, the phenomenological rate equation only applies at high temperatures or high lattice fillings [i.e., $n_A(t=0) \simeq 1$]. The latter are indeed the conditions of previous experiments [6,45]. Our formalism thus allows us to access other regimes of lattice fillings and temperature, which can be explored in future experiments.

In addition, a closer examination of the numerical solution of Eq. (75) shows that the situation is indeed more complex than what can be naively inferred from the above discussion of the validity of the phenomenological rate equation. To see from where the complexity emerges, we have plotted the evolution of the distribution function of Jordan-Wigner fermions in Fig. 3 for an initially half-filled lattice and different values of the effective loss rate $\gamma_{\rm eff}$ (right panels). In the same figure, we also show the evolution of the lattice filling and kinetic energy for different initial temperatures (left panels). Note that for the lattice filling shown on the top left panel, the particle loss dynamics is largely independent of the initial temperature (note the log-log scale). On the other hand, the dynamics of the average kinetic energy (shown in a linear scale) does

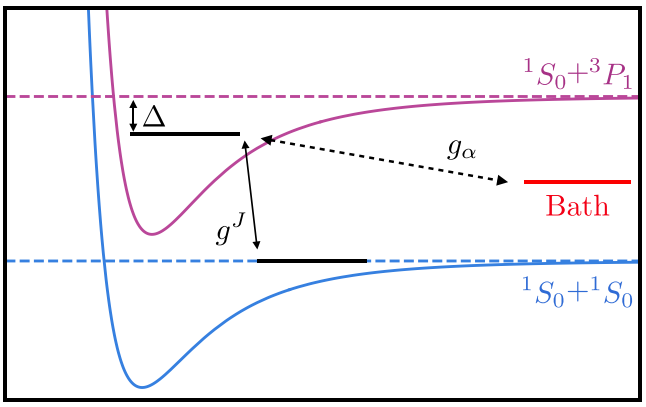

FIG. 4. Scheme of an optical Feshbach resonance: The system consists of N species of spin-F ($N = 2F + 1$) alkaline-earth atoms in the ground state interacting via an s-wave potential. Using a laser colliding pairs of atoms in the ground state are coupled to pairs consisting of one ground state and one excited-state atom in a molecular state. This state has a one-body coupling parametrized by g_α to a bath to which it can be lost or gained.

indeed depend on the initial temperature. As can be seen, the change in kinetic energy induced by the two-body loss is larger in systems with lower initial temperatures.

The dependence on the initial temperature of the kinetic energy, which is the first moment of the momentum distribution, is an indication that the two-body loss drives the system into a nonequilibrium state. To confirm this observation, we focus on the evolution of the full momentum distribution of the Jordan-Wigner fermions, which is shown on the right panels for two values of the initial temperature ($k_BT = J$ and $k_BT = J/10$). Although the initial distribution is assumed to be thermal, it can be seen that under the two-body loss in both cases it rapidly evolves into a nonthermal distribution. Note that the depletion is most effective (especially at low temperatures) for k near $\pm\pi/2$. This is because the form factor C^2_{kp} in Eq. (77) is maximum for $k = -p = \pm\pi/2$, which corresponds to losses of doublons with total zero momentum. The latter are created from Cooper pairs $\sim c_k c_{-k}$ of Wigner fermions. In conclusion, even in cases where evolution of the particle filling stays rather close to the results obtained from the phenomenological rate Eq. (78) down to low temperatures, the system is indeed far from equilibrium as revealed by close examination of other observables like the kinetic energy.

V. LOSSES IN OPTICAL FESHBACH RESONANCE

In this section, we describe a multicomponent mixture of alkaline-earth atoms with emergent SU(N) symmetry [50,51] near an OFR by means of the following model (see Fig. 4):

$$H_c = H_0 + H_{\rm int}, \tag{79}$$

$$H_{\rm int} = \frac{U}{\Omega}\sum_{\mathbf{pkq},\sigma\sigma'} c^\dagger_{\mathbf{p}\sigma} c^\dagger_{\mathbf{k}\sigma'} c_{\mathbf{k}-\mathbf{q}\sigma'} c_{\mathbf{p}+\mathbf{q}\sigma}, \tag{80}$$

$$H_{ca} = \frac{f(t)}{\sqrt{\Omega}}\sum_{\mathbf{pq}}\sum_{\sigma\sigma',JM} g^J_{\mathbf{q}}\langle f f \sigma\sigma'|JM\rangle \times [a^\dagger_{\mathbf{p},JM} c_{\mathbf{p}-\mathbf{q},\sigma} c_{\mathbf{q},\sigma'} + c^\dagger_{\mathbf{q},\sigma} c^\dagger_{\mathbf{p}-\mathbf{q},\sigma'} a_{\mathbf{p},JM}], \tag{81}$$

$$H_a = \sum_{\mathbf{q},JM}(\epsilon^a_{\mathbf{q}} + \Delta) a^\dagger_{\mathbf{q},JM} a_{\mathbf{q},JM}, \tag{82}$$

$$H_{aB} = f(t)\sum_{\mathbf{q},JM,\alpha}[g_\alpha b^\dagger_{\mathbf{q}\alpha} a_{\mathbf{q},JM} + g^*_\alpha a^\dagger_{\mathbf{q},JM} b_{\mathbf{q},\alpha}], \tag{83}$$

$$H_{\rm B} = \sum_{\mathbf{q},\alpha}\omega_{\mathbf{q}\alpha}\, b^\dagger_{\mathbf{q}\alpha} b_{\mathbf{q}\alpha}. \tag{84}$$

Fermions in the ground state $^1S_0$ interact via a weak s-wave potential preserving SU(N) symmetry. The OFR is described by a SU(N) symmetry breaking scattering channel which involves an intermediate bosonic molecular state where one of the colliding atoms is in an optically coupled excited state, i.e., $^1S_0 + {}^3P_0$ (see Fig. 4). The coupling $g^J(q) = g^J(-q)$ is the matrix element of the laser-induced transition between two-particles in the ground state and the molecular excited state, i.e., $\langle {}^1S_0\,{}^3P_0|V_{\rm las}|{}^1S_0\,{}^1S_0\rangle$. In the calculations below, we assume that $g^J \ll \Delta_m, \gamma$, where Δ_m is the detuning from the excited state and γ is the one-body loss rate of the excited molecular state. In this limit, the loss of the molecular state is due to spontaneous emission. The contribution of stimulated emissions is relatively small. The two-body loss of the system particles is described as the one-body loss of the intermediate molecular state, which is coupled to a bath via $H_{\rm aB}$, whose eigenmodes are the bosonic operators $b^\dagger_{\mathbf{q}}$, $b_{\mathbf{q}}$. Since the coupling is linear and the bath is described by a quadratic Hamiltonian, the second-order cumulant expansion is exact and applies even in the limit where the loss rate γ is large. Below we write the Keldysh action after integrating out the bath field following the same steps than in Sec. II. This yields

$$S_{\rm eff} = S_c + S_{a,\rm eff}, \tag{85}$$

$$S_c = \int dt\, \sigma^3_{mn}\Bigg\{\Bigg[\sum_{\mathbf{k}}\bar{c}_{\mathbf{k}\sigma m}(i\partial_t - \epsilon_{\mathbf{k}})c_{\mathbf{k}\sigma n} - H_{\rm int}\Bigg] - \frac{f(t)}{\sqrt{\Omega}}\sum_{\mathbf{p},\mathbf{q}}\sum_{\sigma\sigma' JM} g^J_{\mathbf{q}}\langle f f\sigma\sigma'|JM\rangle[\bar{a}_{\mathbf{p}m}c_{\mathbf{p}-\mathbf{q}\sigma n}c_{\mathbf{q}\sigma' n} + a_{\mathbf{p}m}\bar{c}_{\mathbf{p}-\mathbf{q}\sigma n}\bar{c}_{\mathbf{q}\sigma' n}]\Bigg\},$$

$$S_{a,\rm eff} = \sum_{\mathbf{q},JM}\int dt dt'\,\bar{a}_{\mathbf{q}JMm} G^{-1}_{\mathbf{q},mn}(t,t') a_{\mathbf{q}JMn}, \tag{86}$$

where

$$G^{-1}_{\mathbf{q},mn}(t,t') = \delta(t-t')\big[\sigma^3_{nm}\big(i\partial_{t'} - \epsilon^a_{\mathbf{q}} - \Delta\big) + i\sigma^0_{mn}\gamma(t')/2 - i\sigma^-_{mn}\gamma(t')\big]. \tag{87}$$

In Eq. (86), to lighten the equations, we have adopted the convention that summation over repeated indices $m, n = \pm$ is implied.

Next, we integrate out the molecular field, $\bar{a}_{\mathbf{q},JM}, a_{\mathbf{q},JM}$ using the cumulant expansion. To this end, we need to obtain the Green's function for the molecular fields, i.e., the inverse of the matrix $G^{-1}_0(t,t')$. To take into account the distribution function of the molecules correctly, it is necessary [29] to take a step back and work with the discrete version of the path integral. However, the presence of the dissipative terms make

the inversion of the matrix cumbersome. One way around this difficulty is to realize that in the parameter regime of interest here, i.e., for large $\gamma(t)$ and/or large detuning Δ_m, it is possible to neglect the time derivative part of $G_{\mathbf{q}}^{-1}(t,t')$, which effectively amounts to a Markovian approximation when the molecular field is regarded as a bath itself. To understand this, let us neglect the time dependence of the coupling to the bath γ for the time being. Thus, the matrix $G_{\mathbf{q}}^{-1}$ becomes a function of $t-t'$ and can be Fourier transformed, which yields the following matrix in Keldysh space:

$$G_{\mathbf{q}}^{-1}(\omega) = \sigma^3\big(\omega - \epsilon_{\mathbf{q}}^a - \Delta\big) + i\gamma(\sigma^0/2 - \sigma^-). \tag{88}$$

If $|\omega| \ll \min\{\Delta_m, \gamma\}$, we can neglect the frequency dependence of $G_{\mathbf{q}}^{-1}(\omega)$. Furthermore, assuming a vanishing number of molecules in the initial state, the correlation functions that determine the second-order term in the expansion are given by the inverse of the above matrix with $\omega = 0$. In Eq. (87), this amounts to neglecting the term involving the time derivative, $i\partial_t$, which yields Markovian correlations for the molecular field of the form

$$G_{\mathbf{q},m,n}(t,t') \simeq -\frac{\delta(t-t')}{(\epsilon_{\mathbf{q}}+\Delta)^2 + \gamma^2(t')/4} \times \left[\sigma^3_{nm}\big(\epsilon^a_{\mathbf{q}} + \Delta\big) + \frac{i}{2}\big(\sigma^0_{mn} + \sigma^-_{mn}\big)\gamma(t')\right]. \tag{89}$$

Using the above expression and Eq. (39), we obtain the following effective Keldysh action:

$$\begin{aligned} S_{\text{eff}} = &\sum_{\mathbf{k},\sigma}\int dt\Big\{\sigma^3_{mn}\bar{c}_{\mathbf{k}\sigma m}(i\partial_t - \epsilon_{\mathbf{k}})c_{\mathbf{k}\sigma n} \\ &-\frac{1}{2\Omega}\sum_{\mathbf{pkq}}\sum_{\sigma\sigma'\lambda\lambda',JM}\big[\sigma^3_{mn}U_{\text{eff}}(\mathbf{p},\mathbf{k},\mathbf{q}) - i\sigma^0_{mn}\gamma'(\mathbf{p},\mathbf{k},\mathbf{q})\big] \\ &\times \bar{c}_{\mathbf{p}\sigma m}\bar{c}_{\mathbf{k}\sigma' m}c_{\mathbf{k}+\mathbf{q}\lambda n}c_{\mathbf{p}-\mathbf{q},\lambda',n}\Big\} \\ &- i\int dt\sum_{\mathbf{p},\mathbf{k},\mathbf{q}}\sum_{\sigma\sigma'\lambda\lambda',JM}\frac{\gamma'(\mathbf{p},\mathbf{k},\mathbf{q},t)}{\Omega} \\ &\times \bar{c}_{\mathbf{p}\sigma -}\bar{c}_{\mathbf{k}\sigma' -}c_{\mathbf{k}+\mathbf{q}\lambda +}c_{\mathbf{p}-\mathbf{q}\lambda' +}, \end{aligned} \tag{90}$$

where $U_{\text{eff}}(\mathbf{p},\mathbf{k},\mathbf{q},t) = U_0 + \delta U(\mathbf{p},\mathbf{k},\mathbf{q},t)$ is the renormalized interaction,

$$\begin{aligned} \delta U(\mathbf{p},\mathbf{k},\mathbf{q},t) = &-2g^J_{\mathbf{p}-\mathbf{k}}g^J_{\mathbf{p}-\mathbf{k}-2\mathbf{q}}\langle ff\sigma\sigma'|JM\rangle \\ &\times\langle ff\lambda\lambda'|JM\rangle\frac{\Delta_{\mathbf{p}+\mathbf{k}}f^2(t)}{\Delta^2_{\mathbf{p}+\mathbf{k}} + \frac{\gamma^2(t)}{4}}, \end{aligned} \tag{91}$$

and

$$\begin{aligned} \gamma'(\mathbf{p},\mathbf{k},\mathbf{q},t) = &\, g^J_{\mathbf{p}-\mathbf{k}}g^J_{\mathbf{p}-\mathbf{k}-2\mathbf{q}}\langle ff\sigma\sigma'|JM\rangle \\ &\times\langle ff\lambda\lambda'|JM\rangle\frac{\gamma(t)f^2(t)}{\Delta^2_{\mathbf{p}+\mathbf{k}} + \frac{\gamma^2(t)}{4}} \end{aligned} \tag{92}$$

is the effective two-body loss rate in the limit of strong spontaneous loss on the intermediate molecular states [41,52–57]. In the above expressions, $\Delta_{\mathbf{p}} = \epsilon^a_{\mathbf{p}} + \Delta$ is the energy of the excited molecular state with total momentum $\mathbf{p}$. Note that δU is an SU(N)-symmetry breaking interaction and both quantities are perturbatively small in the limit where $g^J \ll \min\{\Delta_m, \gamma\}$ is of interest here. Hence, perturbation theory to leading order in γ' yields the following rate equation for an quantum degenerate gas:

$$\frac{dn_c(t)}{dt} = -\frac{2}{\Omega^2}\sum_{\mathbf{p},\mathbf{k}}\gamma'(\mathbf{p},\mathbf{k},\mathbf{0})n(\mathbf{p},t)n(\mathbf{k},t), \tag{93}$$

where $n_{\mathbf{p}}(t)$ is the instantaneous momentum distribution and $n_c = \sum_{\mathbf{p}} n_{\mathbf{p}}(t)/\Omega$ is the fermion density. From this result, the phenomenological loss coefficient for a thermal gas can be obtained by replacing the loss coefficient with its thermal average [53,58,59]:

$$\gamma_T = \langle\gamma'(\mathbf{p},\mathbf{k},0)\rangle_T \tag{94}$$

$$= \int \gamma'(\mathbf{p},\mathbf{k},0)f_M(\mathbf{p},T)f_M(\mathbf{k},T)d^3\mathbf{p}d^3\mathbf{k}. \tag{95}$$

Here $f_M(\mathbf{p},T) = (2\pi m k_B T)^{-3/2}\exp(-p^2/2mk_BT)$ denotes the Maxwell distribution at temperature T for particles with mass m normalized to unity. Using the average loss coefficient, the rate equation can be approximated by

$$\frac{dn_c(t)}{dt} = -\gamma_T\,\frac{2}{\Omega^2}\sum_{\mathbf{p},\mathbf{k}}n(\mathbf{p},t)n(\mathbf{k},t) \tag{96}$$

$$= -2\gamma_T\, n_c^2(t), \tag{97}$$

which is the phenomenological two-body loss rate equation, Eq. (1), with $\gamma_1 = 0$ and $\gamma_2 = 2\gamma_T$ for a thermal gas [53,54,56,58,59].

VI. CONCLUSION

In this paper, we have discussed the derivation of the Keldysh path integral for open quantum systems using the second order cumulant expansion. Although we have focused on Markovian baths, the method is not limited to the latter and can be extended to describe effects beyond Markovianity. It also does not require the bath to be noninteracting or the bath coupling to be of a particular form. Formally, it requires that the coupling to the bath is weak enough to be accurately treated using second-order perturbation theory. However, as we have shown above, when describing two-particle losses, the system-bath coupling can be sometimes conveniently reformulated and a strong loss regime can be also described.

Turning to models relevant for ultracold atomic gases, we have studied models of one- and two-body losses. Thus, we have shown how two-body losses caused by photoassociation of doublons in the one-dimensional Bose-Hubbard model can be described within the path-integral formulation, allowing us to obtain a microscopic loss-rate equation. The latter has been compared with a previously derived phenomenological loss equation. The microscopically derived rate equation shows that the phenomenological equation is mostly accurate at high temperatures and/or lattice fillings close to unity. However, we have shown (see Sec. IV) that even in cases where the phenomenological rate equation appears to be sufficiently accurate, the implicit assumption that the system remains in a thermal equilibrium characterized by a temperature T can be incorrect. This has important implications for the calculation

of other physical quantities such as the kinetic energy, for which our theory, which properly handles such deviations from equilibrium, is necessary.

Finally, in Sec. V we have applied the formalism to a model describing an OFR in multicomponent mixture of ultracold alkaline-earth fermions. We have also shown that the phenomenological rate equation is expected to apply to the high-temperature regime. Although we have not fully explored the low-temperature regime yet, using the lessons learned with the much simpler one-dimensional Hubbard model, in the presence of an OFR we expect that if quantum coherence becomes important, deviations from the phenomenological approach will appear.

Finally, let us mention that, in this paper, when dealing with the interactions between the Jordan-Wigner fermions in the lossy one-dimensional Bose-Hubbard model, we have used perturbation theory. This is justified because the latter are weak [47] and the studied temperatures are relatively high. However, in future work it will interesting to revisit this problem to account for the effect of the interactions in the system, which in one dimension can be done nonperturbatively using bosonization [60,61]. Other possible extensions of this paper are, as pointed out above, studying effects beyond Markovianity and the effect of strong interactions and correlations in the bath.

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APPENDIX A: BRIEF RECAP OF THE PATH INTEGRAL APPROACH TO THE LINDBLAD MASTER EQUATION

For an open quantum system described by a Hamiltonian $H = H(a^\dagger, a)$ in contact with a Markovian bath, the time evolution of the reduced density matrix of the system A, ρ_A, reads

$$\partial_t \rho_A = -i[H, \rho_A] + \sum_s \gamma_s \left(L_s \rho_s L_s^\dagger - \frac{1}{2}\{L_s^\dagger L_s, \rho_A\} \right), \quad \text{(A1)}$$

where $[O_1, O_2] = O_1 O_2 - O_2 O_1$ and $\{O_1, O_2\} = O_1 O_2 + O_2 O_1$ and the couplings γ_α along with the Lindblad operators $L_\alpha = L_\alpha(a^\dagger, a)$ and $L_\alpha^\dagger = L_\alpha^\dagger(a^\dagger, a)$ describe the coupling to the bath. Specializing to the case where the Lindbladian describes particle losses (but not gains) and using the resulting time-evolution operator derived from the above master equation, a path integral can be derived following the procedure described in Ref. [30] (see Eqs. (27) and (28) in Ref. [30]). In the notations of this paper where $\psi_\pm \to a_\pm$ and $-i\mathcal{L} \to \mathcal{L}$ describes only the dissipative part of the Keldysh action, etc., the path integral derived in Ref. [30] reads

$$Z = \int D[a_+, \bar{a}_+, a_-, \bar{a}_-]\, e^{iS}, \quad \text{(A2)}$$

$$S = \int dt [\sigma^3_{mn} \bar{a}_m i\partial_t a_n - \sigma^3_{mn} H(\bar{a}_m, a_n) + \mathcal{L}(\bar{a}_+, \bar{a}_-, a_+, a_-)], \quad \text{(A3)}$$

$$\mathcal{L} = -i \sum_s \gamma_s \left[\bar{L}_{s,-} L_{s,+} - \frac{1}{2}(\bar{L}_{s,+} L_{s,+} + \bar{L}_{s,-} L_{s,-}) \right]. \quad \text{(A4)}$$

Note that the quantum jump term $\bar{L}_{s,-} L_{s,+}$ appears in the reversed order compared to the Lindblad master equation [i.e., $L_s \rho L_s^\dagger$ in Eq. (A1)]. This order is important in the fermion case when the Lindblad operator L_s contains an odd number of fermion operators [62]. To make contact with our approach, we shall consider the case of the one-particle losses studied in Secs. II A and III for which the Lindbladian operators are $L_s = a$ and $\gamma_s = \gamma(t)$ and $H(a^\dagger, a) = \epsilon_0\, a^\dagger a$, for the single-mode case (i.e., dropping s, the generalization to the multiple mode corresponds to letting $s = \mathbf{k}$ and $\gamma_{s=\mathbf{k}} = \gamma(t)$ as in Sec. III). Thus, $\mathcal{L}$ given above becomes our Eq. (18).

APPENDIX B: SECOND ORDER CUMULANT EXPANSION

In this Appendix, we provide a short derivation of the second order cumulant expansion in the context of path integral as used in Sec. II A. Consider the derivation of the Feynman-Vernon functional which is obtained by formally integrating out the bath B degrees of freedom, i.e.,

$$\begin{aligned} \log \mathcal{F}[\bar{a}, a] &= \log\left(\left\langle e^{iS_{AB}} \right\rangle_B\right), \\ &= \log\left[\left\langle 1 + iS_{AB} - \tfrac{1}{2}S_{AB}^2 + \cdots \right\rangle_B\right] \\ &\simeq \left[\left\langle iS_{AB} - \tfrac{1}{2}S_{AB}^2 + \cdots \right\rangle_B\right] \\ &\quad - \frac{\left[\langle iS_{AB} - \frac{1}{2}S_{AB}^2 + \cdots \rangle_B\right]^2}{2} + \cdots \\ &= \left[i\langle S_{AB}\rangle_B - \tfrac{1}{2}\left(\left\langle S_{AB}^2\right\rangle_B - \langle S_{AB}\rangle_B^2\right) + \cdots\right], \\ &= i\mathcal{L}[\bar{a}, a], \end{aligned} \quad \text{(B1)}$$

where in the third line we use the expansion $\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - x^2/2 + \cdots$ and kept the terms up to second order in S_{AB}. Note that the second-order term is not just the average of S_{AB}^2 but a cumulant average $\langle S_{AB}^2\rangle_B - \langle S_{AB}\rangle_B^2$ (see, for instance, the discussion below Eq. (2.4) in Ref. [63]).

APPENDIX C: BATH CORRELATIONS IN DILUTE LIMIT AND MARKOVIAN APPROXIMATION

The coupling functions $g^{mn}(t_1 - t_2)$ are expressed in terms of bath correlators. In the limit of very dilute bath excitations, i.e., $n_B(\omega_\alpha) \approx 0$, the effects of interactions in the bath can be neglected and the correlation functions are well approximated by those of a noninteracting system. Before embarking on the calculation, it is important to note that, e.g., the time-ordered correlation takes the form [29]

$$\langle b_{\alpha+}(t_1)\bar{b}_{\alpha+}(t_2)\rangle_B = \{\tilde{\theta}(t_1 - t_2)[1 + z n_B(\omega_\alpha)] + \theta(t_2 - t_1) z n_B(\omega_\alpha)\} e^{-i\omega_\alpha(t_1 - t_2)}. \quad \text{(C1)}$$

In the above expression, we have made explicitly the distinction between the two kinds of step functions resulting from the discrete version of the path integral with $\tilde{\theta}(0) = 1$ and $\theta(0) = 0$ [29]. Note that this prescription differs from the one used in the operator approach to Keldysh perturbation theory, but in this context this regularization is dictated by the discrete form path integral, which is ultimately the mathematically correct form of the latter. As we will show below, the distinction introduced by this regularization is important when taking the Markovian limit where some of the above time arguments become coincident. Furthermore, in the limit where the bath excitations are dilute, the correlations simplify to

$$\langle b_{\alpha+}(t_1)\bar{b}_{\alpha+}(t_2)\rangle_B = \tilde{\theta}(t_1 - t_2)e^{-i\omega_\alpha(t_1-t_2)}, \tag{C2}$$

$$\langle \bar{b}_{\alpha+}(t_2)b_{\alpha+}(t_1)\rangle_B = z\theta(t_1 - t_2)e^{-i\omega_\alpha(t_1-t_2)}, \tag{C3}$$

$$\langle b_{\alpha-}(t_1)\bar{b}_{\alpha-}(t_2)\rangle_B = \tilde{\theta}(t_2 - t_1)e^{-i\omega_\alpha(t_1-t_2)}, \tag{C4}$$

$$\langle \bar{b}_{\alpha-}(t_2)b_{\alpha-}(t_1)\rangle_B = z\theta(t_2 - t_1)e^{-i\omega_\alpha(t_1-t_2)}, \tag{C5}$$

$$\langle b_{\alpha+}(t_1)\bar{b}_{\alpha-}(t_2)\rangle_B = 0, \tag{C6}$$

$$\langle \bar{b}_{\alpha-}(t_1)b_{\alpha+}(t_2)\rangle_B = 0, \tag{C7}$$

$$\langle b_{\alpha-}(t_1)\bar{b}_{\alpha+}(t_2)\rangle_B = e^{-i\omega_\alpha(t_1-t_2)}, \tag{C8}$$

$$\langle \bar{b}_{\alpha+}(t_2)b_{\alpha-}(t_1)\rangle_B = ze^{-i\omega_\alpha(t_1-t_2)}. \tag{C9}$$

The effective coupling are fully determined by Fourier transform of the following spectral density of couplings to the bath:

$$\mathcal{J}_B(\omega) = \sum_\alpha |g_\alpha|^2\delta(\omega - \omega_\alpha). \tag{C10}$$

In terms of the spectral density of couplings to the bath, the functions $g^{\alpha\beta}(t_1, t_2)$ can be written as follows:

$$g^{++}(t_1 - t_2) \simeq \{\tilde{\theta}(t_1 - t_2) + \theta(t_1 - t_2)\}\int \frac{d\omega}{2\pi}\mathcal{J}_B(\omega)e^{-i\omega(t_1-t_2)}, \tag{C11}$$

$$g^{--}(t_1 - t_2) \simeq \{\tilde{\theta}(t_2 - t_1) + \theta(t_1 - t_2)\}\int \frac{d\omega}{2\pi}\mathcal{J}_B(\omega)e^{-i\omega(t_1-t_2)}, \tag{C12}$$

$$g^{-+}(t_1 - t_2) \simeq -2\int \frac{d\omega}{2\pi}\mathcal{J}_B(\omega)e^{-i\omega(t_1-t_2)}, \tag{C13}$$

$$g^{+-}(t_1 - t_2) \simeq 0. \tag{C14}$$

By further assuming that, within a band of width D around $\omega = 0$, $\mathcal{J}_B(\omega)$ is well approximated by a constant, i.e.,

$$\mathcal{J}_B(\omega) = \nu_0|\langle g\rangle|^2 = \text{const} \tag{C15}$$

for $|\omega| < D/2$, where $\langle g\rangle$ is the average coupling strength and $\nu_0 \sim D^{-1}$ is the density of states. We note this constant dependence in the spectral density corresponds to a Markovian bath. Within our formalism, non-Markovianity can be introduced using other types of spectral densities, e.g., power-law forms [64], which yield bath correlations different from the Markovian one. In the Markovian case, using the above expression, we have

$$\int \frac{d\omega}{2\pi}\mathcal{J}_B(\omega)e^{-i\omega t} \simeq \nu_0|\langle g\rangle|^2\frac{\sin(Dt/2)}{\pi t}. \tag{C16}$$

Note that this function is strongly peaked for $t = t_1 - t_2 = 0$ and decreases (oscillates) rapidly for $|t_1 - t_2| \gtrsim D^{-1}$ (the oscillation is an artifact of the hard cutoff). Thus, if the dynamics of the system is characterized by frequencies much smaller than D, we can effectively replace the above function by a Dirac delta function $\delta(t_1 - t_2)$, which yields

$$v^{++}(t_1, t_2) \simeq \nu_0|\langle g\rangle f(t_1)|^2\delta(t_1 - t_2), \tag{C17}$$

$$v^{--}(t_1, t_2) \simeq \nu_0|\langle g\rangle f(t_1)|^2\delta(t_1 - t_2), \tag{C18}$$

$$v^{-+}(t_1, t_2) \simeq -2\nu_0|\langle g\rangle f(t_1)|^2\delta(t_1 - t_2), \tag{C19}$$

$$v^{+-}(t_1, t_2) \simeq 0. \tag{C20}$$

To obtain the above expressions, we have used $\tilde{\theta}(t)\delta(t) = \tilde{\theta}(0)\delta(t) = \delta(t)$ and $\theta(t)\delta(t) = \theta(0)\delta(t) = 0$, as required by the discrete version of the path integral [29]. $\tilde{\theta}(t)$ and $\theta(t)$ are two regularizations of the step function.

APPENDIX D: DERIVATION OF LOSS EQUATION FOR LOSSY 1D BOSE HUBBARD MODEL

The momentum distribution $n_r(t)$ for a particle with momentum r at time t using perturbation expansion to leading order is

$$\begin{aligned} n_r(t) - n_r^0 &\simeq i\langle \bar{c}_{r,-}(t)c_{r,+}(t)\mathcal{L}\rangle_c \\ &= i\Bigg\{\frac{-1}{2M}\sum_{pkq,mn}\int_{-\infty}^{t} dt_1\,\sigma^3_{mn}U_{pkq}(t_1)O_{mn}(t, t_1; r, p, k, q) \\ &\quad + \frac{i}{2M}\sum_{pkq,mn}\int_{-\infty}^{t} dt_1\,\sigma^0_{mn}\Gamma_{pkq}(t_1)O_{mn}(t, t_1; r, p, k, q) \\ &\quad - \frac{i}{M}\sum_{pkq}\int_{-\infty}^{t} dt_1\,\Gamma_{pkq}(t_1)O_{-+}(t, t_1; r, p, k, q)\Bigg\}, \end{aligned} \tag{D1}$$

where $n_r^0 = \langle c^\dagger_{r,-}(t)c_{r,+}(t)\rangle_c$ is the momentum distribution of the Jordan-Wigner fermions in the initial state described by the (noninteracting) Jordan-Wigner fermion action S_c [cf. Eq. (56) or first term on the right-hand side of Eq. (71)] with initial inverse temperature β. In the above expression, $\langle\ldots\rangle_c$ stands for Keldysh time-ordered average with weight e^{iS_c}. In addition, we have also introduced the following notation for the six fermion-operator expectation values:

$$\begin{aligned} &O_{mn}(t, t_1; r, p, k, q) \\ &\quad = \langle \bar{c}_{r,-}(t)c_{r,+}(t)\,\bar{c}_{p,m}(t_1)\bar{c}_{k,m}(t_1)c_{k+q,n}(t_1)c_{p-q,n}(t_1)\rangle_c. \end{aligned} \tag{D2}$$

Expanding the sum including σ^3_{mn} and σ^0_{mn}, we have

$$\sum_{mn}\sigma^3_{mn}O_{mn}(t,t_1;r,p,k,q) = O_{++}(t,t_1;r,p,k,q) - O_{--}(t,t_1;r,p,k,q), \quad \text{(D3)}$$

$$\sum_{mn}\sigma^0_{mn}O_{mn}(t,t_1;r,p,k,q) = O_{++}(t,t_1;r,p,k,q) + O_{--}(t,t_1;r,p,k,q). \quad \text{(D4)}$$

Next, applying Wick's theorem yields

$$\begin{aligned} O_{mn}(t,t_1;r,p,k,q) &= [\langle \bar{c}_{k,m}(t_1)c_{k+q,n}(t_1)\rangle_c \langle c_{r,+}(t)\,\bar{c}_{p,m}(t_1)\rangle_c \\ &\times \langle \bar{c}_{r,-}(t)c_{p-q,n}(t_1)\rangle_c \\ &+ \langle \bar{c}_{p,m}(t_1)c_{p-q,n}(t_1)\rangle_c \langle c_{r,+}(t)\,\bar{c}_{k,m}(t_1)\rangle_c \\ &\times \langle \bar{c}_{r,-}(t)c_{k+q,n}(t_1)\rangle_c] \\ &- [\langle \bar{c}_{k,m}(t_1)c_{p-q,n}(t_1)\rangle_c \langle c_{r,+}(t)\,\bar{c}_{p,m}(t_1)\rangle_c \\ &\times \langle \bar{c}_{r,-}(t)c_{k+q,n}(t_1)\rangle_c \\ &+ \langle \bar{c}_{p,m}(t_1)c_{k+q,n}(t_1)\rangle_c \langle c_{r,+}(t)\,\bar{c}_{k,m}(t_1)\rangle_c \\ &\times \langle \bar{c}_{r,-}(t)c_{p-q,n}(t_1)\rangle_c], \end{aligned} \quad \text{(D5)}$$

which yields

$$\begin{aligned} O_{++}(t,t_1;r,p,k,q) = O_{--}(t,t_1;r,p,k,q) &= \big[\delta_{q,0}\delta_{r,p}n^0_k + \delta_{q,0}\delta_{r,k}n^0_p \\ &- \delta_{q,p-k}\delta_{r,p}n^0_k - \delta_{q,p-k}\delta_{r,k}n^0_p\big] \\ &\times \big[\tilde{\theta}(t-t_1)\big(1-n^0_r\big)n^0_r \\ &- \theta(t_1-t)\big(n^0_r\big)^2\big] \end{aligned} \quad \text{(D6)}$$

and

$$\begin{aligned} O_{-+}(t,t_1;r,p,k,q) = -\big[&\delta_{q,0}\delta_{r,p}n^0_k + \delta_{q,0}\delta_{r,k}n^0_p \\ &- \delta_{q,p-k}\delta_{r,p}n^0_k - \delta_{q,p-k}\delta_{r,k}n^0_p\big]\big(n^0_r\big)^2. \end{aligned} \quad \text{(D7)}$$

Note that the exponential phase dependence on t and t_1 is canceled in the above first order expectation values. Finally, combining Eqs. (D1), (D3), and (D6) with Eq. (D7) and rearranging the momentum indices yields

$$\begin{aligned} n_r(t) - n^0_r &= -\frac{2}{M}\sum_k [\Gamma_{rk,q=0} - \Gamma_{rk,q=r-k}]n^0_r n^0_k \\ &= \frac{-1}{M}\sum_k \int_{-\infty}^{t} \frac{16J^2\gamma(t_1)}{U^2+\gamma^2(t_1)/4}\, n^0_r n^0_k\, dt_1 \\ &\times \sin^2\left(\frac{r-k}{2}\right)\cos^2\left(\frac{r+k}{2}\right). \end{aligned} \quad \text{(D8)}$$

Setting $\gamma(t) = \theta(t)\gamma$ and taking $t \to 0$, we arrive at the loss rate equation given in Eq. (75).

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