Continuum field theory of three-dimensional topological orders with emergent fermions and braiding statistics

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Universal topological data of topologically ordered phases can be captured by topological quantum field theory in continuous space time by taking the limit of low energies and long wavelengths. While previous continuum field-theoretical studies of topological orders in three-dimensional (3D) real space focused on either self-statistics, braiding statistics, shrinking rules, fusion rules, or quantum dimensions, it has yet to systematically put all topological data together in a unified continuum field-theoretical framework. Here, we construct the topological BF field theory with twisted terms (e.g., $AAdA$ and AAB) as well as a K-matrix BB term to simultaneously explore all such topological data and reach anomaly-free topological orders. Following the spirit of the famous K-matrix Chern-Simons theory of two-dimensional topological orders, we present general formulas and systematically show how the K-matrix BB term confines topological excitations, and how self-statistics of particles is transmuted between bosonic one and fermionic one. To reach anomaly-free topological orders, we explore, within the present continuum field-theoretical framework, how the principle of gauge invariance fundamentally influences possible realizations of topological data. More concretely, we present the topological actions of (i) particle-loop braidings with emergent fermions, (ii) multiloop braidings with emergent fermions, and (iii) Borromean rings braidings with emergent fermions, and calculate their universal topological data. Together with previous efforts, our paper paves the way toward a more systematic and complete continuum field-theoretical analysis of exotic topological properties of 3D topological orders. Several interesting future directions are also discussed.

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I. INTRODUCTION

Exploring low-energy long-wavelength effective field theories of quantum many-body systems has a long history in condensed matter physics [1]. For example, the Ginzburg-Landau (GL) field theory, in terms of local order parameters, is applied to symmetry-breaking phases and phase transitions; nonlinear sigma models with topological θ term are applied to quantum spin chains. Since the discovery of the fractional quantum Hall effect in the 1980s, the notion of topological order has been introduced as a route toward exotic phases of matter that cannot be characterized by the mechanism of symmetry breaking. While there has been a broad consensus that the essence of topological order is deeply rooted in patterns of long-range entanglement that is robust against local unitaries of finite depth [2], the original definition of *topological* order really comes from the fact that the low-energy effective field theory of the prototypical topological order—fractional quantum Hall states—is the Chern-Simons theory which is a *topological* quantum field theory (TQFT) [3,4] in continuous spacetime. Along this line of thinking, the common expectation for topological phases of matter—topological robustness against any local perturbations—is achievable by simply noting the fact that correlation functions of all spatially local operators in TQFTs vanish [5], which is in sharp contrast with the GL theory. Particularly, as the most general Abelian formulation, the K-matrix Chern-Simons theory [6,7] whose action is written in terms of $\sim \int \frac{K_{IJ}}{4\pi} A^I dA^J$, serves as the standard TQFT framework of two-dimensional (2D) Abelian topological orders, providing a highly efficient algorithm for computing topological data, such as anyon types, self-statistics, mutual statistics, fusion algebra, chiral central charge, and ground-state degeneracy. Besides topological orders, the K-matrix Chern-Simons theory has also been successfully applied to the study of symmetry-enriched topological phases (SETs) [8,9] and symmetry-protected topological phases [10–14] where global symmetry is nontrivially imposed.

While the K-matrix Chern-Simons theory works very well in 2D topological phases of matter, it is no longer applicable to three-dimensional (3D) and higher, where exotic spatially extended excitations [e.g., loops in no less than 3D space and membranes in no less than four dimensional (4D) space] induce very rich emergent phenomena. Instead, if particles and loops, respectively, carry gauge charges and gauge fluxes of a

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discrete Abelian gauge group $G=\prod_{i=1}^{n}\mathbb{Z}_{N_i}$, one may apply the twisted *BF* field theory [15,16] by properly including twisted terms (denoted as T) [17–21]. This series of TQFTs has been proven to be a very powerful way to efficiently describe various types of nontrivial braiding statistics, such as particle-loop braiding [16,22–26], multiloop braiding [27] (with the twisted terms $\sim AAdA$ and $\sim AAAA$), and particle-loop-loop braiding (i.e., Borromean rings braiding with the twisted term $\sim AAB$) [21]. Recently, the untwisted (twisted) *BF* theory has also been successfully applied to SET [28–32] and SPT [19,33–36] in 3D.

In the twisted *BF* theory (symbolically denoted as $BF+T$), each type of braiding statistics is associated with a particular formulation of TQFT actions, which does not mean that all types of braiding statistics are mutually compatible and can thereby coexist in an anomaly-free topological order. To further examine whether two different types of braiding processes are allowed to compatibly exist in the same topological order, Ref. [37] exhausted all combinations of twisted terms and found that TQFTs of some combinations inevitably violate the principle of gauge invariance. Thus, among all combinations, only a part of combinations are legitimate such that braiding processes can coexist. After TQFTs with mutually compatible braiding processes are obtained, fusion rules and shrinking rules in the TQFTs are further investigated, from which quantum dimensions of both particles and loops are computed [38]. Recently, the ideas of Refs. [37,38] have been subsequently extended to 4D real space [39,40] where membrane excitations are allowed and hierarchy of shrinking rules is definable.

On the other hand, in the untwisted *BF* theory with the inclusion of a *BB* term [41] (symbolically denoted as $BF+BB$), the boson-fermion statistical transmutation of self-statistics (i.e., exchange statistics) of particles in 3D [18] has been studied through equations of motion, where the scenario of Dirac-string-attachment implied by the equations of motion mimics, to some extent, the physics of dyons studied intensively in other contexts [32,42–45]. Along this line, it has become clear that all particles with anyonic statistics (neither fermionic nor bosonic) are exactly confined and thus disappear in the low energy spectrum, which is perfectly consistent with the well-known fact that anyons are impossible in 3D and higher [46–48]. Thanks to the statistical transmutation induced by the *BB* term, we may realize both *emergent fermions* (defined as topologically nontrivial particles that are fermionic) and transparent fermions (defined as topologically trivial particles that are fermionic) in a topological action that is composed of merely bosonic degrees of freedom (i.e., gauge fields). As a side, by definition, once transparent particles are fermionic, the topological order is said to be *fermionic*. In addition to boson-fermion transmutation, the single-component *BB* term also provides a Higgs mechanism that confines either partially or completely the gauge group G set by the coefficient of the *BF* term [49]. In addition, the multicomponent *BB* term was successfully applied to 3D bosonic topological insulators (bosonic SPTs with particle number conservation and time-reversal symmetry) where bulk topological order is trivial but boundary admits anomalous surface topological orders [33].

Logically, once we have understood (i) how to obtain compatible braiding processes via legitimate combinations of twisted terms in the twisted *BF* theory ($BF+T$) and (ii) how to assign self-statistics on particles via the boson-fermion transmutation in the untwisted *BF* theory with the *BB* term ($BF+BB$), it becomes urgent to make a step forward by examining whether braiding statistics is compatible with the assignment of self-statistics on particles in the twisted *BF* theory with the *BB* term (denoted as $BF+T+BB$) within the present continuum-field-theoretical framework. We are motivated to combine all known topological terms in continuous spacetime to achieve a more complete continuum-field-theoretical description of topological data encoded in 3D topological orders with the gauge group G.

In this paper, we first explain the microscopic origin of topological terms via various condensation pictures at the beginning of Sec. II to (i) make the gauge theories more physical in the context of many-body physics and (ii) introduce the critical role of Lagrange multipliers. The topological action of a $\mathbb{Z}_N$ topological order is dual to a standard Abel-Higgs model that describes a boson or vortex-line condensate coupled to gauge field. Other topological twisted terms and *BB* terms can be formally derived through introducing topological interactions among different condensates. In the remaining part of Sec. II, we systematically formulate the untwisted *BF* theory with a K-matrix *BB* term (i.e., $\sim\frac{K_{ij}}{4\pi}B^iB^j$ with a symmetric integer matrix K). In the presence of the K-matrix *BB* term, we present general mathematical formulas that can be applied to efficiently determine (i) excitation contents, i.e., inequivalent Wilson operators of deconfined particles and deconfined loops (Fig. 1), and (ii) self-statistics assignment on particles. In particular, the situation of the single-component *BB* term has been naturally included by regarding the K matrix as an integer. Then, the self-statistics of particles is rigorously derived by computing the expectation values of *framed* Wilson loops (Fig. 2). We obtain the formula Eq. (16) in a compact form that completely fixes self-statistics of the particle labeled by an integer vector, whose usefulness is comparable to the familiar formula in the famous K-matrix Chern-Simons theory of 2D topological orders [6,7]. In Table I, we collect, for our purpose, the most useful properties of the untwisted *BF* theory with a single-component *BB* term. To determine whether a topological order is fermionic or bosonic, we may calculate the self-statistics of trivial (i.e., transparent) particles. To determine whether a topological order supports emergent fermions, we may calculate the self-statistics of particles that carry nontrivial gauge charges of G.

Section III is devoted to studying the interplay of topological data including self-statistics and braiding statistics as well as fusion rules. For this purpose, *BF* theories with *BB* term and different twists are studied, leading to $BF+T+BB$. Three root-braiding processes (coined in Ref. [37]) and their braiding phases are considered: particle-loop braiding (*BF* term), multiloop braiding (*AAdA* and *AAAA* twist), and Borromean rings braiding (*AAB* twist). For completeness, we start our discussions in Sec. III A by considering particle-loop braiding with emergent fermions, which is described by $BF+BB$. Then, we move to a continuum field theory description of the coexistence of emergent fermions and multiloop braidings (Sec. III B) and the coexistence of emergent fermions and Borromean rings braiding (Sec. III C). We present

TABLE I. Properties of $S=\int \frac{N_1}{2\pi}B^1dA^1+\frac{K_{11}}{4\pi}B^1B^1$ with different values of N_1 and K_{11}. Θ_{trivial} is the self-statistics of trivial particle excitation, i.e., those carrying 0 mod N unit of gauge charge. $\Theta_{\text{trivial}}=1(-1)$ means that the trivial particle excitation is a boson (fermion) or, equivalently, the theory is a bosonic (fermionic) one. $\Theta_{e_{1\,\min}}$ is the self-statistics of a particle excitation with $e_{1\,\min}$ units of gauge charge where $e_1\min=N_1/\gcd(\frac{K_{11}}{N_1},N_1)$. $e_{1\,\min}$ is the minimal units of gauge charge carried by a deconfined particle excitation in theory $S=\int \frac{N_1}{2\pi}B^1dA^1+\frac{K_{11}}{4\pi}B^1B^1$. An emergent fermion appears when $\Theta_{\text{trivial}}=1$ while $\Theta_{e_{1\,\min}}=-1$. In other words, there exists nontrivial fermionic particle excitations in a bosonic theory.

N_1	K_{11}	Θ_{trivial}	$\Theta_{e_{1\,\min}}$	Bosonic or fermionic theory?	Emergent fermion?
odd	even	1	1	bosonic	No
odd	odd	-1	-1	fermionic	
even	odd	1	1	bosonic	No
even	even and $\frac{\mathrm{lcm}\left(\frac{K_{11}}{N_1},N_1\right)}{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}\in 2\mathbb{Z}$	1	1	bosonic	No
even	even and $\frac{\mathrm{lcm}\left(\frac{K_{11}}{N_1},N_1\right)}{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}\in 2\mathbb{Z}+1$	1	-1	bosonic	Yes

topological actions, gauge transformations for each case, and calculate topological data including inequivalent Wilson operators, braiding statistics, self-statistics, fusion rules, shrinking rules, and quantum dimensions. We draw our conclusions and make some discussions in Sec. IV.

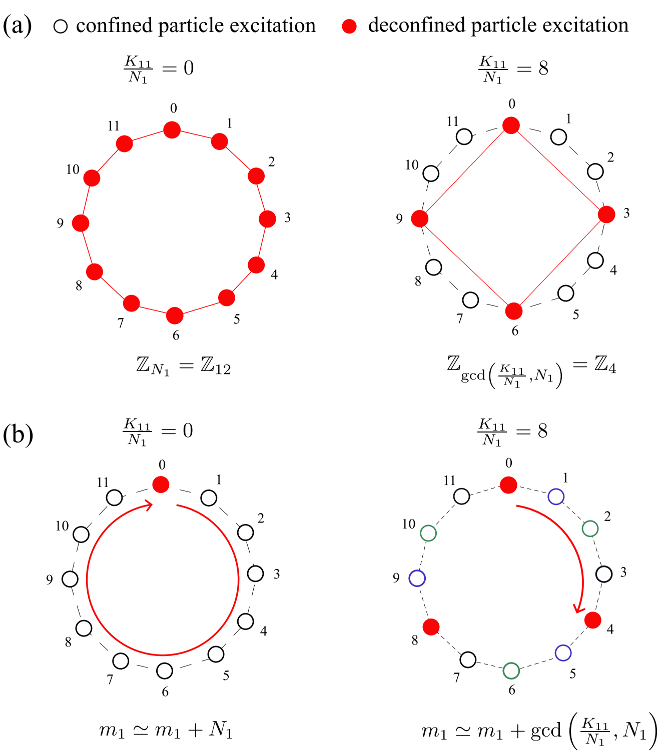

FIG. 1. (a) When $\frac{K_{11}}{N_1}=0$, the charges of deconfined particle excitations of action (18) are labeled by $\mathbb{Z}_{N_1}=\mathbb{Z}_{12}$. Once a $B\wedge B$ term with $\frac{K_{11}}{N_1}$ is added, some particle excitations become confined. Those deconfined are labeled by the unbroken gauge group $\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}=\mathbb{Z}_4$. (b) Period of fluxes for different values of $\frac{K_{11}}{N_1}$. When $\frac{K_{11}}{N_1}=0$, fluxes has a period of N_1. When $\frac{K_{11}}{N_1}=8$, some fluxes are actually equivalent as shown in the same color. In this case, the minimal period of flux is $\gcd(\frac{K_{11}}{N_1},N_1)=4$.

II. CONDENSATION PICTURE AND K-MATRIX BB TERM

A. Condensation picture via Abel-Higgs models with topological interactions

Here we review a condensation picture for TQFT in (3 + 1)D [12,16,19,33,37]. A $\mathbb{Z}_N$ topological order described by a topological action $S=\int \frac{N}{2\pi}B\wedge dA$ can be viewed as the Higgs phase of a Abel-Higgs model. Such a model describes a condensate of charge boson (or flux-threaded vortex lines) coupled to a gauge field. In this section, we will explain this microscopic origin of topological actions. We start from a single-layer condensate of boson or vortex lines. Then, by turning on topological interactions among different condensates, other topological terms emerge after a duality transformation.

We first show how to derive the topological BF term from a boson condensation (or a vortex-line condensation)

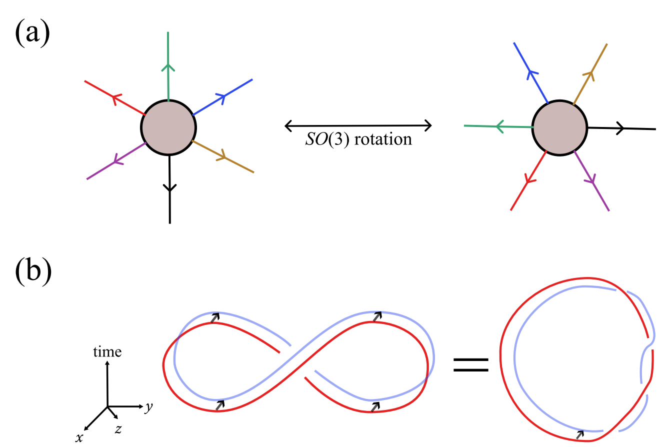

FIG. 2. (a) The physical picture of BB term is to bind flux strings (lines with arrow) to particle excitations (red solid circle); see the Wilson operator of a particle excitation Eq. (9). Different orientations of flux strings can be connected by an SO(3) rotation. (b) Consider a particle excitation attached by a flux string, e.g., Eq. (19), exchanging such two particle excitations can be viewed as a self 2π rotation of a particle. A nontrivial framing of the world line of particle excitation is shown to illustrate this point.

coupled to a gauge field. Consider a condensation of charge-N bosons that couple to a gauge field: $\mathcal{L}=\frac{\rho}{2}(\partial_\mu\theta-NA_\mu)^2+\mathcal{L}_{\text{Maxwell}}$, where A is a $U(1)$ gauge field. This is nothing but the deconfined phase of the Abel-Higgs model. With a Hubbard-Stratonovich auxiliary field j_μ, this Lagrangian is dual to $\mathcal{L}_d=-\frac{1}{2\rho}(j_\mu)^2+j_\mu(\partial_\mu\theta-NA_\mu)$. Integration of θ results in a constraint $\delta(\partial_\mu j_\mu)$ in the path integral measure which can be resolved by introducing a two-form gauge field $B_{\mu\nu}$: $j_\mu=-\frac{1}{4\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\nu B_{\lambda\rho}$. Substituting this solution to the dual Lagrangian and dropping the irrelevant Maxwell terms, we obtain $\mathcal{L}_d\sim\frac{N}{4\pi}\epsilon^{\mu\nu\lambda\rho}A_\mu\partial_\nu B_{\lambda\rho}$. Then the action is $S=\int\mathcal{L}_d dxdt\sim\int\frac{N}{2\pi}A\wedge dB$, where $A=\sum_\mu A_\mu dx^\mu$ and $B=\frac{1}{2!}\sum_{\mu\nu}B_{\mu\nu}dx^\mu dx^\nu$. Through integration by parts and dropping the total derivative term, we reach the topological BF term, $S=\int\frac{N}{2\pi}B\wedge dA$. In this action, B serves as a Lagrange multiplier to enforce $dA=0$ locally.

On the other hand, we can also consider a condensation of flux-N vortex-lines [33] coupled to a two-form gauge field: $\mathcal{L}'=\frac{\rho}{2}(\partial_{[\mu}\Theta_{\nu]}-NB_{\mu\nu})^2+\mathcal{L}'_{\text{Maxwell}}$, where Θ is the phase of vortex-line condensation, B is a $U(1)$ gauge field, and $\partial_{[\mu}\Theta_{\nu]}=\partial_\mu\Theta_\nu-\partial_\nu\Theta_\mu$. This actually is another kind of Abel-Higgs model of a two-form gauge field. Similar to previous discussions, $\mathcal{L}'$ is dual to $\mathcal{L}'_d=-\frac{1}{8\rho}(\Sigma_{\mu\nu})^2+\frac{1}{2}\Sigma_{\mu\nu}(\partial_{[\mu}\Theta_{\nu]}-NB_{\mu\nu})$ with a Hubbard-Stratonovich auxiliary field $\Sigma_{\mu\nu}$. Integrating over Θ leads to a constraint $\partial_\nu\Sigma_{\mu\nu}=0$, which can be resolved by $\Sigma_{\mu\nu}=-\frac{1}{2\pi}\epsilon^{\mu\nu\lambda\rho}\partial_\lambda A_\rho$. Once again, we arrive at $\mathcal{L}'_d\sim\frac{N}{4\pi}\epsilon^{\mu\nu\lambda\rho}B_{\mu\nu}\partial_\lambda A_\rho$ and $S'=\int\mathcal{L}'_d dxdt=\int\frac{N}{2\pi}A\wedge dB+\cdots$, where $\cdots$ includes Maxwell terms and boundary terms that can be dropped. In this case, it is A that plays the role of Lagrange multiplier to enforce $dB=0$ locally. $A\wedge dB$ is also a BF term since it differs from $B\wedge dA$ by a total derivative.

From the above discussion, we have seen that there are two kinds of Abel-Higgs models that can be dual to the BF term. The first (second) one realizes the Higgs phase of a one-form (two-form) gauge theory. This also reveals that a $\mathbb{Z}_N$ topological order has a condensation picture: it originates from either a boson condensate coupled to gauge field or a vortex-line condensate coupled to gauge field. The $\mathbb{Z}_N$ gauge group structure is encoded in the value of Wilson operator of a one-form gauge field or two-form gauge field. This picture can be generalized to a $\prod_{i=1}^n\mathbb{Z}_{N_i}$ topological order. We can derive different topological terms, e.g., $AAdA$, $AAAA$, and AAB ($\wedge$ is omitted), through topological interactions among different condensates.

For a $AAdA$-type topological term, its microscopic origin can be traced back to two-layer or three-layer condensates of charged bosons [19], e.g., $\mathcal{L}=\sum_{i=1}^3\frac{\rho_i}{2}(\partial_\mu\theta^i-N_iA^i_\mu)^2+\mathrm{i}q\epsilon^{\mu\nu\lambda\rho}(\partial_\mu\theta^1-N_1A^1_\mu)(\partial_\nu\theta^2-N_2A^2_\nu)\partial_\lambda A^3_\rho+\mathcal{L}_{\text{Maxwell}}$, where q is a proper coefficient. The theory is dual to $S\sim\int\sum_{i=1}^3\frac{N_i}{2\pi}B^idA^i+qA^1A^2dA^3$ that captures the three-loop braiding. For a $AAAA$-type topological term, it can be derived from a four-layer condensate where each layer is in charge-N_i *boson* condensation [12], e.g., $\mathcal{L}=\sum_{i=1}^4\frac{\rho_i}{2}(\partial_\mu\theta^i-N_iA^i_\mu)^2+\mathrm{i}q\epsilon^{\mu\nu\lambda\rho}\times(\partial_\mu\theta^1-N_1A^1_\mu)\times(\partial_\nu\theta^2-N_2A^2_\nu)\times(\partial_\lambda\theta^3-N_3A^3_\lambda)\times(\partial_\rho\theta^2-N_4A^4_\rho)+\mathcal{L}_{\text{Maxwell}}$. This theory is dual to $S\sim\int\sum_{i=1}^4\frac{N_i}{2\pi}B^idA^i+qA^1A^2A^3A^4$ that corresponds to the four-loop braiding. Such condensation picture applies for AAB and BB topological terms with the caveat that the two-form gauge field B indicates a *vortex-line* condensation. The topological action $S=\int\sum_{i=1}^3\frac{N_i}{2\pi}B^idA^i+qA^1A^2B^3$ can be derived from [37]: $\mathcal{L}=\sum_{i=1}^2\frac{\rho_i}{2}(\partial_\mu\theta^i-N_iA^i_\mu)^2+\frac{(\phi_3)^2}{2}(\partial_{[\mu}\Theta^3_{\nu]}-N_3B^3_{\mu\nu})^2+\mathrm{i}q\epsilon^{\mu\nu\lambda\rho}(\partial_\mu\theta^1-N_1A^1_\mu)(\partial_\nu\theta^2-N_2A^2_\nu)(\partial_{[\lambda}\Theta_{\rho]}-N_3B^3_{\lambda\rho})+\mathcal{L}_{\text{Maxwell}}$ where layers 1 and 2 are in charge-N_1 and -N_2 *boson* condensation while layer 3 is in flux-N_3 *vortex-line* condensation. For the topological action $S=\int\frac{N_1}{2\pi}B^idA^i+\frac{K_{11}}{4\pi}B^1B^1$, where K_{11} is a proper coefficient, it can be derived from a condensate of flux-N_1 vortex line coupled to the gauge field [33]: $\mathcal{L}=\frac{\phi^2}{2}(\partial_{[\mu}\Theta^s_{\nu]}-N_1B^1_{\mu\nu})^2+\mathrm{i}K_{11}\epsilon^{\mu\nu\lambda\rho}(\partial_{[\mu}\Theta^s_{\nu]}-N_1B^1_{\mu\nu})(\partial_{[\lambda}\Theta^s_{\rho]}-N_1B^1_{\lambda\rho})+\mathcal{L}_{\text{Maxwell}}$.

Keeping this condensation picture in mind, we can examine these TQFT actions more carefully. To describe a multi-loop braiding, one can utilize $S\sim\int BdA+AAdA$ (three-loop braiding) or $S\sim\int BdA+AAAA$ (four-loop braiding). In these two actions, the two-form gauge field B's serve as Lagrange multipliers to enforce $dA=0$ locally. A Borromean rings braiding can be described by $S\sim\int\sum_{i=1}^3\frac{N_i}{2\pi}B^idA^i+A^1A^2B^3$. From the above derivation, we note that B^1 and B^2 serve as Lagrange multipliers while A^3 is the Lagrange multiplier for layer 3. Similarly, for the TQFT action $S\sim\int BdA+BB$, one-form gauge field A serves as Lagrange multiplier. We shall emphasize the importance of Lagrange multiplier here. When we consider an Abel-Higgs model of a one-form gauge field (A), a two-form gauge field B emerges as a Lagrange multiplier to encode the constraint $dA=0$ in the path integral and vice versa. In other words, in a BF term $B\wedge dA$, either A or B serves as the Lagrange multiplier, meaning that its microscopic origin can be derived from either a vortex-line condensation or a boson condensation. When we discuss a $\prod_{i=1}^n\mathbb{Z}_{N_i}$ topological order, the BF term is $\sum_{i=1}^n\frac{N_i}{2\pi}B^idA^i$. For each index i, only one of A^i and B^i is a Lagrange multiplier and there must be one Lagrangian multiplier such that a $\mathbb{Z}_{N_i}$ gauge theory can be realized in continuous spacetime. We can draw a conclusion that in our framework of continuum field theory $S=S_{BF}+S_{\text{int}}$, A^i and B^i cannot be simultaneously involved in the interaction term S_{int}, i.e., A^i and B^i cannot show up in twisted terms and BB term at the same time.

B. *K*-matrix *BB* term: Topological action, gauge transformations, coefficient quantization, and periods

The untwisted BF theory with a K-matrix BB term is ($\wedge$ is omitted)

$$S=\int\sum_{i=1}^n\frac{N_i}{2\pi}B^idA^i+\sum_{i,j=1}^n\frac{K_{ij}}{4\pi}B^iB^j,\tag{1}$$

where K is an $n\times n$ symmetric matrix ($K_{ij}=K_{ji}$) whose quantization and periods will be determined shortly. The coefficients $\{N_i\}$ of the first term, i.e., the BF term, determine the gauge group $G=\prod_{i=1}^n\mathbb{Z}_{N_i}$. The $[U(1)]^n\times[U(1)]^n$ gauge transformations of this gauge theory are defined as

$$A^i\to A^i+d\chi^i-\sum_{j=1}^n\frac{K_{ij}}{N_i}V^j,\quad B^i\to B^i+dV^i,\tag{2}$$

where χ^i and V^i are, respectively, zero-form and one-form gauge parameters that satisfy the usual compactness conditions: $\frac{1}{2\pi}\int d\chi^i \in \mathbb{Z}$ and $\frac{1}{2\pi}\int dV^i \in \mathbb{Z}$. It is clear that the BB term, denoted as $S_{BB} = \sum_{i,j=1}^n \frac{K_{ij}}{4\pi} B^i B^j$, induces an extra term $\sum_{j=1}^n \frac{K_{ij}}{N_i} V^j$ compared to the usual gauge transformations of the one-form gauge field A^i.

After the transformations, two additional terms are induced in the BB term: $S_{BB} \to S'_{BB} = S_{BB} + \Delta S^{(1)}_{BB} + \Delta S^{(2)}_{BB}$, where $\Delta S^{(1)}_{BB} = 2\int \sum_{i,j=1}^n \frac{K_{ij}}{4\pi} B^i dV^j$ and $\Delta S^{(2)}_{BB} = \int \sum_{i,j=1}^n \frac{K_{ij}}{4\pi} dV^i dV^j$. In a compact manifold, these two terms vanish if gauge parameters are topologically trivial. But, in general, $\int dV^i$ can be nonzero. Here $\Delta S^{(1)}_{BB}$ can be written as $\Delta S^{(1)}_{BB} = 2\int \sum_{i=1}^n \frac{K_{ii}}{4\pi} B^i dV^i + 2\int \sum_{i<j}^n \frac{K_{ij}}{4\pi} B^i dV^j + 2\int \sum_{i<j}^n \frac{K_{ji}}{4\pi} B^j dV^i$, in which $\frac{1}{2\pi}\int B^i dV^j \in \frac{2\pi}{N_i}\mathbb{Z}$ for arbitrary i and j. Demanding $\Delta S^{(1)}_{BB} \in 2\pi\mathbb{Z}$, we find constraints $\frac{K_{ii}}{N_i} \in \mathbb{Z}$, $\frac{K_{ij}}{N_i} \in \mathbb{Z}$ $(i<j)$, and $\frac{K_{ji}}{N_j} \in \mathbb{Z}$ $(i<j)$. Recall $K_{ij} = K_{ji}$ and we find $\frac{K_{ij}}{N_i} \in \mathbb{Z}$ and $\frac{K_{ij}}{N_j} \in \mathbb{Z}$ for $i \neq j$. In fact, the constraint on K_{ij} is $\frac{K_{ij}}{\mathrm{lcm}(N_i,N_j)} \in \mathbb{Z}$ where $\mathrm{lcm}(N_i,N_j)$ is the least common multiplier of N_i and N_j.

For the calculation of $\Delta S^{(2)}_{BB}$, we need to consider whether a spin structure is taken into account. On a nonspin manifold, $\frac{1}{4\pi^2}\int dV^i dV^i$ is quantized to $\mathbb{Z}$; while on a spin manifold, it is quantized to $2\mathbb{Z}$. For $\frac{1}{4\pi^2}\int dV^i dV^j$ with $i \neq j$, it is quantized to $\mathbb{Z}$ no matter on a spin or nonspin manifold. To keep $\Delta S^{(2)}_{BB} \in 2\pi\mathbb{Z}$ for gauge invariance, we have (i) nonspin manifold: $K_{ii} \in 2\mathbb{Z}, K_{ij} \in \mathbb{Z}\,(i \neq j)$ and (ii) spin manifold: $K_{ii} \in \mathbb{Z}, K_{ij} \in \mathbb{Z}\,(i \neq j)$. Only on a spin manifold *can* the diagonal elements K_{ii} be an odd integer. Indeed, as shown in the following main text, the parity of K_{ii} controls the self-statistics of trivial particle excitations of $\mathbb{Z}_{N_i}$ gauge subgroup. As long as one of the diagonal elements K_{ii} is odd, there must exist a fermionic trivial particle excitation; thus, by definition, theory Eq. (1) describes a fermionic topological order. This is consistent with the fact that a fermionic theory can only be defined on a spin manifold. On the other hand, when all K'_{ii}s are even, this theory Eq. (1) is a bosonic one.

For the period of K_{ij}, we consider $S_{BB} \in \sum_{i=1}^n \frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i}\mathbb{Z} + \sum_{i<j}^n 2 \times \frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j}\mathbb{Z}$. Since $\exp(\mathrm{i}S_{BB})$ should be invariant if we shift either $\frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i}$ or $2 \times \frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j}$ by 2π, we have the following relations (we use $\simeq$ to denote such identification relation): $\frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i} \simeq \frac{K_{ii}}{4\pi}\frac{(2\pi)^2}{N_iN_i} + 2\pi$ and $2 \times \frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j} \simeq 2 \times \frac{K_{ij}}{4\pi}\frac{(2\pi)^2}{N_iN_j} + 2\pi$. Therefore, $K_{ii} \simeq K_{ii} + 2(N_i)^2$ and $K_{ij} \simeq K_{ij} + N_iN_j$, $i<j$. The period for K_{ij} $(i>j)$ is the same as that of $i<j$.

In conclusion, the matrix elements of the symmetric K matrix are simultaneously constrained by the following conditions (TO stands for topological order):

$$\frac{K_{ij}}{N_i} \in \mathbb{Z}, \frac{K_{ij}}{N_j} \in \mathbb{Z}\ (\forall i,j), \tag{3}$$

$$K_{ij} \in \mathbb{Z}\ (i \neq j), \tag{4}$$

$$K_{ii} \simeq K_{ii} + 2(N_i)^2, \tag{5}$$

$$K_{ij} \simeq K_{ij} + N_iN_j\ (i \neq j), \tag{6}$$

$$\text{For a bosonic TO: } K_{ii} \in 2\mathbb{Z}, \tag{7}$$

$$\text{For a fermionic TO: at least one of } K'_{ii}\text{s is odd}. \tag{8}$$

C. Wilson operators and (partial) confinement of gauge group

A particle excitation carrying e_i units of $\mathbb{Z}_{N_i}$ gauge charges can be labeled by a particle vector $\mathbf{l} = (e_1, e_2, \cdots, e_n)^T$ with $e_i \in \mathbb{Z}_{N_i}$, whose Wilson operator is

$$W(\mathbf{l},\gamma) = \exp\left(\int_\gamma \mathrm{i}\sum_{i=1}^n e_iA^i + \sum_{i,j=1}^n \frac{\mathrm{i}e_iK_{ij}}{N_i}\int_{\Sigma_i^j} B^j\right), \tag{9}$$

where Σ_i^j's are Seifert surfaces of γ. The physical picture of Eq. (9) is a particle excitation being attached by flux strings, see Fig. 2(a). The amounts and species of fluxes are controlled by K_{ij}, elements of the K matrix. Seifert surfaces Σ_i^j with different i, j correspond to the world sheets swap by different flux strings. Due to the tension on strings, a particle excitation may be confined. Only those attached by 2π fluxes are deconfined, i.e., $\frac{e_iK_{ij}}{N_i}\int_{\Sigma_i^j} B^j = \frac{e_iK_{ij}}{N_i}\frac{2\pi n_j}{N_j} \in 2\pi\mathbb{Z}$, where n_j is an integer. If a particle excitation labeled by $\mathbf{l}$ is deconfined, it is required that $\frac{e_iK_{ij}}{N_iN_j} \in \mathbb{Z}, \forall i,j \in \{1,\cdots,n\}$. For example, the constraints on e_1 are $\frac{e_1K_{11}}{N_1N_1} \in \mathbb{Z}$, $\frac{e_1K_{12}}{N_1N_2} \in \mathbb{Z}, \cdots, \frac{e_1K_{1n}}{N_1N_n} \in \mathbb{Z}$, which demands $e_1 \in \frac{N_1}{\gcd(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},\cdots,\frac{K_{1n}}{N_n},N_1)}\mathbb{Z}$, where $\gcd(a,b,\cdots)$ is the greatest common divisor of $a, b, \cdots$. In other words, for a deconfined particle excitation carrying $\mathbb{Z}_{N_1}$ gauge charges, the minimal nonzero amount of $\mathbb{Z}_{N_1}$ gauge charges is $e_{1\min} = \frac{N_1}{\gcd(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},\cdots,\frac{K_{1n}}{N_n},N_1)}$. Since e_1 is equivalent to $e_1 + N_1$, the number of nonequivalent values of e_1 is $\gcd(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},\cdots,\frac{K_{1n}}{N_n},N_1)$, i.e., e_1 is labeled by $\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},\cdots,\frac{K_{1n}}{N_n},N_1)}$. This derivation can be applied to any i. As a result, to make the particle labeled by $\mathbf{l}$ deconfined, all e'_is need to satisfy

$$e_{i\min} = \frac{N_i}{\gcd\left(\frac{K_{i1}}{N_1},\frac{K_{i2}}{N_2},\cdots,\frac{K_{in}}{N_n},N_i\right)}. \tag{10}$$

Any particle excitation carrying e_i units of $\mathbb{Z}_{N_i}$ gauge charges with $e_i \notin e_{i\min}\mathbb{Z}$ is confined. To illustrate, an example is shown in Fig. 1, where we consider a BF theory with a single component BB term with $\mathbb{Z}_{N_1} = \mathbb{Z}_{12}$ and $\frac{K_{11}}{N_1} = 8$.

The confinement on $\mathbb{Z}_{N_i}$ gauge charge also alters the period of $\mathbb{Z}_{N_i}$ gauge fluxes. Since the $\mathbb{Z}_{N_i}$ gauge fluxes carried by a loop excitation can be detected by braiding a particle excitation around this loop excitation, we can consider the following particle-loop braiding phase:

$$\Theta_{\mathrm{PL}}(e_{i\min}, m_i) = \exp\left[-\frac{\mathrm{i}2\pi e_{i\min}m_i}{N_i}\right] = \exp\left[-\frac{\mathrm{i}2\pi m_i}{\gcd\left(\frac{K_{11}}{N_1},\frac{K_{12}}{N_2},\cdots,\frac{K_{1n}}{N_n},N_1\right)}\right]. \tag{11}$$

One can see that

$$m_i \simeq m_i + \gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, \cdots, \frac{K_{1n}}{N_n}, N_1\right) \tag{12}$$

in the sense that $\Theta_{\text{PL}}(e_{i\,\text{min}}, m_i)$ differs by an integral multiple of 2π. An illustration for the smaller period of m_i is presented in Fig. 1 where a BF theory with a single component BB term with $\mathbb{Z}_{N_1} = \mathbb{Z}_{12}$ and $\frac{K_{11}}{N_1} = 8$ is considered. In conclusion, the number of deconfined particle excitations and deconfined loop excitations are equivalent, satisfying the general belief of remote detectability of topological excitations in anomaly-free topological orders [50].

D. Self-statistics from the expectation values of framed Wilson operators

Next, we study self-statistics (i.e., exchange statistics) of particle excitations in 3D topological order which turns out to be controlled by the coefficients of the BB term. Furthermore, the expression of self-statistics shares a similar form of that of $(2+1)$D Chern-Simons theory.

In the following, we apply the standard methodology in TQFTs to determine self-statistics of particles: computing the expectation values of framed Wilson operators,

$$\langle W(\mathbf{l},\gamma)\rangle = \left\langle \exp\left(\int_\gamma \mathrm{i}\sum_{i=1}^n e_i A^i + \sum_{i,j=1}^n \frac{\mathrm{i} e_i K_{ij}}{N_i}\int_{\Sigma_i^j} B^j\right)\right\rangle,$$

where $\langle \mathcal{O}\rangle$ is defined as: $\langle \mathcal{O}\rangle = \mathcal{Z}^{-1}\int \mathcal{D}A^i\mathcal{D}B^i \exp(\mathrm{i}S)\mathcal{O}$. The partition function $\mathcal{Z} = \int \mathcal{D}A^i\mathcal{D}B^i \exp(\mathrm{i}S)$ with the action given by Eq. (1).

For this purpose, we integrate out A^i, which results in $B^i = -\frac{2\pi e_i}{N_i}\delta^\perp(\Sigma_i)$ with $\partial\Sigma_i = \gamma$. $\delta^\perp(\Sigma_i)$ is a delta distribution supported on Σ_i, which is two-form valued since B^i is a two-form. Plugging this solution back to the path integral, we have

$$\begin{aligned}\langle W(\mathbf{l},\gamma)\rangle = &\exp\left[\mathrm{i}\sum_{i,j=1}^n \frac{K_{ij}}{4\pi}\frac{2\pi e_i}{N_i}\frac{2\pi e_j}{N_j}\#(\Sigma_i\cap\Sigma_j)\right]\\ &\times \exp\left[-\mathrm{i}\sum_{i,j=1}^n \frac{\mathrm{i} e_i K_{ij}}{N_i}\frac{2\pi e_j}{N_j}\cdot\#\big(\Sigma_i^j\cap\Sigma_j\big)\right],\end{aligned} \tag{13}$$

where $\#(\Sigma_i^j\cap\Sigma_j)$ is the intersection number of two Seifert surfaces Σ_i^j and Σ_j. It equals 1 if γ has a nontrivial framing, see Fig. 2(b). Using $\#((\Sigma_i-\Sigma_i^j)\cap(\Sigma_j-\Sigma_j^i)) = 0$, which is because two closed manifolds (i.e., the difference of two Seifert surfaces) in S^4 has zero intersection number, one has

$$\begin{aligned}&\#(\Sigma_i\cap\Sigma_j) - \#\big(\Sigma_i^j\cap\Sigma_j\big) - \#\big(\Sigma_i\cap\Sigma_j^i\big)\\ &= -\#\big(\Sigma_i^j\cap\Sigma_j^i\big).\end{aligned} \tag{14}$$

Therefore, we have

$$\langle W(\mathbf{l},\gamma)\rangle = \exp\left[-\mathrm{i}\sum_{i,j=1}^n \frac{\pi K_{ij}e_ie_j}{N_iN_j}\#\big(\Sigma_i^j\cap\Sigma_j^i\big)\right], \tag{15}$$

$\#(\Sigma_i^j\cap\Sigma_i^j) = 1$ if a nontrivial framing is introduced. The self-statistics of a particle with e_i charge is $\exp(-\mathrm{i}\frac{\pi K_{ii}e_ie_i}{N_iN_i})$. Off-diagonal terms $\exp(-\mathrm{i}\frac{2\pi K_{ij}e_ie_j}{N_iN_j})$ with $i\neq j$ is the mutual statistics of two particle excitations with e_i units of $\mathbb{Z}_{N_i}$ gauge charges and e_j units of $\mathbb{Z}_{N_j}$ gauge charges. Remember that for a deconfined particle excitation, $e_i \in e_{i\,\text{min}}\mathbb{Z} \mod N_i$, such e_i's guarantee that the self-statistics of a particle is ± 1 and the mutual statistics of two particles are always trivial. In conclusion, for a particle labeled by $\mathbf{l} = (e_1, e_2, \cdots, e_n)^T$, the self- (exchange) statistics is given by

$$\Theta_{\mathbf{l}} = \exp\left(-\mathrm{i}\sum_{i,j=1}^n \frac{\pi K_{ij}e_ie_j}{N_iN_j}\right) = \exp(-\mathrm{i}\pi\mathbf{l}^T\widetilde{K}\mathbf{l}), \tag{16}$$

where $(\widetilde{K})_{ij} = \frac{K_{ij}}{N_iN_j}$. One can recognize that this result is similar to the self-statistics of particles in $(2+1)$D Chern-Simons theory. In the Chern-Simons theory with a K_{CS} matrix, the self-statistics of a particle labeled by a vector $\mathbf{l}^T$ is characterized by

$$\Theta_{\mathbf{l}}^{\text{CS}} = \exp(\mathrm{i}\pi\mathbf{l}^T(K_{\text{CS}})^{-1}\mathbf{l}). \tag{17}$$

In addition, when all diagonal elements of K are *even*, the trivial particle excitation $[\mathbf{l} = (0 \mod N_1, \cdots, 0 \mod N_n)^T]$ is bosonic. When at least one diagonal element is *odd*, this theory admits fermionic trivial particle excitation. This result is similar to that in the K_{CS} Chern-Simons theory. The coefficient matrix K of the BB term plays a similar role as that of the Chern-Simons theory.

So far, we have seen how a K-matrix BB term dramatically changes the number of deconfined operators and exchange statistics of a $\prod_{i=1}^n \mathbb{Z}_{N_i}$ gauge theory. The exchange and mutual statistics can be better explained by the examples of the BF theory with a single- (two-) component BB term. In the single component case, the action is

$$S = \int \frac{N_1}{2\pi}B^1dA^1 + \frac{K_{11}}{4\pi}B^1B^1 \tag{18}$$

and the Wilson operator of a particle excitation carrying e_1 units of $\mathbb{Z}_{N_1}$ gauge charges is

$$W(e_1,\gamma) = \exp\left(\mathrm{i}e_1\int_{\gamma_1} A^1 + \frac{\mathrm{i}e_1K_{11}}{N_1}\int_{\Sigma_1} B^1\right), \tag{19}$$

which describe a particle excitation with one attached flux string. For a deconfined particle excitation, it is required that

$$e_{1\,\text{min}} = \frac{N_1}{\gcd\left(\frac{K_{11}}{N_1}, N_1\right)}. \tag{20}$$

To calculate self-statistics of this particle excitation, we can make use of spin-statistics theorem, see Fig. 2(b). Its expectation value is

$$\langle W(e_1,\gamma)\rangle = \exp\left[-\frac{\mathrm{i}\pi K_{11}e_1e_1}{N_1N_1}\cdot\#(\Sigma_1\cap\Sigma_1)\right]. \tag{21}$$

The value of $\#(\Sigma_1 \cap \Sigma_1)$ depends on whether the framing of γ is nontrivial or not. A framing of γ can be understood as assigning a vector on each point along γ. Actually, we are now considering a particle attached with a flux string. In regularization, the charge and the endpoint of flux string (i.e., monopole) cannot be placed on the same lattice site, i.e., the charge-monopole composite is not isotropic. It is necessary to use a vector to indicate the shape of the composite. Such vectors along the world line of a particle constitute the framing. In $(2+1)$D, there are different ways to equip a vector to each point along the world line. The number of ways to equip is $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ that counts nonequivalent mappings from S_1 (the world line) to SO(2) (2D rotation of vector on each point). $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$ means that in $(2+1)$D there can be anyonic statistics. In $(3+1)$D, the 3D rotation of vector on each point is captured by SO(3) and $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$ means that there are only two kinds of statistics in $(3+1)$D.

For a nontrivial framing of γ, $\#(\Sigma_1 \cap \Sigma_1) = 1$, which also indicates a 2π-rotation of this particle excitation that induces a phase

$$\Theta_{e_1} = \exp\left(-\frac{\mathrm{i}\pi K_{11} e_1 e_1}{N_1 N_1}\right). \tag{22}$$

According to spin-statistics theorem, Θ_{e_1} is the self-statistics of particle excitation with e_1 units of gauge charge. Note that $e_1 \in \frac{N_1}{\gcd(\frac{K_{11}}{N_1}, N_1)}\mathbb{Z}$, and we find $\Theta_{e_{1\,\min}} = \exp[-\frac{\mathrm{i}\pi \operatorname{lcm}(\frac{K_{11}}{N_1}, N_1)}{\gcd(\frac{K_{11}}{N_1}, N_1)}] = \pm 1$ corresponding to bosonic or fermionic statistics.

For a trivial particle excitation, i.e., that with $e_1 = 0 \mod N_1$, its self-statistics is given by

$$\Theta_{\text{trivial}} = \exp(-\mathrm{i}\pi K_{11}). \tag{23}$$

When K_{11} is odd, $\Theta_{\text{trivial}} = -1$, meaning the trivial particle excitation is fermionic, which tells us the theory Eq. (18) is a fermionic theory. Notice that an odd K_{11} can only happen when the theory is defined on a spin manifold. When K_{11} is even, $\Theta_{\text{trivial}} = 1$ indicating that the trivial particle excitation is a boson, i.e., the theory (18) is a bosonic one.

For nontrivial particle excitations, i.e., those with $e_1 \neq 0 \mod N_1$, their self-statistics depends on the values of N_1, $\frac{K_{11}}{N_1}$, and n. Among all possible combinations, it is possible that some particle excitations with $e_1 \neq 0 \mod N_1$ are fermionic while the trivial one is bosonic. We call such particle excitations *emergent fermions* in the sense that they exhibit fermionic statistics in a bosonic theory. Below we summary the properties of theory Eq. (18) for different N_1 and K_{11} in Table I.

The second example is a two-component BB term with the action:

$$S = \int \sum_{i=1}^{2} \frac{N_i}{2\pi} B^i dA^i + \sum_{i,j=1}^{2} \frac{K_{ij}}{4\pi} B^i B^j. \tag{24}$$

Considering a particle excitation carrying two types of gauge charges, denoted by $\mathbf{l} = (e_1, e_2)^T$, its exchange statistics is given by

$$\begin{aligned}\langle W(\mathbf{l}, \gamma)\rangle &= \frac{1}{\mathcal{Z}} \int D[A^i, B^i] \exp(\mathrm{i}S) \\ &\quad \times \exp\left(\int_\gamma \mathrm{i} \sum_{i=1}^{2} e_i A^i + \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\mathrm{i} e_i K_{ij}}{N_i} \int_{\Sigma_i^j} B^j\right) \\ &= \exp\left[-\mathrm{i} \sum_{i,j=1}^{2} \frac{\pi K_{ij} e_i e_j}{N_i N_j} \#(\Sigma_i^j \cap \Sigma_j^i)\right]. \end{aligned} \tag{25}$$

By choosing a nontrivial framing of γ, i.e., $\#(\Sigma_i^j \cap \Sigma_j^i) = 1$, we find the self-statistics of a particle excitation labeled by $\mathbf{l} = (e_1, e_2)^T$ is

$$\Theta_{\mathbf{l}} = \exp\left(-\frac{\mathrm{i}\pi K_{11} e_1 e_1}{N_1 N_1} - \frac{\mathrm{i}\pi K_{22} e_2 e_2}{N_2 N_2} - \frac{\mathrm{i}2\pi K_{12} e_1 e_2}{N_1 N_2}\right), \tag{26}$$

where $\exp(-\frac{\mathrm{i}2\pi K_{12} e_1 e_2}{N_1 N_2})$ is the mutual statistics of $\mathbb{Z}_{N_1}$ charges and $\mathbb{Z}_{N_2}$ charges. Keep in mind that for a deconfined particle excitation, $e_i \in q_{i\,\min}\mathbb{Z} \mod N_i$, where $e_{i\,\min} = \frac{N_i}{\gcd(\frac{K_{i1}}{N_1}, \frac{K_{i2}}{N_2}, N_i)}$. Such e_i's guarantee that the self-statistics of a particle is ± 1 and the mutual statistics of two particles are always trivial. To see this, we consider

$$\Theta_{e_{1\,\min}, e_{2\,\min}} = \exp\left(-\frac{\mathrm{i}2\pi K_{12} e_{1\,\min} e_{2\,\min}}{N_1 N_2}\right). \tag{27}$$

We can see that

$$\begin{aligned}\frac{K_{12} e_{1\,\min} e_{2\,\min}}{N_1 N_2} &= \frac{K_{12}}{N_1 N_2} \frac{N_1}{\gcd\left(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, N_1\right)} \frac{N_2}{\gcd\left(\frac{K_{21}}{N_1}, \frac{K_{22}}{N_2}, N_2\right)} \\ &\in \frac{K_{12}}{N_1 N_2} \frac{N_1}{\gcd\left(\frac{K_{12}}{N_2}, N_1\right)} \frac{N_2}{\gcd\left(\frac{K_{21}}{N_1}, N_2\right)} \mathbb{Z}, \end{aligned} \tag{28}$$

since $\gcd(\frac{K_{12}}{N_2}, N_1) \in \gcd(\frac{K_{11}}{N_1}, \frac{K_{12}}{N_2}, N_1)\mathbb{Z}$ and $\gcd(\frac{K_{21}}{N_1}, N_2) \in \gcd(\frac{K_{21}}{N_1}, \frac{K_{22}}{N_2}, N_2)\mathbb{Z}$. Furthermore, using

$$\begin{aligned}&\frac{K_{12}}{N_1 N_2} \frac{N_1}{\gcd\left(\frac{K_{12}}{N_2}, N_1\right)} \frac{N_2}{\gcd\left(\frac{K_{21}}{N_1}, N_2\right)} \\ &\quad = \frac{K_{12} N_1 N_2}{\gcd(K_{12}, N_1 N_2)\gcd(K_{21}, N_1 N_2)} \\ &\quad = \frac{\operatorname{lcm}(K_{12}, N_1 N_2)}{\gcd(K_{12}, N_1 N_2)}, \end{aligned} \tag{29}$$

we can see that $\Theta_{\text{mutual}}(e_{1\,\min}, e_{2\,\min}) = \exp(-\mathrm{i}2\pi\mathbb{Z}) = 1$. This is consistent with the fact that in 3D space, the mutual statistics (i.e., full braiding) of two particles is topologically trivial.

III. TQFT WITH NONTRIVIAL BRAIDING STATISTICS AND EMERGENT FERMIONS

A. Particle-loop braiding in the presence of emergent fermions ($BF + BB$)

A pure BF theory describes the particle-loop braiding. A BF term is compatible with a BB term to form a legitimate TQFT action. This simplest BF theory with a single component BB term is given by Eq. (18). Emergent fermionic particle excitations are possible provided proper values of N_1 and K_{11}. To explicitly show how the emergent fermion influences particle-loop braiding, we can consider the phase of particle-loop braiding given by

$$\Theta_{\rm PL} = \frac{1}{\mathcal{Z}}\int \mathcal{D}[A^i]\mathcal{D}[B^i]\exp{(\mathrm{i}S)} \times \exp\left(\mathrm{i}e_1\int_\gamma A^1 + \mathrm{i}\frac{e_1K_{11}}{N_1}\int_\Sigma B^1\right)\exp\left(\mathrm{i}m_1\int_\sigma B^3\right), \tag{30}$$

where γ is a closed curve with $\partial\Sigma=\gamma$, σ is a closed surface, e_1 and m_1 are the numbers of charges and fluxes carried by the particle and the loop. γ and σ can be understood as the world line and world sheet of the particle and the loop. The phase of particle-loop braiding is

$$\Theta_{\rm PL} = \exp\left[-\frac{\mathrm{i}\pi K_{11}e_1e_1}{N_1N_1}\#(\Sigma\cap\Sigma) - \frac{\mathrm{i}2\pi e_1m_1}{N_1}\#(\Sigma'\cap\sigma)\right], \tag{31}$$

where $\partial\Sigma'=\gamma$ and $\#(\Sigma'\cap\sigma)$ is the linking number of γ and σ. There are two contributions to this phase. $\exp[-\frac{\mathrm{i}2\pi e_1m_1}{N_1}\#(\Sigma'\cap\sigma)]$ is the usual particle-loop braiding phase due to the particle traveling around the loop. $\exp[-\frac{\mathrm{i}\pi K_{11}e_1e_1}{N_1N_1}\#(\Sigma\cap\Sigma)]$ is just the self-statistics of the particle excitation. As discussed in previous section, the values of e_1 and m_1 are constrained by

$$e_1 = e_{1\,\rm min}\cdot p = \frac{N_1}{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}\cdot p,\ p\in\mathbb{Z}_{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}; \tag{32}$$

$$m_1 \simeq m_1 + \gcd\left(\frac{K_{11}}{N_1},N_1\right). \tag{33}$$

Consider a particle and a loop carrying minimal gauge charge and flux, the phase contributed by a particle-loop braiding is given by

$$\Theta_{\rm PL} = \exp\left(-\frac{\mathrm{i}2\pi e_{1\rm min}m_1}{N_1}\right) = \exp\left(-\frac{\mathrm{i}2\pi m_1}{\gcd\left(\frac{K_{11}}{N_1},N_1\right)}\right), \tag{34}$$

where $m_1\in\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}$. This means that the BF theory with a nontrivial BB term only labels fewer topologically ordered phases than a pure BF theory. This is because a BB term would confine part of the topological excitations, making fewer physical observable braiding phases.

Each topological excitation e can be represented by a gauge invariant Wilson operator $\mathcal{O}_\mathsf{e}$. Using path integral, we can extract fusion rules $\mathsf{a}\otimes\mathsf{b}=\oplus_i N^{\mathsf{ab}}_{\mathsf{e}_i}\mathsf{e}_i$ from [38]

$$\begin{aligned}\langle\mathsf{a}\otimes\mathsf{b}\rangle =& \frac{1}{\mathcal{Z}}\int\mathcal{D}[A^i,B^i]\exp{(\mathrm{i}S)}\times(\mathcal{O}_\mathsf{a}\times\mathcal{O}_\mathsf{b})\\ =& \frac{1}{\mathcal{Z}}\int\mathcal{D}[A^i,B^i]\exp{(\mathrm{i}S)}\times\left(\sum_i N^{\mathsf{ab}}_{\mathsf{e}_i}\mathcal{O}_{\mathsf{e}_i}\right)\\ =& \langle\oplus_i N^{\mathsf{ab}}_{\mathsf{e}_i}\mathsf{e}_i\rangle.\end{aligned} \tag{35}$$

Since emergent fermion can be induced by a proper BB term and we can couple a BB term to other topological terms such that we can study whether and how emergent fermion would influence braiding statistics and fusion rules.

Consider a general topological excitation labeled by (e_1,m_1); when $m_1=0$, it is a pointlike particle excitation; when $e_1=0$, it is a pure loop excitation (a loop excitation without particle attached on it); when $e_1,m_1\neq 0$, it is a decorated loop excitation, i.e., the bound state of a particle and a pure loop. It is straightforward to see that the fusion rule of two topological excitations is given by

$$(e_1,m_1)\otimes(e_1',m_1') = (e_1+e_1',m_1+m_1'). \tag{36}$$

Since both e_1 and m_1 are labeled by $\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}$, the fusion rules can be captured by a $\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}\times\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}$ group. While for a pure BF theory $S=\int\frac{N_1}{2\pi}B^1dA^1$, its fusion rules are captured by a $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_1}$ group.

In conclusion, the coefficient of BB term K_{11} would confine, either partially or completely, the gauge group structure of $\mathbb{Z}_{N_1}BF$ theory, leaving the deconfined gauge group to be $\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}$. Only $\gcd(\frac{K_{11}}{N_1},N_1)$ of N_1 particle excitations are deconfined and the gauge fluxes are labeled by $\mathbb{Z}_{\gcd(\frac{K_{11}}{N_1},N_1)}$. The cyclic structure of the particle-loop braiding phase is described by the deconfined gauge group and so are the fusion rules. The fusion rules are still Abelian.

B. Multi-loop braiding in the presence of emergent fermions ($BF+AAdA/AAAA+BB$)

In 3D topological order, multiloop braiding includes three-loop braiding (described by an $AAdA$ topological term) and four-loop braiding (described by an $AAAA$ topological term). The corresponding simplest TQFT actions are as follows: For a three-loop braiding,

$$S_{\rm 3L} = \int\sum_{i=1}^{2}\frac{N_i}{2\pi}B^idA^i + q_1A^1A^2dA^2 \tag{37}$$

or

$$S_{\rm 3L}' = \int\sum_{i=1}^{2}\frac{N_i}{2\pi}B^idA^i + q_2A^2A^1dA^1, \tag{38}$$

with $G=\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ and a proper coefficient q_1,q_2; for a four-loop braiding:

$$S_{\rm 4L} = \int\sum_{i=1}^{4}\frac{N_i}{2\pi}B^idA^i + q_{\rm 4L}A^1A^2A^3A^4, \tag{39}$$

with $G=\prod_{i=1}^4\mathbb{Z}_{N_i}$ and a proper coefficient $q_{\rm 4L}$. Now we try to consider emergent fermion together with multiloop braiding. Based on the discussion in the previous section, we want to add a BB term to the above topological actions for three-loop braiding or four-loop braiding. However, according to the condensation picture illustrated in Sec. II A, such an attempt would not succeed. The topological action for a multiloop braiding originates from a multilayer Abel-Higgs model in which each layer describes a condensate of *boson* coupled to one-form gauge field (A^i). For example, an $A^1A^2dA^2$ topological term is derived from the interaction of two layers of boson condensation. Introducing a BB term requires that at least one layer is vortex-line condensation. Since a boson condensation is totally different from a vortex-line condensation, it is impossible for a Abel-Higgs model to describe both of them simultaneously. In other words, one of A^i and B^i must be a Lagrange multiplier and they cannot both appear in $AAdA + BB$. We may draw such a conclusion: in bosonic topological orders, multiloop braiding is not compatible with the emergent fermion, based on our condensation picture.

In principle, there seems no reason to forbid multiloop braidings in a system that supports emergent fermions. For example, multiloop braiding is studied in gauged fermionic symmetry-protected topological (fSPT) phases, see, e.g., Refs. [51,52]. Some lattice cocycle models are used to describe multiloop braiding in fermionic systems [51]. A question arises: How do we describe the coexistence of multiloop braidings and emergent fermions in continuum field theory that is believed to be capable for liquidlike phases of matter that have a well-defined thermodynamical limit? We leave this question to future exploration. Here, we come up with some hints: While Wilson operators for fermions should be regularized by framing, loop excitations that correspond to emergent fermions in a given gauge subgroup $\mathbb{Z}_{N_i}$ may also require a regularization of some sort.

C. Borromean rings braiding in the presence of emergent fermions ($BF + AAB + BB$)

A Borromean rings braiding is described by an AAB topological term [21]. A system is equipped with Borromean rings topological order if it supports a Borromean rings braiding. A Borromean ring topological order is featured by non-Abelian fusion rules and loop shrinking rules [38]. Unlike the $AAdA$ term, there is a chance that an AAB term can be compatible with the BB term, so we can consider the following TQFT action:

$$S = \int \sum_{i=1}^{3} \frac{N_i}{2\pi} B^i dA^i + qA^1A^2B^3 + \frac{K_{33}}{4\pi} B^3B^3, \tag{40}$$

where $q = \frac{pN_1N_2N_3}{N_{123}}$ with $N_{123} = \gcd(N_1, N_2, N_3)$, $p \in \mathbb{Z}_{N_{123}}$, and the gauge group is $G = \prod_{i=1}^{3} \mathbb{Z}_{N_i}$. The gauge transformations are

$$A^1 \to A^1 + d\chi^1, \tag{41}$$

$$A^2 \to A^2 + d\chi^2, \tag{42}$$

$$A^3 \to A^3 + d\chi^3 - \frac{K_{33}}{N_3} V^3 - \frac{2\pi q}{N_3}\left(\chi^1 A^2 + \frac{1}{2}\chi^1 d\chi^2\right) + \frac{2\pi q}{N_3}\left(\chi^2 A^1 + \frac{1}{2}\chi^2 d\chi^1\right), \tag{43}$$

$$B^1 \to B^1 + dV^1 - \frac{2\pi q}{N_1}(\chi^2 B^3 - A^2 V^3 + \chi^2 dV^3), \tag{44}$$

$$B^2 \to B^2 + dV^2 + \frac{2\pi q}{N_2}(\chi^1 B^3 - A^1 V^3 + \chi^1 dV^3), \tag{45}$$

$$B^3 \to B^3 + dV^3. \tag{46}$$

The compatibility of the AAB term and BB term indicates that emergent fermion is possible in Borromean rings topological order.

We use $\mathsf{P}_{e_1e_2e_3}$ to denote a particle excitation carrying e_i units of $\mathbb{Z}_{N_i}$ gauge charges and $\mathsf{L}_{m_1m_2m_3}$ to denote a pure loop excitation carrying m_i units of $\mathbb{Z}_{N_i}$ gauge fluxes ($i = 1, 2, 3$). A decorated loop (formed by attaching a particle excitation to a pure loop excitation) is denoted by $\mathsf{L}_{m_1m_2m_3}^{e_1e_2e_3}$. Considering a Borromean rings braiding involving L_{m_100}, L_{0m_20}, and P_{00e_3}, the phase is

$$\Theta_{\mathrm{BR}}(m_1, m_2, e_3) = \exp\left[-\frac{\mathrm{i}2\pi p m_1 m_2 e_3}{N_{123}} \cdot \mathrm{Tlk}\right] \times \exp\left[-\frac{\mathrm{i}\pi K_{33} e_3 e_3}{N_3 N_3} \#(\Sigma \cap \Sigma)\right], \tag{47}$$

where Tlk is the Milnor's triple linking number of the link formed by the two loops' world sheets σ_1, σ_2 and the particle's world line γ, Σ is a Seifert surface of γ. The first term is the phase of Borromean rings braiding Θ_{BR} [21] while the second term is due to the possible self-2π rotation of P_{00e_3} during the braiding process. Since the self-2π rotation of P_{00e_3} would introduce an extra phase of ± 1, depending on its own exchange statistics (spin-statistics theorem), we can just ignore it. Notice that the existence of the BB term will confine some particle excitation carrying $\mathbb{Z}_{N_3}$ gauge charges, the value of e_3 is given by

$$e_3 \in \frac{N_3}{\gcd\left(\frac{K_{33}}{N_3}, N_3\right)} \mathbb{Z}. \tag{48}$$

When $K_{33} = 0$, i.e., no BB term considered, e_3 is labeled by $\mathbb{Z}_{N_3}$ and the minimal e_3 is 1. The Borromean rings braiding phase

$$\Theta_{\mathrm{BR}}(m_1, m_2, 1) = \exp\left(-\frac{\mathrm{i}2\pi p m_1 m_2}{N_{123}}\right) \tag{49}$$

is labeled by $p \in \mathbb{Z}_{N_{123}}$. In the case of $K_{33} \neq 0$, the minimal e_3 cannot be 1 any longer since it would be confined. The minimal value of e_3 is $e_{3\,\min} = \frac{N_3}{\gcd(\frac{K_{33}}{N_3}, N_3)}$. The Borromean rings braiding phase is

$$\Theta_{\mathrm{BR}}(m_1, m_2, e_{3\,\min}) = \exp\left(-\frac{\mathrm{i}2\pi p m_1 m_2 e_{3\,\min}}{N_{123}}\right) = \exp\left(-\frac{\mathrm{i}2\pi p m_1 m_2 N_3}{N_{123}\gcd\left(\frac{K_{33}}{N_3}, N_3\right)}\right). \tag{50}$$

Since $\frac{N_3}{N_{123}} \in \mathbb{Z}$, we can see that p is identified with $p + \gcd(\frac{K_{33}}{N_3}, N_3)$. Combined with $p \in \mathbb{Z}_{N_{123}}$, we find that p is actually labeled by $\mathbb{Z}_{\gcd(N_1, N_2, N_3, \frac{K_{33}}{N_3})}$. We can see that adding a BB term to $S = \int \sum_{i=1}^{3} \frac{N_i}{2\pi} B^i dA^i + qA^1A^2B^3$ may reduce the number of different Borromean rings braiding phases. This result is reasonable since some particle excitations are confined by the BB term and hence cannot contribute to an observable Borromean rings braiding phases.

First, let us find Wilson operators for those topological excitation carrying only *one* kind of gauge charge or flux for the action

$$S=\int\sum_{i=1}^{3}\frac{N_i}{2\pi}B^idA^i+qA^1A^2B^3+\frac{K_{33}}{4\pi}B^3B^3, \tag{51}$$

with $G=\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\mathbb{Z}_{N_3}$. The particle excitations carrying $\mathbb{Z}_{N_1}$ or $\mathbb{Z}_{N_2}$ gauge charges are represented by gauge invariant Wilson operators

$$\mathsf{P}_{e_100}=\mathcal{N}^{e_100}\exp\left(\mathrm{i}e_1\int_\gamma A^1\right), \tag{52}$$

$$\mathsf{P}_{0e_20}=\mathcal{N}^{0e_20}\exp\left(\mathrm{i}e_2\int_\gamma A^2\right), \tag{53}$$

and the pure loop excitations carrying $\mathbb{Z}_{N_3}$ gauge fluxes are represented by

$$\mathsf{L}_{00m_3}=\mathcal{N}_{00m_3}\exp\left(\mathrm{i}m_3\int_\sigma B^3\right), \tag{54}$$

where the factor $\mathcal{N}$'s are to be determined. The operator for pure loop excitation carrying $\mathbb{Z}_{N_1}$ gauge fluxes is

$$\begin{aligned}\mathsf{L}_{m_100}=&\,\mathcal{N}_{m_100}\\&\times\exp\left[\mathrm{i}m_1\int_\sigma B^1+\frac{1}{2}\frac{2\pi q}{N_1}(d^{-1}A^2B^3+d^{-1}B^3A^2)\right]\\&\times\delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right),\end{aligned} \tag{55}$$

where the Kronecker delta function are $\delta(\int_\gamma A^2)=\begin{cases}1, & \int_\gamma A^2=0 \mod 2\pi\\ 0, & \text{else}\end{cases}$ and $\delta(\int_\sigma B^3)=\begin{cases}1, & \int_\sigma B^3=0 \mod 2\pi\\ 0, & \text{else}\end{cases}$ to ensure $d^{-1}A^2$ and $d^{-1}B^3$ are well-defined [17,38,53]. Similarly, the operator for pure loop carrying $\mathbb{Z}_{N_2}$ gauge fluxes is

$$\begin{aligned}\mathsf{L}_{0m_20}=&\,\mathcal{N}_{0m_20}\\&\times\exp\left[\mathrm{i}m_2\int_\sigma B^2-\frac{1}{2}\frac{2\pi q}{N_2}(d^{-1}B^3A^1+d^{-1}A^1B^3)\right]\\&\times\delta\left(\int_\gamma A^1\right)\delta\left(\int_\sigma B^3\right).\end{aligned} \tag{56}$$

The particle excitation carrying $\mathbb{Z}_{N_3}$ gauge charges is represented by

$$\begin{aligned}\mathsf{P}_{00e_3}=&\,\mathcal{N}^{00e_3}\exp\left[\mathrm{i}e_3\int_\gamma A^3+\frac{1}{2}\frac{2\pi q}{N_3}(d^{-1}A^1A^2-d^{-1}A^2A^1)\right.\\&\left.+\mathrm{i}\frac{e_3K_{33}}{N_3}\int_\Sigma B^3\right]\delta\left(\int_\gamma A^1\right)\delta\left(\int_\gamma A^2\right).\end{aligned} \tag{57}$$

These Kronecker delta functions can be expanded by summation of some exponentials, e.g., $\delta(\int_\gamma A^1)=\frac{1}{N_1}\sum_{k=1}^{N_1}\exp(\mathrm{i}k\int_\gamma A^1)$ [38,53]. As mentioned in the previous section, P_{00e_3} is a particle excitation attached by a flux string and may be confined due to the tension on the string. P_{00e_3} is deconfined only when the flux on the string is a multiple of 2π. The minimal e_3 for deconfined P_{00e_3} is

$$e_{3\min}=\frac{N_3}{\gcd\left(\frac{K_{33}}{N_3},N_3\right)}. \tag{58}$$

Note that the limitation of values of e_3 influences the period of $\mathbb{Z}_{N_3}$ gauge fluxes. Since a loop excitation is detected by a particle excitation, we consider the particle-loop braiding phase of $\mathsf{P}_{00e_{3\min}}$ and L_{00m_3}:

$$\begin{aligned}\Theta_{\mathrm{PL}}(e_{3\min},m_3)&=\exp\left[-\frac{2\pi e_{3\min}m_3}{N_3}\right]\\&=\exp\left[-\frac{2\pi N_3m_3}{\gcd\left(\frac{K_{33}}{N_3},N_3\right)}\right].\end{aligned} \tag{59}$$

We immediately see that m_3 is equivalent with $m_3+\gcd(\frac{K_{33}}{N_3},N_3)$. In other words, m_3 has a period of $\gcd(\frac{K_{33}}{N_3},N_3)$. This is important when we discuss the fusion rules in the following text.

So far, we have found Wilson operators for topological excitation carrying only one kind of gauge charge or flux. Other excitations with multiple species of gauge charges or fluxes, e.g., a particle excitation with different $\mathbb{Z}_{N_i}$ gauge charges, is defined by

$$\mathsf{P}_{e_100}\otimes\mathsf{P}_{0e_20}\otimes\mathsf{P}_{00e_3}\equiv\mathsf{P}_{e_1e_2e_3}, \tag{60}$$

or a decorated loop excitation with $\mathbb{Z}_{N_i}$ gauge fluxes and $\mathbb{Z}_{N_j}$ gauge charges is defined by

$$\mathsf{P}_{e_100}\otimes\mathsf{P}_{0e_20}\otimes\mathsf{P}_{00e_3}\otimes\mathsf{L}_{m_100}\otimes\mathsf{L}_{0m_20}\otimes\mathsf{L}_{00m_3}\equiv\mathsf{L}^{e_1e_2e_3}_{m_1m_2m_3}. \tag{61}$$

Next, we need to determine the factors $\mathcal{N}$ for each operator. For an illustration, we consider the loop excitation L_{100} which carries the flux of the $\mathbb{Z}_{N_1}$ gauge subgroup; its Wilson operator is

$$\begin{aligned}\mathsf{L}_{100}=&\,\mathcal{N}_{100}\exp\left[\mathrm{i}\int_\sigma B^1+\frac{1}{2}\frac{2\pi q}{N_1}(d^{-1}A^2B^3+d^{-1}B^3A^2)\right]\\&\times\delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right).\end{aligned} \tag{62}$$

Since L_{100} represents element 1 in group $\mathbb{Z}_{N_1}$, according to the $\mathbb{Z}_{N_1}$ cyclic structure, it is natural to require

$$\underbrace{\mathsf{L}_{100}\otimes\mathsf{L}_{100}\otimes\cdots\otimes\mathsf{L}_{100}}_{N_1\text{ terms}}=\mathsf{1}+\cdots, \tag{63}$$

where $\cdots$ denotes other fusion channels if this fusion is non-Abelian. Here we have made an assumption: For an excitation with only kind of charge or flux, fusing it and its antiexcitation would outputs *one* vacuum. This assumption is reasonable since a pair of particle and antiparticle, or a pair of loop and antiloop, can be created from vacuum and then be annihilated to vacuum. For those with multiple kinds of non-Abelian charges or fluxes, fusing a pair of excitation and antiexcitation may output more than one vacua [38]. In the path integral, the fusion Eq. (63) is written as

$$\begin{aligned}\langle(\mathsf{L}_{100})^{\otimes N_1}\rangle &= (\mathcal{N}_{100})^{N_1}\left\langle \exp\left[\mathrm{i}N_1\int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}(d^{-1}A^2B^3 + d^{-1}B^3A^2)\right]\delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right)\right\rangle \\ &= \frac{(\mathcal{N}_{100})^{N_1}}{N_2N_3}\left\langle 1 + \sum_{e_2=1}^{N_2-1}\exp\left(\mathrm{i}e_2\int_\gamma A^2\right) + \sum_{m_3=1}^{N_3-1}\exp\left(\mathrm{i}m_3\int_\sigma B^3\right) + \sum_{e_2=1}^{N_2-1}\sum_{m_3=1}^{N_3-1}\exp\left(\mathrm{i}e_2\int_\gamma A^2 + \mathrm{i}m_3\int_\sigma B^3\right)\right\rangle, \end{aligned} \quad (64)$$

where we have used

$$\left\langle \exp\left[\mathrm{i}N_1\int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}d^{-1}A^2B^3 + d^{-1}B^3A^2\right]\right\rangle = 1, \quad (65)$$

since B^1 is $\mathbb{Z}_{N_1}$ valued. Since the fusion coefficient of vacuum is 1, it is required that $\frac{(\mathcal{N}_{100})^{N_1}}{N_2N_3} = 1$, i.e., the factor of Wilson operator for L_{100} is $\mathcal{N}_{100} = \sqrt[N_1]{N_2N_3}$. Now we are going to show that the factor $\mathcal{N}_{100}$ is exactly equal to the quantum dimension of L_{100}. Notice that the result in Eq. (64) tells us that the output of fusing N_1 L'_{100}s is

$$\begin{aligned}(\mathsf{L}_{100})^{\otimes N_1} &= \mathbf{1} \oplus \left(\oplus_{e_2=1}^{N_2-1}\mathsf{P}_{0e_20}\right) \oplus \left(\oplus_{m_3=1}^{N_3-1}\mathsf{L}_{00m_3}\right) \\ &\quad \oplus \left(\oplus_{e_2=1}^{N_2-1}\oplus_{m_3=1}^{N_3-1}\mathsf{L}_{00m_3}^{0e_20}\right).\end{aligned} \quad (66)$$

It is easy to see that P_{0e_20} is an Abelian particle excitation whose quantum dimension is 1. This is because

$$\begin{aligned}\langle(\mathsf{P}_{010})^{\otimes N_2}\rangle &= \left\langle(\mathcal{N}^{010})^{N_2}\exp\left(\mathrm{i}N_2\int_\gamma A^2\right)\right\rangle \\ &= \langle(\mathcal{N}^{010})^{N_2}\cdot 1\rangle \\ &= (\mathcal{N}^{010})^{N_2}\cdot \mathbf{1},\end{aligned} \quad (67)$$

where $\mathbf{1}$ denotes the vacuum. Our assumption above requires $(\mathcal{N}^{010})^{N_2} = 1$, thus $\mathcal{N}^{0e_20} = 1, \forall e_2 \in \mathbb{Z}_{N_2}$. Similarly, we know that L'_{00m_3}s and $\mathsf{L}_{00m_3}^{0e_20}$'s are all Abelian excitations. For a fusion rule

$$\mathsf{e}_i \otimes \mathsf{e}_k = \oplus_m N_m^{ik}\mathsf{e}_m,$$

where the quantum dimension of e_i is denoted as d_i, there is a relation of these quantum dimensions (the proof can be found in the Appendix):

$$d_i d_k = \sum_m N_m^{ik} d_m. \quad (68)$$

Let the quantum dimension of L_{100} be d_{100}. Applying Eq. (68) to fusion rule (66), we have

$$(d_{100})^{N_1} = \sum_m N_m^{ik} d_m = N_2N_3, \quad (69)$$

thus the quantum dimension of L_{100} is $d_{100} = \sqrt[N_1]{N_2N_3}$. We can see that the quantum dimension of excitation L_{100} is just the factor of its Wilson operator.

Let us go through this line of thinking again: First, we write the Wilson operator of L_{100} with an unknown factor $\mathcal{N}_{100}$. At this time, we do not know any fusion rules of L_{100} yet. By demanding $\mathsf{L}_{100} \otimes \mathsf{L}_{(N_1-1)00} = \mathbf{1} + \cdots$ from the $\mathbb{Z}_{N_i}$ cyclic structure, we obtain $\mathcal{N}_{100} = \sqrt[N_1]{N_2N_3}$. Meanwhile, by expanding the Kronecker delta functions, we obtain the fusion ruleEq. (66), which tells us the channels are all Abelian excitations. Since the quantum dimension of Abelian excitation is 1, applying Eq. (68) we find the quantum dimension of L_{100} is $d_{100} = \sqrt[N_1]{N_2N_3}$, the same as its Wilson operator's factor. So far, we have seen that for topological excitation carrying only one species of charge or flux, its quantum dimension is the same as the factor of its Wilson operator. For topological excitation carrying charges or fluxes from different $\mathbb{Z}_{N_i}$ subgroups, it is defined by fusion of those with only one kind of charge or flux, see Eqs. (60) and (61). Their quantum dimension can be obtained by Eq. (68) and the factor of their Wilson operator can obtained by the path integral calculation according to Eqs. (60) and (61).

We are ready to discuss how the fusion rules of action Eq. (40) are affected by the BB term. We take an example of $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ and $\frac{K_{33}}{N_3} = 2$. We will compare the two situations of $K_{33} = 0$ and $\frac{K_{33}}{N_3} = 2$. The fusion rules of action Eq. (40) without the BB term in the case of $G = (\mathbb{Z}_2)^3$ are studied in Ref. [38].

We first take a look at the particle excitation P_{00e_3}:

$$\begin{aligned}\mathsf{P}_{00e_3} = \mathcal{N}^{00e_3}\exp\Bigg[&\mathrm{i}e_3\int_\gamma A^3 + \frac{1}{2}\frac{2\pi q}{N_3}(d^{-1}A^1A^2 - d^{-1}A^2A^1) \\ &+ \mathrm{i}\frac{e_3K_{33}}{N_3}\int_\Sigma B^3\Bigg]\delta\left(\int_\gamma A^1\right)\delta\left(\int_\gamma A^2\right).\end{aligned} \quad (70)$$

As shown in the previous discussion, turning on the $\frac{K_{33}}{4\pi}B^3B^3$ term in action Eq. (40) would narrow the choices of e'_3s. When $K_{33} = 0$, e_3 takes values from $\mathbb{Z}_{N_3} = \mathbb{Z}_6$. When $\frac{K_{33}}{N_3} = 2$, there exist a minimal value of e_3, $e_{3\,\min}$, and the charges of deconfined P_{00e_3} should satisfy $e_3 \in e_{3\,\min}\mathbb{Z}$, where

$$e_{3\,\min} = \frac{N_3}{\gcd\left(\frac{K_{33}}{N_3}, N_3\right)} = 3. \quad (71)$$

In the case of $K_{33} = 0$, the charges of P_{00e_3} are labeled by $\mathbb{Z}_6$. This $\mathbb{Z}_6$ cyclicity of e_3 indicates the following fusion rule:

$$(\mathsf{P}_{001})^{\otimes 6} = \mathbf{1} \oplus \mathsf{P}_{100} \oplus \mathsf{P}_{010} \oplus \mathsf{P}_{110}. \quad (72)$$

The quantum dimension of P_{001} is $\mathcal{N}^{001} = \sqrt[6]{1+1+1+1} = 2^{\frac{1}{3}}$. In the case of $\frac{K_{33}}{N_3} = 2$, the charges of deconfined P_{00e_3} are 3 and 6, labeled by $\mathbb{Z}_{\gcd(\frac{K_{33}}{N_3},N_3)} = \mathbb{Z}_2$. By definition, $\mathsf{P}_{00e_{3\,\min}} = (\mathsf{P}_{001})^{\otimes 3} = \mathsf{P}_{003}$ and its operators are

$$\begin{aligned}\mathsf{P}_{00e_{3\,\min}} = \mathcal{N}^{00e_{3\,\min}}\exp\Bigg[&\mathrm{i}e_{3\,\min}\int_\gamma A^3 \\ &+ \mathrm{i}e_{3\,\min}\int_\gamma \frac{1}{2}\frac{2\pi q}{N_3}(d^{-1}A^1A^2 - d^{-1}A^2A^1) \\ &+ \mathrm{i}\frac{e_{3\,\min}K_{33}}{N_3}\int_\Sigma B^3\Bigg]\delta\left(\int_\gamma A^1\right)\delta\left(\int_\gamma A^2\right).\end{aligned} \quad (73)$$

Since P_{00e_3} is labeled by $\mathbb{Z}_2$ when $\frac{K_{33}}{N_3}=2$, from $\langle \mathsf{P}_{00e_{3\min}} \otimes \mathsf{P}_{00e_{3\min}}\rangle$ and requiring the coefficient of vacuum to be 1, we have

$$\mathsf{P}_{00e_{3\min}} \otimes \mathsf{P}_{00e_{3\min}} = \mathsf{1} \oplus \mathsf{P}_{100} \oplus \mathsf{P}_{010} \oplus \mathsf{P}_{110}. \quad (74)$$

Compared to the case of $K_{33}=0$, this is just the fusion rule of two P'_{003}s. Through this example, we see that one of the effects of the *BB* term is to confine some particle excitations, i.e., P_{00e_3} with $e_3 \neq 3\mathbb{Z}$. However, the fusion rules of deconfined particle excitations are unchanged. This result can be understood as that the flux attachment due to *BB* term does not change the particle excitation's internal degrees of freedom that correspond to fusion.

Next, we focus on the loop excitation L_{00m_3}. As aforementioned, the *BB* term makes m_3 have a smaller period than N_3: In the case of $\frac{K_{33}}{N_3}=2$, m_3 is equivalent to m_3+2. In other words, m_3 is labeled by $\mathbb{Z}_2$: for $m_3\in\{0,2,4\}$, L_{00m_3} is equivalent to the vacuum $\mathsf{1}$; for $m_3\in\{1,3,5\}$, L_{00m_3} is equivalent to L_{001}. The corresponding fusion rules are

$$\underbrace{\mathsf{L}_{001}\otimes\mathsf{L}_{001}\otimes\cdots\otimes\mathsf{L}_{001}}_{N_3=6\text{ terms}} = \mathsf{1}, K_{33}=0, \quad (75)$$

$$\mathsf{L}_{001}\otimes\mathsf{L}_{001} = \mathsf{L}_{002}, K_{33}=0, \quad (76)$$

$$\mathsf{L}_{001}\otimes\mathsf{L}_{001} = \mathsf{1}, \frac{K_{33}}{N_3}=2. \quad (77)$$

The last example to show is the non-Abelian loop excitation L_{100}:

$$\mathsf{L}_{100} = \mathcal{N}_{100}\exp\left[\mathrm{i}\int_\sigma B^1 + \frac{1}{2}\frac{2\pi q}{N_1}(d^{-1}A^2B^3 + d^{-1}B^3A^2)\right] \times \delta\left(\int_\gamma A^2\right)\delta\left(\int_\sigma B^3\right). \quad (78)$$

When $K_{33}=0$, these two delta functions can be expanded as

$$\delta\left(\int_\gamma A^2\right) = \frac{1}{2}\left[1+\exp\left(\mathrm{i}\int_\gamma A^2\right)\right], \quad (79)$$

$$\delta\left(\int_\sigma B^3\right) = \frac{1}{6}\sum_{m_3=1}^{6}\exp\left(\mathrm{i}m_3\int_\sigma B^3\right). \quad (80)$$

We can calculate the factor $\mathcal{N}_{100}$ from $\langle \mathsf{L}_{100}\otimes\mathsf{L}_{100}\rangle$: $\mathcal{N}_{100}=\sqrt{2\times 6}=2\sqrt{3}$. As shown in the above discussion, $\mathcal{N}_{100}$ is also the quantum dimension of L_{100}. By setting $\frac{K_{33}}{N_3}=2$, we turn on the *BB* term. Due to the period of m_3, $m_3 \simeq m_3 + \gcd(\frac{K_{33}}{N_3}, N_3)$, the expansion of $\delta(\int_\sigma B^3)$ actually becomes (in the sense of correlation with other operators)

$$\delta\left(\int_\sigma B^3\right) = \frac{1}{2}\left[1+\exp\left(\mathrm{i}\int_\sigma B^3\right)\right]. \quad (81)$$

The factor $\mathcal{N}_{100}$ as well as the quantum dimension of L_{100} then becomes $\mathcal{N}_{100}=\sqrt{2\times 2}=2$.

In summary, the influences of the *BB* term on fusion rules are as follows. First, the *BB* term would confine part of the particle excitations. This, in turn, makes some loop excitations that used to be distinguishable now become equivalent in the sense of correlation with other excitations. As in the above example, L_{00m_3} used to be labeled by $\mathbb{Z}_6$ but is now labeled by $\mathbb{Z}_2$ due to the *BB* term. Consequently, other topological excitations' quantum dimensions are changed. In the above example, the output of fusion two L'_{100}s used to be

$$\mathsf{L}_{100}\otimes\mathsf{L}_{100} = 1 \oplus \mathsf{P}_{010} \oplus \left(\oplus_{m_3=1}^{6}\mathsf{L}_{00m_3}\right) \oplus \left(\oplus_{m_3=1}^{6}\mathsf{L}_{00m_3}^{010}\right), \quad (82)$$

but due to the *BB* term becomes

$$\mathsf{L}_{100}\otimes\mathsf{L}_{100} = 1 \oplus \mathsf{P}_{010} \oplus \mathsf{L}_{001} \oplus \mathsf{L}_{001}^{010}. \quad (83)$$

IV. CONCLUSION AND OUTLOOK

In this paper, we constructed the topological *BF* field theory in the presence of both twisted terms (e.g., *AAdA* and *AAB*) and a *K*-matrix *BB* term. In this TQFT, we are allowed to simultaneously explore the self-statistics of particles, particle-loop braiding, multiloop braiding, Borromean rings braiding, shrinking rules, and fusion rules to reach a more complete continuum-field-theoretical description of anomaly-free 3D topological orders. We carefully explored the effect of *K*-matrix *BB* term in two aspects: (i) self-statistics transmutation and (ii) confinement of excitations. Specially, we illustrated how a general *BB* term with a coefficient matrix *K* alternates the self-statistics of deconfined particle excitations through computing *framed* Wilson loops. We found that the self-statistics of a particle excitation labeled by $\mathbf{l}=(e_1,e_2,\cdots,e_n)^T$ is given by $\Theta_{\mathbf{l}}=\exp(-\mathrm{i}\pi\mathbf{l}^T\widetilde{K}\mathbf{l})$, where $\widetilde{K}_{ij}=\frac{K_{ij}}{N_iN_j}$ as shown in Eq. (16). The expression of this statistical angle is formally very similar to that of *K*-matrix Chern-Simons theory [6,7]. We also examined in what situation, respectively, trivial fermions (fermionic trivial particles) and emergent fermions (fermionic particles that carry nontrivial gauge charges) are possible and how they influence the braiding statistics and fusion rules studied in Ref. [38].

If three-loop braiding and/or BR braiding are considered, the loops are allowed to carry gauge fluxes from different gauge subgroups. We found that for those gauge subgroups whose gauge fluxes take part in three-loop braiding [27] or BR braiding [21], their gauge charges can only be carried by bosonic particle excitations. This result is obtained from the incompatibility between *AAdA* twisted term and *BB* term within our framework of continuum field theory. Physically, this can be interpreted as these two topological terms having different microscopic origins (see Sec. II A). For example, when $G=\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$ and three-loop braidings are considered, all particle excitations are bosonic, i.e., emergent fermions are forbidden. Furthermore, we take BR topological order as an example to see how the emergent fermion influences its fusion rules.

For future directions, it would be interesting to write all compatibility conditions proposed in Ref. [37] and the present paper in a more symbolical way and compare the continuum-field-theoretical analysis and the mathematics of a higher category. Due to the general belief on the bulk-boundary correspondence, we may examine the $(2+1)$D boundary theory by placing TQFTs on an open manifold to understand compatibility from boundary.

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APPENDIX: THE RELATION OF QUANTUM DIMENSIONS IN A FUSION RULE

Let the quantum dimension of e_i be d_i. Now we prove that for

$$\mathsf{e}_i \otimes \mathsf{e}_k = \oplus_m N_m^{ik} \mathsf{e}_m, \tag{A1}$$

one can know

$$d_i d_k = \sum_m N_m^{ik} d_m. \tag{A2}$$

From the associativity of fusion rules, we have

$$(\mathsf{e}_i \otimes \mathsf{e}_j) \otimes \mathsf{e}_k = \mathsf{e}_i \otimes (\mathsf{e}_j \otimes \mathsf{e}_k). \tag{A3}$$

The left-hand side can be written as $\oplus_m N_m^{ij} \mathsf{e}_m \otimes \mathsf{e}_k = \oplus_m N_m^{ij} \oplus_l N_l^{mk} \mathsf{e}_l$ and the right-hand side can be written as $\mathsf{e}_j \otimes (\oplus_m N_m^{ik} \mathsf{e}_m) = \oplus_m N_m^{ik} \oplus_l N_l^{jm} \mathsf{e}_l$. The fusion coefficients N_k^{ij}'s can form a matrix N_i with $(N_i)_{kj} = N_k^{ij}$. Therefore, we have

$$\oplus_m N_m^{ij} N_l^{mk} = \oplus_m N_m^{ik} N_l^{jm}. \tag{A4}$$

Note that $\oplus_m N_m^{ij} N_l^{mk} = \sum_m N_m^{ij} N_l^{mk} = \sum_m (N_i)_{mj} (N_k)_{lm} = (N_i N_k)_{lj}$ and $\oplus_m N_m^{ik} N_l^{im} = \sum_m N_m^{ik} N_l^{jm} = \sum_m N_m^{ik} (N_m)_{lj}$, where we have used $N_c^{ab} = N_c^{ba}$. So, we have a relation between matrices:

$$N_i N_k = \sum_m N_m^{ik} N_m. \tag{A5}$$

Since N_i's are commutative, their largest eigenvalues d_i's, i.e., quantum dimensions of corresponding topological excitations, satisfy

$$d_i d_k = \sum_m N_m^{ik} d_m. \tag{A6}$$

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