Bridging coherence optics and classical mechanics: A generic light polarization-entanglement complementary relation

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While optics and mechanics are two distinct branches of physics, they are connected. It is well known that the geometrical/ray treatment of light has direct analogies to mechanical descriptions of particle motion. However, connections between coherence wave optics and classical mechanics are rarely reported. Here we report links of the two through a systematic quantitative analysis of polarization and entanglement, two optical coherence properties under the wave description of light pioneered by Huygens and Fresnel. A generic complementary identity relation is obtained for arbitrary light fields. More surprisingly, through the barycentric coordinate system, optical polarization, entanglement, and their identity relation are shown to be quantitatively associated with the mechanical concepts of center of mass and moment of inertia via the Huygens-Steiner theorem for rigid body rotation. The obtained result bridges coherence wave optics and classical mechanics through the two theories of Huygens.

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I. INTRODUCTION

Renowned as one of the greatest scientists in history, Huygens made groundbreaking contributions to various branches of natural science, with notable achievements in the fields of optics and mechanics [1]. His remarkable insights and discoveries continue to shape our understanding of these disciplines. Huygens is considered as the starting point of systematic wave explanation of light in the 1670s [2] and the Huygens-Fresnel principle [3] was the basis for the advancement of physical optics, describing coherence phenomena of light including interference, diffraction, polarization, etc. [4], as well as the recently recognized property of vector-space entanglement [5–20] (a direct consequence of a multi-degree-of-freedom amplitude wave theory [17]). On a completely different subject, through the study of pendulum oscillation that led to his invention of the first pendulum clock, Huygens also made pivotal contributions to the development of fundamental mechanical concepts of center of mass (COM) and moment of inertia (MOI) describing rigid body motions, leading to the well-known Huygens-Steiner theorem (also called the parallel-axis theorem) [21]. Although both owe to the contributions of Huygens, almost no links of the two theories have been explored or even anticipated due to their apparent distinctions. To bridge this gap, here we provide an approach that demonstrates the interplay of the two subjects through the analysis of two optical coherence properties: *polarization* and *entanglement.*

As one of the earliest discovered fundamental features of light, polarization was only gradually better understood along with the slow recognition of the light's wave nature [4,22], and it is conventionally understood as the directional property, or degree of freedom (DOF), of light (electromagnetic) wave oscillation. Recently, it has been further shown that polarization coherence needs at least one additional DOF to be fully characterized [9,23]. This allows the discussion of its connection to another two-DOF property, i.e., entanglement [24]. Here we carry out a systematic analysis of both polarization ($\mathcal{P}$) and entanglement ($\mathcal{K}$) for a generic light field and obtain a universal complementary relation $\mathcal{P}^2 + \mathcal{K}^2 = 1$ regardless of the dimensionality.

On the other hand, attempts of geometric understanding of entanglement have been made in various contexts [25–31]. With a geometric mapping of optical coherence parameters to point masses, via the barycentric coordinate system [32], we further establish a surprising quantitative relation between the obtained generic optical polarization-entanglement complementary identity and the rigid body Huygens-Steiner theorem through the specific mechanical concepts of COM and MOI. Our method and results open a systematic avenue for understanding quantitative and conceptual connections between coherence optics and mechanics.

II. POLARIZATION-ENTANGLEMENT COMPLEMENTARY RELATION

We start with the most general form of an arbitrary light field, which can be written as

$$|E\rangle = |x\rangle|E_x\rangle + |y\rangle|E_y\rangle + |z\rangle|E_z\rangle. \tag{1}$$

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Here we have adopted the Dirac vector notation $|\ \rangle$, as in Ref. [9], solely to emphasize the fact that both polarization components $|x\rangle$, $|y\rangle$, $|z\rangle$ and amplitude components $|E_x\rangle$, $|E_y\rangle$, $|E_z\rangle$, describing the remaining degrees of freedom such as temporal mode, spatial mode, etc., are vectors in their respective vector spaces. When normalized by the total light intensity $I = \langle E_x|E_x\rangle + \langle E_y|E_y\rangle + \langle E_z|E_z\rangle$, it becomes

$$|e\rangle = \alpha|x\rangle|e_x\rangle + \beta|y\rangle|e_y\rangle + \gamma|z\rangle|e_z\rangle, \tag{2}$$

where α, β, and γ are real normalized coefficients defined as $\alpha = \sqrt{\langle E_x|E_x\rangle/I}$, $\beta = \sqrt{\langle E_y|E_y\rangle/I}$ and $\gamma = \sqrt{\langle E_z|E_z\rangle/I}$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$. The normalized amplitude vectors are defined as $|e_i\rangle = E_i/\sqrt{\langle E_i|E_i\rangle}$, with $i = x, y, z$. For the generic three-dimensional (3D) field, the cross correlation among amplitude components $|e_{x,y,z}\rangle$ can be arbitrary and, most generally, described by complex values $\delta_1 = \langle e_x|e_y\rangle$, $\delta_2 = \langle e_x|e_z\rangle$, and $\delta_3 = \langle e_y|e_z\rangle$, respectively.

The 3D polarization coherence of the light field can be characterized by the 3×3 coherence matrix [33–37], which can be decomposed into nine Gell-Mann matrices [38] and is obtained as

$$\mathcal{W}_{3D} = \begin{bmatrix} \alpha^2 & \alpha\beta\delta_1 & \alpha\gamma\delta_2 \\ \alpha\beta\delta_1^* & \beta^2 & \beta\gamma\delta_3 \\ \alpha\gamma\delta_2^* & \beta\gamma\delta_3^* & \gamma^2 \end{bmatrix}. \tag{3}$$

To have a systematic analysis for arbitrary dimensions (e.g., arbitrary 2D beam, 3D field, etc.), here we adopt the degree of 3D polarization coherence [33,39,40] as

$$\mathcal{P}_3 = \sqrt{\frac{3}{2}\left(\mathrm{Tr}\mathcal{W}_{3D}^2 - \frac{1}{3}\right)}, \tag{4}$$

which varies between 0 and 1, with 0 meaning complete unpolarization (i.e., $\langle E_x|E_x\rangle = \langle E_y|E_y\rangle = \langle E_z|E_z\rangle \neq 0$ and $\langle E_i|E_j\rangle = 0$ with $i, j = x, y, z$ and $i \neq j$) and 1 indicating fully polarized. Here, the subscript indicates the dimensionality 3. This measure is consistent with the conventional 2D definition of degree of polarization for arbitrary light beams [4]. It means how much the light field is concentrated to a single polarization direction (or vector). Mathematically, it can be reexpressed through the eigenvalues m_1, m_2, m_3 of the coherence matrix (3) as

$$\mathcal{P}_3 = \sqrt{1 - \frac{2 \times 3(m_1 m_2 + m_1 m_3 + m_2 m_3)}{3 - 1}}. \tag{5}$$

Here the normalization of (2) results in $m_1 + m_2 + m_3 = 1$. It is worthwhile to note that the definition of degree of polarization coherence $\mathcal{P}_3$ is equivalent to the normalized purity of the polarization degree of freedom reduced from a quantum state of the form (2) [41].

On the other hand, another coherence quantity, i.e., entanglement, between the polarization space $\{|x\rangle, |y\rangle, |z\rangle\}$ and the amplitude space $\{|E_x\rangle, |E_y\rangle, |E_z\rangle\}$ of the general 3D field represents a 3×3 bipartite pure-state scenario. Therefore, Schmidt analysis [42–44] can be applied with the quantitative Schmidt number measure $K = 1/\sum_{i=1}^{3} \lambda_i^2$. Here, $\sqrt{\lambda_i}$, $i = 1, 2, 3$, are the Schmidt coefficients and can be shown to coincide with the eigenvalues of the normalized polarization coherence matrix (3), i.e., $\lambda_i = m_i$, $i = 1, 2, 3$. The Schmidt number K varies between 1 and 3 for the 3D light field, i.e., $K_3 \in [1, 3]$, where $K_3 = 1$ indicates zero entanglement with only one nonzero Schmidt coefficient and $K_3 = 3$ means maximal entanglement with equal Schmidt coefficients, $m_1 = m_2 = m_3$.

To compare with the normalized 3D degree of polarization (4), entanglement K is also normalized as

$$\mathcal{K}_3 = \sqrt{\frac{3}{2}\left(1 - \frac{1}{K_3}\right)}. \tag{6}$$

Obviously, $\mathcal{K}_3$, called Schmidt weight [29], $\in [0, 1]$ with $\mathcal{K}_3 = 0, 1$ meaning minimum (zero), maximal entanglement, respectively. Some tedious but straightforward calculations show that $\mathcal{K}_3$ can be further expressed in terms of the eigenvalues,

$$\mathcal{K}_3 = \sqrt{3(m_1 m_2 + m_1 m_3 + m_2 m_3)}. \tag{7}$$

By comparing Eqs. (5) and (7), one can immediately arrive at the complementary identity relation,

$$\mathcal{P}_3^2 + \mathcal{K}_3^2 = 1. \tag{8}$$

This is our first major result. It illustrates an intrinsic complementary behavior of polarization coherence with entanglement for arbitrary light fields. It will be shown later that this result can systematically reduce to arbitrary two-dimensional light beams or generalize to any N-dimensional structural fields.

III. CENTER OF MASS

To further understand the optical polarization coherence and entanglement along with their generic complementary relation (8), we now describe a two-step geometric mapping procedure that links to mechanical concepts. We employ the barycentric coordinate system, introduced by Möblus in 1827 [32], in which the location of a point is specified by reference to a regular simplex (an equilateral triangle for points in a plane, a regular tetrahedron for points in three-dimensional space, etc.).

Step 1. Let the polarization coherence matrix eigenvalues m_1, m_2, m_3 represent the values of three point masses.

Step 2: Place these point masses at the vertices of an equilateral triangle inscribed in a unit circle O; see Fig. 1 for illustration. With such a mapping, it is then ready to analyze the connection to mechanical properties.

It is worthwhile to point out that the equilateral triangle is employed due to the fact that its symmetry is consistent with the symmetry in the optical coherence quantities. That is, the distance between the geometric center O and the center-of-mass point M is invariant under permutation of the masses m_1, m_2, m_3. This is aligned with the fact that the degrees of entanglement and polarization are symmetric about the Schmidt coefficients $\sqrt{m_1}$, $\sqrt{m_2}$, $\sqrt{m_3}$. These optical quantities are invariant under local observation basis changes that relate to the permutation of $\sqrt{m_1}$, $\sqrt{m_2}$, $\sqrt{m_3}$.

The three-mass system has a center-of-mass point M that is located inside the two-dimensional triangle $\triangle m_1 m_2 m_3$. Then the coordinates $(X^{(1)}, X^{(2)})$ of M are simply determined as

$$X^{(j)} m_{\mathrm{tot}} = x_1^{(j)} m_1 + x_2^{(j)} m_2 + x_3^{(j)} m_3, \tag{9}$$

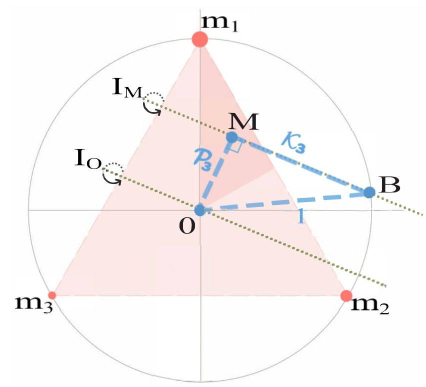

FIG. 1. Geometric illustration of mapping optical polarization coherence and entanglement to mechanical concepts of COM and MOI for arbitrary 3D light fields. The three masses $m_{1,2,3}$ are placed at the vertices of an equilateral triangle (that is inscribed in the circle O) and M is the center-of-mass point. The lengths of $\overline{OM}$, $\overline{MB}$ represent the values of degree of polarization $\mathcal{P}_3$ and entanglement $\mathcal{K}_3$, respectively. Here the line $\overline{MB}$ is perpendicular to $\overline{OM}$ and it intersects with the circle O at point B. The Pythagorean theorem of the right triangle $\triangle OMB$ directly represents the complementary relation (8). I_M, I_O are the moments of inertia that correspond to rotations along the entanglement line $\overline{MB}$ and its parallel partner that passes through point O, respectively. The sizes of the point-mass dots indicate $m_1 \geqslant m_2 \geqslant m_3$ without loss of any generality.

where $m_{\rm tot}$ is the total mass (normalized to 1) and $x_i^{(j)}$ represents the jth coordinate of the ith mass m_i, with $i = 1, 2, 3$ and $j = 1, 2$. When taking O as the origin of the 2D coordinate system, the distance between O and M can be simply determined as $\overline{OM} = \sqrt{(X^{(1)})^2 + (X^{(2)})^2}$. Furthermore, the distance between the mass center M and point B, which is the cross point of the circle O with line $\overline{MB}$ (perpendicular to $\overline{OM}$), can also be obtained directly as $\overline{MB} = \sqrt{1 - (X^{(1)})^2 - (X^{(2)})^2}$. Surprisingly, it can then be shown that

$$\mathcal{P}_3 = \overline{OM} \quad \text{and} \quad \mathcal{K}_3 = \overline{MB}. \tag{10}$$

That is, the degree of 3D polarization $\mathcal{P}_3$ equals the value of the distance between the geometric center O and the mass center M, and the degree of entanglement $\mathcal{K}_3$ equals the value of the distance between mass center M and point B. When the three masses are equal, the center-of-mass point M coincides with the geometric center O so that $\overline{OM} = 0$ ($\overline{MB} = 1$), indicating complete unpolarization $\mathcal{P}_3 = 0$ (maximal entanglement $\mathcal{K}_3 = 1$). When the total mass is concentrated on one point mass (e.g., m_1), the remaining two masses vanish. Then the center of mass M coincides with point m_1, with $\overline{OM} = 1$ ($\overline{MB} = 0$) indicating complete polarization $\mathcal{P}_3 = 1$ (zero entanglement $\mathcal{K}_3 = 0$). This indicates a different application of the barycentric coordinate system, i.e., it can interpret the optical polarization coherence $\mathcal{P}_3$ as the distance of two points: the point of interest (m_1, m_2, m_3) and the completely unpolarized point $(1/3, 1/3, 1/3)$, and similarly for entanglement $\mathcal{K}_3$. The detailed proof of this quantitative connection for the generalized N-dimensional case is given in the Appendix, Sec. 1.

As a result, the polarization-entanglement complementary relation (8) can now be represented by the Pythagorean theorem of the right triangle $\triangle OMB$ that connects the mass center M, geometric center O, and point B, illustrated by the blue dashed lines in Fig. 1,

$$\mathcal{P}_3^2 + \mathcal{K}_3^2 = 1 \quad \Leftrightarrow \quad \overline{OM}^2 + \overline{MB}^2 = \overline{OB}^2. \tag{11}$$

Equations (10) and (11) represent the second major result, showing direct quantitative connections of optical polarization, entanglement, and their complementary relation to the mechanical concept of center of mass.

IV. MOMENT OF INERTIA

The center of mass of a system is related to another mechanical concept, moment of inertia, when the rotation axis passes through the mass center M. Combined with the above discussion, the entanglement line $\overline{MB}$, as shown in Fig. 1, serves as a crucial rotation axis, about which the moment of inertia I_M of the three-mass system can be obtained as $I_M = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$, where r_i, $i = 1, 2, 3$, are the distances of mass m_i to the axis $\overline{MB}$.

The moment of inertia with respect to the parallel line that passes through the geometric center O (see illustration in Fig. 1) can also be achieved as $I_O = m_1 s_1^2 + m_2 s_2^2 + m_3 s_3^2$, where s_i, $i = 1, 2, 3$, are the distances of mass m_i to the parallel axis. Then the Huygens-Steiner theorem [45,46] (also called parallel-axis theorem) reads

$$I_O = I_M + m_{\rm tot} d^2, \tag{12}$$

where d is the distance between the two paralleled axes and $m_{\rm tot} = 1$ due to normalization of (3). This straightforwardly leads to the quantitative relations

$$\mathcal{P}_3 = \sqrt{I_O - I_M} \quad \text{and} \quad \mathcal{K}_3 = \sqrt{1 - I_O + I_M}, \tag{13}$$

which is the third major result. They establish direct quantitative connections between optical coherence quantities and mechanical quantities.

The polarization coherence and entanglement of a generic light field can now be interpreted as the difference between two moments of inertia, I_O and I_M, of the three-mass system. Complete unpolarization $\mathcal{P}_3 = 0$ (or maximal entanglement $\mathcal{K}_3 = 1$) now simply means the moment of inertia I_M coincides with I_O so that $I_O - I_M = 0$, while complete polarization $\mathcal{P}_3 = 1$ (or zero entanglement $\mathcal{K}_3 = 0$) indicates the moments of inertia I_M and I_O are maximally separated with $I_O - I_M = 1$. This provides a different way of understanding and obtaining optical coherence quantities through the mapped point-mass scenario.

On the other hand, the mechanical properties of such a three-mass system (or N-mass system as extended in the following) can also be understood and achieved with the optical polarization coherence and entanglement. These mechanical properties include center of mass, momentum of inertia, as well as their related properties such as angular momentum $L = I\omega$, with ω being rotation frequency, rotational energy $E = I\omega^2/2$, etc.

V. GENERALIZATION TO ARBITRARY DIMENSIONS

The above polarization-entanglement complementary relation (8) along with its connection to the mechanical concepts of center of mass and moment of inertia can be further generalized to arbitrary dimensional tensor structures. In this case, the concept of polarization is not restricted to describe the three wave oscillation directions of the 3D space any longer. It is extended to represent all vectors of a generic vector space of arbitrary dimension [24,47].

A generic N-dimensional two-space (or two-degree-of-freedom) tensor structure can be written as

$$|E\rangle = \sum_{l=1}^{N} |G_l\rangle |Z_l\rangle, \tag{14}$$

where $|G_l\rangle$ are normalized basis vectors of one vector space (e.g., the infinite-dimensional spatial degree of freedom of light) and $|Z_l\rangle$ represent the amplitudes that group all remaining degrees of freedom as a single large vector space (e.g., the combination of wave oscillation directions and temporal modes of light).

The above state is a direct N-dimensional extension of the light field (2), but with the vector space $\{|G_l\rangle\}$ singled out. The extended concept of "polarization" simply means all the basis vectors $|G_l\rangle$ (corresponding to $|x\rangle$, $|y\rangle$, $|z\rangle$ in the general 3D light field case). Then the degree of polarization is directly extended to mean how much this field $|E\rangle$ is concentrated to a single superposed vector in this G space. As a result, this generalized polarization coherence can be systematically defined as [24,47]

$$\mathcal{P}_N = \sqrt{\frac{N}{N-1}\left(\mathrm{Tr}\mathcal{W}_{\mathrm{ND}}^2 - \frac{1}{N}\right)}, \tag{15}$$

which is a direct extension of (4) with

$$\mathcal{W}_{\mathrm{ND}} = \frac{\sum_{k,l}\langle Z_k|Z_l\rangle |G_l\rangle\langle G_k|}{\sum_k \langle Z_k|Z_k\rangle} \tag{16}$$

being the normalized N-dimensional (ND) polarization coherence matrix of the G space, with $k, l = 1, 2, 3, \ldots, N$. Here, $\mathcal{P}_N$ is normalized between 0 and 1, indicating complete unpolarization and polarization, respectively.

Entanglement of the N-dimensional two-space field (14) can be analyzed systematically by Schmidt decomposition as in the 3D case, and quantitatively measured by the Schmidt weight as

$$\mathcal{K}_N = \sqrt{\frac{N}{N-1}\left(1 - \frac{1}{K_N}\right)}, \tag{17}$$

where $K_N = 1/\sum_{i=1}^{N} m_i^2$ is the Schmidt number, with $\sqrt{m_i}$ being the Schmidt coefficients and the m_i are the eigenvalues of the $N \times N$ coherence matrix $\mathcal{W}_{\mathrm{ND}}$. Here, $\mathcal{K}_N$ is bounded between 0 and 1, indicating zero and maximal entanglement, respectively.

Combining the degree of generic polarization and entanglement for the generic ND structure, one is then led to the generalized identity

$$\mathcal{P}_N^2 + \mathcal{K}_N^2 = 1. \tag{18}$$

This generic entanglement-polarization complementary relation suggests that the intrinsic opposite behaviors of polarization and entanglement in a general field are universal for all light fields. The absence of polarization coherence is always accompanied by the display of maximal entanglement, and vice versa. The proof of Eq. (A7) is provided in the Appendix, Sec. 1.

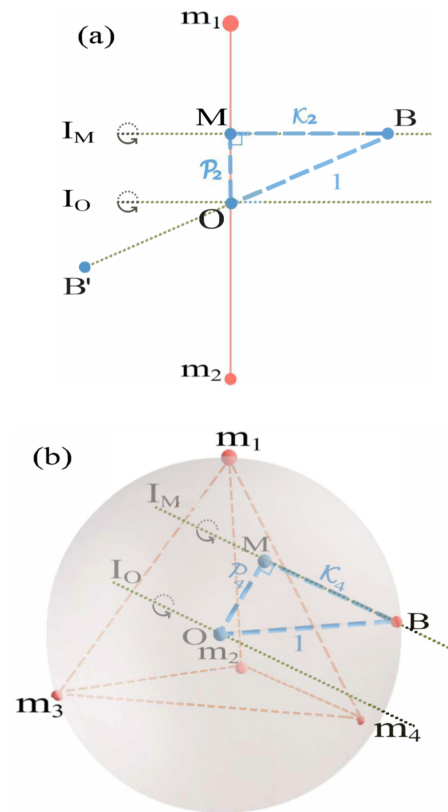

FIG. 2. Geometric illustrations of mapping optical polarization coherence and entanglement to mechanical concepts of COM and MOI for arbitrary 2D beams and 4D generic structural fields. (a) For the 2D case, two masses $m_{1,2}$ are placed at the vertices of the regular 1 simplex (a segment) and the 0 sphere O has two end points B, B'. (b) For the 4D case, four masses $m_{1,2,3,4}$ are placed at the vertices of the regular 3 simplex (a tetrahedron) inscribed in the 2 sphere O. For both cases, M is the center of mass and the lengths of $\overline{OM}$, $\overline{MB}$ represent the values of degree of polarization $\mathcal{P}_N$ and entanglement $\mathcal{K}_N$, respectively. Line $\overline{MB}$ is perpendicular to $\overline{OM}$ and it intersects with the $(N-2)$-sphere O at point B. The Pythagorean theorem of the right triangle $\triangle OMB$ directly represents the complementary relation (A7). I_M, I_O are the moments of inertia that correspond to rotations along the entanglement line $\overline{MB}$ and its parallel partner that passes through O, respectively. The sizes of the point-mass dots indicate $m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4$ without loss of any generality.

It is important to note that when $N = 2$, the field reduces to the arbitrary two-dimensional optical beam case. Then, $\mathcal{P}_2$ is exactly the conventional degree of polarization [4,48] and entanglement $\mathcal{K}_2$ is exactly the well-known entanglement measure concurrence [49]. Figure 2(a) illustrates the 2D complementary relation with triangle $\triangle OMB$. This two-dimensional relation is consistent with wave-particle complementarity relations; see, for example, Refs. [24,50–55].

The connection to mechanical concepts can also be extended systematically following the two-step geometric mapping prescription.

Step 1. Let the eigenvalues (or the square of the Schmidt coefficients) $m_1, \ldots, m_N$ of the polarization coherence matrix (16) represent the values of N point masses.

Step 2. Place these point masses at the N vertices of a regular $(N-1)$ simplex inscribed in a unit $(N-2)$ sphere with origin O.

Consistent with the case of 3D generic light field, the value of the degree of polarization $\mathcal{P}_N$ is exactly the distance from O to the center-of-mass point M, i.e., $\mathcal{P}_N = \overline{OM}$, and the value of degree of entanglement $\mathcal{K}_N = \overline{MB}$ where $\overline{MB} \perp \overline{OM}$ and B is the cross point with the unit $(N-2)$-sphere O; see illustration in Fig. 2. Then the right triangle $\triangle OMB$ with $\overline{OM}^2 + \overline{MB}^2 = \overline{OB}^2$ directly represents the generic polarization-entanglement complementary relation $\mathcal{P}_N^2 + \mathcal{K}_N^2 = 1$. A detailed proof of these generalized results is provided in the Appendix, Sec. 2.

Furthermore, the moment of inertia of the N-mass system with respect to the entanglement line $\overline{MB}$ and the parallel line that passes through O exactly obeys the same quantitative relation as in the 3D case (13), connecting to optical polarization coherence and entanglement as

$$\mathcal{P}_N = \sqrt{I_O - I_M} \quad \text{and} \quad \mathcal{K}_N = \sqrt{1 - I_O + I_M}. \tag{19}$$

To this end, we have shown that all three major results (8), (11), (13) about the generic 3D light field can be reduced to arbitrary 2D beams and extended to arbitrary ND structural fields. The optical polarization-entanglement complementary relation is a generic feature for all light fields. The quantitative connections of optical coherence quantities (polarization and entanglement) with mechanical concepts of center of mass and moment of inertia are also universal for all light fields.

VI. SUMMARY

In summary, we have established a generic polarization-entanglement complementary relation $\mathcal{P}_N^2 + \mathcal{K}_N^2 = 1$ for arbitrary light fields of 2D and 3D polarizations and for general fields of N-dimensional structural polarization. The complementarity suggests that polarization and entanglement are two intrinsically opposite coherence properties of all light fields. The absence of polarization coherence is always accompanied by the display of entanglement, and vice versa. It is worthwhile to point out that such a complementary behavior is general and independent of specific measures. That is, one can also use other measures to characterize this behavior, e.g., the von Neumann entropy S [56] for entanglement in the 2D case; then it will lead to a quantitatively different, but qualitatively similar complementary relation $S + [(1+\mathcal{P}_2)\ln(1+\mathcal{P}_2) + (1-\mathcal{P}_2)\ln(1-\mathcal{P}_2)]/2 = 1$, where S is a monotonically decreasing function of $\mathcal{P}_2$.

A geometric mapping technique based on the barycentric coordinate system is introduced to correspond the eigenvalues (or Schmidt coefficients) of the polarization coherence matrix to point masses that are located at the vertices of a regular simplex. Based on the mapping, optical polarization and entanglement (indication of the Huygens-Fresnel wave theory) are shown to be quantitatively connected to the seemingly unrelated mechanical concepts of center of mass and moment of inertia (result of the Huygens-Steiner theorem).

The obtained quantitative relations in (10), (11), and (13) and their N-dimensional extensions open an avenue that links coherence optics to mechanics. These relations provide a mechanical and geometrical picture to interpret and understand the meaning of coherence optical concepts such as complete polarization, partial polarization, complete unpolarization, separable, partial entanglement, maximal entanglement, etc. Based on the obtained quantitative results, we expect that other relevant optical concepts such as degree of coherence, cross correlation, entropy, etc. can be connected to additional mechanical concepts including angular momentum, rotational energy, etc.

Finally, the tensor structure of the light field (2) or its generalized form (14) is analogous to that of a quantum pure state. Therefore, our analysis of the polarization-entanglement complementarity and the connection to mechanical concepts also applies to quantum systems with respective different physical interpretations.

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APPENDIX

1. Polarization-Entanglement Complementary Relation

This section provides the proof of the polarization-entanglement complementary relation for arbitrary field of N-dimensional (ND) structural polarization. The degree of polarization is defined as

$$\mathcal{P}_N = \sqrt{\frac{N}{N-1}\left(\mathrm{Tr}\mathcal{W}_{\mathrm{ND}}^2 - \frac{1}{N}\right)}, \tag{A1}$$

where $\mathcal{W}_{\mathrm{ND}}$ is the ND polarization coherence matrix. Then it can be rewritten as

$$\mathcal{P}_N^2 = \frac{N\sum_{k=1}^N m_k^2 - 1}{N-1}, \tag{A2}$$

where $m_1, m_2, \ldots, m_N$ are the N eigenvalues of $\mathcal{W}_{\mathrm{ND}}$.

Given the fact that $\mathcal{W}_{\mathrm{ND}}$ is normalized, the sum of all eigenvalues equals identity, i.e., $\sum_k^N m_k = 1$. Then one has

$$\begin{aligned}\mathcal{P}_N^2 &= \frac{N\left(\sum_{k=1}^N m_k^2 - 1\right) + N - 1}{N-1}\\ &= \frac{N\left[\sum_{k=1}^N m_k^2 - \left(\sum_k^N m_k\right)^2\right] + N - 1}{N-1}\\ &= \frac{N\left(-\sum_{i<j} 2m_i m_j\right) + N - 1}{N-1}\\ &= 1 - \frac{N}{N-1}\sum_{i<j} 2m_i m_j.\end{aligned} \tag{A3}$$

On the other hand, the degree of entanglement in terms of the Schmidt weight is defined as

$$\mathcal{K}_N = \sqrt{\frac{N}{N-1}\left(1 - \frac{1}{K_N}\right)}, \tag{A4}$$

where

$$K_N = \frac{1}{m_1^2 + m_2^2 + \cdots + m_N^2} \tag{A5}$$

is the Schmidt number that is defined based on the Schmidt coefficients (which coincide with the eigenvalues of the polarization coherence matrix $\mathcal{W}_{\rm ND}$), $m_1, m_2, \ldots, m_N$. Then the Schmidt weight can be rewritten as

$$\begin{aligned}\mathcal{K}_N^2 &= \frac{N}{N-1}\left[1 - \left(m_1^2 + m_2^2 + \cdots + m_N^2\right)\right] \\ &= \frac{N}{N-1}\sum_{i<j} 2m_i m_j.\end{aligned} \tag{A6}$$

By combining (A3) and (A6), one immediately retrieves the polarization-entanglement complementary identity for arbitrary dimensions,

$$\mathcal{P}_N^2 + \mathcal{K}_N^2 = 1. \tag{A7}$$

2. Degree of Polarization and Center of Mass

This section provides the rigorous proof that the generic ND degree of polarization $\mathcal{P}_N$ equals the distance between the geometric center O and the center-of-mass point M of the N-mass system. The N point masses are placed at the N vertices of a regular $(N-1)$ simplex inscribed in a unit $(N-2)$ sphere with origin O.

A regular $(N-1)$ simplex, or a unit $(N-2)$ sphere, lives in a $(N-1)$-dimensional space. For symmetry purpose, we set O as the origin of this $(N-1)$-dimensional coordinate system. Then the coordinates of all N point masses $m_1, m_2, \ldots, m_N$ can be described as $(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(N-1)})$, $(x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(N-1)})$, $\ldots$, $(x_N^{(1)}, x_N^{(2)}, \ldots, x_N^{(N-1)})$, respectively, where $x_i^{(j)}$ represents the jth coordinate for point mass i.

Then the coordinates of the center-of-mass point $(X^{(1)}, X^{(2)}, \ldots, X^{(N-1)})$ can be determined as

$$X^{(j)} = \frac{\sum_{i=1}^N x_i^{(j)} m_i}{m_{\rm tot}}, \tag{A8}$$

for all $j = 1, 2, \ldots, N-1$. The distance between the origin O and center-of-mass point M can be computed directly as

$$\overline{OM} = \sqrt{\sum_{j=1}^{N-1} |X^{(j)}|^2} = \sqrt{\sum_{j=1}^{N-1}\left|\sum_{i=1}^N x_i^{(j)} m_i\right|^2}, \tag{A9}$$

which can be explicitly expressed as

$$\begin{aligned}\overline{OM}^2 &= \sum_{j=1}^{N-1}\left[\sum_{i=1}^N \left(x_i^{(j)}\right)^2 m_i^2 + \sum_{i<i'} 2x_i^{(j)} m_i x_{i'}^{(j)} m_{i'}\right] \\ &= \sum_{i=1}^N m_i^2 - \sum_{i<i'}\frac{2}{N-1} m_i m_{i'} \\ &= -\sum_{i<i'} 2m_i m_{i'} + 1 - \sum_{i<i'}\frac{2}{N-1} m_i m_{i'} \\ &= 1 - \sum_{i<i'}\frac{N}{N-1} 2m_i m_{i'} \\ &= \mathcal{P}_N^2.\end{aligned} \tag{A10}$$

That is, the distance $\overline{OM}$ is exactly the degree of polarization $\mathcal{P}_N$. Here we have used the fact that the distance from each point mass m_i to the origin O is 1, i.e., $\sum_j (x_i^{(j)})^2 = 1$ for all i, as well as the fact that the distance between any two point masses of a regular $(N-1)$ simplex is $d = \sqrt{2N/(N-1)}$.

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