Asymmetry-induced delocalization transition in the integrable non-Hermitian spin chain

Yuki Ishiguro,[1] Jun Sato,[2] and Katsuhiro Nishinari[3]

[1]*The Institute for Solid State Physics, The University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8581, Japan*
[2]*Faculty of Engineering, Tokyo Polytechnic University, 5-45-1 Iiyama-minami, Atsugi, Kanagawa 243-0297, Japan*
[3]*Research Center for Advanced Science and Technology, The University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153-8904, Japan*

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The emergence of quasiparticles is a universal property in integrable systems. String-type quasiparticles, which are characterized by the string solutions of Bethe equations, play fundamental roles in the analysis of their physics. Through an investigation of the Bethe equations in the asymmetric simple exclusion process, we reveal the existence of string solutions in the presence of non-Hermiticity resulting from asymmetrical hopping. Because of the non-Hermiticity, the string solutions exhibit exotic properties such as the complexification of the center of string solutions and the delocalization of Bethe quantum numbers. In addition, we find the picture of string-type quasiparticles collapses in the strong-asymmetry regime. The collapse of string solutions characterizes the transition of eigenstates from bound states to scattering states.

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I. INTRODUCTION

Non-Hermitian physics has attracted attention as a promising approach for investigating general principles in nonequilibrium systems [1,2]. Since non-Hermiticity effectively describes nonequilibrium phenomena, such as asymmetric hopping and dissipation of particles, non-Hermitian Hamiltonians appear in various situations describing systems out of equilibrium, including self-driven particle systems [3–5] and open quantum systems [6–8]. Quantum integrable systems play crucial roles in understanding interacting many-body systems [9]. Although examples of integrable non-Hermitian systems have been found [3–7], the theoretical understanding for their analysis is less advanced than that for Hermitian systems.

Integrable systems are usually analyzed in terms of quasiparticles. For example, the Heisenberg spin chain is understood through quasiparticles characterized by the string solutions, which are specific solutions for the Bethe equations widely observed among integrable systems [10]. String-type quasiparticles, which are closely related to solitons in classical integrable systems [11–14], allows understanding the physics of integrable systems such as anomalous transport [15,16]. Moreover, through the so-called string hypothesis, they enable us to use the powerful analytical frameworks of integrable systems, including the thermodynamic Bethe ansatz (TBA) [10,17–20] and the generalized hydrodynamics (GHD) [21–25]. Therefore, understanding the quasiparticle picture in integrable non-Hermitian systems is the key to establishing cornerstones for analyzing them.

In this paper we elucidate the string-type quasiparticle picture in the presence of non-Hermiticity caused by asymmetric hopping and present the mechanism of delocalization transition based on this picture. We consider the asymmetric simple exclusion process (ASEP), which is an integrable non-Hermitian spin chain describing asymmetric random walks of self-driven particles with hardcore interactions [3–5, 26–40]. The ASEP provides an excellent field for investigating nonequilibrium physics, such as the KPZ universality class [41,42] and the boundary-induced phase transition [5], and has a wide range of applications including traffic flow [43,44] and biophysics [45,46]. Although the ASEP is exactly analyzable by the Bethe ansatz [4,27–40], the understanding of the properties of eigenstates is still developing because of the complexity of the Bethe equations [4,27–34]. Our main result is to show that the Bethe roots can be understood in terms of the string solutions. Because of the non-Hermiticity induced by asymmetric hopping, the string solutions exhibit intriguing properties, such as the complexification of the center of string solutions and the delocalization of Bethe quantum numbers. One of the most notable phenomena is that the picture of the string solutions collapses in the strong-asymmetry regime. We reveal that the collapse of strings characterizes the transition of eigenstates from bound states to scattering states.

This paper is organized as follows. In Sec. II we introduce the ASEP. The Hamiltonian is exactly diagonalized by the Bethe ansatz. In Sec. III we reveal that the Bethe equations of the ASEP have string solutions. Through the analysis of the Bethe-Takahashi equations, which are the reduced Bethe equations in terms of strings, we discuss the property of the center of string solutions. In Sec. IV we show that the string solutions of the ASEP exhibit exotic properties, such as the collapse of strings and the delocalization of Bethe quantum numbers, because of the non-Hermiticity. In Sec. V we discuss the properties of the eigenstates corresponding to string solutions and clarify that the collapse of strings characterizes the

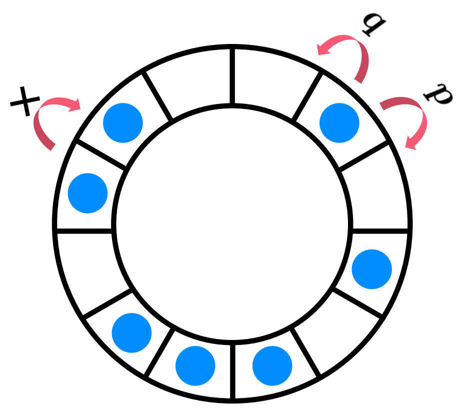

FIG. 1. Asymmetric simple exclusion process with periodic boundary conditions.

delocalization transition. Finally, we summarize our results in Sec. VI.

II. MODEL

The ASEP is a continuous-time Markov process in a one-dimensional lattice defined by the following rule [3–5]. Each particle moves to the nearest right (left) site with the hopping rate p (q). Because of the hardcore interactions, each site contains only a single particle at most. Without loss of generality, we can set $p+q=1$ and $p \geqslant q$. The schematic drawing of the ASEP is shown in Fig. 1. When $q=0$, particles move in only one direction. In this case the model is called the totally asymmetric simple exclusion process (TASEP).

The time evolution of the ASEP is described by the imaginary-time Schrödinger equation

$$\frac{d}{dt}|P(t)\rangle = \mathcal{H}|P(t)\rangle, \tag{1}$$

where $|P(t)\rangle$ is the stochastic state vector of a system at time t. The Hamiltonian $\mathcal{H}$ with L sites is given by

$$\mathcal{H} = \sum_{j=1}^{L}\left[pS_j^+S_{j+1}^- + qS_j^-S_{j+1}^+ + S_j^zS_{j+1}^z - \frac{1}{4}\right], \tag{2}$$

where $S^{x,y,z}$ are half of the Pauli matrices, and $S^{\pm} := S^x \pm iS^y$ are the ladder operators. Here we consider periodic boundary conditions. When hopping rates are symmetric ($p=q$), the Hamiltonian (2) is equivalent to that of the Heisenberg spin-1/2 chain. The asymmetry of hopping rates, which drives current, makes the Hamiltonian non-Hermitian.

The Hamiltonian (2) is exactly diagonalized through the Bethe ansatz [4]. Thus we obtain the corresponding eigenstates if we identify the solutions of the Bethe equations

$$z_j^L = \prod_{\ell=1}^{N}\frac{p - z_j + qz_\ell z_j}{p - z_\ell + qz_\ell z_j} \quad \text{for} \quad j = 1, 2, \ldots, N, \tag{3}$$

where N is the number of particles. The total momentum K and the energy eigenvalue E are described in terms of the Bethe roots $\{z_j\}$ as

$$K = -i\sum_{j=1}^{N}\log z_j, \tag{4}$$

$$E = q\sum_{j=1}^{N}z_j + p\sum_{j=1}^{N}\frac{1}{z_j} - N. \tag{5}$$

Because of the difficulty of solving the Bethe equations (3), many efforts have been made to clarify the distribution of the Bethe roots [4,27–34].

III. STRING SOLUTIONS

A. Variable transformation

Here we reveal that the Bethe equations (3) have the string solutions. First, we introduce new variables $\{\lambda_j\}$ as

$$z_j = \frac{1}{\alpha}\frac{\sin\zeta\left(\lambda_j + \frac{i}{2}\right)}{\sin\zeta\left(\lambda_j - \frac{i}{2}\right)}, \tag{6}$$

where we define α and ζ as

$$\alpha := \sqrt{q/p}, \quad \zeta := -\log\alpha. \tag{7}$$

The parameter α characterizes the non-Hermiticity of the ASEP. When $\alpha = 1$, the Hamiltonian (2) becomes Hermitian. Then the Bethe equations (3) are written as

$$\left[\frac{1}{\alpha}\frac{\sin\zeta\left(\lambda_j + \frac{i}{2}\right)}{\sin\zeta\left(\lambda_j - \frac{i}{2}\right)}\right]^L = \prod_{\ell\neq j}^{N}\frac{\sin\zeta(\lambda_j - \lambda_\ell + i)}{\sin\zeta(\lambda_j - \lambda_\ell - i)} \quad \text{for} \quad j = 1, 2, \ldots, N. \tag{8}$$

These equations are similar to the well-known Bethe equations of the Heisenberg XXZ spin-1/2 chain. The difference is the parameter α. As we will show in the next part, Eq. (8) enable us to formulate string solutions. Note that we cannot define the transformation (6) in the case of the TASEP ($q = 0$). Therefore it is difficult to formulate string solutions in this case. In the following we consider $q \neq 0$.

Here we introduce the Bethe quantum numbers, which allow the classification of eigenstates. Since the Bethe equations (8) have many solutions, it is useful to express them in logarithmic form. We introduce functions $\theta_n(\lambda)$ ($n \in \mathbb{N}$) as

$$\theta_n(\lambda) := \begin{cases} 2\tan^{-1}\left[\coth\left(\frac{n\zeta}{2}\right)\tan\zeta\lambda\right] + 2\pi\left\lfloor\frac{\mathrm{Re}(\zeta\lambda)}{\pi} + \frac{1}{2}\right\rfloor & \text{for} \quad \alpha \neq 1 \\ 2\tan^{-1}\left[\frac{2\lambda}{n}\right] & \text{for} \quad \alpha = 1, \end{cases} \tag{9}$$

where $\lfloor x\rfloor$ is the floor function defined as $\lfloor x\rfloor := \max\{n \in \mathbb{Z}|n \leqslant x\}$. By taking the logarithm of the Bethe equations (8), we obtain

$$\theta_1(\lambda_j) = \frac{2\pi}{L}I_j + \frac{1}{L}\sum_{\ell\neq j}^{N}\theta_2(\lambda_j - \lambda_\ell) + i\log\alpha \quad \text{for} \quad j = 1, 2, \ldots, N, \tag{10}$$

where $\{I_j\}$ are called the Bethe quantum numbers, which are integers when $L-N$ is odd and half-integers when $L-N$ is even. Through a set of Bethe quantum numbers, we can characterize the eigenstates of the Hamiltonian.

In the new variables $\{\lambda_j\}$, the total momentum (4) and the energy eigenvalue (5) are expressed as

$$\begin{aligned} K &= \sum_{j=1}^{N}[\pi - \theta_1(\lambda_j) + i\log\alpha] \\ &= N\pi - \frac{2\pi}{L}\sum_{j=1}^{N} I_j, \end{aligned} \tag{11}$$

$$\begin{aligned} E &= -\frac{1}{2}\tanh\zeta\sum_{j=1}^{N} a_1(\lambda_j) \\ &= \begin{cases} -\sum_{j=1}^{N}\frac{\Delta\tanh^2\zeta}{\Delta-\cos 2\zeta\lambda_j} & \text{for} \quad \alpha\neq 1 \\ -\sum_{j=1}^{N}\frac{2}{4\lambda_j^2+1} & \text{for} \quad \alpha = 1, \end{cases} \end{aligned} \tag{12}$$

where we introduce $\Delta := \cosh\zeta$ and functions $a_n(\lambda)$ $(n\in\mathbb{N})$,

$$\begin{aligned} a_n(\lambda) &:= \frac{d}{d\lambda}\theta_n(\lambda) \\ &= \begin{cases} \frac{2\sinh n\zeta}{\cosh n\zeta - \cos 2\zeta\lambda} & \text{for} \quad \alpha\neq 1 \\ \frac{4n}{4\lambda^2+n^2} & \text{for} \quad \alpha = 1. \end{cases} \end{aligned} \tag{13}$$

B. Formulation of string solutions

The new Bethe equations (8) allow the formulation of the string solutions. We denote a Bethe root as $\lambda = a + ib$ $(a, b\in\mathbb{R})$. The $|\mathrm{LHS}|^{1/L}$ (left-hand side) of Eqs. (8) is classified into three types according to the relation between the real and imaginary parts of the Bethe root as follows (see Appendix A):

$$\left|\frac{1}{\alpha}\frac{\sin\zeta\left(\lambda+\frac{i}{2}\right)}{\sin\zeta\left(\lambda-\frac{i}{2}\right)}\right| \begin{cases} >1 & \text{for} \quad b > c_1(a) \\ =1 & \text{for} \quad b = c_1(a) \\ <1 & \text{for} \quad b < c_1(a), \end{cases} \tag{14}$$

where we introduce functions $c_n(x)$ $(n\in\mathbb{N})$ as

$$c_n(x) = \frac{1}{2\zeta}\log\frac{\cos 2\zeta x}{\cosh n\zeta}. \tag{15}$$

By considering the $L\to\infty$ limit with fixed N, we obtain the string solutions. We assume the Bethe root $\lambda_j = a_j + ib_j$ with $b_j > c_1(a_j)$. From Eq. (14), the $|\mathrm{LHS}|$ of Eq. (8) diverges in the $L\to\infty$ limit. From the consistency of the Bethe equations (8), the denominator of the right-hand side (RHS) should go to 0 in this limit. This implies that there exists an integer ℓ $(1\leqslant \ell\leqslant N,\ \ell\neq j)$ such that $\lambda_\ell = \lambda_j - i$ in $L\to\infty$. Thus if the Bethe root λ_j with $b_j > c_1(a_j)$ exists, another Bethe root exists at the position displaced from λ_j by $-i$. Similarly, the Bethe root λ_j with $b_j < c_1(a_j)$ generates another Bethe root at the position displaced from λ_j by i. From this discussion we obtain a series of Bethe roots:

$$\lambda_A^{n,j} = \lambda_A^n + \frac{i}{2}(n+1-2j) + \delta_A^{n,j} \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{16}$$

These roots are called n-string solutions. A is the index of the string, n is the length of the string, $\delta_A^{n,j}$ is the deviation that vanishes when $L\to\infty$, and λ_A^n is called string center.

The difference from the standard string solutions is that the string center λ_A^n is no longer necessarily real. The string center in the Heisenberg spin chain is restricted to real because of the self-conjugacy of the Bethe roots [47]. Since the non-Hermiticity violates the self-conjugacy, the string center is allowed to have an imaginary part (see Appendix B).

C. Bethe-Takahashi equations

The complexification of the string centers seems to make the analysis difficult due to the increased degrees of freedom. However, the string centers of the ASEP have specific relations between the real and imaginary parts. Therefore the degree of freedom of the position of string centers remains one in the complex plane. The relations are obtained by analyzing the Bethe-Takahashi equations, which are the reduced Bethe equations in terms of strings.

By considering Bethe roots on a string as a quasiparticle, we derive the equations for the string centers. To derive the Bethe-Takahashi equations, we express all Bethe roots by the string solutions (16). That is, N Bethe roots are partitioned as $\sum_{n=1}^{N} nN_n = N$, where N_n is the number of n strings. Then the Bethe equations (8) on an n string are written as

$$\begin{aligned} \left[\frac{1}{\alpha}\frac{\sin\zeta\left(\lambda_A^{n,j}+\frac{i}{2}\right)}{\sin\zeta\left(\lambda_A^{n,j}-\frac{i}{2}\right)}\right]^L &= \prod_{\substack{(m,B)\\ \neq(n,A)}}\prod_{k=1}^{m}\frac{\sin\zeta\left(\lambda_A^{n,j}-\lambda_B^{m,k}+i\right)}{\sin\zeta\left(\lambda_A^{n,j}-\lambda_B^{m,k}-i\right)} \\ &\quad\times\prod_{j'\neq j}^{n}\frac{\sin\zeta\left(\lambda_A^{n,j}-\lambda_A^{n,j'}+i\right)}{\sin\zeta\left(\lambda_A^{n,j}-\lambda_A^{n,j'}-i\right)} \\ &\text{for} \quad j = 1, 2, \ldots, n. \end{aligned} \tag{17}$$

We reduce the Bethe equations on an n string (17) by multiplying them as follows:

$$\begin{aligned} &\prod_{j=1}^{n}\left[\frac{1}{\alpha}\frac{\sin\zeta\left(\lambda_A^{n,j}+\frac{i}{2}\right)}{\sin\zeta\left(\lambda_A^{n,j}-\frac{i}{2}\right)}\right]^L \\ &= \prod_{j=1}^{n}\left[\prod_{\substack{(m,B)\\ \neq(n,A)}}\prod_{k=1}^{m}\frac{\sin\zeta\left(\lambda_A^{n,j}-\lambda_B^{m,k}+i\right)}{\sin\zeta\left(\lambda_A^{n,j}-\lambda_B^{m,k}-i\right)}\right. \\ &\quad\left.\times\prod_{j'\neq j}^{n}\frac{\sin\zeta\left(\lambda_A^{n,j}-\lambda_A^{n,j'}+i\right)}{\sin\zeta\left(\lambda_A^{n,j}-\lambda_A^{n,j'}-i\right)}\right]. \end{aligned} \tag{18}$$

Since the second term of the RHS in Eq. (18) satisfies

$$\prod_{j=1}^{n}\prod_{j'\neq j}^{n}\frac{\sin\zeta\left(\lambda_A^{n,j}-\lambda_A^{n,j'}+i\right)}{\sin\zeta\left(\lambda_A^{n,j}-\lambda_A^{n,j'}-i\right)} = 1, \tag{19}$$

Eq. (18) is given by

$$\prod_{j=1}^{n}\left[\frac{1}{\alpha}\frac{\sin\zeta\left(\lambda_A^{n,j}+\frac{i}{2}\right)}{\sin\zeta\left(\lambda_A^{n,j}-\frac{i}{2}\right)}\right]^L = \prod_{j=1}^{n}\prod_{\substack{(m,B)\\ \neq(n,A)}}\prod_{k=1}^{m}\frac{\sin\zeta\left(\lambda_A^{n,j}-\lambda_B^{m,k}+i\right)}{\sin\zeta\left(\lambda_A^{n,j}-\lambda_B^{m,k}-i\right)}. \tag{20}$$

Here we assume that the deviations of the string solutions (16) vanish, that is $\delta_A^{n,j}\to 0$. By taking the product over j and k in Eq. (20), we obtain the Bethe-Takahashi equations,

$$\left[\frac{1}{\alpha^n}\frac{\sin\zeta\left(\lambda_A^n+\frac{i}{2}n\right)}{\sin\zeta\left(\lambda_A^n-\frac{i}{2}n\right)}\right]^L = \prod_{\substack{(m,B)\\ \neq(n,A)}}\Phi_{nm}\left(\lambda_A^n-\lambda_B^m\right), \tag{21}$$

where

$$\Phi_{nm}(\lambda) := \begin{cases} \frac{\sin\zeta(\lambda+\frac{i}{2}|n-m|)}{\sin\zeta(\lambda-\frac{i}{2}|n-m|)}\left[\frac{\sin\zeta[\lambda+\frac{i}{2}(|n-m|+2)]}{\sin\zeta[\lambda-\frac{i}{2}(|n-m|+2)]}\right]^2 \\ \cdots\left[\frac{\sin\zeta[\lambda+\frac{i}{2}(n+m-2)]}{\sin\zeta[\lambda-\frac{i}{2}(n+m-2)]}\right]^2\frac{\sin\zeta[\lambda+\frac{i}{2}(n+m)]}{\sin\zeta[\lambda-\frac{i}{2}(n+m)]} \\ \qquad\text{for}\quad n\neq m \\ \left[\frac{\sin\zeta(\lambda+i)}{\sin\zeta(\lambda-i)}\right]^2\cdots\left[\frac{\sin\zeta[\lambda+i(n-1)]}{\sin\zeta[\lambda-i(n-1)]}\right]^2\frac{\sin\zeta(\lambda+in)}{\sin\zeta(\lambda-in)} \\ \qquad\text{for}\quad n=m. \end{cases} \tag{22}$$

In logarithmic form, the Bethe-Takahashi equations are written as

$$\theta_n(\lambda_A^n) = \frac{2\pi}{L}I_A^n + \frac{1}{L}\sum_{\substack{(m,B)\\ \neq(n,A)}}\Theta_{nm}\left(\lambda_A^n-\lambda_B^m\right) + in\log\alpha, \tag{23}$$

where

$$\Theta_{nm}(\lambda) := \begin{cases} \theta_{|n-m|}(\lambda)+2\theta_{|n-m|+2}(\lambda)+ \\ \cdots+2\theta_{n+m-2}(\lambda)+\theta_{n+m}(\lambda) \quad\text{for}\quad n\neq m \\ 2\theta_2(\lambda)+2\theta_4(\lambda)\cdots+2\theta_{2n-2}(\lambda)+\theta_{2n}(\lambda) \\ \qquad\text{for}\quad n=m, \end{cases} \tag{24}$$

and $\{I_A^n\}$ are called the Bethe-Takahashi quantum numbers, which are integers when $L-N_n$ is odd and half-integers when $L-N_n$ are even.

The Bethe-Takahashi equations describe the distribution of the string centers. The total momentum K (11) and the energy eigenvalue E (12) are described in terms of string centers $\{\lambda_A^n\}$ as

$$K = \sum_{n,A}\left[\pi-\theta_n\left(\lambda_A^n\right)+in\log\alpha\right], \tag{25}$$

$$E = -\frac{1}{2}\tanh\zeta\sum_{n,A}a_n\left(\lambda_A^n\right). \tag{26}$$

D. Property of string centers

Through the analysis of the Bethe-Takahashi equations (21), we derive the relation between the real and imaginary parts of the string centers. We denote a string center as $\lambda_A^n = a_A^n + ib_A^n$ ($a_A^n, b_A^n \in \mathbb{R}$). The $|\mathrm{LHS}|^{1/L}$ of Eq. (21) is classified into three types according to the relation between the real and imaginary parts of the string center (see Appendix A):

$$\left|\frac{1}{\alpha^n}\frac{\sin\zeta\left(\lambda+\frac{i}{2}n\right)}{\sin\zeta\left(\lambda-\frac{i}{2}n\right)}\right| \begin{cases} >1 & \text{for}\quad b_A^n > c_n\left(a_A^n\right) \\ =1 & \text{for}\quad b_A^n = c_n\left(a_A^n\right) \\ <1 & \text{for}\quad b_A^n < c_n\left(a_A^n\right). \end{cases} \tag{27}$$

As discussed in the derivation of the string solutions (16), we assume the string center λ_A^n that satisfies $b_A^n > c_n(a_A^n)$ $[b_A^n < c_n(a_A^n)]$. From Eq. (27), the $|\mathrm{LHS}|$ of Eq. (21) diverges (goes to 0) in the $L\to\infty$ limit. From the consistency of the Bethe-Takahashi equations (21), the denominator (numerator) of the RHS should go to 0 in this limit. This implies that if the string center λ_A^n with $b_A^n > c_n(a_A^n)$ $[b_A^n < c_n(a_A^n)]$ exists, another string center $\lambda_B^m = \lambda_A^n - \frac{i}{2}\ell$ ($\lambda_B^m = \lambda_A^n + \frac{i}{2}\ell$) also exists in the $L\to\infty$ limit, where $\ell = |n-m|, |n-m|+2, \ldots,$ or $n+m$. However, this contradicts the definition of the string center, which is a representative point of a line parallel to the imaginary axis. Therefore the string centers have to satisfy the relation

$$b_A^n = c_n\left(a_A^n\right). \tag{28}$$

Thus the centers of the n-string solutions are distributed on the curve (15) in the complex plane. Since b_A^n is real, a_A^n is restricted to $-\frac{\pi}{4\zeta} < a_A^n < \frac{\pi}{4\zeta}$. As a result, the string solutions are expressed by a real number a_A^n ($-\frac{\pi}{4\zeta} < a_A^n < \frac{\pi}{4\zeta}$) as

$$\lambda_A^{n,j} = a_A^n + i\left[\frac{n+1-2j}{2}+c_n\left(a_A^n\right)\right]+\delta_A^{n,j}$$
$$\text{for}\quad j=1,2,\ldots,n. \tag{29}$$

IV. NUMERICAL SOLUTIONS

A. Collapse of strings

To confirm the validity of the string solutions, we numerically solve the Bethe equations of the ASEP (8). Although the string solutions are derived in the $L\to\infty$ limit, we can find them even in finite systems. We show the numerical results of the two-string solutions for different hopping rates p in Fig. 2. We confirm that the string solutions with complex string centers exist except for the strong-asymmetry regime.

Figure 2(a) shows the distribution of the two-string solutions. The Bethe roots are distributed approximately i apart on a line parallel to the imaginary axis except for the $p=0.99$ case. In the symmetric case ($p=0.5$), the distribution of the Bethe roots is symmetric about the real axis, and the string centers are real numbers because of the self-conjugacy. When the hopping rate is asymmetric $p>0.5$, the string centers are complexified and satisfy the relation $b_A^2 = c_2(a_A^2)$.

Conversely, in the strong-asymmetry regime, we cannot find the string solutions. Figure 2(b) shows the transition of the two-string solutions as the asymmetry p is increased. The detailed data, which were computed with 1000 digits precision, are shown in Table I. When the asymmetry becomes large ($p=0.9, 0.95$), the difference between the Bethe roots is less than i, and the string center does not satisfy $b_A^2 = c_2(a_A^2)$. Thus the picture of the string solutions collapsed in

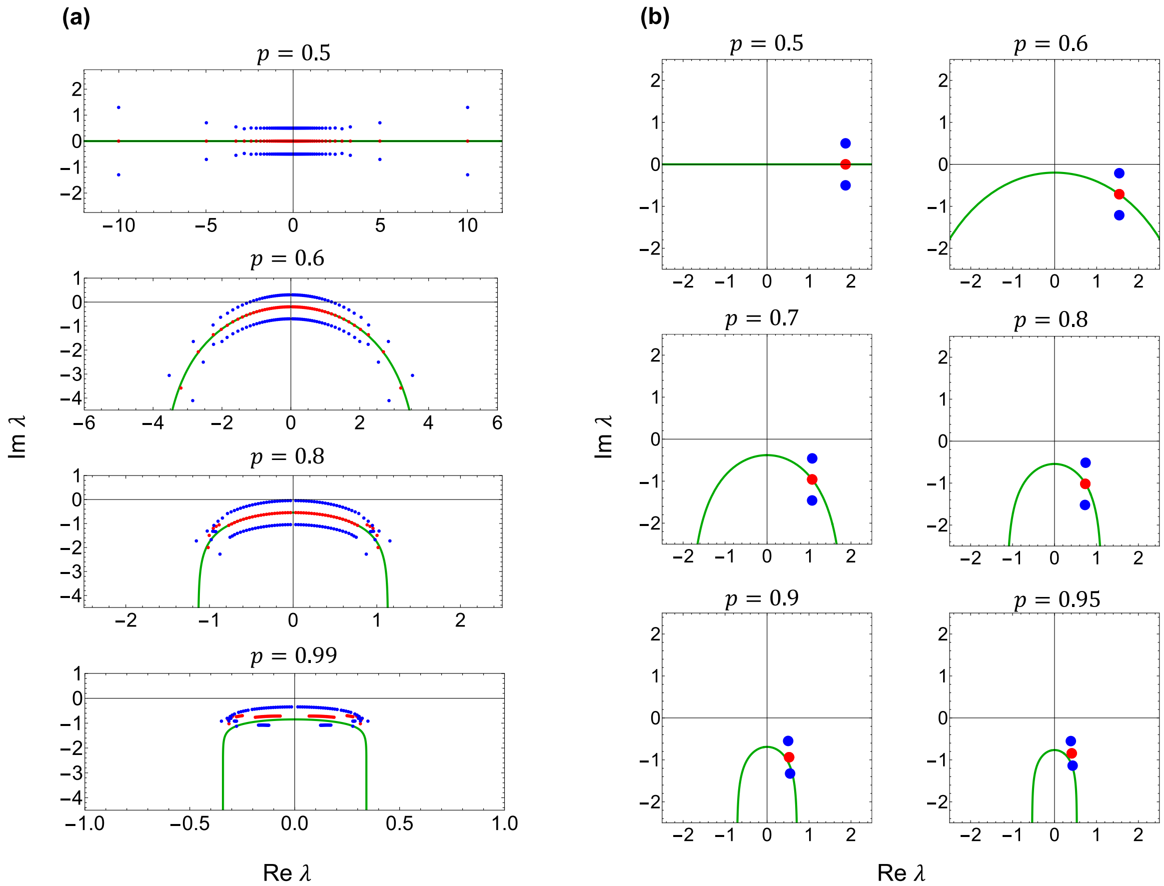

FIG. 2. Two-string solutions of the ASEP for $L = 64$ and $N = 2$. (a) Distribution of the two-string solutions. (b) An example of the collapse of the two-string. Blue dots show Bethe roots, red dots show string centers, and green curves show the relation between the real and imaginary parts of string centers [$b_A^2 = c_2(a_A^2)$].

TABLE I. Two-string solutions of the ASEP for $L = 64$ and $N = 2$.

Hopping rate p	Bethe roots λ	Energy eigenvalue E	Bethe quantum numbers $\{I_1, I_2\}$
0.5	$1.87069731525 + 0.500319967464i$ $1.87069731525 - 0.500319967464i$	-0.222194559412	$\{\frac{53}{2}, \frac{55}{2}\}$
0.55	$1.76812968949 + 0.0795280421456i$ $1.76832342063 - 0.919784365696i$	$-0.224459345125 - 0.0831603910095i$	$\{\frac{53}{2}, \frac{55}{2}\}$
0.6	$1.53829697086 - 0.211511190590i$ $1.53781580701 - 1.21232109213i$	$-0.231075006693 - 0.166256197063i$	$\{\frac{51}{2}, \frac{57}{2}\}$
0.65	$1.29242215650 - 0.376127957552i$ $1.29348654688 - 1.37496941921i$	$-0.242257295822 - 0.249544049466i$	$\{\frac{51}{2}, \frac{57}{2}\}$
0.7	$1.07594641928 - 0.458376716696i$ $1.07361003425 - 1.46011069148i$	$-0.257696964573 - 0.332315028223i$	$\{\frac{49}{2}, \frac{59}{2}\}$
0.75	$0.893248724590 - 0.502127184661i$ $0.899574034235 - 1.49745694729i$	$-0.277788874400 - 0.416702538326i$	$\{\frac{49}{2}, \frac{59}{2}\}$
0.8	$0.741344336499 - 0.514773125989i$ $0.726735163156 - 1.52132494979i$	$-0.302111990569 - 0.496520192138i$	$\{\frac{47}{2}, \frac{61}{2}\}$
0.85	$0.614067763985 - 0.530731832807i$ $0.616101481608 - 1.47435909774i$	$-0.322574664876 - 0.585454599078i$	$\{\frac{47}{2}, \frac{61}{2}\}$
0.9	$0.502576797710 - 0.549017214203i$ $0.548395317443 - 1.32562771882i$	$-0.329804572647 - 0.685161545310i$	$\{\frac{47}{2}, \frac{61}{2}\}$
0.95	$0.389334138196 - 0.552279568789i$ $0.432117780940 - 1.13705859096i$	$-0.330820590099 - 0.779273420786i$	$\{\frac{47}{2}, \frac{61}{2}\}$

this regime. The cause of the collapse of strings is explained as follows. As discussed before, the Bethe equations of the TASEP do not have string solutions. Therefore strings collapse in the TASEP limit ($p \to 1$). In the numerical results, the strings collapse near the TASEP ($p < 1$). This is due to the finite-size effect. String solutions are derived in the $L \to \infty$ limit. Since we consider finite systems in the numerical calculations, the collapse of strings occurs at $p < 1$. We numerically observed that as L increases, strings are not prone to collapse. In the $L \to \infty$ limit, the collapse of strings occurs only in the TASEP ($p = 1$).

B. Delocalization of Bethe quantum numbers

In addition to the collapse of strings, we also find the intriguing phenomenon about the Bethe quantum numbers. In the Heisenberg spin chain, the Bethe quantum numbers of the string solutions tend to be localized [48]. The Bethe quantum numbers of a two-string solution in two-body systems $\{I_1, I_2\}$ $(I_1 \leqslant I_2)$ generally satisfy

$$|I_1 - I_2| \leqslant 1. \tag{30}$$

The string solutions with $|I_1 - I_2| = 1$ are called wide strings, whereas those with $|I_1 - I_2| = 0$ are called narrow strings.

However, the numerical solutions in Table I show that the Bethe quantum numbers of the string solutions in the ASEP might become $|I_1 - I_2| > 1$. In the weak-asymmetry regime, the difference between the Bethe quantum numbers is $|I_1 - I_2| = 1$. As the asymmetry increases, the difference becomes large. For example, when the hopping rate is $p = 0.6$, the difference is $|I_1 - I_2| = 3$. Thus the delocalized Bethe quantum numbers are allowed in the presence of the non-Hermiticity.

V. DELOCALIZATION TRANSITION

Here we elucidate the properties of eigenstates based on the string-type quasiparticle picture through the analysis of the two-body problem. In the case of the Heisenberg spin chain ($p = q$), the Bethe equations have two types of solutions: Real solutions (one-string solutions), which correspond to scattering states, and complex solutions (n-string solutions ($n \geqslant 2$)), which correspond to bound states [49].

The difference between real and complex solutions appears in the spectra of the Hamiltonian. Figure 3 shows the dispersion relation of the Heisenberg spin chain for $L = 32$ and $N = 2$ [49]. For comparison, we also show the dispersion relation of the free magnons. The distribution of the spectra of real solutions is similar to that of the free magnons. The difference between the spectra of real solutions and those of the free magnons is due to the effect of interactions. Conversely, the distribution of the spectra of complex solutions is completely different from those of real solutions and the free magnons. In this sense, complex solutions form the eigenstates where the effect of interactions is prominent.

Similarly, the difference in the spectra between one-string and two-string solutions was also observed in the ASEP. Figure 4 shows the complex spectra of the ASEP for different hopping rates p. We find two types of complex spectra of the ASEP, except near $p = 1$. One type of spectra is regularly distributed inside the large ellipse. The other is distributed on the small ellipse. We call the former type I and the latter type II.

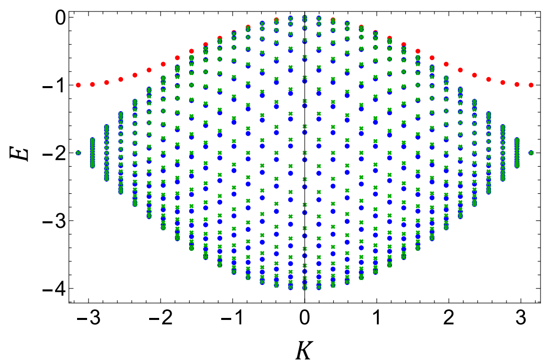

FIG. 3. Dispersion relations of the Heisenberg spin chain and the free magnons for $L = 32$ and $N = 2$. Blue dots show the spectra of real solutions, red dots show the spectra of complex solutions, and green crosses show the spectra of free magnons.

The type-II spectra correspond to the two-string solutions. From Eqs. (26) and (29), the energy eigenvalues of the two-string solutions with the assumption that the deviations vanish ($\delta_A^{n,j} \to 0$) are analytically expressed by the real part of the string center x $(-\frac{\pi}{4\zeta} < x < \frac{\pi}{4\zeta})$ as

$$E(x) = \frac{8\cosh 2\zeta \sinh^2\zeta \cos^2 2\zeta x}{\cos 4\zeta x - \cosh 4\zeta}(1 + i\tan 2\zeta x). \tag{31}$$

This describes an ellipse in a complex plane. Figure 4 shows that the ellipse of Eq. (31) coincides with the distribution of type-II spectra except in $p = 0.99$. Conversely, the spectra do not distribute on the ellipse of the two-string solutions (31)

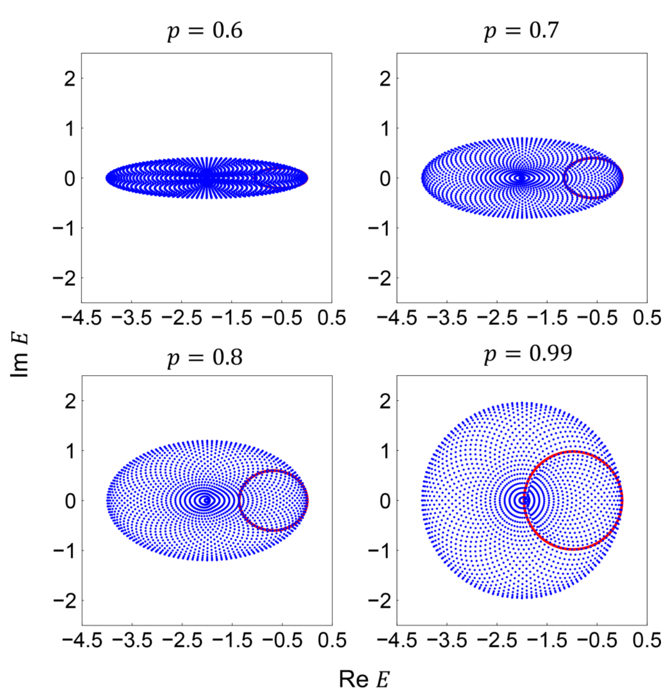

FIG. 4. Complex spectra of the ASEP for $L = 64$ and $N = 2$. Blue dots show the spectra obtained by numerical diagonalization, and red curves show the spectra of two-string solutions [Eq. (31)].

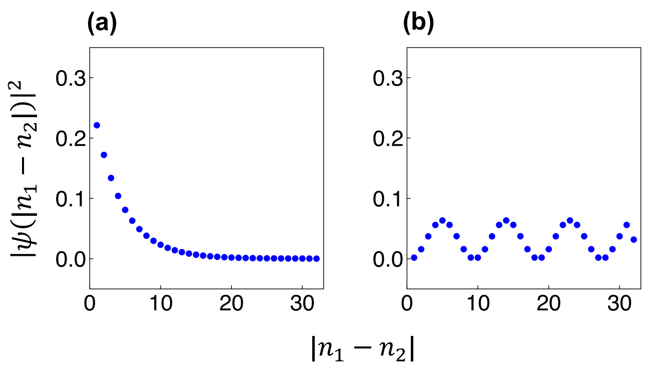

FIG. 5. Weight distributions of the eigenstates corresponding to (a) the complex solution (bound state) and (b) the real solution (scattering state) against the distance between particles in the Heisenberg spin chain.

near the TASEP where the strings collapse [Fig. 4 ($p = 0.99$)]. Generally, as the asymmetry increases the ellipse of type-II spectra becomes larger. Near $p = 1$ the ellipse of type-II spectra blends into the distribution of type-I spectra and becomes hard to distinguish. The indistinguishability of type-I and type-II spectra corresponds to the collapse of strings.

In the Heisenberg spin chain, complex solutions form bound states where particles tend to be localized, while real solutions correspond to scattering states where particles do not. These properties are understood through the weight distributions against the distance between particles as shown in Fig. 5. The weight distribution is calculated as follows. We express an eigenstate $|\psi\rangle$ by the basis vectors $|n_1, n_2\rangle := S^-_{n_1} S^-_{n_2}|0\rangle$ as $|\psi\rangle = \sum_{1\leqslant n_1<n_2\leqslant N} \psi(n_1, n_2)|n_1, n_2\rangle$, where $|0\rangle$ is the vacuum. The weight distribution is defined as $|\psi(\ell)|^2 := \sum_{|n_1-n_2|=\ell} |\psi(n_1, n_2)|^2$.

We find that the two-string solutions of the ASEP also form bound states. Figure 6 shows the weight distributions of the eigenstates of two-string solutions for different hopping rates p. The weight distribution increases as the distance between particles decreases, except near the TASEP. Thus the two-string solutions of the ASEP form bound states. Conversely, in the strong-asymmetry regime where the string solutions collapse, the localization of particles disappears. This indicates that the eigenstate of the string solutions transition from the bound state to the scattering state through the collapse of strings. This delocalization transition is qualitatively explained as follows. In the case of the Heisenberg spin chain, the effect of interactions in scattering states is smaller than that in bound states [49]. The interactions of the ASEP are hardcore interactions. In the case of the TASEP, particles move in only one direction. The collisions of particles are less likely to occur in the TASEP than in the ASEP ($p \neq 1$), and the effect of the interactions is smaller. Therefore the string solutions collapse in the TASEP, and bound states transition to scattering states.

VI. CONCLUSION

We have formulated the string solutions of the Bethe equations in the ASEP and elucidated the delocalization transition induced by the asymmetric hopping based on the picture of string-type quasiparticles. The perspective of the asymmetry-induced transition by the collapse of strings would be applicable to a variety of situations, since the strings are universal quasiparticles widely observed in quantum integrable systems. Recently, related work was proposed where quasiparticles under dissipations are discussed in the context of quantum open systems [50]. Understanding the quasiparticle picture in the general non-Hermitian condition is an important challenge for the future.

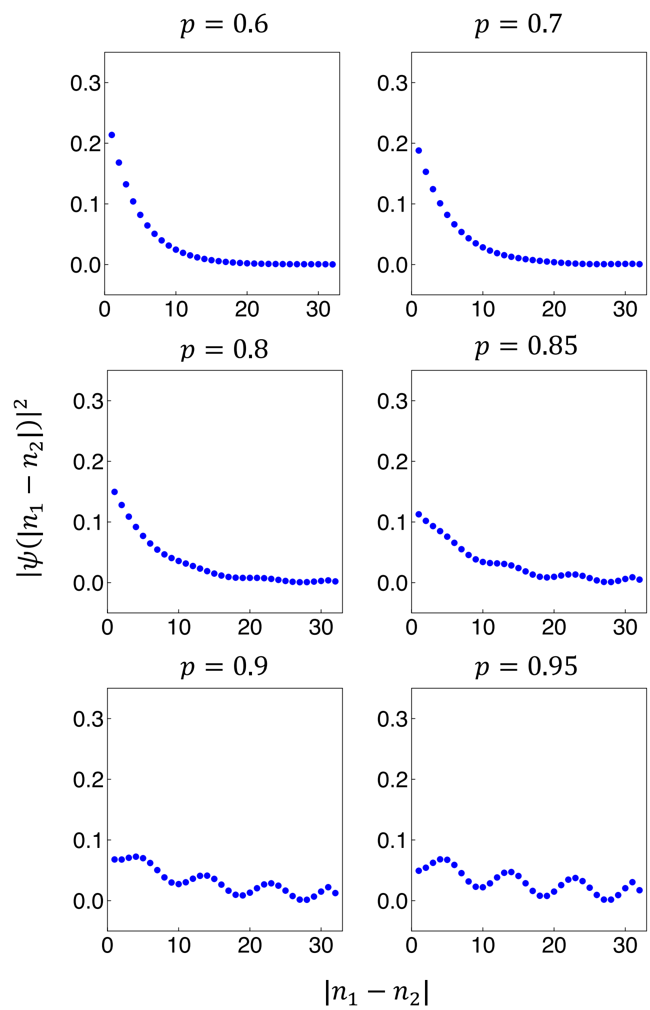

FIG. 6. Weight distributions of the eigenstates of two-string solutions against the distance between particles in the ASEP for $L = 64$ and $N = 2$.

Integrable systems have celebrated analytical frameworks such as the TBA [10,17–20] and the GHD [21–25]. These frameworks are not restricted to standard quantum systems. They are also applicable to classical integrable systems [51,52] and integrable cellular automata [53,54], since the underlying quasiparticle picture is common in integrable systems. By extending these methods to non-Hermitian systems, we aim to establish analytical frameworks for nonequilibrium systems, including stochastic processes and open quantum systems. The results of this work are expected to contribute to the realization of this goal.

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APPENDIX A: PROOF OF THE CLASSIFICATION BY THE BETHE ROOTS [EQS. (14) AND (27)]

Here we prove Eqs. (14) and (27), which provide the classification of the LHS of the Bethe equations (8) and the Bethe-Takahashi equations (21) in the ASEP. Since Eq. (14) is a special case of Eq. (27) with $n = 1$, we consider the proof of Eq. (27).

When the hopping rates are symmetric ($\zeta = 0$), the ASEP is equivalent to the Heisenberg spin chain. In this case, $c_n(x) = 0$ and Eq. (15) is obviously satisfied. In the following we consider $\zeta > 0$. We introduce the functions $f_n(x)$ ($n \in \mathbb{N}$) as

$$\begin{aligned} f_n(b) :&= \left|\sin\zeta\left[a + i\left(b + \frac{n}{2}\right)\right]\right|^2 - \left|\alpha^n \sin\zeta\left[a + i\left(b - \frac{n}{2}\right)\right]\right|^2 \\ &= e^{-\zeta n}\sinh n\zeta\,(e^{2\zeta b}\cosh\zeta n - \cos 2\zeta a). \end{aligned} \tag{A1}$$

The sign of $f_n(b)$ corresponds to the classification of the LHS of Eq. (27). When $f_n(b) > 0$ (<0), the LHS of Eq. (27) is larger (smaller) than 1. From Eq. (A1), $f_n(b)$ is a monotonically increasing function for $\zeta > 0$. The unique solution of $f(c_n) = 0$ is given by

$$c_n = \frac{1}{2\zeta}\log\frac{\cos 2\zeta a}{\cosh n\zeta}. \tag{A2}$$

Therefore $f_n(b) < 0$ for $b < c_n$ and $f_n(b) > 0$ for $b > c_n$. These results lead to Eq. (27).

APPENDIX B: COMPLEXIFICATION OF STRING CENTERS

Here we elucidate the violation of the self-conjugacy of the Bethe roots, which allows the string centers of the ASEP to become complex. The Bethe equations of the Heisenberg spin chain, which correspond to Eq. (8) for the symmetric case ($p = q$), are given by

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}\right)^L = \prod_{\ell\neq j}^{N}\frac{\lambda_j - \lambda_\ell + i}{\lambda_j - \lambda_\ell - i} \quad \text{for} \quad j = 1, 2, \ldots, N. \tag{B1}$$

If $\{\lambda_j\}$ is a solution of the Bethe equations (B1), its complex conjugate $\{\overline{\lambda_j}\}$ is also a solution. The self-conjugacy of the Bethe roots indicates

$$\{\overline{\lambda_j}\} = \{\lambda_j\}. \tag{B2}$$

Since the Bethe equations of the Heisenberg spin chain exhibit this property [47], the distribution of the Bethe roots is symmetric about the real axis, and the string centers become real.

On the other hand, the Bethe equations of the ASEP ($p \neq q$) do not have this property. The complex conjugate of Eqs. (8) are given by

$$\left(\alpha\frac{\sin\zeta\left(\overline{\lambda_j} + \frac{i}{2}\right)}{\sin\zeta\left(\overline{\lambda_j} - \frac{i}{2}\right)}\right)^L = \prod_{\ell\neq j}^{N}\frac{\sin\zeta\left(\overline{\lambda_j} - \overline{\lambda_\ell} + i\right)}{\sin\zeta\left(\overline{\lambda_j} - \overline{\lambda_\ell} - i\right)} \quad \text{for} \quad j = 1, 2, \ldots, N. \tag{B3}$$

Thus $\{\overline{\lambda_j}\}$, which is a complex conjugate of the Bethe roots, is not a solution to the Bethe equations due to the asymmetry α. In this case, $\{\overline{\lambda_j}\}$ is a solution to the Bethe equations where the asymmetry is reversed, $\alpha \to \alpha^{-1}$. Because of the violation of the self-conjugacy, the distribution of the Bethe roots is not symmetric about the real axis. Therefore the string centers of the ASEP are complexified.

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