Random networks with q-exponential degree distribution

Cesar I. N. Sampaio Filho,[1] Marcio M. Bastos,[1] Hans J. Herrmann,[1,2] André A. Moreira,[1] and José S. Andrade, Jr.[1]
[1]*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Brazil*
[2]*PMMH, ESPCI, CNRS UMR 7636, 7 quai St. Bernard, 75005 Paris, France*

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We use the configuration model to generate random networks having a degree distribution that follows a q-exponential, $P_q(k) = (2-q)\lambda[1-(1-q)\lambda k]^{-1/(q-1)}$, for arbitrary values of the parameters q and λ. Typically, for small values of λ, this distribution crosses over from a plateau at small k's to a power-law decay at large values of the node degrees. Furthermore, by sufficiently increasing λ, we can continuously narrow this plateau, getting closer and closer to a pure power-law degree distribution. As a generalization of the pure scale-free networks, therefore, q-exponentials display a rich variety of behavior in terms of their topological and transport properties. This is substantiated here by investigating their average degree, assortativity, small-world behavior, resilience to random and malicious attacks, and k-core decomposition. Our results show that the more the degree distribution resembles a pure power law, the less well connected the networks. As a consequence, their average degree follows $\langle k\rangle \sim \lambda^{-1}$ for $\lambda < 1$, and the expected average degree k_{nn} of the nearest neighbors of a given node with degree k generally decreases with λ. Moreover, random q-exponential networks exhibit small-world behavior for any λ, but with an average shortest path that becomes smaller as λ decreases and q increases. Finally, q exponentials become more resilient to random and malicious attacks as their degree distribution systematically deviates from the pure power law. Being at the same time well-connected and robust, networks with q-exponential degree distribution exhibit scale-free and small-world properties, making them a particularly suitable model for applications in several systems.

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I. INTRODUCTION

In a large variety of fields, one finds experiments, numerical results, and theoretical models that fairly well agree with q exponentials. This includes applications in fully developed turbulence [1], anomalous diffusion in plasmas [2], statistics of cosmic rays [3], econometry [4,5], biophysics [6], and many others. In particular, many empirical complex networks have been found to follow q-exponential degree distributions [7–9]. The same behavior has also been detected in several proposed model networks [10], most of them generated through growth models based on the preferential attachment principle [11–14]. These models produce empirically q-exponential distributions, which in some cases can even be confirmed analytically [8,14]. Here we use the configuration model [15] to systematically generate and study the properties of random networks with arbitrarily chosen q-exponential degree distribution.

Networks are called scale-free if the distribution of degrees follows a power law with exponent γ, $P(k) \sim k^{-\gamma}$, where k is the degree of a node, defined as the number of connection it has with other nodes. The q-exponential distribution given by

$$P_q(k) = (2-q)\lambda[1-(1-q)\lambda k]^{-1/(q-1)}, \tag{1}$$

has two parameters, $q \geqslant 1$ and $\lambda \geqslant 0$. It is a generalization of a power-law distribution, since for large k ($k \gg \lambda^{-1}$) it decays like $k^{-\gamma}$ with $\gamma = 1/(q-1)$. For small degrees ($k \ll \lambda^{-1}$), however, it tends to a plateau distribution of height $(2-q)\lambda$. The parameter λ^{-1} therefore determines the crossover between these two regimes. The distribution Eq. (1) represents a fundamental ingredient in the mathematical formalism of the generalized thermostatistics and its applications [16–24]. Although most properties observed in complex networks with heavy-tailed distributions are determined by the shape of the tail, deviations from the power law at smaller degrees alter the occurrence of the least connected nodes, resulting in structural changes that can affect the main properties and processes taking place on these networks. In what follows, we take a closer look at these effects.

Different from previous works based on preferential attachment models of growth to obtain networks with degree distributions mimicking the q-exponential behavior, here, as already mentioned, we employ the configuration model [15] to build our random networks. In this way, we assure that the particular topological properties observed, including intrinsic node correlations [25], are not induced by the growth process, but are rather a direct consequence of the q-exponential form of the degree distribution. The remainder of the paper is organized as follows. In Sec. II, we describe the construction of q-exponential networks using the configuration model. In Sec. III, we investigate, as a function of parameters q and λ, the main properties of these networks, namely, their intrinsic assortativity, average minimum path, and robustness to random failure and malicious attack targeted by the highest degree. Finite-size scaling analysis is also used to investigate the k-core decomposition of the q-exponential networks. We conclude in Sec. IV.

II. GENERATING q-EXPONENTIAL NETWORKS

We start by assigning to each node i a prescribed degree k_i, drawn from a q-exponential distribution. More precisely, since the degrees are integers, we chose randomly a number x_i from a q-exponential distribution and define k_i as the largest integer smaller than x_i. To avoid obtaining many small disconnected clusters, we only consider nodes with degree $k_i \geqslant 2$. We visualize the yet unconnected k_i degrees on node i by k_i stubs. Then we proceed to connect nodes pairwise. To do this, we choose two different nodes with probabilities proportional to their number of stubs. If these two nodes are not yet connected, a link is placed between them and the number of stubs is decreased for each of the two nodes by one. This process continues until all stubs have been connected or, in case that at the end some stubs remain disconnected, these stubs are removed and the degree originally assigned to the corresponding nodes is adjusted accordingly. Furthermore, a maximum degree $k_{\max}$ exists due to the fact that the number of nodes in the network is finite. When $\gamma < 3$, the second moment $\langle k^2 \rangle$ of the distribution diverges and one can show that $\langle k^2 \rangle \sim k_{\max}^{3-\gamma}$, i.e., that the second moment is controlled by the most connected node. This property can have profound effects on the behavior of processes taking place in the network, including fragility to targeted attacks and resilience to random failure [26–28].

III. RESULTS AND DISCUSSION

As shown in Fig. 1, the degree distributions obtained by this method follow very closely the expected q-exponential form of Eq. (1). For $\lambda \ll 1$, one observes a pronounced plateau for small k. On the other hand, when $\lambda \gg 1$, effectively, our degree distribution will be identical to a scale-free distribution with the same $k_{\min}$. Therefore, by decreasing the parameter λ, we can continuously move away from a scale-free degree distribution and widen the plateau. Thus, varying λ will allow us to identify the effect of the deviations from pure scale-freeness that are particular to q-exponential distributions.

In Fig. 2, we show how the average degree $\langle k \rangle$ depends on λ for different values of the parameter q. As expected for a q-exponential network, for small values of λ, the average degree is proportional to λ^{-1}. However, for sufficiently large values of λ, the degree distribution turns into a power law. In this last regime, the average degree becomes independent on λ. Therefore, on one hand scale-free networks have many more least-connected nodes than networks having q-exponential degree distribution with small λ. The smaller λ, the denser the networks become, increasing the number and degree of their hubs. Topological differences like these can lead to substantial changes in structural properties of complex networks as well as in the static and dynamical behavior of models when implemented on these substrates.

It has been shown [25,29,30] that random networks with heavy-tailed degree distribution often display intrinsic assortativity. This is also the case for q-exponential networks. In Fig. 3, we show the expected average degree $k_{\mathrm{nn}}(k)$ of the nearest neighbors of a given node with degree k. As can be seen, the most connected nodes have a smaller k_{nn}, pointing towards negative intrinsic assortativity. The reason for this dissortative behavior in random networks is that the most connected nodes can neither connect to themselves nor have

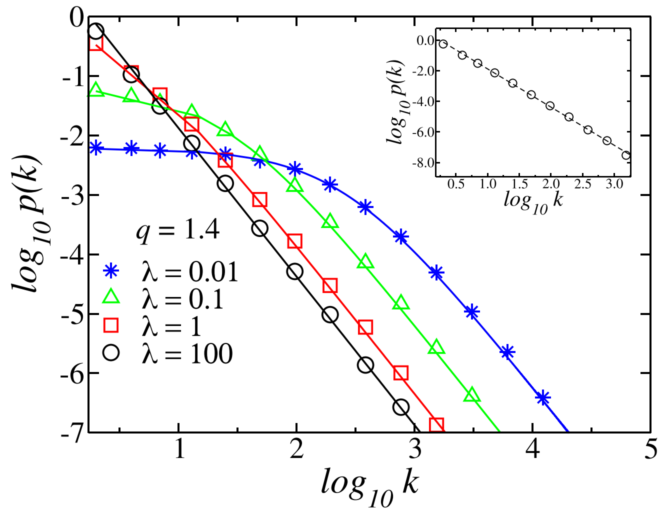

FIG. 1. Degree distribution of q-exponential networks (symbols) compared with the expected distribution of Eq. (1) (solid lines) for $q = 1.4$ and for $\lambda = 0.01$ (blue stars), $\lambda = 0.1$ (green triangles), $\lambda = 1$ (red squares), and $\lambda = 100$ (black circles). The inset shows a comparison between the degree distributions of a q exponential with $\lambda = 100$ and a pure scale-free distribution (dashed black line) with $q = 1.4$ ($\gamma = 2.5$). These results are obtained for networks with size $N = 500\,000$ by averaging over 100 samples. As can be seen, for small λ the distributions attain a plateau at small degrees and the power-law regime becomes larger with increasing λ.

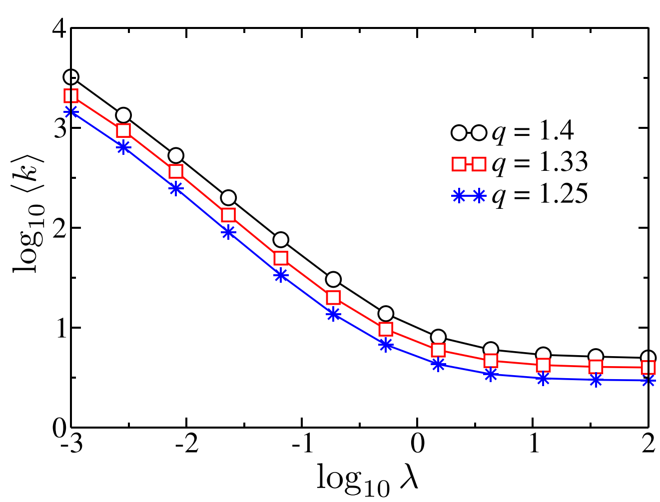

FIG. 2. Dependence of the average degree $\langle k \rangle$ on the parameter λ for different values of q. These curves correspond to $q = 1.4$, (black circles), $q = 1.33$, (red squares), and $q = 1.25$, (blue stars). For values of $\lambda < 1$, the average degree follows $\langle k \rangle \sim \lambda^{-1}$, as expected for q-exponential distributions. In the pure power-law limit ($\lambda \gg 1$), the average degree saturates at a value independent of λ.

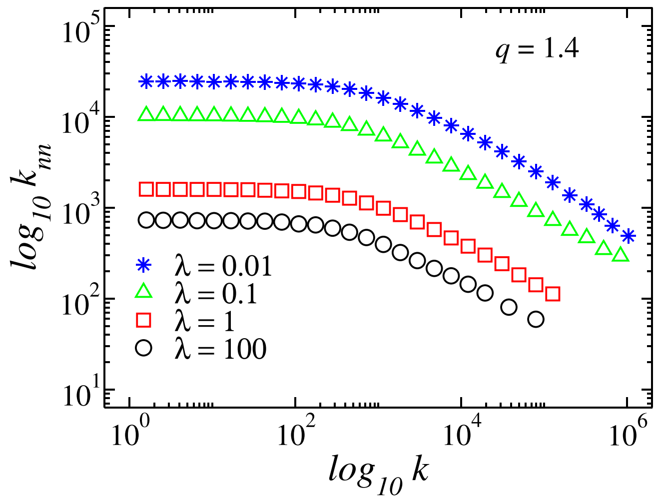

FIG. 3. Dependence of the average nearest-neighbor degree k_{nn} of vertices on the degree k for networks with size $N = 10^7$ nodes, $q = 1.4$, and $\lambda = 100$ (black circle), $\lambda = 1$ (red square), $\lambda = 0.1$ (green triangle), and $\lambda = 0.01$ (blue star). The plateaus of each curve correspond to the value $\langle k^2\rangle/\langle k\rangle$ [25]. All curves point towards intrinsic dissortative behavior for sufficiently large values of k.

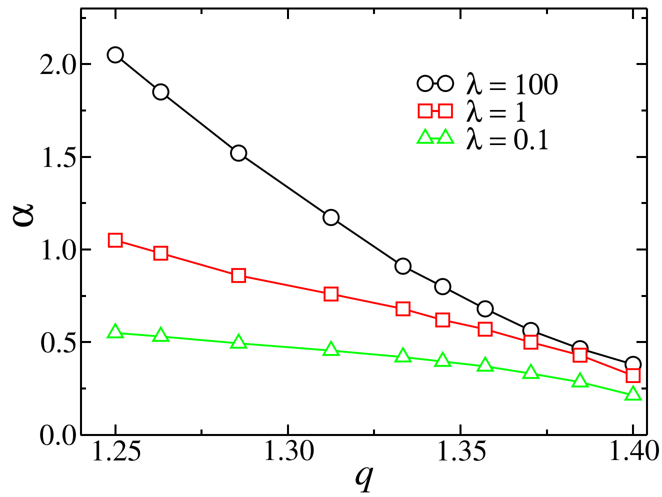

FIG. 5. The prefactor α as a function of q for $\lambda = 100$ (black circles), $\lambda = 1$ (red squares), and $\lambda = 0.1$ (green triangles). Each point is obtained from a least-squares fit to the data, $\langle\ell\rangle \sim \alpha \log_{10} N$, with error bars being smaller than the symbols. The continuous lines represent guides to the eye.

multiple connections between them [25]. We also see from Fig. 3 that k_{nn} becomes larger the more the distribution deviates from a pure scale-free one.

Random q-exponential networks also exhibit small-world behavior, which means that their average shortest path increases logarithmically with network size, $\langle\ell\rangle = \alpha \log_{10} N$, as shown in Fig. 4. Clearly, the shortest path becomes considerably shorter as the degree distributions systematically deviate from the pure power law, namely, for sufficiently large values of λ. Figure 5 shows the variation of the prefactor α for different values of q in the range $1.25 \leqslant q \leqslant 1.4$, and $\lambda = 0.1$, 1, and 100. For all practical purposes, this range of q, which corresponds to $2.5 \leqslant \gamma \leqslant 4.0$, covers all interesting power-law decays. As already mentioned, the case $\lambda = 100$ is equivalent to a scale-free network with $P(k) \sim k^{-\gamma}$ and $k_{\mathrm{min}} = 2$.

A particularly important property for practical purposes is the robustness of networks against random failures. In Fig. 6, we plot the density of nodes in the largest cluster

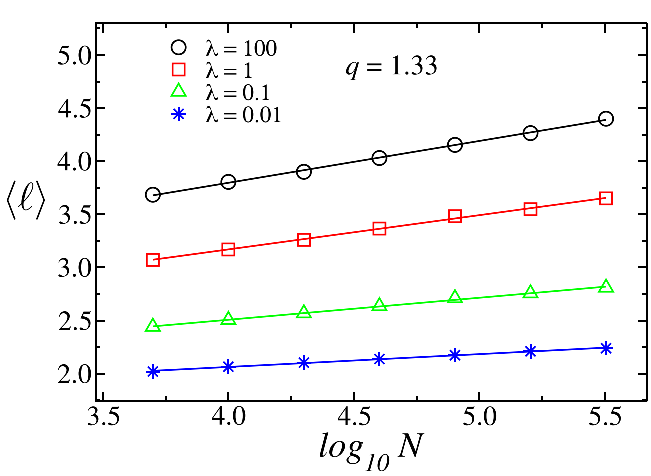

FIG. 4. Size dependence of the mean length of the shortest-path $\langle\ell\rangle$ for $q = 1.33$ and for $\lambda = 0.01$ (blue stars), $\lambda = 0.1$ (green triangles), $\lambda = 1$ (red squares), and $\lambda = 100$ (black circles). The results are obtained by averaging over 10^5, 10^4, 10^4, 10^4, 10^4, 10^3, and 10^3 samples of size $N = 5000$, 10 000, 20 000, 40 000, 80 000, 160 000, and 320 000, respectively.

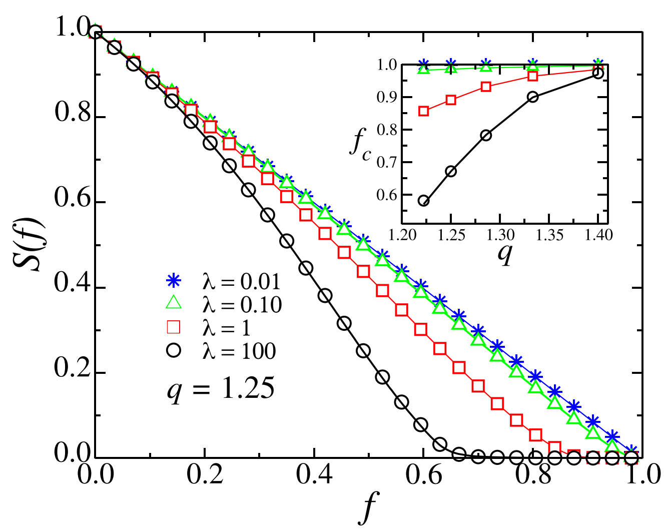

FIG. 6. The density of nodes in the largest cluster $S(f)$ as a function of the fraction f of removed nodes for random failures in q-exponential networks with $q = 1.25$, $\lambda = 0.01$ (blue stars), $\lambda = 0.1$ (green triangles), $\lambda = 1$ (red squares), and $\lambda = 100$ (black circles). These results correspond to networks of size $N = 320000$ by averaging over 200 samples. The continuous lines for $\lambda = 0.01$, 0.1, and $\lambda = 1$ represent guides to the eye, while for $\lambda = 100$ the black solid line corresponds to the result for random attacks on a pure scale-free network. The inset shows the critical fraction f_c determined by the Molloy-Reed criteria [32].

$S(f)$ as a function of the fraction f of randomly removed nodes, for $N = 320\,000$, $q = 1.25$, and various values of λ. For large $\lambda \gg 1$, this degree distribution becomes a power law with $\gamma = 4$. Considering that $k_{\min} = 2$, this network should exhibit a critical point $f_c < 1$ [31]. That is precisely what one sees in Fig. 6 for $\lambda = 100$. For smaller values of λ, the network becomes more robust, with $f_c \to 1$ as $\lambda \to 0$. In the inset of Fig. 6 we see how f_c depends on q for different values of λ. Clearly, the closer a q-exponential network resembles a scale-free, namely, for large values of λ, the more it is fragile against random failures. Furthermore, this effect is dramatically amplified with the decrease of the parameter q.

In the case of malicious attack, we show in Fig. 7(a) the variation of the density of the largest cluster S as a function of the fraction f of removed nodes targeted by the highest degree, for $q = 1.4$ ($\gamma = 2.5$) and different values of λ. Here the degree distribution is updated after each node removal, as described in Refs. [28,33]. As depicted, the q-exponential networks become less and less resilient as the crossover λ increases, since a larger fraction of nodes needs to be removed before the critical point is achieved. This behavior persists up to a point at which the value of λ is sufficiently large, so the scale-free behavior of the degree distribution dominates. As a result, the q-exponential curve S versus f for $\lambda = 100$ and the corresponding one generated from networks with pure scale-free distribution are perfectly coincident.

At this point, we argue that the basis for comparing the robustness of scale-free and q-exponential networks is not obvious, in the sense that the first can be viewed as a particular case of the second if we consider the same values of q and $k_{\min}$ and for a sufficiently large value of the parameter λ. The situation becomes quite different for low values of λ, since these q-exponential networks have substantially larger values of $\langle k \rangle$, being therefore more resilient. In order to obtain scale-free networks having the same average degree as these q exponentials, we therefore have to change their $k_{\min}$. In any case, it is still interesting to compare the resilience of q-exponential and scale-free networks with similar average degrees. As an example, if we fix $q = 1.4$ and consider q-exponentials generated with $k_{\min} = 2$ and $\lambda = 10^{-1}$, and scale-free networks with $k_{\min} = 8$, both of them of size $N = 10^6$, their resulting average degrees become quite close, namely, $\langle k \rangle = 24$ and 25, respectively. In the case of random attacks, the q-exponential and the scale-free networks respond in a similar fashion, being necessary the removal of most of the nodes to completely degrade their largest clusters (not shown). This strong resilience should be expected, since the parameter $q = 1.4$ ($\gamma = 2.5$) falls into the regime where even scale-free networks are robust to random failure [31]. The results presented in Fig. 7(b) show that, in these particular conditions, q-exponential networks (with small λ) are less resilient for malicious attacks than scale-free ones (or q-exponentials with large λ). Precisely, the fraction of the largest cluster, $S(f)$, of these q-exponential networks decays faster with f than that of their scale-free counterparts with similar average degree, with the largest cluster vanishing after a smaller fraction of the nodes are removed. For comparison, we also show the response to malicious attacks of the less robust scale-free networks generated with $k_{\min} = 2$, leading to $\langle k \rangle = 6$.

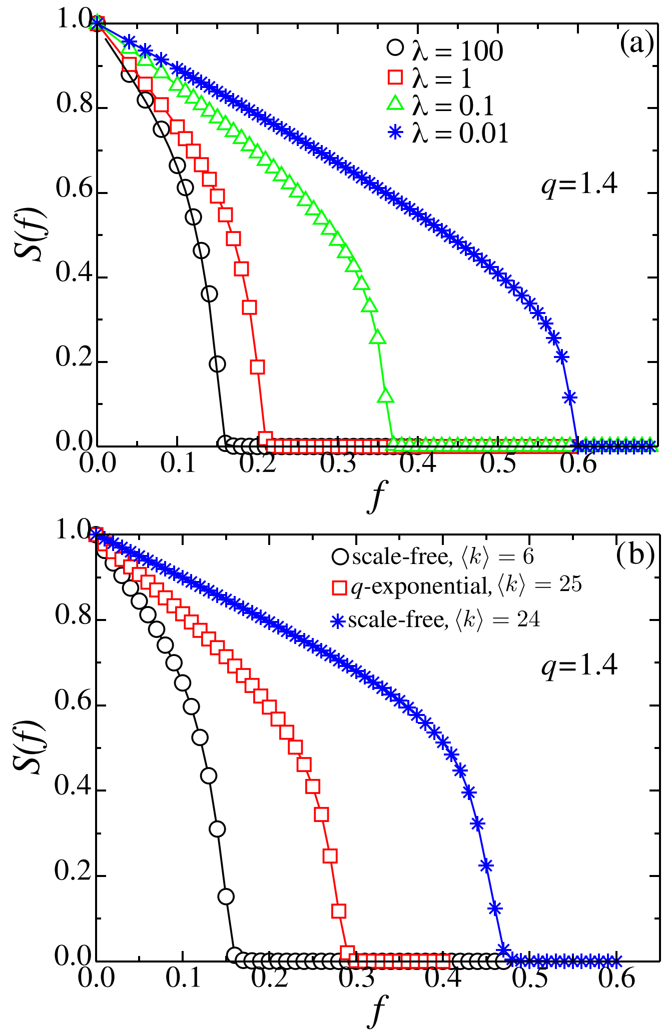

FIG. 7. (a) The density of nodes in the largest cluster $S(f)$ as a function of the fraction f of removed nodes for malicious attacks targeted by the highest degree, for $q = 1.4$ and $\lambda = 0.01$ (blue stars), $\lambda = 0.1$ (green triangles), $\lambda = 1$ (red squares), and $\lambda = 100$ (black circles). The continuous lines for $\lambda = 0.01, 0.1$, and $\lambda = 1$ represent guides to the eye, while for $\lambda = 100$ the black solid line corresponds to the result for malicious attack on a pure power-law network. These results correspond to networks of size $N = 320\,000$ by averaging over 200 samples. (b) The density $S(f)$ as a function of the fraction f of networks generated with $q = 1.4$ and subjected to malicious attacks targeted by highest degree. The results correspond to scale-free networks with $k_{\min} = 2$ and $\langle k \rangle = 6$ (black circles), $k_{\min} = 8$ and $\langle k \rangle = 24$ (blue stars), and for q-exponential networks with $\lambda = 10^{-1}$, $k_{\min} = 2$, and $\langle k \rangle = 25$ (red squares). They were obtained with networks of size $N = 10^6$ by averaging over 100 samples.

Figure 8 shows the dependence of the critical fraction f_c on the parameter q, as determined by Molloy-Reed's criterion [32] and for different values of q. It is interesting to note that the behavior of f_c changes substantially with λ. For instance, considering $\lambda = 0.1$ we see a plateau up to $q \approx 4/3$, followed by a decay. On the other hand, for $\lambda = 100$ we observe a clear maximum in f_c at $q \approx 4/3$. As mentioned, for $\lambda = 100$ the degree distribution approaches the form of a power law, and this maximum in the critical condition is consistent with the

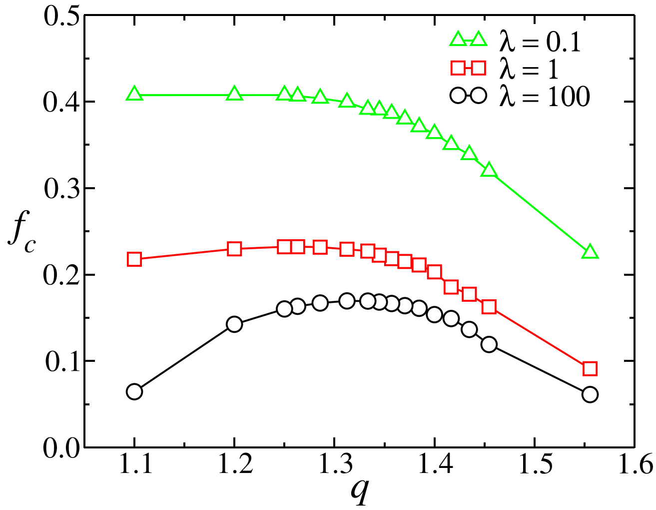

FIG. 8. The critical fraction f_c for malicious attacks as a function of q for $\lambda = 0.1$ (green triangles), $\lambda = 1$ (red squares), and $\lambda = 100$ (black circles). The results are obtained for networks of size $N = 500\,000$ by averaging over 2000 samples. The fraction from which the largest cluster does not obey Molloy-Reed's criterion is the critical fraction f_c. The q-exponential networks with smaller values of λ are clearly more robust, as a larger fraction of nodes needs to be removed to attain the critical point. The continuous lines represent guides to the eye.

expected for pure scale-free networks [34]. We note that this maximum is due to a compromise between two effects. For $q > 4/3$, the degree distribution decays asymptotically as a power law with controlling exponent $\gamma < 3$, reaching $\gamma = 2$ as q approaches $3/2$. At this limit, removing just a few hubs results in a total breakdown of the network. At the other limit, as γ diverges when $q \to 1$, the degree distribution is no longer heavy-tailed, decreasing rapidly. Moreover, by setting $k_{\min} = 2$, the generated network is already near the critical state. This behavior can be suppressed by imposing $k_{\min} \geqslant 3$ in the case of pure scale-free networks [34], or by using small values of λ in q-exponential networks.

Next we extend the analysis of the topology of q-exponential networks by investigating their hierarchical structure in terms of the k-core decomposition method [35–41]. The k-core of a graph G is the largest connected subgraph in which all its nodes have a degree larger than or equal to k. To obtain the k core, we remove all nodes with a degree less than k. Next, we scan if some nodes still have a current degree less than k and remove them. We repeat this check until no additional removal is possible. From this decomposition, we can define for each node a rank in the network, such that a node will be the more peripheral the smaller is its k. k-core subgraphs are resilient against failure, since they preserve their convexity after $(k-1)$ random rewirings of edges or nodes. This kind of robustness tends to increase for the innermost nodes. Here we analyze the finite-size dependence of the highest k core, k_h, and its mass, M_h, for q-exponential networks with $q = 1.4$ and different values of the parameter λ. From Ref. [40], we expect to obtain finite-size scaling as

$$k_h \sim N^{\delta} \tag{2}$$

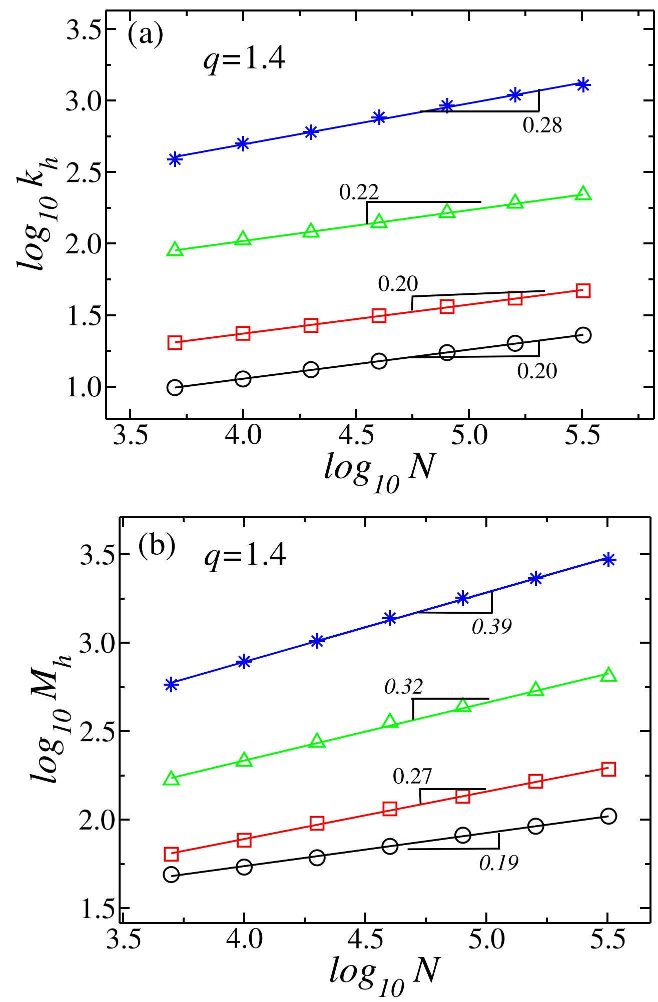

FIG. 9. (a) Highest k-core and (b) mass of the highest k-core versus N for $q = 1.4$ and $\lambda = 0.01$ (blue stars), $\lambda = 0.1$ (green triangles), $\lambda = 1$ (red squares), and $\lambda = 100$ (black circles). The results are obtained by averaging over 10^5, 10^4, 10^4, 10^4, 10^4, 10^3, and 10^3 samples of size $N = 5000$, 10 000, 20 000, 40 000, 80 000, 160 000, and 320 000, respectively.

for the highest k-core, and

$$M_h \sim N^{\Delta} \tag{3}$$

for the mass of the highest k core. We find that the exponents δ and Δ are functions of the distribution parameters q and λ. Figure 9(a) shows the dependence on the network size N of k_h in a double-logarithmic plot for $q = 1.4$ and different values of λ. As compared to the moderate dependence of the exponent δ on λ, the prefactor of the relation of Eq. (2), however, increases considerably with λ^{-1}. On the other hand, as shown in Fig. 9(b), the dependence of M_h on N indicates that both the exponent Δ and the prefactor of Eq. (3) decrease substantially with λ. This shows that networks with a larger plateau (smaller λ) have a highest k core that is bigger and has a larger k. This is in accordance with our previous findings that q-exponential networks become more robust the smaller λ.

IV. CONCLUSIONS

In conclusion, we have generated unbiased complex networks exhibiting q-exponential degree distributions with arbitrary parameter values. These structures generalize the scale-free networks in the sense that pure power-law degree distributions can be obtained through the control of the crossover parameter λ determining the extent of the q-exponential plateau at low values of node degree. Our results show that this additional degree of freedom gives the q-exponential networks great flexibility with respect to topological and transport properties, as investigated here for their assortativity, small-world behavior, resilience to random and malicious attacks, and finite-size scaling of the k-core decomposition elements. In the future, it would be interesting to also study dynamical properties of these networks, like synchronization, epidemic spreading, and opinion formation.

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