Bosonic Gaussian channel and Gaussian witness entanglement criterion of continuous variables

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We use quantum entanglement witnesses derived from Gaussian operators to study the separable criteria of continuous variable states. For bipartite system, we transform the validity of a Gaussian witness to a bosonic Gaussian channel problem. It follows that the maximal means of two-mode and some four-mode Gaussian operators over product pure states are achieved by vacuum (or coherent states and squeezed states) according to the properties of bosonic Gaussian channels. We demonstrate necessary and sufficient criteria of separability for two-mode and some four-mode Gaussian quantum states. We also propose multipartite Gaussian witnesses based on numeric evidence. The entanglement and multipartite entanglement detecting powers of the Gaussian witness criteria are illustrated by the mixture of two Gaussian states. We apply the Gaussian witness criterion to a Gaussian Fock state, which is prepared by passing a product Fock state through bosonic Gaussian channel. We show that the necessary criterion of separability for a Gaussian Fock state does not rely on the initial product Fock state, it relies on the channel.

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I. INTRODUCTION

Quantum entanglement is a key role in the rapidly developing field of quantum information processing. Its characterization and detection have been a topic of crucial importance that has attracted a great deal of effort, both theoretically and experimentally. Many entanglement criteria have been proposed to detect entanglement [1,2]. They are positive partial transpose (PPT) criterion [3,4], uncertainty principle criterion [5], entropy criterion [6], and so on. The criteria for continuous variable system were first proposed for two-mode Gaussian states [5,7], they are necessary and sufficient. Further developments of the entanglement criteria are either for Gaussian states [8–10] or non-Gaussian states [11–19], including photon addition and subtraction from a Gaussian state [20–22]. Non-Gaussian states are essential for universal quantum computations [23]. The ability of non-Gaussian entanglement detection is the focus of a new entanglement criterion.

Entanglement can also be detected with entanglement witnesses [2,24]. An entanglement witness $\hat{W}$ is a Hermitian operator, which has non-negative means on all separable states $\hat{\rho}_s$, namely, $\mathrm{Tr}(\hat{\rho}_s\hat{W})\geqslant 0$ and has a negative mean at least on one entangled state $\hat{\rho}$, namely, $\mathrm{Tr}(\hat{\rho}\hat{W})<0$. Let $\hat{W}=\Lambda\hat{I}-\hat{M}$, where $\hat{I}$ is the identity operator, $\hat{M}$ is a Hermitian operator (we call it detect operator). The optimization of the inequality $\mathrm{Tr}(\hat{\rho}_s\hat{W})\geqslant 0$ over separable state set fixes one of the parameters of the witness [25,26], we have $\Lambda=\sup_{\hat{\rho}_s}\mathrm{Tr}(\hat{\rho}_s\hat{M})$. The witness then is called weakly optimal [2].

Hence for a given detect operator $\hat{M}$, there is a necessary criterion of separability

$$\mathrm{Tr}(\hat{\rho}\hat{M})\leqslant\Lambda. \tag{1}$$

The weak optimization of a witness was applied to detect the entanglement structure of a six-mode continuous variable state [27], with the witness chosen as the quadratic combination of position and momentum operators. In general, it is rather ad hoc to choose a witness operator. The weak optimization problem is to minimize the mean of a witness over the set of product states. Although this can be transformed to the multipartite (or bipartite) separability eigenvalue equation [26], the weak optimization of a witness is in fact a process of iterative calculation. The overall performance of the multipartite separability equation was presented by the entanglement and genuine entanglement detection of a W state mixed with white noise [26], it is weaker than the other criteria [28,29] for this special example. However, it is possible to enhance the entanglement detecting ability of the witness method. A full optimization [30–32] of all the parameters of a witness will lead to a matched entanglement witness. The matched entanglement witness is capable of detecting different types of entanglement in multipartite entanglement [30]. The necessary and sufficient conditions of entanglement and genuine entanglement for a W state mixed with white noise were derived with the matched entanglement witness [32].

For a given state $\hat{\rho}$, we define $\mathcal{L}:=\inf_{\hat{M}}\frac{\Lambda}{\mathrm{Tr}(\hat{\rho}\hat{M})}$ with $\Lambda>0$ and $\mathrm{Tr}(\hat{\rho}\hat{M})>0$. The refined separable criterion is

$$\mathcal{L}\geqslant 1. \tag{2}$$

Notice that (2) is a necessary and also sufficient criterion for separability. The sufficiency comes from that if there is an entangled state $\hat{\rho}$ such that $\mathcal{L}\geqslant 1$, it means there is no witness that can detect this entangled state. However, it is known that "for each entangled state $\hat{\rho}$, there exists an entanglement witness detecting it" [2,4].

II. BOSONIC GAUSSIAN CHANNEL AND GAUSSIAN WITNESS

A Gaussian state of n modes is completely characterized by a $2n \times 2n$ real, positive, and symmetric matrix γ called covariance matrix (CM) up to its mean. The covariance matrix should fulfill a further condition of $\gamma - \frac{i}{2}\sigma \geqslant 0$, where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus n}$. For a general quantum state $\hat{\rho}$, the mean of position and momentum operator vector $\hat{R} = (\hat{x}_1, \hat{p}_1, \dots, \hat{x}_n, \hat{p}_n)^T$ is defined as the vector $m_j = \mathrm{Tr}(\hat{\rho}\hat{R}_j)$. The covariance matrix is

$$\gamma_{ij} = \tfrac{1}{2}\mathrm{Tr}[\hat{\rho}(\hat{R}_i\hat{R}_j + \hat{R}_j\hat{R}_i)] - m_i m_j. \tag{3}$$

The mean can be removed by proper local displacement. A local displacement does not affect the entanglement, which implies that we only need to consider the case of zero mean. The characteristic function of a state is defined as $\chi(z) = \mathrm{Tr}[\hat{\rho}\exp(iz\hat{R})]$, where $\exp(iz\hat{R})$ is called Weyl operator. For a Gaussian state, we have $\chi(z) = \exp(-\frac{1}{2}z\gamma z^T)$ when the mean is set to zero. Sometimes, it is more convenient to deal with the annihilation operator and creation operator. Let the Weyl operator $\exp(iz\hat{R})$ be equal to the displacement operator $\mathcal{D}(\mu) = \exp[\Sigma_{j=1}^n(\mu_j\hat{a}_j^\dagger - \mu_j^*\hat{a}_j)]$, where $\hat{a}_j = \frac{1}{\sqrt{2}}(\hat{x}_j + i\hat{p}_j)$ and $\hat{a}_j^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_j - i\hat{p}_j)$ are the annihilation and creation operators of the jth mode, respectively. Then, we have $\mu_j = \frac{1}{\sqrt{2}}(-z_{2j} + iz_{2j-1})$. The characteristic function, $\chi(z) = \chi(\mu) = \exp[-\frac{1}{2}(\mu, \mu^*)\tilde{\gamma}(\mu, \mu^*)^T]$, with complex covariance matrix $\tilde{\gamma}$ being a transform of the CM γ.

We choose a zero-mean Gaussian operator $\hat{M}$ as the detect operator. We call the corresponding witness operator $\hat{W}$ Gaussian witness for simplicity. For the validity of the witness, we need to calculate the maximal mean of the detect operator $\hat{M}$ over product states. We have $\Lambda = \sup_{\hat{\rho}_A,\hat{\rho}_B}\mathrm{Tr}[\hat{M}(\hat{\rho}_A \otimes \hat{\rho}_B)]$, if the system is divided into A and B subsystems. It suffices to consider $\hat{\rho}_A$ and $\hat{\rho}_B$ to be pure states, so $\Lambda = \sup_{\psi_A,\psi_B}\langle\psi_A|\langle\psi_B|\hat{M}|\psi_A\rangle|\psi_B\rangle$.

We deal with $\langle\psi_B|\hat{M}|\psi_B\rangle$ as the output matrix of a map $\hat{M}$ when the input is the pure state $|\psi_B\rangle$. The output matrix then has a largest eigenvalue $\Lambda_{|\psi_B\rangle}$ with corresponding eigenvector $|\phi\rangle$. Let $|\psi_A\rangle = |\phi\rangle$, our problem of maximizing the mean of $\hat{M}$ over the product pure state reduces to the problem of maximizing the largest eigenvalue $\Lambda_{|\psi_B\rangle}$ of the output matrix of map $\hat{M}$ with respect to input state $|\psi_B\rangle$. If the map $\hat{M}$ represents one of the four quantum bosonic Gaussian channels [33–35], then it is known that vacuum state as the input will minimize the output entropy [35,36], furthermore it will maximize the output majorization [37,38]. For the majorization of a state $\hat{\rho}$, we say $\hat{\rho}_2$ majorizes $\hat{\rho}_1$ if

$$\sum_{j=1}^{k}\lambda_j^{\hat{\rho}_1} \leqslant \sum_{j=1}^{k}\lambda_j^{\hat{\rho}_2}, \quad \forall k \geqslant 1, \tag{4}$$

where $\lambda_j^{\hat{\rho}}(j = 1, \dots, k)$ are the eigenvalues of $\hat{\rho}$ in descending order. Clearly, if $\hat{\rho}_2$ majorizes $\hat{\rho}_1$, then the largest eigenvalue of $\hat{\rho}_2$ is not less than that of $\hat{\rho}_1$. Hence, vacuum state input maximizes the largest eigenvalue of the bosonic Gaussian channel output. As far as we prove that the map $\hat{M}$ represents a bosonic Gaussian channel, the maximal mean of $\hat{M}$ over product states will be shown to be achieved by vacuum (or coherent states and squeezed states). In the following, we will prove that the 1×1 (one mode at each party) and some 2×2 (with two modes at both parties) Gaussian detect operators are bosonic Gaussian channels.

A. Two-mode Gaussian detect operator

For a 1×1 system, consider a detect operator $\hat{M}$ with standard form of CM, γ_M, without loss of generality. The CM has the form of

$$\gamma_M = \begin{pmatrix} \mathcal{M}_\mathcal{A} & \mathcal{M}_\mathcal{C} \\ \mathcal{M}_\mathcal{C} & \mathcal{M}_\mathcal{B} \end{pmatrix}, \tag{5}$$

where $\mathcal{M}_\mathcal{A} = \mathrm{diag}(M_1, M_1)$, $\mathcal{M}_\mathcal{B} = \mathrm{diag}(M_3, M_3)$, $\mathcal{M}_\mathcal{C} = \mathrm{diag}(M_5, -M_6)$.

Let $|\psi_B\rangle = \sum_k a_k|k\rangle$, then the application of map $\hat{M}$ on the basis $|k\rangle\langle m|$ leads to output element $(|k\rangle\langle m|)_{\mathrm{out}} = \mathrm{Tr}_B(\hat{M}|k\rangle\langle m|)$. Let the characteristic function of the basis $|k\rangle\langle m|$ and $(|k\rangle\langle m|)_{\mathrm{out}}$ be $\chi_{\mathrm{in}}(|k\rangle\langle m|, \mu)$ and $\chi_{\mathrm{out}}(|k\rangle\langle m|, \mu)$, respectively. Then

$$\chi_{\mathrm{out}}(|k\rangle\langle m|, \nu_1) = \frac{1}{M_3'(1 - \frac{1}{M_3'})^{(m+k)/2}} \times \chi_{\mathrm{in}}\left(|k\rangle\langle m|, -\frac{\tau^*}{\sqrt{M_3'(M_3'-1)}}\right) \times \exp\left[-M_1|\nu_1|^2 + \frac{M_3|\tau|^2}{M_3'(M_3'-1)}\right]. \tag{6}$$

Where $\tau = \frac{1}{2}[(M_5 + M_6)\nu_1 + (M_5 - M_6)\nu_1^*]$, $M_3' = M_3 + \frac{1}{2}$ (see Appendix A for the details).

The definition of a bosonic Gaussian channel is that its characteristic function undergoes a transformation of [33]

$$\chi_{\mathrm{out}}(z) = \chi_{\mathrm{in}}(zK)e^{-\frac{1}{2}z\alpha z^T}, \tag{7}$$

where the real vector, $z = (z_1, z_2)$, is related to our complex variable ν_1 through $\nu_1 = \frac{1}{\sqrt{2}}(-z_2 + iz_1)$. Here, K is a linear transformation in symplectic space and α is a Hermitian matrix. For one mode channel, when $\det K > 0/\det K < 0$, it can always be transformed to gauge covariant/contravariant channel by proper symplectic transformations of input and output states [34]. Clearly, when M_3 tends to infinite, the factor $(1 - \frac{1}{M_3'})$ can be removed from (6), the map $\hat{M}$ then is a bosonic Gaussian channel. The complete positivity of the channel will lead to an inequality on α and K. This inequality turns out to be that $\hat{M}$ is a Gaussian quantum state for gauge covariant channel, and $\hat{M}$ is a separable Gaussian quantum state for contravariant channel.

Next, we will consider a more general Gaussian detect operator $\hat{M}$ with six-parameter CM. The CM has still the form of (5), but with $\mathcal{M}_\mathcal{A} = \mathrm{diag}(M_1, M_2)$, $\mathcal{M}_\mathcal{B} = \mathrm{diag}(M_3, M_4)$, $\mathcal{M}_\mathcal{C} = \mathrm{diag}(M_5, -M_6)$ instead. With a local squeezing operation $\hat{S}_A \otimes \hat{S}_B$, the detect operator $\hat{M}$ can be transformed to another detect operator $\hat{M}_s = \hat{S}_A \otimes \hat{S}_B\hat{M}\hat{S}_A^\dagger \otimes \hat{S}_B^\dagger$ with a standard form of CM. We have $\mathrm{Tr}(\hat{M}\hat{\rho}_A \otimes \hat{\rho}_B) =$

$\mathrm{Tr}(\hat{M}_s\hat{S}_A\hat{\rho}_A\hat{S}_A^\dagger \otimes \hat{S}_B\hat{\rho}_B\hat{S}_B^\dagger)$. So that the six-parameter detect operator $\hat{M}$ is a bosonic Gaussian channel if its parameters tend to infinite. It follows that its maximal mean over pure product states is

$$\Lambda = \frac{1}{\inf_{\gamma_A,\gamma_B}\sqrt{\det(\gamma_M+\gamma_A\oplus\gamma_B)}}, \tag{8}$$

where $\gamma_A = \frac{1}{2}\mathrm{diag}(x, 1/x)$, $\gamma_B = \frac{1}{2}\mathrm{diag}(y, 1/y)$ are the CMs of squeezed vacuum states of the two subsystems.

B. Bound entangled Gaussian detect operator

The CM of Werner-Wolf bounded entangled 2×2 state [8] can be extended to a more general form of (the state then is called generalized Werner-Wolf state)

$$\gamma = \begin{pmatrix} A & 0 & 0 & 0 & E & 0 & 0 & 0\\ 0 & B & 0 & 0 & 0 & 0 & 0 & -F\\ 0 & 0 & A & 0 & 0 & 0 & -E & 0\\ 0 & 0 & 0 & B & 0 & -F & 0 & 0\\ E & 0 & 0 & 0 & C & 0 & 0 & 0\\ 0 & 0 & 0 & -F & 0 & D & 0 & 0\\ 0 & 0 & -E & 0 & 0 & 0 & C & 0\\ 0 & -F & 0 & 0 & 0 & 0 & 0 & D \end{pmatrix}. \tag{9}$$

The system is divided as the first two modes for Alice versus the last two modes for Bob. Proper local squeezing operations will transform γ to a standard form CM with four parameters. For the four-parameter CM state, we consider a standard γ_{Ms}, which has the same structure of γ with (A, B, C, D, E, F) substituted by $(M_1, M_1, M_3, M_3, M_5, M_6)$, respectively. It is convenient to work with complex covariance matrix in proving $\hat{M}$ to be a bosonic Gaussian channel. The complex covariance matrix, $\tilde{\gamma}_{Ms}$, of $\hat{M}$ is

$$\begin{pmatrix} 0 & 0 & -N_5 & N_6 & M_1 & 0 & N_5 & N_6\\ 0 & 0 & N_6 & N_5 & 0 & M_1 & N_6 & -N_5\\ -N_5 & N_6 & 0 & 0 & N_5 & N_6 & M_3 & 0\\ N_6 & N_5 & 0 & 0 & N_6 & -N_5 & 0 & M_3\\ M_1 & 0 & N_5 & N_6 & 0 & 0 & -N_5 & N_6\\ 0 & M_1 & N_6 & -N_5 & 0 & 0 & N_6 & N_5\\ N_5 & N_6 & M_3 & 0 & -N_5 & N_6 & 0 & 0\\ N_6 & -N_5 & 0 & M_3 & N_6 & N_5 & 0 & 0 \end{pmatrix},$$

with $N_5 = M_5/2, N_6 = -M_6/2$. The characteristic function of $\hat{M}$ will be $\chi(\mu) = \exp[-\frac{1}{2}(\mu,\mu^*)\tilde{\gamma}_{Ms}(\mu,\mu^*)^T]$, with $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$.

With the similar process as in 1×1 case, we obtain the output characteristic function $\chi_{\mathrm{out}}(\mu_1,\mu_2)$ when the input is a pure state at Bob's hand with characteristic function $\chi_{\mathrm{in}}(\mu_3,\mu_4)$ in the case of infinite M_3. We have

$$\begin{aligned}\chi_{\mathrm{out}}(\mu_1,\mu_2) = {}& \frac{1}{M_3'^2}\exp\left[-\left(M_1 - \frac{M_6^2M_3}{M_3'(M_3'-1)}\right)(\mu_{1R}^2+\mu_{2R}^2)\right]\exp\left[-\left(M_1 - \frac{M_5^2M_3}{M_3'(M_3'-1)}\right)(\mu_{1I}^2+\mu_{2I}^2)\right]\\ &\times \chi_{\mathrm{in}}\left(-\frac{M_6\mu_{2R}-iM_5\mu_{1I}}{\sqrt{M_3'(M_3'-1)}}, -\frac{M_6\mu_{1R}+iM_5\mu_{2I}}{\sqrt{M_3'(M_3'-1)}}\right),\end{aligned} \tag{10}$$

where μ_{jR}, μ_{jI} are the real and imaginary parts of μ_j. Thus $\hat{M}$ is a bosonic Gaussian channel [34,39]. A critical point is that we should prove that this channel is equivalent to tensor product of one-mode channels. That is, the matrices K and α of the channel should be simultaneously diagonalizable [37]. By comparing (10) with (7), we have

$$K = \frac{1}{\sqrt{M_3'(M_3'-1)}}\begin{pmatrix} M_5 & 0 & 0 & 0\\ 0 & 0 & 0 & -M_6\\ 0 & 0 & -M_5 & 0\\ 0 & -M_6 & 0 & 0 \end{pmatrix} \tag{11}$$

and $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$, with $\alpha_1 = M_1 - \frac{M_5^2M_3}{M_3'(M_3'-1)}$, $\alpha_2 = M_1 - \frac{M_6^2M_3}{M_3'(M_3'-1)}$. So K and α commute with each other, they can be simultaneously diagonalized. Then $\hat{M}$ represents a tensor product of one-mode gauge covariant (or contracovariant) channels. The theorem that pure Gaussian input maximizes the channel output majorization can be applied [37]. Hence, for the four-parameter four-mode Gaussian detect operator $\hat{M}$, we have proven that its maximal mean over product pure states is achieved by the product of Gaussian pure states.

We then consider a six-parameter detect operator with CM, γ_M, being the same structure of γ in (9), with (A, B, C, D, E, F) substituted by $(M_1, M_2, M_3, M_4, M_5, M_6)$, respectively. Then such a detect operator is also a bosonic Gaussian channel, because it can be transformed into a detect operator with four-parameter standard form of CM using local squeezing. Then, we have the following result. For the new six-parameter Gaussian detect operator of 2×2 system, the maximal mean of $\hat{M}$ over the product pure states is Λ, with

$$\begin{aligned}\frac{1}{\Lambda} = \inf_{x,y}&\left[\left(M_1+\frac{x}{2}\right)\left(M_3+\frac{y}{2}\right)-M_5^2\right]\\ &\times\left[\left(M_2+\frac{1}{2x}\right)\left(M_4+\frac{1}{2y}\right)-M_6^2\right].\end{aligned} \tag{12}$$

III. GAUSSIAN WITNESS AND NECESSARY CRITERION

Given a Gaussian detect operator $\hat{M}$, we have shown that the maximal mean of $\hat{M}$ over product state is achieved by pure product Gaussian state in two cases. One is the 1×1 standard state system with $\Lambda = \sup_{\hat{\rho}_s}\mathrm{Tr}(\hat{\rho}_s\hat{M})$ obtained in (8), the other is the multimode system, which could be treated with the technique of product bosonic Gaussian channels. As a representation of the multimode case, we present a special 2×2 system with $\Lambda = \sup_{\hat{\rho}_s}\mathrm{Tr}(\hat{\rho}_s\hat{M})$ obtained in (12). It follows the necessary criteria of separability for the Gaussian states. The criteria will be presented in next section together with sufficiency in Proposition 2 (proved in Appendix B) and Proposition 3.

For a Gaussian detect operator $\hat{M}$, and Λ obtained with (8) or (12) for different systems, we have a necessary criterion of separability (1) for any state $\hat{\rho}$. Notice that (1) is also the necessary criterion for different kinds of separability in multipartite systems, as far as Λ is defined as the maximal of $\Lambda_{\mathcal{I}_k}$, with partition $\mathcal{I}_k \in \mathcal{I}$ and $\mathcal{I}$ is the set of partitions [30].

A. Mixture of different Gaussian states

For Gaussian detect operator with CM, γ_M, of form (5), where $\mathcal{M}_\mathcal{A} = \mathrm{diag}(M_1, M_2)$, $\mathcal{M}_\mathcal{B} = \mathrm{diag}(M_3, M_4)$, $\mathcal{M}_\mathcal{C} = \mathrm{diag}(M_5, -M_6)$ with all $M_i \to \infty$ for 1×1 system, consider the mixture of several two-mode Gaussian states

$$\hat{\rho} = \sum_{j=1}^{m} p_j \hat{\rho}_j, \tag{13}$$

where p_j is the probability of state $\hat{\rho}_j$, the state is non-Gaussian in general. The CM of Gaussian state $\hat{\rho}_j$ is γ_j, it is supposed to be in the following standard form:

$$\gamma_j = \begin{pmatrix} \mathcal{A}_j & \mathcal{C}_j \\ \mathcal{C}_j & \mathcal{B}_j \end{pmatrix}, \tag{14}$$

where $\mathcal{A}_j = \mathrm{diag}(a_j, a_j)$, $\mathcal{B}_j = \mathrm{diag}(b_j, b_j)$, $\mathcal{C}_j = \mathrm{diag}(c_j, -c_j')$. Using criterion (1), the separable condition for the state is

$$\sum_{j=1}^{m} p_j \frac{\inf_{\gamma_A,\gamma_B} \sqrt{\det(\gamma_M + \gamma_A \oplus \gamma_B)}}{\sqrt{\det(\gamma_j + \gamma_M)}} \leqslant 1, \tag{15}$$

where $\gamma_A = \frac{1}{2}\mathrm{diag}(x, 1/x)$, $\gamma_B = \frac{1}{2}\mathrm{diag}(y, 1/y)$.

An especially simple case is the mixture of two mode squeezed thermal states,

$$\hat{\rho} = (1-p)\hat{\varrho}_1 + p\hat{\varrho}_2, \tag{16}$$

where $p \in (0, 1)$ and $\hat{\varrho}_j$ $(j = 1, 2)$ are two-mode squeezed thermal states characterized by CM (14) with $\mathcal{A}_j = \mathcal{B}_j = a_j I_2$, $\mathcal{C}_j = c_j\sigma_3$, with I_2 is the 2×2 identity matrix and $\sigma_3 = \mathrm{diag}(1, -1)$ is the third Pauli matrix.

A proper choice of the detect operator is that it is a two-mode squeezed state of squeezing parameter q, which is characterized by CM (5) with $\mathcal{M}_\mathcal{A} = \mathcal{M}_\mathcal{B} = \frac{1}{2}\cosh(2q)I_2$, $\mathcal{M}_\mathcal{C} = \frac{1}{2}\sinh(2q)\sigma_3$. When $q \to \infty$, inequality (15) leads to necessarily separable condition

$$(1-p)\frac{w_1}{w_1 + \frac{1}{2}} + p\frac{w_2}{w_2 + \frac{1}{2}} \geqslant 0, \tag{17}$$

where $w_j = a_j - c_j - \frac{1}{2}$. A two-mode squeezed thermal state $\hat{\varrho}$ can be produced by squeezing a two-mode thermal state $\hat{\tau}_N^{\otimes 2}$ with two-mode squeeze operator $\hat{S}_2(r) = e^{r(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1\hat{a}_2)}$ of squeezing parameter r, where the single-mode thermal state is $\hat{\tau}_N = \frac{1}{N+1}(\frac{N}{N+1})^{\hat{a}^\dagger\hat{a}}$, with N the average photon number of the thermal state. The relationships of the parameters are $a = (N + \frac{1}{2})\cosh(2r)$, $c = (N + \frac{1}{2})\sinh(2r)$. We will use normalized parameters $v = \frac{N}{N+1}$, $\lambda = \tanh(r)$ instead of N, r to specify a two-mode squeezed thermal state. Then $a - \frac{1}{2} = \frac{v+\lambda^2}{(1-v)(1-\lambda^2)}$, $c = \frac{(1+v)\lambda}{(1-v)(1-\lambda^2)}$. Correspondingly, we use λ_j, v_j to specify the state $\hat{\varrho}_j$. The necessarily separable condition (17) is

$$(1-p)\frac{v_1 - \lambda_1}{(1+v_1)(1-\lambda_1)} + p\frac{v_2 - \lambda_2}{(1+v_2)(1-\lambda_2)} \geqslant 0. \tag{18}$$

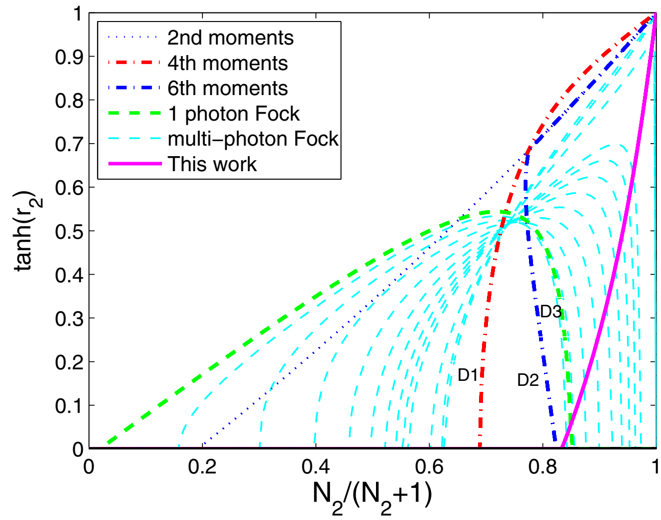

FIG. 1. Comparison of Gaussian witness criterion with Shchukin-Vogel PPT criterion (moment criterion) and Fock space PPT criterion. The state is an equal probability mixture of two two-mode squeezed thermal states, $\hat{\rho} = \frac{1}{2}(\hat{\varrho}_1 + \hat{\varrho}_2)$. Here $\hat{\varrho}_1$ is an entangled two-mode squeezed thermal state with $\lambda_1 = \tanh(r_1) = 0.8$, $v_1 = N_1/(N_1+1) = 0.65$. $\hat{\varrho}_2$ is a two-mode squeezed thermal state specified with $v_2 = N_2/(N_2+1)$ and $\lambda_2 = \tanh(r_2)$. The multiphoton Fock criterion includes the cases of Fock subspace of photon number being 2,3, 4,5, 6,7, 8,9, 10, 20, 50, 100, respectively. Each of them draws a region of necessarily separable state set. The overlap of the sets is the necessarily separable state set (the union of D_1, D_2, and D_3 regions in the figure) of Fock space PPT criterion. The second-moment criterion does not detect entangled states from the these regions. The fourth-moment criterion detects entangled states from the D_1 region, and the sixth-moment criterion further detect entangled states from the D_2 region. The Gaussian witness criterion does detect almost all of the entangled states in the union of D_1, D_2, and D_3.

The Gaussian witness criterion (18) is shown in Fig. 1 and Fig. 2 with solid lines.

To compare the entanglement detecting ability of Gaussian witness criterion with other criteria, we will calculate the necessarily separable conditions from Shchukin-Vogel PPT criterion [11] and Fock space PPT criterion [12] for state (16). It is not necessary to consider the local operator criterion [18] since Gaussian matched entanglement witness is better than the local operator criterion [18] of Gaussian type by definition. It was explicitly shown that matched entanglement criterion is stronger than the local operator criterion for two-mode Gaussian states [22]. Also we will not consider the computable cross norm and realignment criterion, because it is weaker than the local operator criterion [18].

In Fock space, the elements of a two-mode squeezed thermal state with parameters λ, v are [12]

$$\hat{\varrho}_{k_1k_2,m_1m_2} = \kappa\mathcal{N}\frac{\partial^{k_1+k_2+m_1+m_2}}{\partial t_1^{k_1}\partial t_2^{k_2}\partial t_1'^{m_1}\partial t_2'^{m_2}} \exp[\varepsilon(t_1t_1' + t_2t_2') + \zeta(t_1t_2 + t_1't_2')]|_{t_1=t_1'=t_2=t_2'=0}, \tag{19}$$

where $\kappa = (k_1!k_2!m_1!m_1!)^{-\frac{1}{2}}$, $\mathcal{N} = \frac{(1-v)^2(1-\lambda^2)}{1-v^2\lambda^2}$, $\varepsilon = \frac{v(1-\lambda^2)}{1-v^2\lambda^2}$, $\zeta = \frac{(1-v)^2\lambda}{1-v^2\lambda^2}$. A necessary criterion of separability for the state

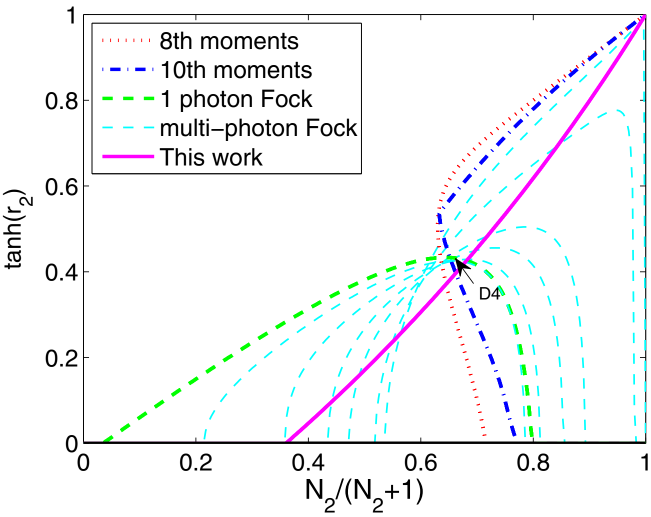

FIG. 2. Further comparison of Gaussian witness criterion with Shchukin-Vogel PPT criterion and Fock space PPT criterion. $\lambda_1 = \tanh(r_1) = 0.65$, $\upsilon_1 = N_1/(N_1+1) = 0.51$. The multiphoton Fock criterion includes the cases of Fock subspace of photon number being 2, 3, 4, 5, 10, 50, respectively. The district D_4 as indicated by the arrow in the figure is a delta district enclosed by tenth-moment criterion, two-photon Fock criterion, and the Gaussian witness criterion. The entangled states in district D_4 are detected neither by the moment criterion up to 10th order nor by Fock criterion, but can be detected with Gaussian witness criterion.

(16) is $\hat{\rho}_{0n,0n} \geqslant \hat{\rho}_{00,nn}$, which turns out to be

$$(1-p)\mathcal{N}_1(\varepsilon_1^n - \zeta_1^n) + p\mathcal{N}_2(\varepsilon_2^n - \zeta_2^n) \geqslant 0. \qquad (20)$$

Although we can utilize the necessarily separable condition of $\hat{\rho}_{ln,ln} \geqslant \hat{\rho}_{ll,nn}$, numerical calculation shows that $l = 0$ suffices. We will call (20) the n photon Fock criterion.

The higher order moment Shchukin-Vogel criterion can be obtained from the submatrix of the infinite matrix given in Ref. [11]. The symmetry of the states greatly simplifies the calculation. The positivity of a submatrix provides a necessarily separable criterion. We calculate the higher-order moment criterion of fourth, sixth, eighth, and tenth moments. The results are shown in Fig. 1 and Fig. 2, respectively.

Each separable criterion shown in the figures is a curve, which splits the (υ, λ) region into two districts, the district with the upper-left corner and the district without the upper-left corner. A state in the latter district is necessarily separable. We are interested in the overlap of all the necessarily separable districts from Shchukin-Vogel criterion and Fock space criterion. The overlap district in Fig. 1 is D_3 district. The states in D_3 district are necessarily separable from Shchukin-Vogel criterion up to sixth-order moments and Fock space criterion. However, the states in D_3 (except those below the solid line) are detected to be entangled by Gaussian witness criterion. The states in D_4 district of Fig. 2 are necessarily separable from Shchukin-Vogel criterion up to tenth-order moments and Fock space criterion, and they are detected to be entangled by Gaussian witness criterion. We can conclude that Gaussian witness criterion can detect some entangled states that are not detected by Fock space criterion together with Shchukin-Vogel criterion at least up to tenth-order moments.

B. Gaussian Fock states

In Ref. [22], the entanglement of a non-Gaussian state prepared by adding photon to or/and subtracting photon from a Gaussian state can also be precisely (necessarily and sufficiently) detected with witness based on a Gaussian detect operator $\hat{M}$. We here present the entanglement detection of Gaussian witness on another kind of non-Gaussian states that are closely related with Gaussian states.

Let $\hat{\rho}_G$ be a Gaussian state with CM γ, then γ can be diagonalized to $\gamma_d = \text{diag}\{N_1+\frac{1}{2}, N_1+\frac{1}{2}, N_2+\frac{1}{2}, N_2+\frac{1}{2}, \ldots, N_n+\frac{1}{2}, N_n+\frac{1}{2}\}$ by some symplectic transformation S^{-1}, namely, $\gamma = S^T\gamma_d S$. The CM γ_d corresponds to a product thermal state $\bigotimes_{j=1}^n \hat{\tau}_{N_j}$. A thermal state $\hat{\tau}_N$ can be produced by passing vacuum through thermal channel $\mathcal{E}_N$, where

$$\mathcal{E}_N(\hat{\omega}) = \frac{1}{N}\int \frac{d^2\mu}{\pi} \exp\left[-\frac{|\mu|^2}{N}\right]\mathcal{D}(\mu)\hat{\omega}\mathcal{D}(\mu)^\dagger. \qquad (21)$$

Our bosonic Gaussian channel is $\mathcal{BG}$ with $\mathcal{BG}(\hat{\omega}) = \hat{U}(S)\mathcal{E}_{N_1N_2\ldots N_n}(\hat{\omega})\hat{U}(S)^\dagger$, where $\hat{U}(S)$ is the unitary transform corresponds to the symplectic transform, $\mathcal{E}_{N_1N_2\ldots N_n} = \bigotimes_{j=1}^n \mathcal{E}_{N_j}$ is the parallel thermal channel. Then $\hat{\rho}_G = \mathcal{BG}(|0^{\otimes n}\rangle\langle 0^{\otimes n}|)$. A Gaussian Fock state is defined as

$$\hat{\rho} = \mathcal{BG}(|l\rangle\langle l|), \qquad (22)$$

where $l = l_1 l_2 \ldots l_n$. We call $\hat{\rho}_G$ the adjoint Gaussian state of Gaussian Fock state $\hat{\rho}$.

At the limit of infinite γ_M, that is, all $M_i \to \infty$ for $i = 1, 2, \ldots, 6$, we can show that (see Appendix C for details)

$$\text{Tr}(\hat{\rho}\hat{M})_{\gamma_M\to\infty} \to \frac{1}{\sqrt{|\det(\tilde{\gamma}_G + \tilde{\gamma}_M)|}} = \frac{1}{\sqrt{|\det(\gamma_G + \gamma_M)|}}. \qquad (23)$$

Then we have the following proposition.

Proposition 1. The necessary criterion of separability for a Gaussian Fock state of 1×1 system and some 2×2 system is

$$\inf_{\gamma_M} \sup_{\gamma_A,\gamma_B} \frac{\det(\gamma_G + \gamma_M)}{\det(\gamma_A \oplus \gamma_B + \gamma_M)} \geqslant 1, \qquad (24)$$

for γ_M with infinite parameters M_i. Herein, γ_G is the CM of adjoint Gaussian state of Gaussian Fock state $\hat{\rho}$, γ_G is either a 1×1 CM, or a 2×2 CM in the form of (9), γ_A and γ_B are the CMs of the subsystem pure Gaussian states with $\det(2\gamma_A) = \det(2\gamma_B) = 1$.

Proof. The necessary criterion (24) comes from (23) and (1), with $\Lambda = [\det(\gamma_A \oplus \gamma_B + \gamma_M)]^{-\frac{1}{2}}$ in (1) that has already been established for a Gaussian detect operator in 1×1 and some 2×2 continuous variable systems in (8) and (12), respectively.

The criterion (24) means that the necessary criterion of separability for a Gaussian Fock state can be reduced to the necessary and sufficient criterion of separability of its adjoint Gaussian state under the condition of infinite γ_M. The application of (24) leads to the following corollary.

Corollary 1. For a Gaussian Fock state $\hat{\rho} = \hat{U}(S)\mathcal{E}_{N_1N_2}(|l_1l_2\rangle\langle l_1l_2|)\hat{U}(S)^\dagger$ with its adjoint Gaussian state described by standard CM of (37) and a 2×2 Gaussian Fock state with its adjoint Gaussian state characterized by CM of

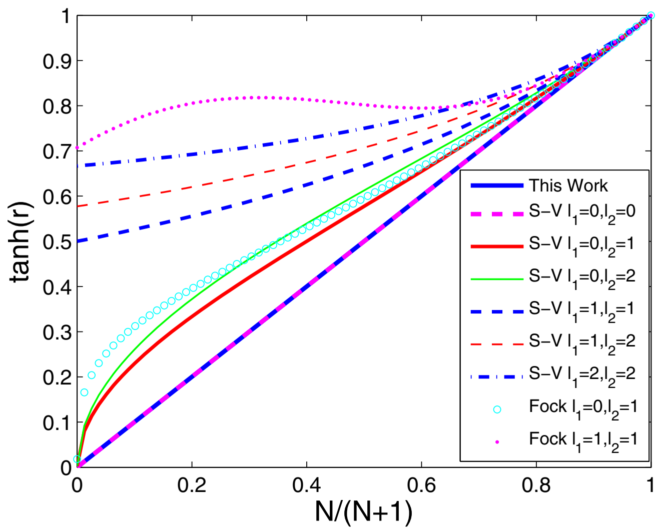

FIG. 3. Comparison of Gaussian witness criterion with second moment of Shchukin-Vogel PPT criterion and Fock space PPT criterion for the Gaussian Fock state with initial Fock product state $|l_1 l_2\rangle$ and bosonic Gaussian channel characterized with squeezing parameter r and thermal noise N. We show the necessary second moment and Fock space criteria in $\lambda = \tanh(r)$, $v = N/(N+1)$ plane. The Gaussian witness necessary criterion of separability is $\lambda \leqslant v$.

(9), the necessary criteria of separability are (B6) and (40), respectively, regardless what the initial product Fock states are. Violation of them implies entanglement.

In Fig. 3, the Gaussian witness gives a necessary criterion of separability $\lambda \leqslant v$, it is better than the second moment criterion, the latter is

$$\cosh(2r) \leqslant \frac{\left(N + l_1 + \frac{1}{2}\right)\left(N + l_2 + \frac{1}{2}\right) + \frac{1}{4}}{N + \frac{1}{2}(l_1 + l_2 + 1)}. \tag{25}$$

We also show the result of Fock space criterion for the Gaussian Fock state. The Gaussian witness criterion is also better than the Fock space criterion. When $l_1 = l_2 = 0$, the state is Gaussian, all the three criteria yield the same separable condition of $\lambda \leqslant v$.

C. Detecting multipartite entanglement of non-Gaussian states

In multipartite entanglement detection, given a multipartite Gaussian operator $\hat{M}$, to find a valid witness is to find the maximal mean of $\hat{M}$ over the defined separable state set. For a tripartite system with parties A, B, and C, there are two kinds of separabilities. A biseparable state $\hat{\rho}_{bs}$ can be written as the probability mixture of the states $\hat{\rho}_{A|BC}$, $\hat{\rho}_{B|AC}$, $\hat{\rho}_{C|AB}$, where $\hat{\rho}_{A|BC} = \sum_i p_i \hat{\rho}_A^{(i)} \otimes \hat{\rho}_{BC}^{(i)}$ and so on. A state $\hat{\rho}_{A|BC}$ is bipartite separable under partition $A|BC$, it is usually not separable under partition $B|AC$. A biseparable state $\hat{\rho}_{bs} = q_1 \hat{\rho}_{A|BC} + q_2 \hat{\rho}_{B|AC} + q_3 \hat{\rho}_{C|AB}$ is usually not separable under partition $A|BC$. Thus biseparability is a multipartite concept strictly different from bipartite separability. A fully separable state can be written as $\hat{\rho}_s = \sum_i p_i \hat{\rho}_A^{(i)} \otimes \hat{\rho}_B^{(i)} \otimes \hat{\rho}_C^{(i)}$. Where p_i is a probability. The problem of obtaining the maximal mean of $\hat{M}$ over product mixed state set can be reduced to over product pure state set. We have the valid witness $\hat{W} = \Lambda \hat{I} - \hat{M}$ where $\Lambda = \sup_{|\psi\rangle} \langle\psi|\hat{M}|\psi\rangle$, with $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle|\psi_C\rangle$ for full separability, and

$$\Lambda = \max\left(\Lambda_1, \Lambda_2, \Lambda_3\right), \tag{26}$$

where $\Lambda_i = \sup_{|\psi_i\rangle} \langle\psi_i|\hat{M}|\psi_i\rangle$, with $|\psi_1\rangle = |\psi_A\rangle|\psi_{BC}\rangle$, $|\psi_2\rangle = |\psi_B\rangle|\psi_{AC}\rangle$, $|\psi_3\rangle = |\psi_{AB}\rangle|\psi_C\rangle$ for biseparability, respectively.

Consider a three-mode Gaussian operator $\hat{M}$ to be a bosonic Gaussian channel, then we have to decide the input mode(s) and output mode(s). In either of the fully separable or biseparable cases, the numbers of mode for input and output of the channel are not equal. Thus the matrix K in (7) is a 2×4 or 4×2 matrix. The commutation of K and α is not well defined. Using the method of bosonic Gaussian channel to prove the validity of Gaussian witness of three modes is not easy if not impossible. Alternatively, we will consider numerical method and characteristic function equation method [22] in the following.

1. Numerical method for validity of witness

A direct calculation of the maximal mean of $\hat{M}$ over product state set is to randomly generate product state $|\psi\rangle = |\psi_A\rangle|\psi_B\rangle|\psi_C\rangle$ for fully separable case (or $|\psi\rangle = |\psi_A\rangle|\psi_{BC}\rangle$ for biseparable case), maximizing the mean with iterations, and compare the numeric maximal mean with $\Lambda = \sup_{\hat{V}} \langle 000|\hat{V}^\dagger \hat{M} \hat{V}|000\rangle$, where $\hat{V}$ is the product of local squeezing operators. The suitable basis for calculation is in Fock space. The Gaussian operator $\hat{M}$ then is expressed in Fock basis [12,22]. It is the most time consuming step in the numeric calculation. After $\hat{M}$ is built, we find the eigenvector of the matrix $\langle\psi_B|\langle\psi_C|\hat{M}|\psi_B\rangle|\psi_C\rangle$ as the next round $|\psi_A\rangle$ and similarly we have the next round $|\psi_B\rangle$, $|\psi_C\rangle$ in the fully separable case. While in the biseparable case, we iteratively obtain the next round of $|\psi_A\rangle$, $|\psi_{BC}\rangle$ as the eigenvectors of $\langle\psi_{BC}|\hat{M}|\psi_{BC}\rangle$, $\langle\psi_A|\hat{M}|\psi_A\rangle$, respectively.

For simplicity, we consider the fully mode symmetric Gaussian operator $\hat{M}$ with CM [We use the order of $\hat{R} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)^T$ to simplify the expression of CM in this section]

$$\gamma_M = H(M_1, M_2) \oplus H(M_3, -M_4). \tag{27}$$

Where $H(x, y)$ is a 3×3 matrix whose diagonal elements are x and all other elements are y. Numeric calculation shows that the maximal mean of Gaussian operator $\hat{M}$ is achieved by the product of local squeezed states, which are $(\hat{S}(q)|0\rangle)^{\otimes 3}$ for fully separable state set and $\hat{S}(q_1)|0\rangle \hat{S}_2(s)\hat{S}(q_2)^{\otimes 2}|00\rangle$ for biseparable state set with some parameters q, q_1, q_2, s. Where $\hat{S}$, $\hat{S}_2$ are single-mode and two-mode squeezing operators, respectively. For randomly generated initial product states, after the iterations, the mean of $\hat{M}$ approaches Λ but never exceeds Λ. We use a cutoff on Fock space. The number state $|k\rangle$ is limited to $k < k_{\text{cut}}$. The complexity of the computation is proportional to k_{cut}^{15} after using the presqueezing technique [22] to simplify the calculation. On a current laptop or desktop, the computational overhead of numerically constructing each $\hat{M}$ in Fock space is about half an hour when $k_{\text{cut}} = 5$, while it is about a minute when $k_{\text{cut}} = 4$. The computational overhead for the iteration of the product states can be neglected.

The continuous variable GHZ state $|\psi_{\text{GHZ}}\rangle$ has a CM

$$\gamma_{\text{GHZ}} = H(a, b) \oplus H(a + b, -b), \tag{28}$$

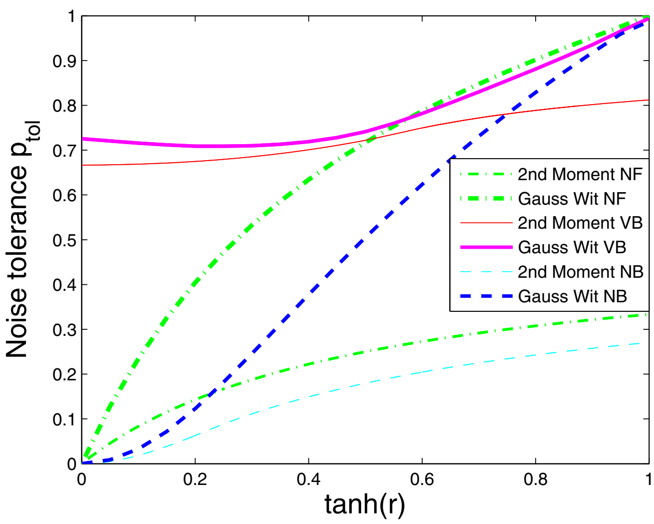

FIG. 4. The noise tolerance of three-mode continuous variable GHZ state with squeezing parameter r. When $p \geqslant p_{\rm tol}$, the mixed state is not detected to be entangled (or genuinely entangled) by the corresponding criterion. F and B stand for full separability and biseparability, respectively. N for noise of thermal state with $N = 1$ and V for vacuum. '2nd moment' for Werner-Wolf criterion and 'Gauss Wit' for Gaussian witness criterion of this work.

where $a = \frac{1}{6}(e^{2r} + 2e^{-2r}), b = \frac{1}{6}(e^{2r} - e^{-2r})$, with r being the squeezing parameter of the state. Consider the continuous variable GHZ state mixing with a three-mode thermal state

$$\hat{\rho} = (1-p)|\psi_{\rm GHZ}\rangle\langle\psi_{\rm GHZ}| + p\hat{\tau}_N^{\otimes 3}. \tag{29}$$

The CM of the mixed state is

$$\gamma = H(a', b') \oplus H(a' + b', -b'). \tag{30}$$

where $a' = (1-p)a + p(N + \frac{1}{2}), b' = (1-p)b$.

The Werner-Wolf necessary conditions of full separability and biseparability for a three-mode continuous variable state with such a CM are known [22]. We yield the separable conditions in the form of noise tolerance $p_{\rm tol}$ as a function of $\tanh(r)$ in Fig. 4. Noise tolerance is the largest fraction of noise that can be added to the system while the entanglement (or genuine entanglement) of the mixed state can still be detected by a necessary criterion of separability.

The necessary conditions of full separability and biseparability from Gaussian witness are

$$\sup_{\gamma_M} \frac{1}{\Lambda}\left[\frac{(1-p)}{\sqrt{\det(\gamma_{\rm GHZ} + \gamma_M)}} + \frac{p}{\sqrt{\det\left[(N+\frac{1}{2})I_6 + \gamma_M\right]}}\right] \leqslant 1, \tag{31}$$

where I_m is the $m \times m$ identity matrix. Notice that Λ for full separability is different from Λ for biseparability. More concretely,

$$\Lambda = \begin{cases} \sup_q \frac{1}{\sqrt{\gamma_M + \frac{1}{2}(e^q I_3 \oplus e^{-q} I_3)}}, & \text{for full separability;} \\ \sup_{q_1,q_2,s} \frac{1}{\sqrt{\gamma_M + \gamma_{bs}}}, & \text{for biseparability,} \end{cases}$$

where $\gamma_{bs} = \frac{1}{2}[e^{q_1} \oplus e^{q_2}\begin{pmatrix}\cosh(2s) & \sinh(2s)\\ \sinh(2s) & \cosh(2s)\end{pmatrix} \oplus e^{-q_1} \oplus e^{-q_2} \oplus \begin{pmatrix}\cosh(2s) & -\sinh(2s)\\ -\sinh(2s) & \cosh(2s)\end{pmatrix}]$.

The numerical results of Gaussian witness separable criterion are shown in Fig. 4. Gaussian witness criterion is always better than the Werner-Wolf criterion for detecting entanglement and genuine entanglement of the states studied.

2. Characteristic function equation method

A pure four-mode Gaussian state is proposed [13] to elucidate the genuine entanglement detection ability of the entropic criterion. The Gaussian state $|\psi(r)\rangle$ can be characterized with its CM, $\gamma = \gamma_x \oplus \gamma_p$, where $\gamma_x = \beta(r), \gamma_p = \frac{1}{4}\beta(r)^{-1}$, with

$$\beta(r) = \begin{pmatrix} 2c & -\sqrt{2}s & -\sqrt{2}s & 0 \\ -\sqrt{2}s & 2c & 0 & \sqrt{2}s \\ -\sqrt{2}s & 0 & 2c & -\sqrt{2}s \\ 0 & \sqrt{2}s & -\sqrt{2}s & 2c \end{pmatrix}, \tag{32}$$

where $c = \cosh(2r), s = \sinh(2r)$. The state then is mixed with the vacuum state to form a mixed state

$$\hat{\rho} = (1-p)|\psi(r)\rangle\langle\psi(r)| + p(|0\rangle\langle 0|)^{\otimes 4} \tag{33}$$

for genuine entanglement detection. The state $\hat{\rho}$ is a mixture of two Gaussian states (the vacuum state is also Gaussian). The noise tolerance of genuine entanglement for the state $\hat{\rho}$ is about $p = 0.308$ for $r = 2$ using entropic criterion [13]. We will show that the state $\hat{\rho}$ is genuinely entangled when $p < \frac{e^{4r}-2}{e^{4r}-1}$. That is $p < 0.99966$ when $r = 2$.

We choose a detect operator $\hat{M} = |\psi(q)\rangle\langle\psi(q)|$, with CM $\gamma_M = \beta(q) \oplus \frac{1}{4}\beta(q)^{-1}$. To prove that the maximal mean of $\hat{M}$ over bipartite product state set is achieved by Gaussian product states, we may try to use parallel bosonic Gaussian channels for the partitions of two modes in both parties. However, it is difficult to deal with partitions of one mode in a party and three modes in another. A convenient method of proving is to use characteristic eigenfunction for the present detect operator [22]. We should consider all seven partitions. For each partition, we write $\gamma_M = \begin{pmatrix}\gamma_1 & \gamma_3\\ \gamma_3^T & \gamma_2\end{pmatrix}$, where γ_1, γ_2 are the CMs of the first and second parts, γ_3 is the correlation of the parts. For each partition, we have checked that γ_1 commutes with $\gamma_3\gamma_2\gamma_3^T$, and the CMs of product Gaussian states (denoted as γ_A, γ_B) obtained from the equations in Ref. [22] are pure state CMs. Thus the conditions in Ref. [22] are fulfilled. Then the maximal mean of $\hat{M}$ is achieved by Gaussian pure product states. We have

$$\Lambda_{A|B} = \sup_{\hat{\rho}_A, \hat{\rho}_B} [\hat{M}(\hat{\rho}_A \otimes \hat{\rho}_B)] = [\det(\gamma_M + \gamma_A \oplus \gamma_B)]^{-\frac{1}{2}}, \tag{34}$$

for partition $A|B \in \mathcal{I}$, where the partition set $\mathcal{I} = \{12|34, 13|24, 14|23, 1|234, 2|134, 3|124, 4|123\}$. A detailed calculation shows that

$$\Lambda = \max_{A|B\in\mathcal{I}} \Lambda_{A|B} = \Lambda_{12|34} = 4[1 + \sqrt{1 + \cosh(2q)^2/2}]^{-2}. \tag{35}$$

Furthermore, $\langle\psi(r)|\hat{M}|\psi(r)\rangle = [\det(\gamma_M + \gamma)]^{-\frac{1}{2}} = 4[1 + \cosh(2q - 2r)]^{-2}$, $\langle 0^{\otimes 4}|\hat{M}|0^{\otimes 4}\rangle = [\det(\gamma_M + \frac{1}{2}I)]^{-\frac{1}{2}} = 4[\cosh(2q) + 17/8]^{-2}$. In the limit of $q \to \infty$, the separable condition (1) is

$$(1-p)e^{4r}/2 + p/2 \leqslant 1, \tag{36}$$

namely, the necessary condition of biseparability is $p \geqslant \frac{e^{4r}-2}{e^{4r}-1}$. The Gaussian witness criterion has a very strong genuine multipartite entanglement detection ability comparing with entropic criterion in this example.

It seems unbelievable that the state $|\psi(r=2)\rangle$ has a noise tolerance of $p = 0.99966$. However, it is reasonable. As a comparison, let us consider a mixture of the two-mode squeezed vacuum and the vacuum state. The left-hand side of (17) is $(1-p)(1-e^{2r})$, which is always negative for nonzero r and $p \neq 1$. The inequality (17) is always violated, thus the mixed state is always entangled. It had been shown that a mixture of the two-mode squeezed vacuum and number state is always inseparable [9] regardless of the squeezing parameter and the ratio of the number state.

IV. SUFFICIENT CRITERION

The sufficient criterion of separability is simply $\mathcal{L} \geqslant 1$ as in (2) with $\mathcal{L} = \inf_{\hat{M}} \frac{\Lambda}{\mathrm{Tr}(\hat{\rho}\hat{M})}$. In the following, we will show that the minimization over general detect operator $\hat{M}$ can be reduced to Gaussian detect operator for some continuous variable states.

A. 1 × 1 Gaussian state

A 1×1 Gaussian state can always be transformed by local operations to its standard form with CM [5,7]

$$\gamma = \begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \mathcal{C} & \mathcal{B} \end{pmatrix}, \tag{37}$$

where $\mathcal{A} = \mathrm{diag}(a, a)$, $\mathcal{B} = \mathrm{diag}(b, b)$, $\mathcal{C} = \mathrm{diag}(c, -c')$. From the 1×1 Gaussian witness $\hat{W} = \Lambda\hat{I} - \hat{M}$, with Gaussian detect operator $\hat{M}$ as a density matrix described by CM γ_M of six parameters and Λ are shown as (8), we then have the following proposition from the definition of the witness.

Proposition 2. For a 1×1 Gaussian state with CM of standard form γ, the necessary and sufficient criterion of separability is

$$\mathcal{L}_G^2 := \inf_{\gamma_M} \frac{\det(\gamma + \gamma_M)}{\inf_{\gamma_A,\gamma_B} \det(\gamma_A \oplus \gamma_B + \gamma_M)} \geqslant 1, \tag{38}$$

for γ_M with infinite parameters M_i $(i = 1, \ldots, 6)$ and $\gamma_M \geqslant \frac{i}{2}\sigma$ since $\hat{M}$ is a Gaussian state, and $\det(2\gamma_A) = \det(2\gamma_B) = 1$ for pure Gaussian subsystem states.

This is the Gaussian witness entanglement criterion for a two-mode Gaussian state. Notice that the necessary and sufficient criterion for a two-mode Gaussian state is well known [5,7], we should check if Proposition 2 lead to the same explicit condition of separability (see Appendix B for the proof of Proposition 2).

B. Bounded entangled Gaussian state

For generalized Werner-Wolf states with CM in (9), we have the following necessary and sufficient condition for the separability.

Proposition 3.

$$\inf_{\gamma_M} \sup_{x,y} \frac{(M_1 + A)(M_3 + C) - (M_5 + E)^2}{\left(M_1 + \frac{x}{2}\right)\left(M_3 + \frac{y}{2}\right) - M_5^2} \times \frac{(M_2 + B)(M_4 + D) - (M_6 + F)^2}{\left(M_2 + \frac{1}{2x}\right)\left(M_4 + \frac{1}{2y}\right) - M_6^2} \geqslant 1, \tag{39}$$

where $M_i \to \infty$ $(i = 1, \ldots, 6)$ are the elements of CM, γ_M, being the same structure of γ in (9), with (A, B, C, D, E, F) substituted by $(M_1, M_2, M_3, M_4, M_5, M_6)$, respectively. Furthermore $\gamma_M \geqslant \frac{i}{2}\sigma$.

Proof. The necessity of (39) comes from (12) and (1). The optimization in (39) can be worked out. It is very similar to the case of the 1×1 Gaussian state. The solution is almost the same. We arrive at the explicit necessary and sufficient criterion of separability for the generalized Werner-Wolf state as follows:

$$(AC - E^2)(BD - F^2) - \tfrac{1}{2}|EF| - \tfrac{1}{4}(CD + AB) + \tfrac{1}{16} \geqslant 0. \tag{40}$$

The sufficiency of the separability criterion (39) and (40) can be seen from the fact that $(A - \frac{x}{2})(C - \frac{y}{2}) - E^2 \geqslant 0$ and $(B - \frac{1}{2x})(D - \frac{1}{2y}) - F^2 \geqslant 0$ derived in the optimization of the necessary part of condition (39). These two inequalities mean that γ in (9) is larger than the CM of a four-mode product squeezed state specified with (x, y), so the generalized Werner-Wolf state can be generated from the product state with local operations (classical probability distribution of the first moment or displacement), thus, is separable. ■

Werner and Wolf have constructed the five-parameter series of 2×2 Gaussian states [8] and have shown that these states are bounded entangled with a quite sophisticated way. A direct calculation using (40) will show that these states are entangled. The five parameters are $a, b, c, d, e > 0$ (we abuse a and b here for a while) with $ad - bc > 0$ and $ce - a > 0$ [8]. Then $A = \frac{de-b}{2(ce-a)}$, $B = \frac{a}{2b}$, $C = \frac{c(da-bc)}{2(ce-a)}$, $D = \frac{eb+d}{2b(ad-bc)}$, $E = \frac{ad-bc}{2(ce-a)}$, $F = \frac{1}{2b}$ from the state description. The left-hand side of (40) is equal to $-\frac{(ad-bc)}{16b(ce-a)}$, which is always negative. Therefore, the states are entangled.

V. CONCLUSION

We use a Gaussian operator to build Gaussian witness for detecting the entanglement of either Gaussian states or non-Gaussian states. Necessary and sufficient separable criteria are given for 1×1 Gaussian, generalized Werner-Wolf 2×2 Gaussian states, respectively. The validity of the Gaussian witness is to show that the maximal mean of Gaussian operator over product states is achieved by product of Gaussian pure states, they are vacuum, coherent states and squeezed states in the cases considered. We show the validity using the known properties of the bosonic Gaussian channel of gauge covariant or gauge contracovariant. We transform the maximal mean problem to a bosonic Gaussian channel problem when the covariance matrix of the Gaussian detect operator tends to infinite. It is an analytical proof for the validity of Gaussian witness in contrast to the proofs in Ref. [22], where the proofs are either with some conditions on the parameters of witness (thus not powerful) or numerical. The necessary criterion of separability (may not be sufficient) of Gaussian witness has also been applied to detect the entanglement of non-Gaussian states, which is a probability mixture of Gaussian states. The proposed Gaussian witness criterion can detect entanglement of the mixture of two of two-mode squeezed thermal states that can not be detected by Fock space PPT criterion of all subspaces together with Shchukin-Vogel PPT criteria up to tenth moments. For a two-mode (or some four-mode) Gaussian Fock state generated by passing product Fock state through bosonic Gaussian channel, the necessary criterion of separability from Gaussian witness is derived, it is equivalent to the criterion derived for the adjoint Gaussian state of the Gaussian Fock state.

For multipartite entanglement detection, based on numeric calculation, we proposed three-mode Gaussian witnesses which are symmetric on the modes. The application of the Gaussian witness criterion shows that it significantly rises the noise tolerance of the entanglement and genuine entanglement of the continuous variable GHZ state comparing with Werner-Wolf second-moment criterion.

The Gaussian witness criterion was also exhibited to be significantly more powerful than entropic criterion in detecting the genuine entanglement of a four-mode Gaussian state mixed with the vacuum. Further progresses could be made for multimode Gaussian witnesses as far as that the corresponding bosonic Gaussian channels are diagonalizable [36].

ACKNOWLEDGMENT

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APPENDIX A: PROOF OF EQ. (6)

From the definition of characteristic function, we have

$$\chi_{\text{in}}(|k\rangle\langle m|,\mu) = \langle m|\mathcal{D}(\mu)|k\rangle = \frac{(-1)^k}{\sqrt{m!k!}} e^{-\frac{|\mu|^2}{2}} H_{mk}(\mu,\mu^*), \tag{A1}$$

with Hermitian polynomial

$$\begin{aligned} H_{mk}(\mu,\mu^*) &= \sum_{l=0}^{\inf(m,k)} \frac{m!k!}{(m-l)!(k-l)!l!}(-1)^l \mu^{m-l}\mu^{*k-l} \\ &= \frac{\partial^{m+k}}{\partial t^m \partial t'^k} \exp[-tt' + \mu t + \mu^* t']|_{t=t'=0}. \end{aligned}$$

Denote $\nu = (\nu_1, \nu_2)$, the characteristic function of $\hat{M}$ is $\chi_M(\nu)$, then $(|k\rangle\langle m|)_{\text{out}}$ is

$$\begin{aligned} &\mathrm{Tr}\int \chi_M(\nu)\mathcal{D}(-\nu)\chi_{\text{in}}(|k\rangle\langle m|,\mu)\mathcal{D}(-\mu)\left[\frac{d^2\nu}{\pi}\right]\frac{d^2\mu}{\pi} \\ &= \int \chi_M(\nu)\mathcal{D}(-\nu_1)\chi_{\text{in}}(|k\rangle\langle m|,-\nu_2)\left[\frac{d^2\nu}{\pi}\right]. \end{aligned} \tag{A2}$$

The integral on ν_2 can be carried out by using (A1) and interchanging the order of integral and partial derivatives. The characteristic function of the output basis element $(|k\rangle\langle m|)_{\text{out}}$ is

$$\chi_{\text{out}}(|k\rangle\langle m|,\nu_1) = \exp\left[-M_1|\nu_1|^2 + \frac{1}{M_3'}|\tau|^2\right]\frac{(-1)^k}{M_3'\sqrt{m!k!}}\frac{\partial^{m+k}}{\partial t^m\partial t'^k}\exp\left[-tt'\left(1-\frac{1}{M_3'}\right) - \frac{\tau^* t}{M_3'} - \frac{\tau t'}{M_3'}\right]|_{t=t'=0}, \tag{A3}$$

where $\tau = \frac{1}{2}[(M_5+M_6)\nu_1 + (M_5-M_6)\nu_1^*]$, $M_3' = M_3 + \frac{1}{2}$. Let $s = \sqrt{1-\frac{1}{M_3'}}t$, $s' = \sqrt{1-\frac{1}{M_3'}}t'$, then we can put the characteristic function in the following form:

$$\begin{aligned} \chi_{\text{out}}(|k\rangle\langle m|,\nu_1) &= \frac{(-1)^k \exp\left[-M_1|\nu_1|^2 + \frac{1}{M_3'}|\tau|^2\right]}{M_3'\left(1-\frac{1}{M_3'}\right)^{(m+k)/2}\sqrt{m!k!}} H_{mk}\left(-\frac{\tau^*}{\sqrt{M_3'(M_3'-1)}}, -\frac{\tau}{\sqrt{M_3'(M_3'-1)}}\right) \\ &= \frac{1}{M_3'\left(1-\frac{1}{M_3'}\right)^{(m+k)/2}}\chi_{\text{in}}\left(|k\rangle\langle m|, -\frac{\tau^*}{\sqrt{M_3'(M_3'-1)}}\right)\exp\left[-M_1|\nu_1|^2 + \frac{M_3|\tau|^2}{M_3'(M_3'-1)}\right]. \end{aligned} \tag{A4}$$

APPENDIX B: PROOF OF PROPOSITION 2

The necessary criterion (1) with Λ in (8) leads to the following necessary criterion of separability for 1×1 Gaussian state with standard CM, γ,

$$\frac{\det(\gamma+\gamma_M)}{\inf_{\gamma_A,\gamma_B}\det(\gamma_A\oplus\gamma_B+\gamma_M)} \geqslant 1. \tag{B1}$$

Minimizing the left-hand side with respect to Gaussian detect operator $\hat{M}$ with CM, γ_M yields a more tight necessary criterion of separability. Hence the necessary part of Proposition 2 is demonstrated.

For the sufficient part of Proposition 2, we use the six-parameter γ_M described above with diagonal elements M_1, M_2, M_3, M_4, and off-diagonal elements $M_5, -M_6$. Then

$$\mathcal{L}_G^2 = \inf_{\hat{M}}\sup_{x,y} f_1(M_1,M_3,M_5,x,y)f_2(M_2,M_4,M_6,x,y), \tag{B2}$$

with

$$f_1(M_1,M_3,M_5,x,y) = \frac{(M_1+a)(M_3+b)-(M_5+c)^2}{\left(M_1+\frac{x}{2}\right)\left(M_3+\frac{y}{2}\right)-M_5^2},$$

$$f_2(M_2,M_4,M_6,x,y) = \frac{(M_2+a)(M_4+b)-(M_6+c')^2}{\left(M_2+\frac{1}{2x}\right)\left(M_4+\frac{1}{2y}\right)-M_6^2}.$$

We then change the order of minimization and maximization in (B2). We have $(M_1+a)(M_3+b)-(M_5+c)^2 = A_1+A_2+A_3$ with $A_1=(M_1+\frac{x}{2})(M_3+\frac{y}{2})-M_5^2$, $A_2=(M_1+\frac{x}{2})(b-\frac{y}{2})+(M_3+\frac{y}{2})(a-\frac{x}{2})-2M_5c$, $A_3=(a-\frac{x}{2})(b-\frac{y}{2})-c^2$. Notice that $A_2\geqslant A_2' := 2(\sqrt{(M_1+\frac{x}{2})(M_3+\frac{y}{2})}\sqrt{(a-\frac{x}{2})(b-\frac{y}{2})}-M_5c)$, we may use A_2' to substitute A_2 in the minimization of f_1 with respect to M_1, M_3, M_5. For sufficiently large M_1, M_3, M_5, we just omit A_3 term, then $\inf f_1 = 1+\inf_{M_1,M_3,M_5}\frac{A_2'}{A_1}$. Notice that f_1 and f_2 are independent in minimizations, we should keep both $\inf f_1\geqslant 1$ and $\inf f_2 \geqslant 1$ to preserve (B2). We should keep $A_2'\geqslant 0$, which is only possible when $\sqrt{(a-\frac{x}{2})(b-\frac{y}{2})}-c\geqslant 0$. Otherwise, we can make M_5 approach $\sqrt{(M_1+\frac{x}{2})(M_3+\frac{y}{2})}$ to violate $A_2'\geqslant 0$. Thus, we have the following two conditions derived from (38).

$$\left(a-\frac{x}{2}\right)\left(b-\frac{y}{2}\right)-c^2\geqslant 0, \tag{B3}$$

$$\left(a-\frac{1}{2x}\right)\left(b-\frac{1}{2y}\right)-c'^2\geqslant 0. \tag{B4}$$

conditions (B3) and (B4) are equivalent to

$$\gamma-\frac{1}{2}\mathrm{diag}\left(x,\frac{1}{x}\right)\oplus\frac{1}{2}\mathrm{diag}\left(y,\frac{1}{y}\right)\geqslant 0, \tag{B5}$$

which is the sufficient criterion of separability [8]. Hence Proposition 2 is proved.

Furthermore, the equalities in (B3) and (B4) are drawn as two curves in (x, y) plane. If the two curves intersect with each other, we have solution for the two inequalities. The combination of the two equalities in (B3) and (B4) leads to a quadratic equation for x (or y). The condition for the existence of the real x solution can be obtained easily. So the existence of the solution (x, y) for both inequalities (B3) and (B4) leads to the following condition:

$$(ab-c^2)(ab-c'^2)-\tfrac{1}{2}|cc'|-\tfrac{1}{4}(a^2+b^2)+\tfrac{1}{16}\geqslant 0. \tag{B6}$$

It is just the necessary and sufficient separable condition derived by Simon [7] with PPT criterion. Notice that the existence of even a point (x, y) as the solution of inequalities (B3) and (B4) means that we have a product squeezed state specified by x and y at hand, the two-mode Gaussian state with CM γ can be generated from this product squeezed state by applying displacement operations with classical probability distribution. So, inequalities (B3) and (B4) are also sufficient conditions for separability. We have shown that $\mathcal{L}_G^2\geqslant 1$ leads to condition (B6). On the other hand, if the condition (B6) is fulfilled, the two inequalities (B3) and (B4) are true for point (x, y) in some range of the plane. Then, we have $\mathcal{L}_G^2\geqslant 1$. Hence, the Proposition 2 is equivalent to the result of PPT criterion for a two-mode Gaussian state.

APPENDIX C: PROOF OF EQ. (23)

The characteristic function of a Gaussian Fock state is

$$\chi(\mu)=\mathrm{Tr}[\hat\rho\mathcal{D}(\mu)]. \tag{C1}$$

With the thermal channel defined in (21), after some algebra, we have

$$\chi(\mu)=\exp\left(-\sum_{j=1}^{n}N_j|\xi_j|^2\right)\langle l|\mathcal{D}(\xi)|l\rangle, \tag{C2}$$

where $\xi=(\xi_1,\xi_2,\dots,\xi_n)$, $(\xi,\xi^*)=(\mu,\mu^*)\tilde S^T$, and $\tilde S$ is a transform of S to adapt the complex variable characteristic function. We then express the characteristic function as [12]

$$\chi(\mu)=\mathcal{O}\exp\left[-\sum_{j=1}^{n}\left(N_j+\frac{1}{2}\right)|\xi_j|^2+t^Tt'+\xi t-\xi^*t'\right], \tag{C3}$$

where $\mathcal{O}=\frac{1}{\prod_j l_j!}\frac{\partial^{2|l|}}{\partial t^l\partial t'^l}|_{t=t'=0}$, with $|l|=\sum_{j=1}^n l_j$ and $\partial t^l=\partial t_1^{l_1}\partial t_2^{l_2}\dots\partial t_n^{l_n}$. Notice that $\exp[-\sum_{j=1}^n(N_j+\frac{1}{2})|\xi_j|^2]=\gamma_G$, we thus have

$$\begin{aligned}\mathrm{Tr}(\hat\rho\hat M)&=\int\chi(\mu)\chi_M(-\mu)\left[\frac{d^2\mu}{\pi}\right]\\&=\mathcal{O}\int\left[\frac{d^2\mu}{\pi}\right]\exp\left[-\frac{1}{2}(\mu,\mu^*)(\tilde\gamma_G+\tilde\gamma_M)\right.\\&\qquad\left.\times(\mu,\mu^*)^T+t^Tt'+\xi t-\xi^*t'\right]\\&=\frac{\mathcal{O}\exp[tt'+f(t,t')]}{\sqrt{|\det(\tilde\gamma_G+\tilde\gamma_M)|}}\end{aligned} \tag{C4}$$

with $f(t,t')=\frac{1}{2}(t^T,-t'^T)\tilde S(\tilde\gamma_G+\tilde\gamma_M)^{-1}\tilde S^T\binom{t}{-t'}$. In the limit of $\tilde\gamma_M\to\infty$ (also denoted as $\gamma_M\to\infty$), that is, all the parameters, M_i, in the γ_M tend to infinite or negative infinite, we have $f(t,t')\to 0$. Further notice that $\mathcal{O}\exp(t^Tt')=1$. Then equation (23) follows.

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