Fractional quantum Hall states on $\mathbb{CP}^2$ space

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We study four-dimensional fractional quantum Hall states on $\mathbb{CP}^2$ geometry from microscopic approaches. While in 2d the standard Laughlin wave function, given by a power of Vandermonde determinant, admits a product representation in terms of the Jastrow factor, this is no longer true in higher dimensions. In 4d, we can define two different types of Laughlin wave functions, the determinant-Laughlin and Jastrow-Laughlin states. We find that they are exactly annihilated by, respectively, two-particle and three-particle short-ranged interacting Hamiltonians. We then mainly focus on the ground state, low-energy excitations, and the quasihole degeneracy of determinant-Laughlin state. The quasihole degeneracy exhibits an anomalous counting, indicating the existence of multiple forms of quasihole wave functions. We argue that these are captured by the mathematical framework of "commutative algebra of Npoints in the plane." The microscopic wave functions and Hamiltonians studied in this work pave the way for a systematic study of a high-dimensional topological phase of matter that is potentially realizable in cold atom and optical experiments.

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I. INTRODUCTION

Searching for and understanding exotic phases of matter is a long standing goal of modern condensed matter physics. In three or lower spatial dimensions, many materials exhibiting topological or exotic properties have been synthesized. Higher dimensions, although realizable using optical and cold atom experimental techniques, are less well understood. It is both theoretically and experimentally interesting to ask about the possible many-body phenomena in four and higher dimensions [1,2].

The quantum Hall effect is the most studied phase of matter exhibiting topological properties [3,4]. It was discovered forty years ago in electron gases confined in two-dimensional semiconductor heterostructure in the presence of an ultra strong magnetic field [3]. In magnetic fields, an electron gas reorganizes itself into completely dispersionless Landau levels, in which the complete quench of kinetic energy paves the way for purely interaction driven physics. Interactions give raise to exotic many-body states including the Laughlin phase [5], which exhibits fractional Hall conductivity, and fractionalized particles or anyons [6]. Theoretically Landau levels have been predicted to exist in even dimensions [1,2,7–12], and examples have been realized in optical and cold atom experiments [13–17].

Motivated by these studies, in this work, we explore interacting effects in four and higher dimensions with uniform magnetic fields by initiating the study of the fractional quantum Hall (FQH) problem on $\mathbb{CP}^n$ geometry. We start by microscopically describing the $\mathbb{CP}^2$ geometry and the Landau levels. We then begin the discussion of many-body wave functions with a focus on Laughlin states and their quasihole descendants. In particular, we found there exist multiple types of Laughlin wave functions that are characterized by two-particle or three-particle clustering properties on the $\mathbb{CP}^2$ space:

$$\Psi^D_\beta = (\det_{1\leqslant i,j\leqslant N} f_{i,j})^\beta, \quad f_{i,j} \equiv \prod_{a=1}^{3} (z_i^a)^{p_j^a}, \tag{1}$$

$$\Psi^J_\gamma = \left(\prod_{1\leqslant i<j<k\leqslant N} \epsilon_{abc} z_i^a z_j^b z_k^c\right)^\gamma. \tag{2}$$

Here each particle's position on $\mathbb{CP}^2$ is specified by homogeneous coordinates, i.e., a complex triplet $\boldsymbol{z}_i = (z^1, z^2, z^3)_i$ where subscript $i = 1, \ldots, N$ labels the particle. In Eq. (1), $\det f$ is a "Slater," or "generalized Vandermonde," determinant, determined by a set of N triples of non-negative integers $\boldsymbol{p}_j = (p^1, p^2, p^3)_j$. Notations are specified in detail in Sec. II.

The constructions of Eqs. (1) and (2) can be intuitively thought of as follows: while the Jastrow factor equals the Vandermonde determinant in two dimensions (for instance on

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$\mathbb{CP}^1$), this is not true in higher dimensions. Viewing the $\mathbb{CP}^1$ Laughlin states as determinants or as Jastrow factors leads to distinct generalizations to $\mathbb{CP}^2$. Thereby, we term Eq. (1) as determinant-type Laughlin wave function (Det-Laughlin) and Eq. (2) as a Jastrow-type Laughlin wave function (Jas-Laughlin). We further show they are exact zero energy ground states for short-ranged two-particle and three-particle repulsive interactions. In the end, we discuss generalized Haldane pseudopotentials in high dimensions [18–22].

The quasihole excitations show interesting differences when compared with two dimensions. We found huge degeneracies in the quasihole space on $\mathbb{CP}^2$ manifold, which might be useful for storing quantum information [23]. Mathematically, the degeneracy of quasihole wave functions is interesting, and we will compare our results with relevant work of Haiman *et al.* [24].

The many-body wave function and model Hamiltonian developed here serves as an explicitly solvable model of a higher dimensional topologically ordered system [25–27], less well known than its lower dimensional analogs [28–30], yet potentially realizable in cold atom and optical experiments [13–17]. Further studying collective modes and numerically detecting non-point-like excitations [31,32] are just one of the interesting future directions.

The paper is organized as follows. We begin in Sec. II with an introduction to $\mathbb{CP}^2$ Landau levels. We also define coherent state representation [18,19,33] of single-particle and multi-particle bound-state wave functions in this section. In Sec. III, we define two types of Laughlin wave functions and discuss their parent Hamiltonians. In Sec. IV, we numerically study the low-energy excitations as well as the quasihole excitations of the Det-Laughlin state. In the subsequent section, Sec. V, we discuss symmetric interaction and the higher dimensional generalization of Haldane-pseudopotentials [18–22]. Finally in Sec. VI, we discuss the connection between our studies to the commutative algebra in the mathematical literature. We list open questions and interesting future directions in Sec. VII.

II. $\mathbb{CP}^2$ LANDAU LEVEL

The two-sphere $S^2 \simeq \mathbb{CP}^1$ geometry is one of the most useful geometries for studying two-dimensional quantum Hall physics [18,19]. The discussion of higher dimensional quantum Hall physics was initiated by S.C. Zhang and J. Hu in Ref. ([1]), who gave single particle Landau level wave functions on the four-sphere S^4. Soon after, D. Karabali and V. Nair, generalized the S^2 Landau levels to the $2n$-dimensional complex projective spaces $\mathbb{CP}^n$ in Ref. [2] and the generic case of a compact Kähler manifold with a uniform magnetic field was treated in Ref. [10]. In this section, we first review Landau levels on $\mathbb{CP}^2$ space, introduce a diagrammatic representation, and discuss coherent states which turn out to be useful for discussing interactions.

A. $\mathbb{CP}^2$ geometry and lowest Landau level states

The $\mathbb{CP}^2$ manifold is conveniently parameterized by three complex variables, satisfying a real constraint and identifying points which are equivalent up to a U(1) phase:

$$\boldsymbol{z} = (z^1, z^2, z^3) = (u, v, w), \quad |u|^2 + |v|^2 + |w|^2 = 1, \quad \boldsymbol{z} \sim \boldsymbol{z} e^{i\theta}, \tag{3}$$

where a subscript $i = 1, \dots, N$ will be added to label particles when discussing many-particle physics. Throughout this work, we will use superscript $a = 1, 2, 3$ to denote the inner index of SU(3) quantum number, and we will use (z^1, z^2, z^3) and (u, v, w) interchangeably.

$\mathbb{CP}^2$ has an SU(3) symmetry under which these coordinates transform in the fundamental representation. There is a unique metric (up to the overall scale) which respects this symmetry, the Fubini-Study metric [10]. The analogous expression with an $(n+1)$ component vector parameterizes $\mathbb{CP}^n$, with $SU(n+1)$ symmetry.

$\mathbb{CP}^2$ can also be obtained as the homogeneous space $\mathbb{CP}^2 = SU(3)/[SU(2) \times U(1)]$. This construction defines a natural background $SU(2) \times U(1)$ gauge field with minimal magnetic charge under the Dirac quantization condition and which respects the SU(3) symmetry. The U(1) part of this field has magnetic field strength given by a two-form $\mathcal{F}_{ab}$, which stands in a simple mathematical relationship to the Fubini-Study metric: it is the Kähler form for this metric. It is this relation, which is responsible for the simple form of the lowest Landau level (LLL) wave functions, and comparable results could be obtained for any Kähler manifold, as derived in Ref. [2].

In this work, we will consider particles with charge $e = S$ under the U(1) Abelian magnetic field, and with zero SU(2) charge. Equivalently, we can think of charge $e = 1$ particles under the influence of a magnetic field with flux S.

Following Ref. [2], the LLL wave functions are holomorphic functions of $\boldsymbol{z} = (u, v, w)$, with no dependence on the complex conjugation $\bar{\boldsymbol{z}}$. A complete orthonormal basis for such functions is

$$\psi_{S,\boldsymbol{p}} = u^{p^1} v^{p^2} w^{p^3}, \tag{4}$$

where $\boldsymbol{p} = (p^1, p^2, p^3)$ is a vector of non-negative integers and we define $p^1 + p^2 + p^3 = S$. The subset with fixed S are the wave functions with that U(1) charge, of total number $D = (S+1)(S+2)/2$.

Instead of $\boldsymbol{p}$, we sometimes use the labels (i, y), i.e., "isospin" and "hypercharge," defined by

$$\mathrm{i} = p^1 - p^2, \quad \mathrm{y} = p^1 + p^2 - 2p^3.$$

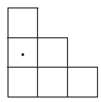
For an N-particle many-body state, we use capital $I = \sum_{i=1}^N \mathrm{i}_i$, $Y = \sum_{i=1}^N \mathrm{y}_i$ to label the total "isospin" and "hypercharge." The total $\sum_{i=1}^N p_i^{1,2} = (2NS + Y \pm 3I)/6$ are integers. Therefore,

$$2NS + Y \pm 3I \in 6\mathbb{Z}. \tag{5}$$

B. Diagrammatic representation and bivariate Vandermonde determinant

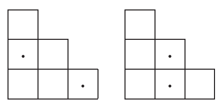
Here we introduce a diagrammatic representation for $\mathbb{CP}^2$ states. We represent the quantum numbers p^1 and p^2 by horizontal and vertical axes respectively. Since $p^{1,2,3} \in [0, S]$, each $\mathbb{CP}^2$ state is represented as a box in a triangle of base and height $S+1$. We then represent a particular N particle state by distributing N dots among the boxes, one per particle. If the particles are fermions, the Pauli principle is enforced by allowing at most one dot per box.

For example, the one particle state with $S = 2$ and $\boldsymbol{p} = (0, 1, 1)$ is depicted by



(6)

As another example, consider $N = 2$ particles and the $S = 2$, $I = 1$, $Y = 1$ subspace. There are two basis states, depicted by



(7)

The fermionic wave function corresponding to a diagram can be written as a *bivariate determinant*, a two variable generalization of the usual Vandermonde determinant. Denoting the set of filled boxes as $\{\boldsymbol{p}^1, \dots, \boldsymbol{p}^N\}$, the corresponding N particle wave function is

$$\Delta_{\{\boldsymbol{p}\}} = \det_{i,j}\left(z_i^{\boldsymbol{p}^j}\right),$$

$$= \det \begin{bmatrix} u_1^{p_1^1} v_1^{p_1^2} w_1^{p_1^3} & \dots & u_1^{p_N^1} v_1^{p_N^2} w_1^{p_N^3} \\ \vdots & & \vdots \\ u_N^{p_1^1} v_N^{p_1^2} w_N^{p_1^3} & \dots & u_N^{p_N^1} v_N^{p_N^2} w_N^{p_N^3} \end{bmatrix}. \tag{8}$$

Equation (8) is indeed "bivariate" (rather than tri-variate) because u, v, w are constrained by $|u|^2 + |v|^2 + |w|^2 = 1$. To see this more explicitly, one can turn "homogeneous coordinates" $\boldsymbol{z} = (u, v, w)$ into "projective coordinates":

$$\tilde{u} \equiv u/w, \quad \tilde{v} \equiv v/w, \tag{9}$$

and Eq. (8) can be rewritten as

$$\Delta_{\{\boldsymbol{p}\}} = \left(\prod_{i=1}^{N} w_i^S\right) \det \begin{bmatrix} \tilde{u}_1^{p_1^1} \tilde{v}_1^{p_1^2} & \dots & \tilde{u}_1^{p_N^1} \tilde{v}_1^{p_N^2} \\ \vdots & & \vdots \\ \tilde{u}_N^{p_1^1} \tilde{v}_N^{p_1^2} & \dots & \tilde{u}_N^{p_N^1} \tilde{v}_N^{p_N^2} \end{bmatrix}. \tag{10}$$

We sometimes use captical $Z \equiv (\tilde{u}, \tilde{v})$ to denote the projective coordinate.

C. Coherent states

Instead of discrete integer-valued quantum numbers (p_1, p_2, p_3) or (i, y), coherent states provide an overcomplete basis with continuous parameters [33]. They are useful for considering multi-particle interactions [18,19].

In the following discussion, we use the notation (z^1, z^2, z^3) in place of (u, v, w). Bold face $\boldsymbol{z}$ is still used to represent a vector, while $\bar{\boldsymbol{z}}$ represents the vector obtained by complex conjugating each component. The dot product represents the standard Euclidean inner product, for example, $\bar{\boldsymbol{\alpha}} \cdot \boldsymbol{z} = \bar{\alpha}^1 z^1 + \bar{\alpha}^2 z^2 + \bar{\alpha}^3 z^3$.

1. One-particle coherent state

The single-particle coherent state wave function is parameterized by a $\mathbb{CP}^2$ point $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$,

$$\psi_{S,\boldsymbol{\alpha}}^{(1)}(\boldsymbol{z}) = (\bar{\boldsymbol{\alpha}} \cdot \boldsymbol{z})^S, \tag{11}$$

where just as in Eq. (3), $\boldsymbol{\alpha}$ is normalized to one. The point corresponding to an equivalence class of $\boldsymbol{\alpha}$ under U(1) phase rotation is the point on which the coherent state is maximized, and in the limit $S \to \infty$ the state is localized at this point.

The SU(3) symmetry acts on the coherent state by moving its center $\boldsymbol{\alpha}$ while keeping its shape invariant. Varying the polarization vector $\boldsymbol{\alpha}$ continuously yields an Abelian Berry phase corresponding to the U(1) magnetic flux described earlier.

2. Three-particle coherent state

We next seek the coherent representations of bound states. A bound state has a center of mass $\boldsymbol{\alpha}$ which transforms as a fundamental of the SU(3) global symmetry, and internal variables which can also transform. Given $\boldsymbol{\alpha}$ there is a "little group" SU(2) × U(1) ⊂ SU(3) which preserves $\boldsymbol{\alpha}$, which can be used to define two internal quantum numbers. One is the spin J_1 under this U(1), which acts as [34–36]

$$\left(\sum_{i=1}^{8} \lambda_i(\boldsymbol{\alpha}) \cdot \hat{\boldsymbol{\Lambda}}_i\right)\phi \equiv J_1 \phi,$$

where $\hat{\boldsymbol{\Lambda}}_i$ are the Gell-Mann matrices, $\lambda_i = \sum_{a,b=1}^{3} \bar{\alpha}^a (\hat{\boldsymbol{\Lambda}}_i)_{ab} \alpha^b$, and ϕ is a three-component vector. The second is the SU(2) Casimir J_2:

$$\left(\sum_{i=1}^{8} \hat{\boldsymbol{\Lambda}}_i\right)^2 \phi \equiv J_2 \phi.$$

To construct bound states with definite values of these quantum numbers, we use invariant tensors. Now the only invariant tensor which couples fundamental representation of SU(3) is the three-index tensor ϵ^{abc}. Because of this, on $\mathbb{CP}^2$ the three particle bound state is simpler than the two particle bound state.

We define the "three particle coherent state" to be

$$\psi_{S,J,\boldsymbol{\alpha}}^{(3)}(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3) \equiv \left(\epsilon_{abc} z_1^a z_2^b z_3^c\right)^{S-J} \prod_{i=1}^{3} (\bar{\boldsymbol{\alpha}} \cdot \boldsymbol{z}_i)^J.$$

As its relative part $\epsilon_{abc} z_1^a z_2^b z_3^c$ is invariant under the little group, both of the internal quantum numbers are determined by J:

$$J_1 = J; \qquad J_2 = \tfrac{1}{3} J(J+3). \tag{12}$$

3. Two-particle coherent state

This expression can be adapted to describe two particles on $\mathbb{CP}^2$ by replacing the position $\boldsymbol{z}_3$ of the third particle with a constant vector $\boldsymbol{\beta}$. In some sense it describes the precession of the relative coordinate.

We define the "two particle coherent state" as

$$\psi_{S,J,\boldsymbol{\alpha},\boldsymbol{\beta}}^{(2)}(\boldsymbol{z}_1, \boldsymbol{z}_2) \equiv \left(\epsilon_{abc} z_1^a z_2^b \beta^c\right)^{S-J} \prod_{i=1}^{2} (\bar{\boldsymbol{\alpha}} \cdot \boldsymbol{z}_i)^J. \tag{13}$$

It is not invariant under the little group. An invariant state can be made by averaging over the relative vector $\boldsymbol{\beta}$, and it has the same internal quantum numbers Eq. (12).

We will use this decomposition to revisit the pseudopotential formalism and discuss generic symmetric interactions in Sec. V.

III. LAUGHLIN WAVE FUNCTIONS

We proceed to discuss interacting physics. An N-particle Laughlin wave function will be defined to be a totally antisymmetric (for β odd) or symmetric (for β even) state on the LLL, which vanishes to order β when any pair of particles coincides.

On $\mathbb{CP}^1$, the N-particle Laughlin wave function is defined with $S = \beta(N-1)$ total degree. Its wave function reads

$$\Psi_\beta = \prod_{i<j}^{N}(u_i v_j - u_j v_i)^\beta, \tag{14}$$

$$= \det \begin{bmatrix} u_1^0 v_1^{N-1} & \dots & u_1^{N-1} v_1^0 \\ \vdots & & \vdots \\ u_N^0 v_N^{N-1} & \dots & u_N^{N-1} v_N^0 \end{bmatrix}^\beta . \tag{15}$$

Equations (14) and (15) are respectively written in a form of the Jastrow factor and the Vandermonde determinant. The equivalence between the Jastrow factor and Vandermonde determinant, however, is no longer true in higher dimensions. As will be seen in this section, this leads to two types of Laughlin wave functions on $\mathbb{CP}^{n>1}$.

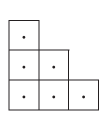
Nor is it *a priori* clear that either type of wave function completely exhausts the possible Laughlin wave functions. In Sec. VI, we will show that the determinantal wave functions do cover all of the Laughlin wave functions.

Throughout this work, when saying "Laughlin," we implicitly refers to the "Det-Laughlin," i.e., we will use the terminology "Det-Laughlin" and "Laughlin" interchangeably. For the Jastrow-type Laughlin states, we will explicitly term them as "Jas-Laughlin."

A. Type I: determinant-Laughlin wave function

We first discuss generalizing Laughlin wave function to four dimensions by using bivariate Vandermonde determinants. For degree S, the filled Landau level has in total $N = (S+1)(S+2)/2$ particles. We denote the filled Landau level wave function by $\Delta_{\mathrm{FLL},S}$. For instance, the filled Landau level wave function for $S = 2$ is represented as the following diagram, which involves $N = 6$ particles:

$$\Delta_{\mathrm{FLL},S2} = \Psi_{N6S2} = \tag{16}$$



We define the determinant-type Laughlin wave function at filling fraction $\nu = \beta^{-1}$ as

$$\Psi_\beta^D = (\Delta_{\mathrm{FLL},S/\beta})^\beta, \tag{17}$$

which occurs when

$$N = (S/\beta + 1)(S/\beta + 2)/2. \tag{18}$$

The Det-Laughlin wave functions vanishes in power of β when any two particles approach each other. Following similar arguments of Trugman and Kivelson [37], it is the exact zero energy ground state for short-ranged repulsive interactions. The concrete form of interacting Hamiltonian is given as follows in Eq. (19), which is generalized from the two-dimensional form proposed by Wen *et al.* [30]:

$$H_\beta^D = -\sum_{i<j}\sum_{l=0}^{\beta-1} V_l (\partial^\dagger)^l \delta(Z_i, Z_j)(\partial)^l, \tag{19}$$

where V_l are arbitrary non-negative potentials and the conjugation is defined with respect to an appropriate L^2 structure on many-body states. The $Z = (\tilde{u}, \tilde{v})$ are the projective coordinates introduced in Eq. (9).

B. Type II: Jastrow-Laughlin wave function

Alternatively, the $\mathbb{CP}^1$ Laughlin wave function can be regarded as a Jastrow factor as shown in Eq. (14). Generalizing the singlet $(u_i v_j - u_j v_i)$ to $\mathbb{CP}^2$ requires three particles. Considering three particles labeled by 1,2,3, the singlet wave function is

$$u_{123} \equiv \epsilon^{ijk} u_i v_j w_k; \quad i, j, k = 1, 2, 3.$$

We define the Jastrow-type Laughlin wave function as

$$\Psi_\gamma^J = \left(\prod_{i<j<k} u_{ijk}\right)^\gamma, \tag{20}$$

which occurs when

$$S/\gamma = (N-1)(N-2)/2. \tag{21}$$

To be concrete, the $N = 3, 4, 5$ particle Jas-Laughlin wave functions are

$$\begin{aligned} S/\gamma &= 1, \quad \Psi_\gamma^J(\boldsymbol{u}_{1,\dots,3}) = (u_{123})^\gamma, \\ S/\gamma &= 3, \quad \Psi_\gamma^J(\boldsymbol{u}_{1,\dots,4}) = (u_{123}\, u_{124}\, u_{134}\, u_{234})^\gamma, \\ S/\gamma &= 6, \quad \Psi_\gamma^J(\boldsymbol{u}_{1,\dots,5}) = (u_{123}\, u_{124}\, u_{125}\, u_{134}\, u_{135}\, u_{145} \\ &\qquad\qquad u_{234}\, u_{235}\, u_{245}\, u_{345})^\gamma. \end{aligned} \tag{22}$$

Note that we used γ to parameter the wave function Eq. (20). The two-particle vanishing power of Jas-Laughlin wave function, defined as the vanishing power when any pair of particles approach, is

$$\beta = \gamma(N-2). \tag{23}$$

Therefore, for both Det-Laughlin and Jas-Laughlin, β determines the statistics: the wave function is fermionic if β is odd, and bosonic if β is even.

The vanishing power β for Jas-Laughlin is tricky, as it dependents on the particle number N, therefore for fixed γ there is no thermodynamic definition for the Jas-Laughlin based on β. The γ instead defines the three-particle clustering property of Jas-Laughlin, and this phase is thereby characterized by three, rather than two, particle properties.

TABLE I. The particle number N and the associated degree S for the Det-Laughlin Ψ^D_β and the Jas-Laughlin Ψ^J_γ, following Eqs. (18) and (21). "Statistics" is 0 (1) means that the Laughlin state is realized by interacting bosons (fermions). For $N = 3$ particles, Det-Laughlin is equivalent as the Jas-Laughlin. The $N = 4$ Jas-Laughlin is the simplest nontrivial example which are zero energy ground state of $\sum_{i<j} \delta(Z_i - Z_j)$ interactions but cannot be written as a product of determinants.

Det-Laughlin Ψ^D_β			Jas-Laughlin Ψ^J_γ		
N	S/β	statistics	N	S/γ	statistics
3	1	$\beta \mod 2$	3	1	$\gamma \mod 2$
6	2	$\beta \mod 2$	4	3	0
10	3	$\beta \mod 2$	5	6	$\gamma \mod 2$
…	…	…	…		

C. Comparing two types of Laughlin wave functions

We end this section by tabulating the particle number N and the associated total degree that the two Laughlin wave functions can occur in Table. I. Note that for $N = 3$ particles, the two types of Laughlin wave function coincide. Generally speaking, Ψ^D is denser than Ψ^J as seen from Fig. 1.

The Det-Laughlin wave function has been proved to be the unique zero energy ground state for two-particle short-ranged interaction [20]. We verify this result numerically in the next section. For this reason, we term the $(N, S/\beta)$ of the left column as the "commensurate parameter" for Det-Laughlin wave function. Similarly, the $(N, S/\gamma)$ of the right column can be termed as the commensurate parameter for Jas-Laughlin wave functions.

As discussed earlier, the two-particle vanishing power β for Det-Laughlin is particle number independent. Thereby, the Det-Laughlin admits a well defined two-particle pseudopotential parent Hamiltonian for all system size given fixed β. Although the Jas-Laughlin also vanishes with certain power when any two-particle cluster as seen from Eq. (23), the vanishing power is particle number dependent. Therefore, for Jas-Laughlin, only three-particle pseudopotential parent Hamiltonian admits a system size independent definition.

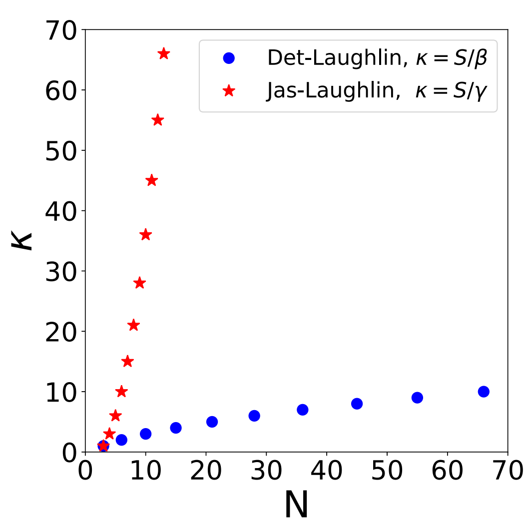


FIG. 1. Illustration of Table I, from which we see directly Det-Laughlin generally is much denser than Jas-Laughlin. Moreover, it is interesting to observe that in the thermodynamic limit $N \to \infty$, the flux per particle approaches 0 and ∞ for Det-Laughlin and Jas-Laughlin, respectively. This is in contrast to the $\mathbb{CP}^1$ case where the flux density approaches constant.

In the next section, we numerically study the parent Hamiltonian of the Det-Laughlin, with a focus on its ground state, low-lying excitations and quasihole degeneracies.

IV. NUMERICAL STUDIES

A. Numerical diagonalization

Laughlin wave functions are characterized by their clustering behavior: on $\mathbb{CP}^1$, the Laughlin wave function Ψ_β vanishes in power of β when any two particles coincide. This yields the consequence that Laughlin wave functions are exact zero energy ground states for any repulsive interactions as $\sum_{l<\beta} -V_l \partial^{\dagger l} \delta(Z_i, Z_j) \partial^l$ where $V_l > 0$.

In the following sections, we numerically study the Laughlin state and their quasihole descendent for repulsive two-body interaction at various (N, S). For fermions, we use

$$H = -\sum_{i<j} \partial^\dagger \delta(Z_i, Z_j) \partial, \tag{24}$$

and for bosons, we use

$$H = \sum_{i<j} \delta(Z_i, Z_j). \tag{25}$$

As reviewed in the first section, orthonormal noninteracting single-particle states in this Hilbert space are labeled by Eq. (4). We first derive the two-body interaction element,

$$V_{\boldsymbol{q}\boldsymbol{q}';\boldsymbol{p}\boldsymbol{p}'} \equiv \langle \boldsymbol{q}\boldsymbol{q}'|H|\boldsymbol{p}\boldsymbol{p}'\rangle, \tag{26}$$

where $|\boldsymbol{p}\boldsymbol{p}'\rangle$ is a two-particle wave function, that is antisymmetrized for fermions and symmetrized for bosons. With the matrix elements Eq. (26), the second quantized Hamiltonian reads:

$$H = \sum_{\boldsymbol{p}_1\boldsymbol{p}_2;\boldsymbol{p}_3\boldsymbol{p}_4} V_{\boldsymbol{q}\boldsymbol{q}';\boldsymbol{p}\boldsymbol{p}'} \delta_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{p}'+\boldsymbol{q}'} c^\dagger_{\boldsymbol{q}} c^\dagger_{\boldsymbol{q}'} c_{\boldsymbol{p}} c_{\boldsymbol{p}'}, \tag{27}$$

where $c^\dagger_{\boldsymbol{p}}$ creates a single-particle wave function $\psi_{S,\boldsymbol{p}}$ as seen in Eq. (4). The δ function above stems from the SU(3) quantum number conservation since the interaction is SU(3) symmetric (corresponding to transnational invariant in the thermodynamic limit). Diagonalizing the second quantized Hamiltonian Eq. (27) gives many-body wave functions and energies. The matrix elements are straightforwardly derived using the single particle wave functions.

For all (N, S) listed in the left-panel of Table I for Vandermonde Laughlin Ψ^D_β, numerically we found single degenerate many-body zero mode for interaction Eq. (24). They correspond to the *fermionic Det-Laughlin wave functions* at $\beta = 3$. They are fermionic because the vanishing power β is odd.

We also found a single zero mode at $N = 4$, $S = 3$ for interaction Eq. (25). This corresponds to the four-particle *bosonic Jas-Laughlin wave function* of $\gamma = 1$ listed in the second line of Eq. (22). It is bosonic because its vanishing power β, according to Eq. (23), is even. Interestingly, this

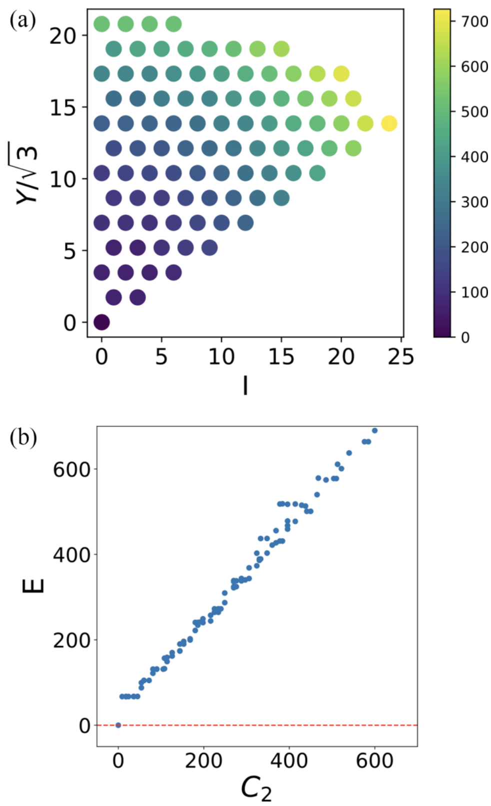


FIG. 2. Ground state and low-energy excitations in $(N, S) = (6, 6)$ sector. (a) shows the lowest energy in each quantum number sector. The color indicates energy. (b) plots the same data as (a) but in terms of the quadratic Casimir C_2 of SU(3). In (b), a unique zero energy ground state at $I = Y = 0$ is clearly seen. Moreover, (b) indicates the low-energy excitations are well approximated by a quadratic dispersion form.

wave function is the simplest wave function which exhibits the anomalous counting of $\mathbb{CP}^2$ which we will discuss more in Sec. VI: this wave function *cannot* be represented by a product of Vandermonde determinant, but rather is a linear combination of determinants where the high order vanishing powers has cancellation.

At $(N, S) = (5, 6)$, we observed multiple zero modes for Eq. (24). One of them is the $N = 5, \gamma = 1$ Jastrow wave function, i.e., the last line of Eq. (22). Besides, the zero mode space also include quasihole descendants of $(N, S) = (6, 6)$ Det-Laughlin as $(N, S) = (5, 6)$ can be obtained from $(N, S) = (6, 6)$ by removing one particle.

B. Low-energy excitation of Det-Laughlin

In this section, we focus on the ground state and low-energy excitation at the commensurate filling fraction for $N = 6$ particle Det-Laughlin at $S = 6$. As shown in Fig. 2, the numerically observed unique zero energy ground state and a finite energy excitation may indicate the Det-Laughlin is an in-compressible state. Careful numerical studies about the finite size scaling of the gap are required in the future.

Since the interaction Eq. (24) is SU(3) invariant, it will take the same value on every state in an irreducible SU(3) representation. This value can be written as a function of two Casimir invariants, the quadratic Casimir C_2 and the cubic Casimir C_3. For a given irreducible representation, there is a unique highest weight state (largest I and Y), and one can write the Casimirs as functions of its quantum numbers. Particularly,

$$C_2 = 3I^2 + Y^2 + 6(I + Y). \tag{28}$$

As shown in Fig. 2, the lowest energy in each quantum number sector (I, Y) displays a good linearity, if plotting the quantum number in terms of the quadratic Casimir C_2. This indicates a quadratic low-energy dispersion which may be captured by a high-dimensional generalization of the magneto-roton theory [38,39], which we leave for future studies.

TABLE II. Zero mode space dimension for Eq. (24) interaction in $(N, S) = (3, 4)$ sector. All zero modes can be written as a product of three determinants of degree $S = 1 + 1 + 2$. Empty grids are invalid quantum numbers as they are constrained by Eq. (5).

	$I = 0$	$I = \pm 1$	$I = \pm 2$	$I = \pm 3$
$Y = 0$	2		2	
$Y = \pm 3$		2		1
$Y = \pm 6$	1		0	

C. Quasihole degeneracies

We have numerically studied the energy spectrum of short-ranged interaction Eq. (24) in the Hilbert space (N, S) listed in the Table I. We found the Det-Laughlin are the unique ground states. While Jas-Laughlin are not generally the unique $E = 0$ states for two-particle interactions, they are for three-particle interactions as we will discuss later.

In this section, we introduce one extra flux quanta from the commensurate parameter of Det-Laughlin, and study the ground state degeneracies of two-particle interaction Eq. (24). The ground state degeneracies correspond to the dimension of quasihole wave functions. We found anomalous counting in the quasihole degeneracy, which has close connection to the mathematical subject discussed in Sec. VI.

1. N3S4

We start with the simplest quasihole state at $(N, S) = (3, 4)$, which descends from the simplest Det-Laughlin wave function at $(N, S) = (3, 3)$. In Table II, we list the dimension of the zero-modes in each quantum number (I, Y) sector which counts the dimension of quasihole wave functions.

In fact, in this case, quasihole zero modes are all given by the wave function which is written as a product of three determinants of degree $S = 1 + 1 + 2$:

$$\Psi_{N3S4} = (\Delta_{\text{FLL},S1})^2 \times \Psi_{N3S2}, \tag{29}$$

where $\Delta_{\text{FLL},S}$ is the fully filled $\mathbb{CP}^2$ Landau level wave function defined above Eq. (16). The Ψ_{N3S2} has three particles filled in space of degree $S = 2$, and it is diagrammatically represented as shown in Eq. (7). From Eq. (7), we also see

TABLE III. Zero mode space dimension of $(N, S) = (6, 7)$. *Not all* zero modes can be written as a product of three determinants of degree $S = 2 + 2 + 3$: the number outside the bracket labels the dimension of three determinant product wave functions, while that inside the bracket marks the dimension of additional zero modes observed in numerical calculations. The total dimension of three determinant product wave functions is 210, while the total zero mode space dimension is 266. Empty grids are invalid quantum numbers as they are constrained by Eq. (5).

	$I = 0$	$I = \pm 1$	$I = \pm 2$	$I = \pm 3$	$I = \pm 4$	$I = \pm 5$	$I = \pm 6$	$I = \pm 7$
$Y = 0$	12 (5)		10 (3)		6 (2)		2	
$Y = 3$		10 (3)		6 (2)		3		
$Y = -3$		10 (3)		8 (2)		3 (1)		1
$Y = 6$	8 (2)		6 (2)		3		1	
$Y = -6$	6 (2)		6 (2)		3 (1)		1 (1)	
$Y = 9$		3 (1)		2				
$Y = -9$		3		2		1		
$Y = 12$	1 (1)		1					
$Y = -12$	1							

why the dimension of $(I, Y) = (0, 0)$ sector in Table. II is two. Degeneracies in other quantum number sectors can be worked out diagrammatically straightforwardly following the same spirit. This case does not show any anomalous properties as the dimensions of zero modes are all expected.

2. N6S7

We next study the zero mode dimension in the $(N, S) = (6, 7)$ sector, which descends from the $(N, S) = (6, 6)$ Det-Laughlin by adding one extra flux quanta. Analogies to Eq. (29), we first write down the quasihole wave functions as a product of determinants of degree $S = 2 + 2 + 3$:

$$\Psi_{N6S7} = (\Delta_{\mathrm{FLL},S2})^2 \times \Psi_{N6S3}, \tag{30}$$

where Ψ_{N6S3} represents diagrams of $N = 6$ dots filled in degree $S = 3$.

However, as we see in Table III, this type of quasihole wave functions is *not* enough to explain the degeneracies observed numerically: in each (I, Y) sector, we label the dimension of quasihole wave function of type Eq. (30) as the number outside the bracket, and we list in the bracket the additional zero modes dimension observed numerically. The total zero mode space dimension seen numerically is 266, which has in total 56 more zero modes than the total dimension of wave function Eq. (30) which is only 210.

The unexpected extra zero modes indicates the quasihole wave function on $\mathbb{CP}^2$ has more than one expression in sharp contrast to the $\mathbb{CP}^1$ case. We noted the $\gamma = 1$ Jas-Laughlin at $(N, S) = (4, 3)$ is an example of zero mode which cannot be written purely as a product of several determinants. We anticipate besides Eq. (30), general forms of quasihole wave function should also include those with linear combination of determinants and cancellations, which we decide to discuss more extensively in Sec. VI.

V. SU(3) PSEUDOPOTENTIALS

Interactions Eqs. (24) and (25) are important short-ranged interactions. How do we classify generic SU(3) symmetric interaction in high dimension? Here, we generalize the Haldane-pseudopotential to higher dimensions, for both two-particle and three-particle interactions. The two-particle pseudopotential was initially derived based on group theoretical analysis by Chyh-Hong Chern *et al.* in Ref. ([20]). We use a different approach by using coherent state representations developed in Sec. II.

Considering two-particle Hilbert space, as shown in Eq. (13), such space is block diagonalized by two-particle coherent states labeled by non-negative integer $J \in [0, S]$. We define the J-space projector as $\hat{P}^{(2)}_{S,J}$. The action of any SU(3) symmetric interaction can be block-diagonalized into actions within the J subspaces as follows:

$$H = \sum_{J=0}^{S} V^{(2)}_J \hat{P}^{(2)}_{S,J}, \tag{31}$$

where $V^{(2)}_J$ is the interaction decomposition coefficient, which can be defined as the *two-particle SU(3) pseudopotential*.

The symmetries are manifest in the Hamiltonian Eq. (31). To implement practical calculations, one needs to convert it into the second quantized form such as Eq. (27). The matrix elements $v_{\boldsymbol{qq'};\boldsymbol{pp'}}$ are straightforwardly derived:

$$V_{\boldsymbol{qq'};\boldsymbol{pp'}} = V^{(2)}_J \langle \boldsymbol{qq'} | \hat{P}^{(2)}_{S,J} | \boldsymbol{pp'} \rangle = V^{(2)}_J \sum_{\boldsymbol{\alpha}} C^{J;\boldsymbol{\alpha}}_{\boldsymbol{qq'}} C^{J;\boldsymbol{\alpha}}_{\boldsymbol{pp'}},$$

where $C^{J,\boldsymbol{\alpha}}_{\boldsymbol{pp'}}$ is the SU(3) Clebsch-Gordan coefficient. The above expression was initially derived in Ref. ([20]).

The three-particle coherent state wave function Eq. (12) explicitly shows that the three-particle Hilbert space is block-diagonalized by index J for SU(3) symmetric three-particle interaction. Consequently, we have

$$H = \sum_J V^{(3)}_J \hat{P}^{(3)}_{S,J}, \tag{32}$$

where $\hat{P}^{(3)}_J$ is the three-particle projector that projects three-particle bound states into J subspace. The $V^{(3)}_J$ are defined as the *three-particle SU(3) pseudopotential*. And the projector can be similarly represented by the CG coefficients.

VI. COMPARISON WITH MATHEMATICAL RESULTS

It turns out that the problem we are discussing, of fermionic or bosonic wave functions for the Laughlin states in four dimensions, fits nicely into the mathematical framework of commutative algebra of N points in the plane, as studied by M. Haiman [24].

By "plane" here one means a space parameterized by two complex coordinates, so this is the first relation: one can parametrize almost all of $\mathbb{CP}^2$ by taking general (u, v) in Eq. (3), and solving the constraint to determine w. Thus we can regard an N-particle wave function as a function of the $2N$ complex coordinates $u_1, v_1, \ldots, u_N, v_N$. From Eq. (4) the wave functions of interest are polynomial in these variables. Thus we ask the following question.

Question VI.0.1. Characterize the polynomials in the variables $u_1, v_1, \ldots, u_N, v_N$ which vanish whenever $u_i = u_j$ and $v_i = v_j$ for some (i, j).

The space of such polynomials is an ideal $I = \cap_{i<j}(u_i - u_j, v_i - v_j)$ in the polynomial ring $\mathbb{C}[u_1, v_1, \ldots, u_N, v_N]$ in $2N$ variables. Now it is easy to see that the analog of the bivariate determinants Eq. (8) in two variables (equivalently, solving for the w_i's) are polynomials with this property, but it is not so obvious that all such polynomials can be obtained this way (more precisely, are sums of bivariate determinants). Theorem 1.1 of Ref. ([24]) is precisely this fact, that I coincides with the ideal generated by the bivariate Vandermondes for N points, and indeed the author states that "this is not an easy theorem." We will not even try to explain the proof here, but instead cite further relevant results from this work.

First, let us compare with the case of complex dimension one. There the analogous statement was true, namely all polynomials in $u_1, \ldots, u_N$ which vanish for any $u_i = u_j$ can be obtained as sums of determinants $\det_{i,j} u_i^{p_j}$ each multiplied by a polynomial. But a much simpler statement was also true, namely, all such polynomials can be obtained by multiplying the Vandermonde determinant $\Delta(u) = \prod_{i<j}(u_i - u_j)$ (the special case with $p_j = j - 1$) by a single arbitrary polynomial f. In other words, the ideal $J = \cap_{i<j}(u_i - u_j) \subseteq \mathbb{C}[u_1, \ldots, u_N]$ is a principal ideal, i.e., an ideal generated by a single polynomial $\Delta(u)$. In standard algebraic notations, $J = (\Delta(u))$, where the notation on the right stands for the set of polynomials obtained by multiplying a given polynomial Δ by an arbitrary polynomial f.

To restrict this to totally antisymmetric functions, one need only restrict f to be totally symmetric. Physically, this is closely related to exact bose-fermi equivalence in one dimension—the bosonic operators (totally symmetric between particles) act naturally on the free fermion Hilbert space.

Could there be a similar simplification in two variables? We have translated our question into: is I a principal ideal? According to theorem 1.2 of Ref. ([24]), no: the situation is more complicated. Fortunately we can broaden our definitions as follows: let $(\Delta_1, \Delta_2, \ldots, \Delta_k)$ be the space of polynomials obtained by taking an arbitrary linear combination $\sum_a f_a \Delta_a$ where Δ_a are bivariate Vandermondes and the f_a's are general polynomials. This is a general ideal, and general results tell us that this is possible.

In fact we can be more precise: define the generators of I to be a basis of elements which cannot be obtained as $\sum_a f_a \Delta_a$ where the $\Delta_a \in I$, but the f_a have no constant part (so, they can be the variables u_i, v_i or higher order polynomials). Theorem 1.2 tells us that the dimension of this basis is the N'th Catalan number,

$$C_N = \frac{1}{N+1}\binom{2N}{N}.$$

However, the proof is nonconstructive, and no explicit choice for this basis is known except for $N = 2, 3$. In physics terms, this tells us that if there is an exact bosonization in two variables, it will not suffice to let the bosonic operators act on a unique ground state; to get the entire fermionic Hilbert space one will need to start from several (though a finite) number of distinct states.

Let us turn to discuss the FQHE states for the filling fraction $1/\beta$, where $\beta \in \mathbb{N}^*$. Here we have two possible definitions–the physics and the algebraic one.

Definition (*algebraic*). Laughlin states are the polynomials in $\mathbb{C}[u_1, v_1, \ldots, u_N, v_N]$, which are

(1) symmetric (for even β), or antisymmetric (for odd β) wrt exchanging the pairs of coordinates of N points $(u_1, v_1), \ldots, (u_N, v_N)$;

(2) of partial degree $S \in \mathbb{N}$, where partial degree is a sum of top degrees in u_j and v_j [this is independent of the choice of index j due to (1)];

(3) belonging to I^β.

The latter condition means that a Laughlin state can be written a linear combination of the form $\sum_{a_1,\ldots,a_\beta} f_{a_1,\ldots,a_\beta} \Delta_{a_1} \cdots \Delta_{a_\beta}$, where $f_{a_1,\ldots,a_\beta}$ is a (necessarily symmetric) polynomial.

Definition (*physics*). Laughlin states are the polynomials in $\mathbb{C}[u_1, v_1, \ldots, u_N, v_N]$, satisfying (1) and (2) above, which are also

(3') exact ground states of Hamiltonian (19).

The condition (3') can be reformulated as follows. Let $I^{\langle\beta\rangle}$ denote the ideal of all polynomials in $\mathbb{C}[u_1, v_1, \ldots, u_N, v_N]$, which belong to I together with all of their partial derivatives of order $\beta - 1$ and less, i.e.,

$$I^{\langle\beta\rangle} = \left\langle f \,\middle|\, \frac{\partial^{\mathbf{r}} f}{\partial u_1^{r_1} \ldots \partial v_{2N}^{r_{2N}}} \in I \text{ for all } \mathbf{r} \in \mathbb{N}^{2N}, \sum r_n < \beta \right\rangle$$

then (3') is equivalent to (3") Laughlin state is a polynomial in $I^{\langle\beta\rangle}$.

It is easy to see that in complex dimension one the two definitions coincide. Luckily, the same property holds in complex dimension two, at least as long as we allow all mixed derivatives of the order up to $\beta - 1$ in Hamiltonian (19). We have $I^\beta = I^{\langle\beta\rangle}$. The proof is indirect and is a consequence of the property 1.7. in Ref. [24] of the coincidence of the powers of I with symbolic powers (a notion we do not define here), and the coincidence of symbolic and differential powers (Zariski-Nagata theorem) for the radicalr ideals [40].

Now we would like to pose anothe question.

Question VI.0.2. Compute dimensions of the Hilbert spaces of Laughlin states quasihole states, as a function of N, S, β.

In a complex dimension, one we know that all of these states can be obtained by multiplying the Laughlin state Δ^β by a symmetric function, so it would suffice to count the dimensions of the spaces of symmetric polynomials of given degree. In particular, on a Riemann surface of genus-g, we know that there are no Laughlin states for $N > S/\beta + 1 - g$, at $N = S/\beta + 1 - g$ their number is β^g and the degenaracies for $N < S/\beta + 1 - g$ have been computed as well [41].

What is the situation in two variables? Our results show that the situation is rather complicated, and far from being fully understood. In some sense, this question is addressed by theorem 1.8 and its corollary 1.9. This states that the ideal I^β, defined as the product of β polynomials each taken from I, is just the β'th power of the ideal generated by the bivariate determinants. In other words, every polynomial which vanishes to at least order β when $u_i = u_j$ and $v_i = v_j$ for any pair (i, j), can be obtained as a sum of terms, each of which is a product of β bivariate determinants multiplied by some (unconstrained) polynomial.

Now, the discussion in Ref. [24] concerns all polynomials, with no symmetry or antisymmetry imposed. Furthermore the quantum number S is an additional feature of our problem. It is tempting to adapt corollary 1.1 to our physical situation by making the following conjectures:

Conjecture VI.0.1. Every symmetric polynomial (for β even) or antisymmetric polynomial (for β odd) of degree S in each variable, which vanishes to at least order β when $u_i = u_j$ and $v_i = v_j$ for any pair (i, j), can be obtained as a sum of terms, each of which is a product of β bivariate determinants multiplied by a symmetric polynomial.

Conjecture VI.0.2. Furthermore each determinant is homogeneous, and the sum of their degrees $\sum_a S_a = S$.

These conjectures, if true, would give us a general construction of the FQHE states on $\mathbb{CP}^2$. The first step is to list all partitions of the U(1) charge of ther form:

$$S = \sum_{a=1}^{\beta} S_a.$$

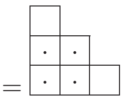
We can then enumerate all of the bivariate determinants of each required degree S_a, of total number $\binom{(S_a+1)(S_a+2)/2}{N}$, using the diagrammatic method. Finally we combine the choices, taking into account equivalences which arise if any $S_a = S_b$ for $a \neq b$.

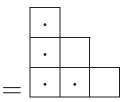
It took us some time to realize that while conjecture VI.0.1 is true (it follows from the corollary 1.9, Ref. [24]), conjecture VI.0.2 is in fact false. A simple counterexample is the following: consider the four particle Jastrow wave function Ψ_γ^J with $\gamma = 1$ and thus vanishing order $\beta = 2$; see the second line of Eq. (22). According to the conjecture, it should be in the ideal generated by products of two bivariate determinants, schematically $\sum f_a \Delta\Delta$. Now since $S = 3$, if the determinants are homogeneous, then one of them must have degree $S_a < 2$. But since there are only three independent states with $S = 1$, all such four particle determinantsr vanish.

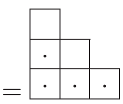
In fact the required sum of products of determinants (which must exist by corollary 1.1) is

$$u_{123}\, u_{124}\, u_{134}\, u_{234} = \Delta_1^2 - \Delta_2\Delta_3, \tag{33}$$

where the determinants are defined using the following sets of four indices:

$\Delta_1 =$  (34)

$\Delta_2 =$  (35)

$\Delta_3 =$  (36)

Each of these determinants has $S = 2$, and the individual products which appear in Eq. (33) indeed have terms with $S = 4$. However, these terms cancel in the difference, resolving the contradiction.

We found a similar mismatch at $N = 6$ and $S = 7$ between the total count of $\beta = 3$ states (266) and the number of states (210) which can be realized as a products of three determinants with $S = 2 + 2 + 3$. We believe that the resolution is analogous, that the extra 56 states are sums of products of three determinants in which the higher degree terms cancel.

Thus, the construction of the FQHE states on $\mathbb{CP}^2$ which we outlined above, does not produce all of the states. To fix this, one would need some understanding of the cancellations we observed, and at the very least a bound on the maximal individual degrees S_a's.

To summarize, in this section we have emphasized the validity of the two key elements behind the analytic construction of Laughlin states in two complex dimensions. The first is that the Laughlin state will have a generic form of the product of bivariate Vandermonde determinants. The second element is the equivalnce of the algebraic and physics definitions, which establishes the isomorphism between the ground states of the Haldane-Wen *et al.* Hamiltonian (19) and the Laughlin-style algebraic definition in terms of holomorphic polynomials. In this respect, the situation is similar to the case of complex dimension one.

Finally we note our question about Laughlin states can be posed for any compact Kähler manifold, replacing the degree-S polynomials by holomorphic sections of the line bundle [10] and bivariate Vandermonde determinant by the corresponding Slater determinant [42–44].

VII. CONCLUSION AND FUTURE DIRECTIONS

We have discussed the FQH effect on $\mathbb{CP}^2$ manifold with uniform U(1) background magnetic field. We defined two types of Laughlin wave function, one of the determinant type and one of the Vandermonde type. They are respectively shown to be the exact zero energy ground states for short-ranged two- and three-body interactions, for the (N, S) specified from their wave functions such as those listed in Table I. The quasihole space degeneracy shows anomalous behavior indicating quasihole wave function has more than one form, which is different compared to the $\mathbb{CP}^1$ usual FQH effects. There are few future research directions that the theory and techniques developed in this work could be useful. For instance, a further detailed study of the compressibilities of the two types of Laughlin wave functions, as well as their low-energy excitations [38] are interesting. In contrast to the two-dimension, high dimension may support membrane-like excitations [31,32]. It may also be possible to extend this work to other quantum Hall states including the paired states [45] and gapless states [46–51]. A thorough mathematical understanding of quasihole wave functions besides Sec. VI is an open questions. Last but not least, searching for experimental realizations of interacting physics in high-dimensional Landau levels [14–16] are important future directions.

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