Jarzynski-like equality of nonequilibrium information production based on quantum cross-entropy

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The two-time measurement scheme is well studied in the context of quantum fluctuation theorem. However, it becomes infeasible when the random variable determined by a single measurement trajectory is associated with the von Neumann entropy of the quantum states. We employ the one-time measurement scheme to derive a Jarzynski-like equality of nonequilibrium information production by proposing an information production distribution based on the quantum cross-entropy. The derived equality further enables one to explore the roles of the quantum cross-entropy in quantum communications, quantum machine learning, and quantum thermodynamics.

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I. INTRODUCTION

Quantum thermodynamics explores the laws of the thermodynamics in the nanoscale from the perspective of quantum information science [1–10]. On such scales, statistical fluctuations become more significant, and have principally been accounted for by the fluctuation theorem [11–15]. The discovery of the fluctuation theorem is one of the most important accomplishments in thermodynamics to date [16]. The fluctuation theorem can be regarded as a first principle in thermodynamics, from which many fundamental principles of thermodynamic phenomena can be derived, such as arrow of time [17] and response theory [18,19].

More recently, fluctuation theorems have equally been used to characterize information processing tasks. For example, Sagawa and Ueda [20] and Fujitani and Suzuki [21] related the fluctuation theorem with an efficacy of the feedback control for the manipulation of the total entropy production via measurements. The relation between the fluctuation theorem and the adiabaticity of the process was revealed by considering the state distinguishability [22,23]. In the context of quantum computing and communications, Gardas and Deffner [24] demonstrated that the fluctuation theorem can be used to determine the dynamics of the quantum systems and the susceptibility to the thermal noise. Also, Kafri and Deffner [25] related the fluctuation theorem and the Holevo information [26–28], which upper bounds the amount of classical information that can be transmitted through the quantum channel.

A standard approach to the fluctuation theorem in the quantum regime is the two-time measurement (TTM) scheme [29–38], in which the distribution of the measurement outcomes is constructed by the projection measurements on the system before and after the process. The first measurement corresponds to the state preparation of the input state, which is an ensemble of the eigenstates of the first measurement weighted by the probabilities of obtaining the corresponding outcomes, while the second measurement can be independent from the output state [25,39].

While this scheme corresponds to the classical approach in stochastic thermodynamics [40], in the quantum regime, it is considered to be inconsistent because it does not take into account the quantum coherence [41] and the informational contribution of back-action of projection measurements [22]. Particularly, for the information production (namely, the von Neumann entropy gain), one needs to fully obtain the information of the output state because the second measurement is strictly dependent on the principal components of the output state, which requires the quantum state tomography. Therefore, from a practical and conceptual perspective, the TTM scheme is infeasible when we want to deal with information production in the context of fluctuation theorem. Also, while there are other approaches beyond the TTM scheme, such as the Bayesian method [42,43] and quasiprobability [44,45], in order to deal with information production, they all strictly require the quantum state tomography; therefore, we need to find an alternative approach to deal with the information production.

To solve this problem, we employ a so-called one-time measurement (OTM) scheme, which was proposed by Deffner *et al.* in Ref. [22]. In this scheme, similar to the TTM scheme,

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we perform a projection measurement initially, which corresponds to the state preparation of the input state. However, what differs from the TTM scheme is that the second projection measurement is avoided, so that the corresponding distribution of the measurement outcomes is determined by the conditional expectation of the observable of interest given the initial measurement outcome. This quantity can be estimated if the postmeasurement state of the initial measurement and the dynamics are known. Particularly, for the information production, we do not have to diagonalize the output state in the OTM scheme, so that the OTM scheme is the *only* option.

This paper is organized as the following. In Sec. II, we first propose an information production distribution for an input state of rank r and a quantum channel. Then, we derive the Jarzynski-like equality and the lower bound on the total information production, which particularly becomes significant when we need to consider the information flow of the system in the quantum processes [46,47]. We demonstrate that the lower bound is characterized by the quantum cross-entropy. While there was less attention on the quantum cross-entropy, recently, the relations of the quantum cross-entropy with the maximum likelihood principle in the machine learning [48] and the quantum source coding [49] have been explored. In our paper, we further explore the roles of quantum cross-entropy in various protocols. In Sec. III, we discuss the applications of our result to quantum communications, quantum machine learning, and quantum thermodynamics by focusing on the quantum autoencoder (QAE) protocol [50,51], which is a quantum data compression protocol assisted by the variational quantum algorithms [52–57], and the maximum available work theorem [58] in the quantum thermodynamic systems, followed by the conclusion in Sec. IV.

II. MAIN RESULTS

Let us consider a Hilbert space $\mathcal{H}$ of dimension $d \equiv \dim(\mathcal{H})$. Let $\mathcal{B}(\mathcal{H})$ denote the set of the density matrices acting on $\mathcal{H}$. We initially prepare a quantum state $\rho_0 \in \mathcal{B}(\mathcal{H})$, and perform a measurement with an observable $P \equiv \sum_{i=1}^{d} a(p_i)\Pi_i$, where $\Pi_i \equiv |p_i\rangle\langle p_i|$ are the projectors on the eigenbasis of P. Suppose that the outcome is $a(p_i)$. Then, the postmeasurement state is given by $|p_i\rangle\langle p_i| = \Pi_i\rho_0\Pi_i/p_i$ with $p_i \equiv \mathrm{Tr}[\rho_0\Pi_i]$. Then, in general, the input state is given by [25,39]

$$\rho_{\mathrm{in}} = \sum_{i=1}^{r} p_i |p_i\rangle\langle p_i|, \tag{1}$$

where $r \equiv \mathrm{rank}(\rho_{\mathrm{in}})$ denotes the rank of the input state. The state ρ_{in} is an ensemble of the eigenbasis of the initial measurement P weighted by the probabilities of obtaining the outcomes $a(p_i)$; therefore, we can regard the initial measurement as a protocol of the state preparation of ρ_{in}. In this case, p_i satisfies the conditions $0 < p_i \leqslant 1$ $(1 \leqslant i \leqslant r)$, $p_i = 0$ $(r+1 \leqslant i \leqslant d)$, and $\sum_{i=1}^{r} p_i = 1$.

Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ be a quantum channel, which is a completely positive and trace-preserving (CPTP) map [59]. Through this channel, the output state $\rho_{\mathrm{out}} \in \mathcal{B}(\mathcal{H}')$ is given by

$$\rho_{\mathrm{out}} \equiv \Phi(\rho_{\mathrm{in}}). \tag{2}$$

The total information production is defined as

$$\Delta S \equiv S(\rho_{\mathrm{out}}) - S(\rho_{\mathrm{in}}), \tag{3}$$

where $S(\rho) \equiv -\mathrm{Tr}[\rho \ln \rho]$ denotes the von Neumann entropy of the quantum state ρ.

Here, we propose the following information production distribution in the OTM scheme [60]:

$$\widetilde{P}(\sigma) \equiv \sum_{i=1}^{r} p_i \delta(\sigma - C[\Phi(|p_i\rangle\langle p_i|), \rho_{\mathrm{out}}] - \ln p_i), \tag{4}$$

where $C(\rho_1, \rho_2) \equiv -\mathrm{Tr}[\rho_1 \ln \rho_2]$ denotes the quantum cross-entropy of ρ_1 with respect to ρ_2. Let $\mathrm{supp}(\rho)$ denote the support of a quantum state ρ. Then, note that $C(\rho_1, \rho_2) < \infty$ $[\mathrm{supp}(\rho_1) \subseteq \mathrm{supp}(\rho_2)]$ and $C(\rho_1, \rho_2) = \infty$ (otherwise). Also, by definition, we have $C(\rho, \rho) = S(\rho)$. In Eq. (4), due to $\rho_{\mathrm{out}} = \Phi(\rho_{\mathrm{in}}) = \sum_{i=1}^{r} p_i \Phi(|p_i\rangle\langle p_i|)$, we have $\mathrm{supp}[\Phi(|p_i\rangle\langle p_i|)] \subseteq \mathrm{supp}(\rho_{\mathrm{out}})$, so that $C[\Phi(|p_i\rangle\langle p_i|), \rho_{\mathrm{out}}] < \infty$.

With this distribution, the average of σ with respect to $\widetilde{P}(\sigma)$ becomes the exact information production

$$\langle \sigma \rangle_{\widetilde{P}} = S(\rho_{\mathrm{out}}) - S(\rho_{\mathrm{in}}) = \Delta S, \tag{5}$$

where we used the linearity on the first argument of the quantum cross-entropy [48] and Eq. (2). Let $r' \equiv \mathrm{rank}(\rho_{\mathrm{out}})$ be the rank of the output state. Then, we can interpret the random variable σ as follows. Let $\{q_j, |q_j\rangle\}_{j=1}^{r'}$ denote an eigensystem of ρ_{out}. Let us define the transition probability $P(j|i) \equiv \langle q_j|\Phi(|p_i\rangle\langle p_i|)|q_j\rangle$. Then, in the OTM scheme, σ randomly takes $\sum_j(-\ln q_j)P(j|i) + \ln p_i = \sum_j(-\ln q_j + \ln p_i)P(j|i)$, which is the conditional expectation of the information production given the initial measurement outcome.

Here, σ can be also identified to be a random variable as a source of the information production ΔS. Therefore, $\widetilde{P}(\sigma)$ is a *good* definition. Averaging the exponentiated information production with respect to the distribution in Eq. (4), we can obtain our main result.

Theorem 1 (Jarzynski-like equality of nonequilibrium information production). The Jarzynski-like equality of nonequilibrium information production is

$$\langle e^{-\sigma} \rangle_{\widetilde{P}} = \sum_{i=1}^{r} e^{-C[\Phi(|p_i\rangle\langle p_i|), \rho_{\mathrm{out}}]}, \tag{6}$$

which results in

$$\Delta S \geqslant L_{\mathrm{OTM}}, \tag{7}$$

where L_{OTM} is defined as

$$L_{\mathrm{OTM}} \equiv -\ln\left(\sum_{i=1}^{r} e^{-C[\Phi(|p_i\rangle\langle p_i|), \rho_{\mathrm{out}}]}\right). \tag{8}$$

Proof. From Eqs. (1) and (4), we have

$$\begin{aligned} \langle e^{-\sigma} \rangle_{\widetilde{P}} &= \int d\sigma\, \widetilde{P}(\sigma) e^{-\sigma} \\ &= \sum_{i=1}^{r} p_i e^{-C[\Phi(|p_i\rangle\langle p_i|), \rho_{\mathrm{out}}]} e^{-\ln p_i} \\ &= \sum_{i=1}^{r} e^{-C[\Phi(|p_i\rangle\langle p_i|), \rho_{\mathrm{out}}]}, \end{aligned} \tag{9}$$

which proves Eq. (6). From Jensen's inequality $\langle e^{-\sigma}\rangle_{\widetilde{P}} \geqslant e^{-\langle\sigma\rangle_{\widetilde{P}}}$ and Eq. (5), we obtain Eq. (7) [61]. ∎

When $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is particularly a unital map [i.e., $\Phi(\mathbb{1}) = \mathbb{1}$, where $\mathbb{1}$ denotes the identity matrix acting on $\mathcal{H}$], it is well known that we have $\Delta S \geqslant 0$ [62]. However, we can obtain a tighter bound as demonstrated in the following corollary.

Corollary 1 (Lower bound from a OTM scheme under a unital map). When $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a unital map, $L_{\rm OTM}$ is a tighter bound on ΔS as

$$\Delta S \geqslant L_{\rm OTM} \geqslant 0. \tag{10}$$

Proof. Given an input state $\rho_{\rm in} = \sum_{i=1}^{r} p_i|p_i\rangle\langle p_i|$ of rank r, let $\Pi_{\rm in} \equiv \sum_{i=1}^{r} |p_i\rangle\langle p_i|$ ($\overline{\Pi}_{\rm in} \equiv \mathbb{1} - \Pi_{\rm in}$) be the projectors onto the support (null space) of $\rho_{\rm in}$. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital map, i.e., $\Phi(\mathbb{1}) = \mathbb{1}$. Because the quantum cross-entropy can be lower bounded by using the state overlap [48], we have

$$C[\Phi(|p_i\rangle\langle p_i|), \rho_{\rm out}] \geqslant -\ln {\rm Tr}[\Phi(|p_i\rangle\langle p_i|)\rho_{\rm out}]. \tag{11}$$

Then, due to the linearity of the CPTP map, we can obtain

$$\sum_{i=1}^{r} e^{-C[\Phi(|p_i\rangle\langle p_i|),\rho_{\rm out}]} \leqslant {\rm Tr}[\Phi(\Pi_{\rm in})\rho_{\rm out}]. \tag{12}$$

Because $\Pi_{\rm in} + \overline{\Pi}_{\rm in} = \mathbb{1}$, from

$$\Phi(\mathbb{1}) = \Phi(\Pi_{\rm in}) + \Phi(\overline{\Pi}_{\rm in}) = \mathbb{1}, \tag{13}$$

we can obtain

$${\rm Tr}[\Phi(\Pi_{\rm in})\rho_{\rm out}] = 1 - {\rm Tr}[\Phi(\overline{\Pi}_{\rm in})\rho_{\rm out}] \leqslant 1. \tag{14}$$

Therefore, from Eqs. (8), (12), and (13), we obtain Eq. (10), which proves Corollary 1. ∎

III. EXAMPLES

In this section, we illustrate two applications of our result: quantum autoencoder and quantum thermodynamics.

A. Quantum autoencoder

As our first example, we demonstrate the application of our result in the QAE proposed by Romero *et al.* in Ref. [50]. The QAE is a quantum analog of the (classical) variational autoencoder [63]. In the QAE, the encoding and decoding operations are described by a parametrized quantum circuit. The original quantum data is compressed to the latent system by tracing over the other subsystem. Then, one prepares the fresh qubits, and decompresses the quantum data through the decoding operation acting on the fresh-qubit system and the latent system. The goal of the protocol is to recover the quantum data in the output, implying that a low-dimensional feature quantum state is well extracted through the encoding process; thus, we can use the resulting decoding process as a generative model to produce a quantum state outside the training quantum dataset by fluctuating the feature state. The cost function dependent on these tunable parameters, which measures the distance between the output and the input state, is constructed by the quantum computer, and the set of the parameters is optimized through training the cost function with the classical computers. Recently, as a practical near-term quantum algorithm, the QAE has been widely explored both theoretically and experimentally [64–75].

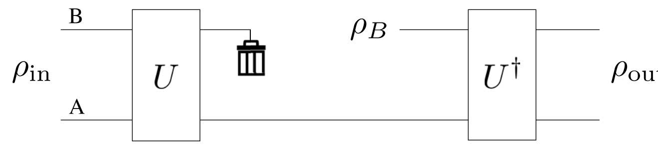

FIG. 1. Quantum autoencoder: We use U to compress the input state $\rho_{\rm in}$ into the reduced Hilbert space $\mathcal{H}_A$, and use the state ρ_B of the fresh qubits in $\mathcal{H}_B$ to decompress the data by applying the unitary $U^\dagger$ to generate the output state $\rho_{\rm out}$.

Let us describe the setup of the QAE below. We consider a composite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ ($\mathcal{H}_B$) denotes the Hilbert space of the reduced quantum system A (B). For the following, let us regard $\mathcal{H}_A$ as the latent Hilbert space, into which we compress our quantum data. Also, let us write d_j as the dimension of the reduced Hilbert space $\mathcal{H}_j$, $d_j \equiv \dim(\mathcal{H}_j)$ ($j = A, B$), so that the dimension of the total system is given by $d = d_A d_B$. Following Ref. [50], we consider the following scenario (see Fig. 1). In this setup, we apply a parametrized unitary U to the input state $\rho_{\rm in}$ and perform the partial trace over $\mathcal{H}_B$ to compress the quantum data into the latent Hilbert space $\mathcal{H}_A$. Then, we use the fresh qubits prepared in the state $\rho_B \in \mathcal{B}(\mathcal{H}_B)$ to decompress the data by applying the unitary $U^\dagger$ to generate the output state $\rho_{\rm out}$. In this case, we have

$$\rho_{\rm out} = \Phi(\rho_{\rm in}) \equiv U^\dagger({\rm Tr}_B[U\rho_{\rm in}U^\dagger] \otimes \rho_B)U. \tag{15}$$

To discuss the Jarzynski-like equality, it is convenient to define the compressed states

$$\rho_A \equiv {\rm Tr}_B[U\rho_{\rm in}U^\dagger], \tag{16}$$

$$\rho_A^{(i)} \equiv {\rm Tr}_B[U|p_i\rangle\langle p_i|U^\dagger]. \tag{17}$$

Therefore, we can write $\rho_A = \sum_{i=1}^{r} p_i\rho_A^{(i)}$, so that we have ${\rm supp}(\rho_A^{(i)}) \subseteq {\rm supp}(\rho_A)$.

Given the setup above, we can relate $L_{\rm OTM}$ to the classical information transmission and the cost function in QAE, which demonstrates the roles of the quantum cross-entropy in quantum communications and quantum machine learning in the framework of QAE protocol.

Let us first derive the expression of $L_{\rm OTM}$ in QAE. From Eq. (15), we have

$$C[\Phi(|p_i\rangle\langle p_i|), \rho_{\rm out}] = S(\rho_B) + C\big(\rho_A^{(i)}, \rho_A\big). \tag{18}$$

Therefore, we can write

$$\langle e^{-\sigma}\rangle_{\widetilde{P}} = e^{-S(\rho_B)} \sum_{i=1}^{r} e^{-C(\rho_A^{(i)},\rho_A)}, \tag{19}$$

so that $L_{\rm OTM}$ is given by

$$L_{\rm OTM} = S(\rho_B) - \ln\left(\sum_{i=1}^{r} e^{-C(\rho_A^{(i)},\rho_A)}\right). \tag{20}$$

An important observation is that $\langle e^{-\sigma}\rangle_{\widetilde{P}}$ includes two terms which characterize the protocols of the QAE. One is the von Neumann entropy $S(\rho_B)$, which is the informational contribution from the state preparation protocol in the fresh-qubit system $\mathcal{H}_B$. The other one is associated with the quantum cross-entropy $C(\rho_A^{(i)}, \rho_A)$ with respect to the latent Hilbert space $\mathcal{H}_A$. This quantity can be regarded as a term characterizing the compression protocol of the QAE. In the following, we explore the roles of the quantum cross-entropy in the quantum communications and quantum machine learning from the relation of the lower bound $L_{\rm OTM}$ to the loss of Holevo information and the global cost function of the QAE.

1. Relation to the loss of Holevo information in QAE

Here, we explore the relation between $L_{\rm OTM}$ and the entropic disturbance. Entropic disturbance is the loss of Holevo information through a given quantum channel Φ [46,76]. Hence, it quantifies the loss of the maximum amount of classical information transmittable through the quantum channel. In Ref. [46], a lower bound on $\Delta\chi$ was derived. Here, for a given quantum channel Φ, we provide an *upper* bound on the entropic disturbance by using $L_{\rm OTM}$ to provide an operational meaning to the quantum cross-entropy in terms of classical information transmission.

The entropic disturbance is defined as follows. Let $\mathcal{L} \equiv \{p_i, \rho_i\}_{i=1}^r$ denote an ensemble of input state $\rho_{\rm in} \equiv \sum_{i=1}^r p_i\rho_i$ and $\Phi(\mathcal{L}) \equiv \{p_i, \Phi(\rho_i)\}_{i=1}^r$ denote the ensemble of output state $\rho_{\rm out} \equiv \Phi(\rho_{\rm in}) = \sum_{i=1}^r p_i\Phi(\rho_i)$. Entropic disturbance is defined as $\Delta\chi \equiv \chi(\mathcal{L}) - \chi[\Phi(\mathcal{L})]$, where $\chi(\mathcal{L}) \equiv S(\rho_{\rm in}) - \sum_{i=1}^r p_iS(\rho_i)$ and $\chi[\Phi(\mathcal{L})] \equiv S(\rho_{\rm out}) - \sum_{i=1}^r p_iS[\Phi(\rho_i)]$ are the Holevo information of $\rho_{\rm in}$ and $\rho_{\rm out}$, respectively. In our case, we have $\rho_i = |p_i\rangle\langle p_i|$, so that $S(\rho_i) = S(|p_i\rangle\langle p_i|) = 0$. Therefore, due to $\Delta S \geqslant L_{\rm OTM}$ and Eq. (8), we can obtain

$$\Delta\chi \leqslant \ln\left(\sum_{i=1}^r e^{-C[\Phi(|p_i\rangle\langle p_i|),\rho_{\rm out}]}\right) + \sum_{i=1}^r p_iS[\Phi(|p_i\rangle\langle p_i|)], \tag{21}$$

which shows that the upper bound of the entropic disturbance can be characterized by the quantum cross-entropy [77].

Now, let us consider the case of QAE, in which Φ satisfies Eq. (15). Due to $S[\Phi(|p_i\rangle\langle p_i|)] = S(\rho_A^{(i)}) + S(\rho_B)$, we have $\sum_{i=1}^r p_iS[\Phi(|p_i\rangle\langle p_i|)] = \sum_{i=1}^r p_iS(\rho_A^{(i)}) + S(\rho_B)$, so that the upper bound on $\Delta\chi$ in QAE is given by

$$\Delta\chi \leqslant \sum_{i=1}^r p_iS\big(\rho_A^{(i)}\big) + \ln\left(\sum_{i=1}^r e^{-C(\rho_A^{(i)},\rho_A)}\right). \tag{22}$$

Therefore, the information and the quantum cross-entropy of the compressed states contribute to setting an upper bound on the entropic disturbance in the QAE protocol. Also, note that for the QAE, $\Delta\chi$ can be explicitly written as

$$\Delta\chi = S(\rho_{\rm in}) - S(\rho_A) - \sum_{i=1}^r p_iS\big(\rho_A^{(i)}\big), \tag{23}$$

which implies that the loss of the maximum amount of classical information in the QAE protocol is *independent* of the choice of ρ_B but strictly dependent on the compressed and input states.

2. Relation to the global cost function of QAE

The lower bound $L_{\rm OTM}$ can be also related to the performance of the QAE, which can be characterized by its global cost function. The cost function of the QAE is well defined when the fresh-qubit state ρ_B is a pure state

$$\rho_B = |\psi\rangle\langle\psi|. \tag{24}$$

In this case, from Eq. (20) and $S(\rho_B) = S(|\psi\rangle\langle\psi|) = 0$, we have

$$L_{\rm OTM} = -\ln\left(\sum_{i=1}^r e^{-C(\rho_A^{(i)},\rho_A)}\right). \tag{25}$$

Let η_B denote the reduced state $\eta_B \equiv {\rm Tr}_A[U\rho_{\rm in}U^\dagger]$. Then, from Refs. [50,78], the global cost function $\mathscr{C}$ can be given by

$$\mathscr{C} \equiv 1 - \langle\psi|\eta_B|\psi\rangle, \tag{26}$$

which satisfies $0 \leqslant \mathscr{C} \leqslant 1$. The ultimate goal of this protocol is to find an optimal unitary U_* to realize $\rho_{\rm in} = \rho_{\rm out}$. In this optimal case, the global cost function is $\mathscr{C} = 0$. Then, by using d_B the dimension of $\mathcal{H}_B$ and the global cost function, we can obtain the following inequality (see Appendix A for the proof):

$$L_{\rm OTM} \leqslant \Delta S \leqslant 2\ln\big(\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}}\big). \tag{27}$$

From Eq. (25), we can finally obtain

$$\sum_{i=1}^r e^{-C(\rho_A^{(i)},\rho_A)} \geqslant \left(\frac{1}{\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}}}\right)^2, \tag{28}$$

which shows that the quantum cross-entropy plays a role as an informational contribution of the compressed state to the performance of the QAE protocol. Here, note that, due to $0 \leqslant \mathscr{C} \leqslant 1$, we have $0 \leqslant 2\ln(\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}}) \leqslant \ln(d_B)$, where $\mathscr{C} = 0$ (i.e., $\eta_B = |\psi\rangle\langle\psi|$) leads to the minimum, and $\mathscr{C} = 1 - 1/d_B$ (i.e., $\eta_B = \mathbb{1}_B/d_B$) leads to the maximum.

We can also check the consistency of Eq. (27) by considering the optimal case. The optimal unitary U_* is a disentangling gate [75], so that U_* satisfies

$$\begin{aligned} U_*\rho_{\rm in}U_*^\dagger &= \rho_A \otimes |\psi\rangle\langle\psi|, \\ U_*|p_i\rangle\langle p_i|U_*^\dagger &= \rho_A^{(i)} \otimes |\psi\rangle\langle\psi|. \end{aligned} \tag{29}$$

Then, by using the unitary invariance of the quantum cross-entropy, we obtain

$$\begin{aligned} C\big(\rho_A^{(i)}, \rho_A\big) &= C\big(\rho_A^{(i)} \otimes |\psi\rangle\langle\psi|, \rho_A \otimes |\psi\rangle\langle\psi|\big) \\ &= C(|p_i\rangle\langle p_i|, \rho_{\rm in}) \\ &= -\ln p_i. \end{aligned} \tag{30}$$

In this way, we have $L_{\rm OTM} = 0$. By definition, in the optimal case, we have $\mathscr{C} = 0$; therefore, $\ln(\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}}) = 0$. From $L_{\rm OTM} \leqslant \Delta S \leqslant 2\ln(\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}})$, we get the expected result $\Delta S = 0$ for the optimal unitary case.

B. Maximum available work theorem

For the second example, we explore the role of quantum cross-entropy in work extraction from the quantum thermodynamic systems by the relation between L_{OTM} and the maximum available work theorem [58].

In Ref. [58], a generic quantum thermodynamic system is regarded as a tripartite system composed of the system, work reservoir, and heat bath. In this setup, the work $\langle W\rangle$, the internal energy change of the system ΔE_s, and the internal energy change of the heat bath ΔE_b satisfy the first law of thermodynamics $\Delta E_s + \Delta E_b = \langle W\rangle$. Also, when ΔS and ΔS_b denote the von Neumann entropy change of the system and heat bath, respectively, the second law of thermodynamics states $\Delta S + \Delta S_b \geqslant 0$. Because the heat reservoir is so large, which can be regarded as being always in equilibrium at inverse temperature β, we can write $\Delta S_b = \beta\Delta E_b$. Then, the maximum available work theorem states

$$\langle W\rangle \geqslant \Delta E_s - \beta^{-1}\Delta S \equiv \Delta\mathcal{E}, \tag{31}$$

where $\mathcal{E}$ is called exergy or availability, which quantifies the maximally available work.

In this setup, from Theorem 1, we can obtain the upper bound on the exergy $\Delta\mathcal{E}$ as

$$\Delta\mathcal{E} \leqslant \Delta E_s + \beta^{-1}\ln\left(\sum_{i=1}^{r} e^{-C[\Phi(|p_i\rangle\langle p_i|),\rho_{\mathrm{out}}]}\right). \tag{32}$$

This demonstrates the informational contribution of the quantum cross-entropy in extracting maximally available work in quantum thermodynamic systems. If there is no work reservoir, i.e., $\langle W\rangle = 0$, the corresponding maximum available work theorem becomes $\Delta S \geqslant -\beta\Delta E_b$. However, when the system undergoes the energy-emitting process ($\Delta E_b \geqslant 0$) described by a unital evolution, from Corollary 1, we have a tighter bound as

$$\Delta S \geqslant L_{\mathrm{OTM}} \geqslant -\beta\Delta E_b. \tag{33}$$

A good example of the energy-emitting unital evolution is the spin-boson model [79]. Let us consider a system $\mathcal{H}_s$ initially prepared in ρ_{in} coupled to a heat bath $\mathcal{H}_b$, whose initial state is prepared in the Gibbs state

$$\rho_b^{\mathrm{eq}} \equiv \frac{e^{-\beta H_b}}{Z}, \tag{34}$$

where $Z \equiv \mathrm{Tr}[e^{-\beta H_b}]$ is the canonical partition function with inverse temperature β and H_b the time-independent bare Hamiltonian of the bath. Then, when Φ is a thermal operation [4–6] from $t = 0$ to $t = \tau$,

$$\rho_{\mathrm{out}} = \Phi(\rho_{\mathrm{in}}) = \mathrm{Tr}_b\big[U_\tau\big(\rho_{\mathrm{in}}\otimes\rho_b^{\mathrm{eq}}\big)U_\tau^\dagger\big]. \tag{35}$$

In the quantum thermodynamic setup of the spin-boson model, $\mathcal{H}_s$ and $\mathcal{H}_b$ usually describes a two-level atomic system and the bosonic heat bath, respectively (see Fig. 2), and the atomic system in Eq. (35) undergoes the dephasing process, which is described by a unital map. This model can be descried by the following time-independent Hamiltonian (we set $\hbar = 1$):

$$H = \frac{\omega_0}{2}\sigma_z + H_b + \sigma_z\otimes\sum_k(g_k a_k + g_k^* a_k^\dagger), \tag{36}$$

where

$$H_b \equiv \sum_k \omega_k a_k^\dagger a_k \tag{37}$$

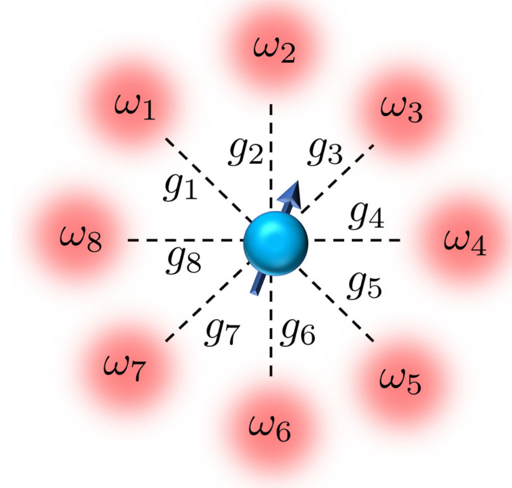

FIG. 2. Spin-boson model: A two-level atom is interacting with multiple boson modes. Each boson mode is decoupled from each other, and has different angular frequency ω_k. The atomic system is coupled to each mode with different interaction strength g_k. The Hamiltonian of the spin-boson model is described in Eq. (36).

is the bare Hamiltonian of the boson fields, and $\sigma_z \equiv \mathrm{diag}(1, -1)$ is Pauli's Z operator acting on the atom. ω_0 and ω_k are the angular frequencies of the atom and the kth boson mode, and g_k denotes the coupling strength between the atom and the kth boson mode. Here, in general g_k is a complex number, and g_k^* denotes the complex conjugate of g_k. Also, a_k ($a_k^\dagger$) is the annihilation (creation) operator of the kth mode of the boson fields. Equation (36) describes an interaction between atom and boson fields with multiple modes, which leads to the dephasing process of the atomic system. In the interaction picture, we have

$$H(t) = \sigma_z\otimes\sum_k(g_k a_k e^{-i\omega_k t} + g_k^* a_k^\dagger e^{+i\omega_k t}). \tag{38}$$

During the evolution from $t = 0$ to $t = \tau$, the internal energy change of the heat bath becomes (see Appendix B for the proof)

$$\begin{aligned}\Delta E_b &\equiv \mathrm{Tr}\big[U_\tau\big(\rho_{\mathrm{in}}\otimes\rho_b^{\mathrm{eq}}\big)U_\tau^\dagger H_b\big] - \mathrm{Tr}\big[\rho_b^{\mathrm{eq}} H_b\big]\\ &= \sum_k \omega_k|g_k|^2\left(\frac{\sin(\omega_k\tau/2)}{\omega_k/2}\right)^2 \geqslant 0,\end{aligned} \tag{39}$$

which shows that the system undergoes the energy-emitting process, which can be verified from the energy conservation of the total system. Considering the noise spectral density $J(\omega) = \sum_k |g_k|^2\omega\delta(\omega - \omega_k)$, we can obtain

$$\Delta E_b = \int_{-\infty}^{\infty} J(\omega)\left(\frac{\sin(\omega\tau/2)}{\omega/2}\right)^2 d\omega. \tag{40}$$

Since we have $\lim_{\tau\to\infty}\frac{\sin(\omega\tau/2)}{\omega/2} = \delta(\omega/2) = 2\delta(\omega)$, where we used the relation $\lim_{\tau\to\infty}(\tau\frac{\sin(\omega\tau)}{\omega\tau}) = \delta(\omega)$, we obtain

$$\lim_{\tau\to\infty}\Delta E_b = \int_{-\infty}^{\infty} 4J(\omega)\delta^2(\omega)d\omega = 4J(0)\delta(0) = 0. \tag{41}$$

This is intuitively consistent because there will be no energy exchange between a small system and a large bath for the dephasing process at time $\tau \to \infty$.

In the spin-boson model, the work reservoir is not taken into account. Therefore, we have $\Delta S \geqslant -\beta \Delta E_b$. Since the two-level atom undergoes the unital evolution, the total information production has to be $\Delta S \geqslant 0$. However, from Eq. (39), we have $-\beta \Delta E_b \leqslant 0$, which shows that the lower bound in the maximum available work theorem for the spin-boson model is not tight enough. Instead, from Corollary 1, we can find that $L_{\rm OTM}$ is the tighter bound as demonstrated in Eq. (33). Because $L_{\rm OTM}$ is characterized by the quantum cross-entropy, the quantum cross-entropy becomes a more meaningful quantity to characterize the quantum process of a system induced from its interaction with a heat bath.

IV. CONCLUSION

In conclusion, we have proposed a distribution of an information production for a quantum state of arbitrary rank and a quantum channel by adopting the one-time measurement scheme. The derived Jarzynski-like equality and the lower bound on the total information production are characterized by the quantum cross-entropy, which further enables one to explore the roles of quantum cross-entropy in quantum communications, quantum machine learning, and quantum thermodynamics. By focusing on the quantum autoencoder, we have explored the informational contributions of the quantum cross-entropy of the compressed states to the loss of the maximum classical information transmittable through the circuit and the performance of the protocol characterized by the global cost function. We have also demonstrated the application of our result in the quantum thermodynamics by exploring the relation between the quantum cross-entropy and the maximum available work theorem. Our result can provide insights of the quantum cross-entropy to assist in designing quantum information processing protocols which utilize the quantum cross-entropy as a resource to achieve some tasks. As a valuable future direction, we will explore the reverse process in the OTM scheme, which still remains open.

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APPENDIX A: PROOF OF EQ. (27)

Here, we provide a detailed proof of Eq. (27). First, by using the cost function $\mathscr{C}$, we can obtain an upper bound on ΔS. From

$$\rho_{\rm out} = U^\dagger(\rho_A \otimes |\psi\rangle\langle\psi|)U \tag{A1}$$

with $\rho_A \equiv {\rm Tr}_B[U\rho_{\rm in}U^\dagger]$, because the von Neumann entropy is unitarily invariant, we have

$$\begin{aligned} S(\rho_{\rm out}) &= S(\rho_A \otimes |\psi\rangle\langle\psi|) = S(\rho_A), \\ S(\rho_{\rm in}) &= S(U\rho_{\rm in}U^\dagger), \end{aligned} \tag{A2}$$

which leads to

$$\Delta S = S(\rho_A) - S(U\rho_{\rm in}U^\dagger). \tag{A3}$$

Because $\eta_B \equiv {\rm Tr}_A[U\rho_{\rm in}U^\dagger]$, from Araki-Lieb inequality [80]

$$|S(\rho_A) - S(\eta_B)| \leqslant S(U\rho_{\rm in}U^\dagger), \tag{A4}$$

we have

$$\Delta S \leqslant S(\eta_B). \tag{A5}$$

By using d_B the dimension of the Hilbert space $\mathcal{H}_B$, $S(\eta_B)$ can be upper bounded as

$$\begin{aligned} S(\eta_B) &= \ln(d_B) - S\left(\eta_B \middle|\middle| \frac{\mathbb{1}_B}{d_B}\right) \\ &\leqslant \ln(d_B) - S_{\rm min}\left(\eta_B \middle|\middle| \frac{\mathbb{1}_B}{d_B}\right), \end{aligned} \tag{A6}$$

where

$$S_{\rm min}(\rho_1||\rho_2) \equiv -\ln\left(F[\rho_1,\rho_2]\right) \tag{A7}$$

denotes the sandwiched minimum relative entropy of ρ_1 with respect to ρ_2 [81–83] with the standard quantum fidelity defined as

$$F[\rho_1,\rho_2] \equiv \left({\rm Tr}\left[\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}\right]\right)^2. \tag{A8}$$

Therefore, we have

$$S(\eta_B) \leqslant \ln\left(d_B F\left[\eta_B, \frac{\mathbb{1}_B}{d_B}\right]\right). \tag{A9}$$

Here, we consider so-called generalized quantum fidelity [84–87], which is defined as

$$\widetilde{F}[\sigma_1,\sigma_2] \equiv \left(\sqrt{F[\sigma_1,\sigma_2]} + \sqrt{(1-{\rm Tr}[\sigma_1])(1-{\rm Tr}[\sigma_2])}\right)^2. \tag{A10}$$

Here, note that σ_1 and σ_2 are the subnormalized states, i.e., $0 \leqslant {\rm Tr}[\sigma_1] \leqslant 1$ and $0 \leqslant {\rm Tr}[\sigma_2] \leqslant 1$. Because applying the projection operator $|\psi\rangle\langle\psi|$ is described by the completely positive trace nonincreasing (CPTNI) map [87], from the monotonicity of the generalized quantum fidelity under the CPTNI maps [84–87], we have

$$\begin{aligned} F\left[\eta_B, \frac{\mathbb{1}_B}{d_B}\right] &= \widetilde{F}\left[\eta_B, \frac{\mathbb{1}_B}{d_B}\right] \\ &\leqslant \widetilde{F}\left[|\psi\rangle\langle\psi|\eta_B|\psi\rangle\langle\psi|, \frac{1}{d_B}|\psi\rangle\langle\psi|\right]. \end{aligned} \tag{A11}$$

Since we have

$$\widetilde{F}\left[|\psi\rangle\langle\psi|\eta_B|\psi\rangle\langle\psi|, \frac{1}{d_B}|\psi\rangle\langle\psi|\right] = \left(\sqrt{\frac{1-\mathscr{C}}{d_B}} + \sqrt{\mathscr{C}\left(1-\frac{1}{d_B}\right)}\right)^2, \quad \text{(A12)}$$

we can obtain

$$\Delta S \leqslant S(\eta_B) \leqslant 2\ln(\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}}), \quad \text{(A13)}$$

which states that the information production in the quantum autoencoder with the pure fresh-qubit state can be upper bounded by using $\mathscr{C}$. Therefore, from Theorem 1, we can finally arrive at Eq. (27):

$$L_{\mathrm{OTM}} \leqslant \Delta S \leqslant 2\ln(\sqrt{1-\mathscr{C}} + \sqrt{(d_B-1)\mathscr{C}}). \quad \text{(A14)}$$

APPENDIX B: PROOF OF EQ. (39)

We demonstrate the detailed proof of Eq. (39) based on Refs. [88,89]. The Hamiltonian of the spin-boson model is

$$H = \frac{\omega_0}{2}\sigma_z + H_b + \sigma_z \otimes \sum_k (g_k a_k + g_k^* a_k^\dagger), \quad \text{(B1)}$$

where we define $\sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and a_k ($a_k^\dagger$) as the annihilation (creation) operator of the kth mode of the boson heat bath. The annihilation and creation operators satisfy the commutation relation

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0. \quad \text{(B2)}$$

Also, H_b is defined as

$$H_b \equiv \sum_k \omega_k a_k^\dagger a_k. \quad \text{(B3)}$$

Then, the Hamiltonian in the interaction picture becomes

$$H(t) = \sigma_z \otimes \sum_k (g_k a_k e^{-i\omega_k t} + g_k^* a_k^\dagger e^{i\omega_k t}). \quad \text{(B4)}$$

Using Magnus expansion, the propagator becomes

$$U_t = \exp[-it(\overline{H}_0 + \overline{H}_1)], \quad \text{(B5)}$$

where the higher terms are vanishing because $[H(t_1), H(t_2)]$ becomes just a number [see Eq. (B10)]. Here, we define

$$\overline{H}_0 \equiv \frac{1}{t}\int_0^t H(t_1)dt_1 \quad \text{(B6)}$$

and

$$\overline{H}_1 \equiv -\frac{i}{2t}\int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]. \quad \text{(B7)}$$

More explicitly, $\overline{H}_0$ can be written as

$$\overline{H}_0 = \sigma_z \otimes \sum_k (G_k(t)a_k + G^*(t)a_k^\dagger), \quad \text{(B8)}$$

where

$$G_k(t) \equiv g_k \frac{\sin(\omega_k t/2)}{\omega_k t/2} e^{-i\omega_k t/2}. \quad \text{(B9)}$$

For $\overline{H}_1$, because we have

$$[H(t_1), H(t_2)] = -2i\sum_k |g_k|^2 \sin[\omega_k(t_1-t_2)] \quad \text{(B10)}$$

and

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \sin[\omega_k(t_1-t_2)] = \frac{1}{\omega_k}\left(t - \frac{1}{\omega_k}\sin(\omega_k t)\right), \quad \text{(B11)}$$

we can write

$$\overline{H}_1 = -\sum_k \frac{|g_k|^2}{\omega_k}\left(1 - \frac{\sin(\omega_k t)}{\omega_k t}\right) \in \mathbb{R}, \quad \text{(B12)}$$

which is just a real number.

Therefore, from Eqs. (B5), (B8), and (B12), the propagator becomes

$$U_t = \exp\left[-it\sum_k (\sigma_z \otimes [G_k(t)a_k + G_k^*(t)a_k^\dagger])\right] e^{-it\overline{H}_1}. \quad \text{(B13)}$$

With this propagator, due to

$$\left[\sum_{k'} \sigma_z \otimes [G_{k'}(t)a_{k'} + G_{k'}^*(t)a_{k'}^\dagger], a_k\right] = -G_k^*(t)\sigma_z,$$
$$\left[\sum_{k'} \sigma_z \otimes [G_{k'}(t)a_{k'} + G_{k'}^*(t)a_{k'}^\dagger], a_k^\dagger\right] = G_k(t)\sigma_z, \quad \text{(B14)}$$

from Baker-Hausdorff-Campbell's formula, we have

$$U_t^\dagger a_k U_t = a_k - itG_k^*(t)\sigma_z,$$
$$U_t^\dagger a_k^\dagger U_t = a_k^\dagger + itG_k(t)\sigma_z. \quad \text{(B15)}$$

Therefore, we can write

$$U_t^\dagger H_b U_t = \sum_k \omega_k (U_t^\dagger a_k^\dagger U_t)(U_t^\dagger a_k U_t)$$
$$= H_b + it\sum_k \omega_k \sigma_z \otimes [G_k(t)a_k - G_k^*(t)a_k^\dagger]$$
$$+ \sum_k \omega_k |G_k(t)|^2 t^2. \quad \text{(B16)}$$

Let ρ_{in} be the input state of the system with a rank r, and the initial state of the boson heat bath be the Gibbs state

$$\rho_b^{\mathrm{eq}} = \frac{e^{-\beta H_b}}{Z}. \quad \text{(B17)}$$

Note that, with a_k and $a_k^\dagger$, for all k, we have

$$\mathrm{Tr}[\rho_b^{\mathrm{eq}} a_k] = \mathrm{Tr}[\rho_b^{\mathrm{eq}} a_k^\dagger] = 0. \quad \text{(B18)}$$

We assume that the two-level atomic system and bosonic field are initially decoupled. Therefore, the initial state of the total system is $\rho_{\mathrm{in}} \otimes \rho_b^{\mathrm{eq}}$, so that the evolution of the atomic system from $t = 0$ to $t = \tau$ is described by the following thermal operation:

$$\rho_{\mathrm{out}} = \Phi(\rho_{\mathrm{in}}) = \mathrm{Tr}_b[U_\tau(\rho_{\mathrm{in}} \otimes \rho_b^{\mathrm{eq}})U_\tau^\dagger]. \quad \text{(B19)}$$

The internal energy change of the heat bath during the evolution can be defined as the difference in the average energy of the heat bath at $t = \tau$ and $t = 0$:

$$\Delta E_b \equiv \mathrm{Tr}\big[U_\tau\big(\rho_{\mathrm{in}} \otimes \rho_b^{\mathrm{eq}}\big)U_\tau^\dagger H_b\big] - \mathrm{Tr}\big[\rho_b^{\mathrm{eq}} H_b\big]. \tag{B20}$$

From Eqs. (B9), (B16), and (B18), ΔE_b can be explicitly written as

$$\Delta E_b = \sum_k \omega_k |g_k|^2 \left(\frac{\sin(\omega_k \tau/2)}{\omega_k/2}\right)^2 \geqslant 0. \tag{B21}$$

APPENDIX C: SECOND-LAW-LIKE INEQUALITY INVOLVING GUESSED HEAT

The information production distribution $\widetilde{P}(\sigma)$ can be related to the distribution of the internal energy difference in the OTM scheme $\widetilde{P}(\Delta E_s)$ in a very special case, which leads to a second-law-like inequality involving the guessed heat introduced in Ref. [89]. Let $H_s(t)$ be the system's bare Hamiltonian, which is time dependent. Also, suppose that the system is initially decoupled from the heat bath, which is initially prepared in a Gibbs state

$$\rho_b^{\mathrm{eq}} = \frac{e^{-\beta H_b}}{Z_b}, \tag{C1}$$

where H_b is the bath's bare Hamiltonian, which is time independent. Here, $Z_b \equiv \mathrm{Tr}[e^{-\beta H_b}]$ is the partition function defined by H_b. Let H_{int} be the interaction Hamiltonian. Then, the unitary operator U_t describing the time evolution of the total system follows the Schrödinger's equation $\partial_t U_t = -i[H_s(t) + H_b + H_{\mathrm{int}}]U_t$ with $U_0 \equiv \mathbb{1}$. Evolving the total system from $t = 0$ to $t = \tau$ and focusing on the system alone, we have

$$\rho_{\mathrm{out}} = \Phi(\rho_{\mathrm{in}}) = \mathrm{Tr}_b\big[U_\tau\big(\rho_{\mathrm{in}} \otimes \rho_b^{\mathrm{eq}}\big)U_\tau^\dagger\big]. \tag{C2}$$

Let $\{E_i, |E_i\rangle\}_{i=1}^d$ be an eigensystem of $H_s(0)$. When we have

$$\rho_{\mathrm{in}} = \rho_s^{\mathrm{eq}}(0) \equiv \frac{e^{-\beta H_s(0)}}{Z_0}, \quad \rho_{\mathrm{out}} = \rho_s^{\mathrm{eq}}(\tau) \equiv \frac{e^{-\beta H_s(\tau)}}{Z_\tau}, \tag{C3}$$

where $Z_0 \equiv \mathrm{Tr}[e^{-\beta H_s(0)}]$ and $Z_\tau \equiv \mathrm{Tr}[e^{-\beta H_s(\tau)}]$ are the partition functions defined by $H_s(0)$ and $H_s(\tau)$, respectively, from Eq. (4), the information production distribution in the OTM scheme becomes

$$\widetilde{P}(\sigma) = \frac{1}{\beta}\sum_{i=1}^d \frac{e^{-\beta E_i}}{Z_0}\delta\left(\frac{\sigma}{\beta} + \Delta F - \Delta\widetilde{E}(E_i)\right), \tag{C4}$$

where

$$\Delta\widetilde{E}(E_i) \equiv \mathrm{Tr}[\Phi(|E_i\rangle\langle E_i|)H_s(\tau)] - E_i, \tag{C5}$$

and

$$\Delta F \equiv -\beta^{-1}\ln\left(\frac{Z_\tau}{Z_0}\right) \tag{C6}$$

is the equilibrium Helmholtz free-energy difference. From Ref. [89], $\widetilde{P}(\Delta E_s)$ is given by

$$\widetilde{P}(\Delta E_s) = \sum_{i=1}^d \frac{e^{-\beta E_i}}{Z_0}\delta(\Delta E_s - \Delta\widetilde{E}(E_i)). \tag{C7}$$

Therefore, we can write

$$\widetilde{P}(\Delta E_s) = \beta\widetilde{P}(\sigma) \tag{C8}$$

with the random variable σ being

$$\sigma = \beta(\Delta E_s - \Delta F). \tag{C9}$$

Following Ref. [89], we have

$$\langle e^{-\beta\Delta E_s}\rangle_{\widetilde{P}} = e^{-\beta\Delta F}e^{-\beta\langle\widetilde{Q}\rangle_b}e^{-S[\Theta_{sb}(\tau)||\rho_s^{\mathrm{eq}}(\tau)\otimes\rho_b^{\mathrm{eq}}]}, \tag{C10}$$

where

$$\Theta_{sb}(\tau) \equiv \sum_{i=1}^d \frac{e^{-\beta\mathrm{Tr}[\Phi(|E_i\rangle\langle E_i|)H_s(\tau)]}}{\widetilde{Z}_\tau}U_\tau\big(|E_i\rangle\langle E_i| \otimes \rho_b^{\mathrm{eq}}\big)U_\tau^\dagger \tag{C11}$$

is called "guessed state," and $\langle\widetilde{Q}\rangle_b$ is a heatlike quantity called "guessed heat" defined as

$$\langle\widetilde{Q}\rangle_b \equiv \mathrm{Tr}\big[H_b\rho_b^{\mathrm{eq}}\big] - \mathrm{Tr}[H_b\Theta_{sb}(\tau)]. \tag{C12}$$

This heatlike quantity describes an energy dissipation from the heat bath as if its final state is $\mathrm{Tr}_s[\Theta_{sb}(\tau)]$, the reduced state of the guessed state. From Eqs. (C8) and (C9), we can obtain

$$\langle e^{-\beta\Delta E_s}\rangle_{\widetilde{P}} = e^{-\beta\Delta F}\langle e^{-\sigma}\rangle_{\widetilde{P}}. \tag{C13}$$

From Eq. (C10), we can finally write

$$\langle e^{-\sigma}\rangle_{\widetilde{P}} = e^{-\beta\langle\widetilde{Q}\rangle_b}e^{-S[\Theta_{sb}(\tau)||\rho_s^{\mathrm{eq}}(\tau)\otimes\rho_b^{\mathrm{eq}}]}. \tag{C14}$$

By using Jensen's inequality and the non-negativity of the quantum relative entropy, we can arrive at

$$\Delta S - \beta\langle\widetilde{Q}\rangle_b \geqslant 0, \tag{C15}$$

which is the second-law-like inequality involving the guessed heat.

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[61] Note that our main claims are the derivation of the general integrated fluctuation theorems, which hold for any states and quantum channels, and its potential of characterizing the quantum protocol with the quantum cross-entropy. We leave the comparison between the tight bound derived in Refs. [46,47] and our bound derived in the OTM scheme as an open problem.
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