

# Pair correlation function of the one-dimensional Riesz gas

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(Received 6 December 2022; accepted 9 February 2023; published 27 February 2023)

A method from random-matrix theory is used to calculate the pair correlation function of a one-dimensional gas of  $N \gg 1$  classical particles with a power-law repulsive interaction potential  $u(x) \propto |x|^{-s}$  (a so-called Riesz gas). An integral formula for the covariance of single-particle operators is obtained, which generalizes known results in the limits  $s \rightarrow -1$  (Coulomb gas) and  $s \rightarrow 0$  (log-gas). As an application, we calculate the variance of the center of mass of the Riesz gas, which has a universal large- $N$  limit that does not depend on the shape of the confining potential.

DOI: [10.1103/PhysRevResearch.5.013152](https://doi.org/10.1103/PhysRevResearch.5.013152)

## I. INTRODUCTION

The one-dimensional (1D) Riesz gas [1–4] describes  $N$  classical particles that move on a line (the  $x$  axis) with a repulsive interaction potential  $u(x)$  of the form

$$u(x) = \begin{cases} \text{sign}(s)|x|^{-s} & \text{for } s > -2, \\ -\ln|x| & \text{for } s = 0. \end{cases} \quad (1.1)$$

The particles are prevented from moving off to infinity by a confining potential  $V(x)$ .

The cases  $s = 0$  and  $s = -1$  are also referred to as log-gas and Coulomb gas, respectively. The log-gas plays a central role in random-matrix theory (RMT) [5]. Experimentally, an interaction potential with an adjustable exponent  $s \lesssim 1$  has been realized in a chain of trapped ion spins [6].

Thermal averages  $\langle \cdots \rangle$ , at inverse temperature  $\beta$ , are defined with respect to the Gibbs measure

$$P(x_1, x_2, \dots, x_N) = Z^{-1} \exp \left( -\beta \left[ J \sum_{i < j=1}^N u(x_i - x_j) + \sum_{i=1}^N V(x_i) \right] \right), \quad (1.2)$$

where  $Z$  normalizes the distribution to unity. The interaction strength is parameterized by  $J > 0$ . The average density is

$$\rho(x) = \left\langle \sum_{i=1}^N \delta(x - x_i) \right\rangle, \quad (1.3)$$

normalized to the particle number,

$$\int_{-\infty}^{\infty} \rho(x) dx = N. \quad (1.4)$$

Much is known about the dependence of  $\rho(x)$  on the range of the interaction [7–10], in the three regimes  $s > 1$  (short-range repulsion),  $-1 < s < 1$  (weakly long-range repulsion), and  $-2 < s < -1$  (strongly long-range repulsion). In what follows we consider the second regime,  $-1 < s < 1$ , which includes the log-gas at  $s = 0$  and the Coulomb gas in the limit  $s \rightarrow -1$ .

For  $N \gg 1$  the density has a compact support, which may consist of multiple disjoint intervals. Considering a single interval  $(a, b)$ , an end point may be  $N$  independent, fixed by a hard wall, or it may be  $N$  dependent, freely adjustable in a smooth potential. The density vanishes as a power law at a free end point, while it diverges (with an integrable singularity) at a fixed end point.

For example, in the case of a quadratic confinement  $V(x) \propto x^2$ , there are two free  $N$ -dependent end points at  $b = -a \propto N^{1/(2+s)}$  and the density profile is [9]

$$\rho(x) \propto (b^2 - x^2)^{(s+1)/2}, \quad |s| < 1. \quad (1.5)$$

The density profile becomes flat in the Coulomb gas limit, while the log-gas has the Wigner semicircle law [11]. Alternatively, if we set  $V \equiv 0$  and confine the Riesz gas by a hard wall at  $x = a$  and  $x = b$ , then the density diverges on approaching a fixed end point [10],

$$\rho(x) \propto (x - a)^{(s-1)/2} (b - x)^{(s-1)/2}, \quad |s| < 1. \quad (1.6)$$

Knowledge of the density allows us to calculate by integration the average  $\langle F \rangle$  of a single-particle observable  $F = \sum_{i=1}^N f(x_i)$  (also known as a “linear statistic”). For  $N \gg 1$  we can restrict the integral to the interval  $(a, b)$ ,

$$\langle F \rangle = \int_a^b \rho(x) f(x) dx. \quad (1.7)$$

Going beyond the first moment, Flack, Majumdar, and Schehr recently obtained [12] a strikingly simple formula for

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the large- $N$  limit of the variance of  $F$  in the Coulomb case [13],

$$\text{Var } F = \frac{1}{2\beta J} \int_a^b [df(x)/dx]^2 dx \quad \text{for } s = -1. \quad (1.8)$$

The corresponding formula in the RMT case  $s = 0$  is known [14,15], but results for other values of the interaction parameter  $s$  are not known. The aim of this paper is to provide that information for the entire range  $-1 < s < 1$ .

For that purpose one needs the connected pair correlation function

$$R(x, y) = \left\langle \sum_{i,j=1}^N \delta(x - x_i) \delta(y - x_j) \right\rangle - \rho(x) \rho(y), \quad (1.9)$$

which gives the variance upon integration,

$$\text{Var } F = \int_a^b dx \int_a^b dy R(x, y) f(x) f(y). \quad (1.10)$$

In RMT there are basically two methods to compute  $R(x, y)$ . The method of orthogonal polynomials [16] applies to specific confining potentials (typically linear or quadratic) and then gives results for any  $N$ . The alternative method of functional derivatives [17] (equivalently, the method of loop equations [18]) takes the large- $N$  limit, but then works generically for any form of confinement. Since the latter method does not assume a logarithmic repulsion, it is the method of choice in what follows.

## II. PAIR CORRELATION FUNCTION

For  $N \gg 1$  the pair correlation function oscillates rapidly on the scale of the interparticle spacing  $\delta x \simeq (b - a)/N$ . These oscillations are irrelevant for the computation of the variance of an observable that varies smoothly on the scale of  $\delta x$ , so that in the large- $N$  limit it is sufficient to know the smoothed correlation function.

The method of functional derivatives starts from the exact representation

$$R(x, y) = -\frac{1}{\beta} \frac{\delta \rho(x)}{\delta V(y)}. \quad (2.1)$$

The variation of the density is to be carried out at constant particle number,

$$\int_a^b \delta \rho(x) dx = 0. \quad (2.2)$$

The integration interval  $(a, b)$  is the support of the smoothed particle density  $\rho(x)$  in the large- $N$  limit. (We assume that the confining potential produces a support in a single interval.)

In the regime  $-1 < s < 1$  of a weakly long-ranged repulsion, and for  $J \gtrsim 1/\beta$ , variations in the smoothed density  $\rho$  and in the confining potential  $V$  are related by the condition of mechanical equilibrium [9,19],

$$J \int_a^b u(x - y) \delta \rho(y) dy + \delta V(x) = \text{constant}, \quad a < x < b. \quad (2.3)$$

Taking the derivative with respect to  $x$  we have a singular integral equation,

$$J(|s| + \delta_{0,s}) \mathcal{P} \int_a^b dy \delta \rho(y) \frac{\text{sign}(x - y)}{|x - y|^{s+1}} = \frac{d}{dx} \delta V(x), \quad a < x < b, \quad (2.4)$$

which we need to invert in order to obtain the functional derivative (2.1). (The symbol  $\mathcal{P}$  indicates the principal value of the integral.)

For  $-1 < s < 1$  the general solution to Eq. (2.4) is given by the Sonin inversion formula [20–22] (see Appendix A),

$$J(|s| + \delta_{0,s}) \delta \rho(x) = C_{\delta V} [(x - a)(b - x)]^{s-} - C_1 \mathcal{S}_{\delta V}(x), \quad (2.5a)$$

$$\begin{aligned} \mathcal{S}_{\delta V}(x) = & -(x - a)^{s+} \frac{d}{dx} \int_x^b dt \frac{(t - x)^{s-}}{(t - a)^s} \frac{d}{dt} \\ & \times \int_a^t dy (y - a)^{s-} (t - y)^{s+} \frac{d}{dy} \delta V(y), \end{aligned} \quad (2.5b)$$

$$C_1 = \frac{\sin(\pi s_+) \Gamma(s + 1)}{\pi s_+ \Gamma(s_+)^2}, \quad s_{\pm} = (s \pm 1)/2. \quad (2.5c)$$

The coefficient  $C_{\delta V}$  is fixed by the constraint (2.2),

$$C_{\delta V} = \frac{C_1}{C_2} \int_a^b dx \mathcal{S}_{\delta V}(x), \quad (2.6a)$$

$$C_2 = \int_a^b dx [(x - a)(b - x)]^{s-} = \frac{(b - a)^s \sqrt{\pi} \Gamma(s_+)}{2^s \Gamma(1 + s/2)}. \quad (2.6b)$$

The function  $\Gamma(x)$  is the usual gamma function.

The pair correlation function  $R(x, y)$  is obtained from Eq. (2.1) as a distribution, defined by its action on a test function  $g(y)$  upon integration over  $y$ . Using the functional-derivative identities

$$\int_a^b dy \frac{\delta \mathcal{S}_{\delta V}(x)}{\delta V(y)} g(y) = \mathcal{S}_g(x), \quad (2.7a)$$

$$\int_a^b dy \frac{\delta C_{\delta V}}{\delta V(y)} g(y) = \frac{C_1}{C_2} \int_a^b dx \mathcal{S}_g(x), \quad (2.7b)$$

we find the following expression:

$$\begin{aligned} J\beta(|s| + \delta_{0,s}) \int_a^b dy R(x, y) g(y) \\ = C_1 \mathcal{S}_g(x) - [(x - a)(b - x)]^{s-} \frac{C_1}{C_2} \left( \int_a^b dx \mathcal{S}_g(x) \right). \end{aligned} \quad (2.8)$$

## III. COVARIANCE OF SINGLE-PARTICLE OBSERVABLES

Equation (2.8) provides a formula for the covariance of the two observables  $F = \sum_{i=1}^N f(x_i)$  and  $G = \sum_{i=1}^N g(x_i)$ ,

$$\begin{aligned} \text{CoVar}(F, G) &= \langle FG \rangle - \langle F \rangle \langle G \rangle \\ &= \int_a^b dx \int_a^b dy R(x, y) f(x) g(y). \end{aligned} \quad (3.1)$$

The covariance is given by integrals over  $f(x)$  and  $\mathcal{S}_g(x)$ ,

$$J\beta(|s| + \delta_{0,s}) \text{CoVar}(F, G) = C_1 \int_a^b dx f(x) \mathcal{S}_g(x) - \frac{C_1}{C_2} \left( \int_a^b dx \mathcal{S}_g(x) \right) \times \int_a^b dx [(x-a)(b-x)]^{s-} f(x). \quad (3.2)$$

The variance of a single observable then follows from  $\text{Var } F = \text{CoVar}(F, F)$ .

For general functions  $f, g$  the integrals in Eq. (3.2) may be carried out numerically (see Appendix A). Closed-form expressions can be obtained for polynomial functions. It is convenient to shift the origin of the coordinate system so that  $(a, b) \rightarrow (0, L)$ . We define  $X_p = \sum_i x_i^p$ ,  $p \geq 1$ , and obtain the covariance

$$J\beta(|s| + \delta_{0,s}) \text{CoVar}(X_p, X_q) = \frac{2\pi L^{p+q+s} \Gamma(s)}{\Gamma(-p-s_-) \Gamma(p+s+1) \Gamma(-q-s_-) \Gamma(q+s+1)} \times \frac{pqs \sin(\pi s_+)}{(p+q+s)(\cos[\pi(p+q+s)] + \cos[\pi(p-q)])}. \quad (3.3)$$

For the variance this reduces to

$$J\beta(|s| + \delta_{0,s}) \text{Var } X_p = \frac{2\pi L^{2p+s} p^2 s \Gamma(s) \sin(\pi s_+)}{(2p+s) \Gamma(-p-s_-)^2 \Gamma(p+s+1)^2 (1 + \cos[\pi(2p+s)])}. \quad (3.4)$$

We have checked that the general formula (3.3) agrees with the known formulas in the Coulomb gas limit [12] (see also Appendix B),

$$J\beta \text{CoVar}(F, G) = \frac{1}{2} \int_0^L \frac{df(x)}{dx} \frac{dg(x)}{dx} dx, \quad \text{for } s = -1, \Rightarrow J\beta \text{CoVar}(X_p, X_q) = \frac{pqL^{p+q-1}}{2(p+q-1)}, \quad (3.5)$$

and in the log-gas limit [23,24],

$$J\beta \text{CoVar}(F, G) = \frac{1}{\pi^2} \mathcal{P} \int_0^L dx \int_0^L dy \frac{g(y) df(x)/dx}{y-x} \sqrt{\frac{x(L-x)}{y(L-y)}}, \quad \text{for } s = 0, \quad (3.6)$$

$$\Rightarrow J\beta \text{CoVar}(X_p, X_q) = \frac{\pi L^{p+q}}{(p+q) \Gamma(\frac{1}{2}-p) \Gamma(p) \Gamma(\frac{1}{2}-q) \Gamma(q) \cos(\pi p) \cos(\pi q)}.$$

#### IV. VARIANCE OF THE CENTER OF MASS

By way of illustration, we compute the variance of the center of mass  $M = N^{-1} \sum_{i=1}^N x_i$  of the Riesz gas. Equation (3.4) for  $p = 1$  gives

$$\text{Var } M = \frac{C_s \sqrt{\pi}}{2^{s+3} \Gamma(\frac{1}{2}-s/2) \Gamma(2+s/2)}, \quad C_s = \frac{L^{s+2}}{N^2 J \beta} \frac{1}{|s| + \delta_{0,s}}. \quad (4.1)$$

The dimensionless coefficient  $C_s$  is  $N$  independent if the system is scaled at constant interparticle spacing  $\delta x = L/N$  with interaction strength  $J \propto N^s$ .

The dependence of  $\text{Var } M$  on the interaction exponent  $s$  is plotted in Fig. 1. At the upper limit  $s \rightarrow 1$  we find  $C_s^{-1} \text{Var } M \rightarrow 0$ , to leading order in  $1/N$ . This is consistent with the fact that the repulsion is short range for  $s > 1$ , so we would expect the positions of the particles to fluctuate independently. The variance of the center of mass (rescaled by  $C_s$ ) would then be of order  $1/N$ .

The theory applies to the interval  $-1 < s < 1$  of a weakly long-range repulsion. The dashed curve in Fig. 1 is the analytical continuation of Eq. (4.1) to the interval  $-2 < s < -1$  of a strongly long-range repulsion. We surmise that the formula remains valid in that regime.

#### V. CONCLUSION

In conclusion, we have computed the pair correlation function of a 1D system of classical particles with a long-range power-law repulsion. In the thermodynamic limit (particle number  $N$  and system size  $L$  to infinity at fixed interparticle spacing  $\delta x = L/N$ ), and upon smoothing

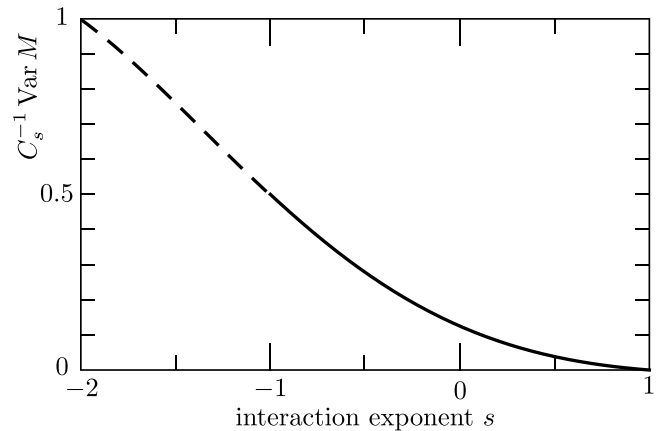


FIG. 1. Variance of the center of mass of the Riesz gas, computed from Eq. (4.1). The calculation applies to the interval  $|s| < 1$ , the dashed curve is the analytical continuation of Eq. (4.1) to smaller  $s$ .

over  $\delta x$ , the pair correlation function becomes a *universal* function of the power law exponent  $s \in (-1, 1)$ —independent of the shape of the confining potential  $V(x)$  for a given single-interval support  $(a, b)$  of the average density. So it does not matter if the Riesz gas is confined to the interval  $(0, L)$  by a soft parabolic potential or by a hard-wall confinement—the density fluctuations are the same even though the average density profile  $\rho(x)$  is very different in the two cases [10].

Our result (3.2) for the covariance of a pair of single-particle observables generalizes old results for a logarithmic repulsion [14,15] and a very recent result for a linear repulsion [12]. We rely on a solution of an integral equation that requires  $|s| < 1$  (weakly long-range regime), but the method of functional derivatives that we have used can be applied also outside of this interval. The independence of the pair correlation function  $R(x, y) = -\beta^{-1} \delta \rho(x) / \delta V(y)$  on the shape of the confining potential follows directly from the linearity of the relation between  $\rho$  and  $V$ , so we expect a universal result also for  $-2 < s < -1$ . In the short-range regime  $s > 1$ , in contrast, the  $\rho$ - $V$  relation is nonlinear [9] and no universal answer is expected.

## ACKNOWLEDGMENT

The author receives funding from the European Research Council (Advanced Grant No. 832256).

## APPENDIX A: SONIN INVERSION FORMULA

We summarize results from Refs. [20–22] on the solution of the singular integral equation

$$\mathcal{P} \int_a^b dy S(y) \frac{\text{sign}(x-y)}{|x-y|^{s+1}} = g'(x), \quad x \in (a, b), \quad |s| < 1. \quad (\text{A1})$$

The homogenous integral equation, with zero on the right-hand side, has the solution

$$S_0(x) = [(x-a)(b-x)]^{s-}. \quad (\text{A2})$$

The Sonin inversion formula gives two particular solutions to the inhomogeneous integral equation,

$$S_-(x) = -C_1(x-a)^{s-} \frac{d}{dx} \int_x^b dt \frac{(t-x)^{s+}}{(t-a)^s} \frac{d}{dt} \int_a^t dy (y-a)^{s+} (t-y)^{s-} g'(y), \quad (\text{A3})$$

$$S_+(x) = C_1(x-a)^{s+} \frac{d}{dx} \int_x^b dt \frac{(t-x)^{s-}}{(t-a)^s} \frac{d}{dt} \int_a^t dy (y-a)^{s-} (t-y)^{s+} g'(y). \quad (\text{A4})$$

For the general solution we take either one of these two particular solutions and add it to an arbitrary multiple of the homogeneous solution,

$$S(x) = S_{\pm}(x) + \text{constant} \times S_0(x), \quad (\text{A5})$$

where “constant” means independent of  $x$ . Since, by construction, the difference of two particular solutions solves the homogeneous integral equation, it does not matter for the general solution, which particular solution we choose.

In the main text we chose  $S_+(x)$ . This has the benefit over  $S_-(x)$  that the derivatives can be eliminated upon partial integration,

$$\begin{aligned} \int_a^b dx f(x) S_+(x) &= -C_1 \int_a^b dx [f'(x)(x-a) + s_+ f(x)](x-a)^{s-} \\ &\quad \times \int_x^b dt \frac{(t-x)^{s-}}{(t-a)^s} \int_a^t dy s_+(y-a)^{s-} (t-y)^{s-} g'(y). \end{aligned} \quad (\text{A6})$$

The three definite integrals of Eq. (A6) are in a form that can be evaluated numerically, without the need to take derivatives.

## APPENDIX B: COULOMB GAS LIMIT

The Coulomb gas limit  $s = -1$  can be obtained from the general formulas for  $|s| < 1$  by means of the identities [25]

$$\lim_{\epsilon \searrow 0} \int_x^b \frac{\epsilon f(y)}{(y-x)^{1-\epsilon}} dy = f(x), \quad (\text{B1a})$$

$$\lim_{\epsilon \searrow 0} \int_a^b \frac{\epsilon f(x)}{[(b-x)(x-a)]^{1-\epsilon}} dx = \frac{f(a) + f(b)}{b-a}. \quad (\text{B1b})$$

We start from the derivative-free representation (A6) of the integrals in Eq. (3.2), and apply Eq. (B1) first to the integral over  $y$ ,

$$I_1(t) = \lim_{s \searrow -1} \int_a^t dy s_+(y-a)^{s-} (t-y)^{s-} g'(y) = \frac{g'(t) + g'(a)}{t-a}, \quad (\text{B2})$$

then to the integral over  $t$ ,

$$I_2(x) = \lim_{s \searrow -1} \int_x^b dt \frac{(t-x)^{s-}}{(t-a)^s} I_1(t) = \frac{1}{s_+} [g'(x) + g'(a)], \quad (\text{B3})$$

and finally to the integral over  $x$ ,

$$\begin{aligned} I_3 &= \lim_{s \searrow -1} \int_a^b dx [f'(x)(x-a) + s_+ f(x)](x-a)^{s-} I_2(x) \\ &= \frac{1}{s_+} \int_a^b dx f'(x) [g'(x) + g'(a)] + \frac{2}{s_+} f(a) g'(a). \end{aligned} \quad (\text{B4})$$

Moreover, since  $1/C_2 \rightarrow \frac{1}{2}s_+(b-a)$  for  $s \rightarrow -1$ , we have

$$\lim_{s \searrow -1} \frac{1}{C_2} \int_a^b dx [(x-a)(b-x)]^{s-} f(x) = \frac{1}{2} [f(b) + f(a)]. \quad (\text{B5})$$

Using also  $C_1 = \frac{1}{2}s_+ + \mathcal{O}(s+1)^2$  we thus arrive at

$$\begin{aligned} \lim_{s \searrow -1} C_1 \int_a^b dx f(x) \mathcal{S}_g(x) \\ = \frac{1}{2} \int_a^b dx f'(x) [g'(x) + g'(a)] + f(a)g'(a), \end{aligned} \quad (\text{B6a})$$

$$\begin{aligned} \lim_{s \searrow -1} \frac{C_1}{C_2} \left( \int_a^b dx \mathcal{S}_g(x) \right) \int_a^b dx [(x-a)(b-x)]^{s-} f(x) \\ = \frac{1}{2} g'(a) [f(b) + f(a)]. \end{aligned} \quad (\text{B6b})$$

Substitution of Eq. (B6) into Eq. (3.2) gives

$$\lim_{s \searrow -1} \text{CoVar}(F, G) = \frac{1}{2J\beta} \int_a^b dx f'(x)g'(x), \quad (\text{B7})$$

in accord with Ref. [12].

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- [25] Equation (B1) is a special case of Ramanujan's master theorem, for a proof see <https://mathoverflow.net/q/435723/11260>.