Manipulation and enhancement of Einstein-Podolsky-Rosen steering between two mechanical modes generated by two Bogoliubov dissipation pathways

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We consider a three-mode optomechanical system in which two mechanical oscillators are independently coupled to a cavity mode driven by two controllable lasers. By controlling the two-tone driving, one can prepare the entanglement and Einstein-Podolsky-Rosen (EPR) steering of two mechanical modes, in which the cavity mode acts as a single reservoir to cool two Bogoliubov modes. We find that the direction of EPR steering can be manipulated effectively by adjusting the damping rates and the thermal noises of two mechanical modes. In addition, we show that the entanglement and EPR steering between two mechanical modes can be enhanced by adding a parametric amplifier (PA) into the cavity. The effects of the strength and phase of the PA on the mechanical entanglement and EPR steering are analyzed and discussed in detail. Meanwhile, the additional PA can also expand the region of the one-way steering and strengthen the robustness of the entanglement and EPR steering against the thermal noise. The present scheme may provide effective resources for quantum information processing.

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I. INTRODUCTION

Quantum entanglement and Einstein-Podolsky-Rosen (EPR) steering, initially introduced by Schrödinger [1] to discuss the EPR paradox [2,3], are significant features of quantum mechanics, and nowadays play an important role in quantum information processing. Quantum entanglement can be used to test the fundamental limits of quantum mechanics [4–8] and also has potential applications in many quantum technologies, such as quantum networking [9–11] and quantum metrology [12–15]. EPR steering is a kind of nonclassical correlation that is stronger than entanglement but weaker than Bell nonlocality [16], which provides a novel insight on quantum nonlocality [17], and describes the ability of one party to remotely control the other party's states through local measurements. Apart from its fundamental physical significance, due to the asymmetrical characteristics, EPR steering has important practical applications in quantum information, such as the one-sided device-independent quantum key distribution protocols [18–20], quantum teleportation [21], and so on. A lot of work in both theory and experiment are devoted to the generation and quantification [22,23] of EPR steering for the continuous variable in various physical systems such as cavity optomechanical systems [24–31], magnonic systems [32–35], and others [36–42].

In recent years, based on the optomechanical system, a number of schemes for entanglement generation have been proposed by using a variety of methods, such as modulating external driving field [43–46], combining auxiliary systems [47,48], and adding a nonlinear medium [49–52]. However, entanglement is easily disturbed by environmental noise in these schemes; the system dissipation is detrimental to the generation of entanglement. Tan [53] and Wang *et al.* [54,55] independently proposed an approach for the generation of nonclassical states in optomechanical systems by using dissipation and it has recently attracted wide interest in reservoir engineering. By introducing two-tone driving in the reservoir-engineering method, one can construct two Bogoliubov dissipation pathways to prepare high entanglement between two mechanical modes or between a mechanical mode and a cavity mode [48,55–57]. Reservoir engineering has been proved to be a very promising method for preparing entanglement [48,55,56,58–60], which is beneficial for experimental implementation due to its independence from the initial state of the system. The macroscopic entanglement between two mechanical oscillators has been implemented experimentally by using reservoir engineering [61,62].

On the other hand, EPR steering of macroscopic and massive objects has attracted much interest, and many different methods for preparing EPR steering have been proposed in the optomechanical system, such as adding a three-level atom [26,27], a controllable phase [4,28], and thermal noise [29–31]. Very recently, the EPR steering in a cavity magnonic

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system has also attracted the attention of many researchers [32–35]. Compared with entanglement and Bell nonlocality, one significant feature of EPR steering is its asymmetrical characteristics. The steerability of the two entangled parts can be different; even the steering can exist in one direction but not in the other, i.e., one-way steering. This feature gives the EPR steering many important potential applications in quantum cryptography and quantum information. Therefore, the manipulation of the steering direction and the generation of the one-way steering become the focus of many researchers. The existing methods for generating asymmetrical EPR steering usually relies on the asymmetry of the system's intrinsic mechanism [26–28,34,35] or the asymmetry induced by the external environment [29–31]. What is more, the experimental study for EPR steering has been widely reported and the one-way steerability has been observed in recent years [38–42].

Motivated by these works above, in this paper, we investigate the generation and manipulation of the EPR steering in a three-mode optomechanical system consisting of two mechanical oscillators and a cavity driven by two-tone lasers. By optimizing the ratio of the two-tone driving, we can structure two Bogoliubov dissipation pathways to prepare the entanglement and two-way EPR steering in much broader parameter regions. We numerically study the manipulation of the EPR-steering direction and find that the asymmetrical and one-way EPR steering can be achieved by adjusting the damping rates and the thermal noises of two mechanical modes. In addition, it is shown that the entanglement and EPR steering between two mechanical modes can be enhanced via adding a parametric amplifier (PA) into the cavity. Meanwhile, the introduction of the PA will strengthen the robustness of the entanglement and EPR steering against the thermal noise, which will relax the requirements for experimental conditions and make the scheme more practical.

This paper is organized as follows. In Sec. II, we describe the basic model of the scheme in detail and obtain the effective Hamiltonian. In Sec. III, we show the generation of mechanical entanglement and two-way EPR steering by two Bogoliubov dissipation pathways. In Sec. IV, we study the way to manipulate the direction of the EPR steering via adjusting the damping rates and the thermal noises of the mechanical modes. In Sec. V, by adding a PA into the cavity we show that the entanglement and EPR steering between the two mechanical modes can be enhanced. Finally, conclusions are presented in Sec. VI.

II. DESCRIPTION OF THE MODEL

As shown in Fig. 1, we consider a cavity optomechanical system involving two mechanical oscillators and a cavity that is driven by two controllable lasers. The Hamiltonian (in the unit of $\hbar = 1$) is

$$H = \omega_c a^\dagger a + \omega_{b_1} b_1^\dagger b_1 + \omega_{b_2} b_2^\dagger b_2 + g_1 a^\dagger a(b_1^\dagger + b_1) + g_2 a^\dagger a(b_2^\dagger + b_2) + H_{dr}, \quad (1)$$

with

$$H_{dr} = (\varepsilon_+ a e^{i\omega_+ t} + \varepsilon_+^* a^\dagger e^{-i\omega_+ t}) + (\varepsilon_- a e^{i\omega_- t} + \varepsilon_-^* a^\dagger e^{-i\omega_- t}). \quad (2)$$

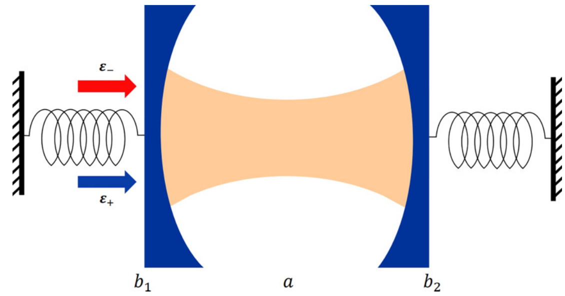

FIG. 1. Schematic representation of the system considered. Two mechanical oscillators b_1 and b_2 coupled to a cavity mode driven by two lasers with different amplitudes $\varepsilon_\pm$ and frequencies $\omega_\pm$.

Here a ($a^\dagger$) is the annihilation (creation) operator of the cavity mode with frequency ω_c and decay rate κ; b_j ($b_j^\dagger$) ($j = 1, 2$) is the annihilation (creation) operator of the mechanical modes with frequency ω_{b_j} and decay rate γ_j; g_j is the single-photon optomechanical coupling strength between the cavity and the jth mechanical oscillator; and $\omega_\pm$ are the frequencies of two lasers. H_{dr} denotes the driving of the electromagnetic mode with the amplitudes $\varepsilon_+, \varepsilon_-$.

In the following, we assume that the single-photon optomechanical coupling rates are equal, i.e., $g_1 = g_2 = g$. In the strong driving case, each Heisenberg operator can be rewritten as $O = \langle O\rangle + \delta O$ ($\langle O\rangle = \langle a\rangle, \langle b_1\rangle, \langle b_2\rangle$) ($\delta O = a, b_1, b_2$), where δO is the zero-mean quantum fluctuation operator around the classical c-number first moments $\langle O\rangle$. Remarkably, the mean value of the cavity mode meets $\langle a\rangle = \overline{a}_+ e^{-i\omega_+ t} + \overline{a}_- e^{-i\omega_- t}$ under the conditions of two-laser driving, in which $\overline{a}_\pm$ is the coherent light field amplitude. Meanwhile, as long as $|\langle a\rangle| \gg 1$, standard linearization techniques can be applied to Eq. (1). When the single-photon optomechanical coupling g_j is small, the small frequency shift of the cavity $g(\langle b_i\rangle + \langle b_i^*\rangle) \ll \omega_c$ can be neglected. We define $H_0 = \omega_c a^\dagger a + \Delta b_1^\dagger b_1 + \Delta b_2^\dagger b_2$, and $\pm\Delta = \omega_\pm - \omega_c$. The linearized Hamiltonian in the rotating frame with respect to H_0 is

$$H_{\rm lin} = \Delta_1 b_1^\dagger b_1 + \Delta_2 b_2^\dagger b_2 + G_+(b_1 a + b_1^\dagger a^\dagger) + G_+(b_2 a + b_2^\dagger a^\dagger) + G_-(b_1 a^\dagger + b_1^\dagger a) + G_-(b_2 a^\dagger + b_2^\dagger a), \quad (3)$$

where $\Delta_1 = \omega_{b_1} - \Delta$, $\Delta_2 = \omega_{b_2} - \Delta$, and $G_\pm = g\overline{a}_\pm$ is the optomechanical coupling strength. Note that we have made the rotating-wave approximation in Eq. (3) by omitting the effects of all high-frequency terms, which is valid in the resolved sideband regime. The Hamiltonian can be use to generate a two-mode squeezed state based on the reservoir-engineering approach [54,55,58]. For the two bosonic modes b_1 and b_2, one can introduce delocalized Bogoliubov-mode annihilation operators

$$\beta_1 = S(r) b_1 S^\dagger(r) = b_1 \cosh r + b_2^\dagger \sinh r,$$
$$\beta_2 = S(r) b_2 S^\dagger(r) = b_2 \cosh r + b_1^\dagger \sinh r. \quad (4)$$

Here $S(r) = \exp[r(b_1 b_2 - b_1^\dagger b_2^\dagger)]$ is the two-mode squeezing operator with squeezing parameter $r = \text{arctanh}(G_+/G_-)$. It is well known that the joint ground state of β_1 and β_2 is the two-mode squeezed state of the two mechanical modes. Therefore, the entanglement of two mechanical oscillators can be achieved by cooling the modes β_1 and β_2 to their ground states. In terms of the Bogoliubov modes defined in Eq. (4), the Hamiltonian in Eq. (3) can be written as

$$\begin{aligned} H_{\text{lin}} &= \beta_1^\dagger \beta_1(\Delta_1 \cosh^2 r + \Delta_2 \sinh^2 r) \\ &+ \beta_2^\dagger \beta_2(\Delta_1 \sinh^2 r + \Delta_2 \cosh^2 r) \\ &- \frac{\sinh 2r}{2}(\beta_1^\dagger \beta_2^\dagger + \beta_1 \beta_2)(\Delta_1 + \Delta_2) \\ &+ \mathcal{G}(a\beta_1^\dagger + a^\dagger \beta_1) + \mathcal{G}(a\beta_2^\dagger + a^\dagger \beta_2), \end{aligned} \tag{5}$$

where $\mathcal{G} = \sqrt{G_-^2 - G_+^2}$. In order to explain the physical mechanism clearly, we here set $\Delta_1 = -\Delta_2 = \Omega$, i.e., $\omega_m = \Delta$, where $\Omega = \frac{\omega_{b_1} - \omega_{b_2}}{2}$ denotes the frequency difference between the two mechanical oscillators and $\omega_m = \frac{\omega_{b_1} + \omega_{b_2}}{2}$ is the average of the two mechanical frequencies. Then the parametric amplification term between the Bogoliubov modes, i.e., the third term in the equation above, can be eliminated, and the system Hamiltonian becomes

$$H_{\text{lin}} = \Omega(\beta_1^\dagger \beta_1 - \beta_2^\dagger \beta_2) + \mathcal{G}(a\beta_1^\dagger + a^\dagger \beta_1) + \mathcal{G}(a\beta_2^\dagger + a^\dagger \beta_2). \tag{6}$$

In the following, we focus on the regime $G_+ < G_-$, such that the dynamics is stable [56,64]. From the Hamiltonian above, we can see that β_i $(i = 1, 2)$ and a are coupled by the beam-splitter-like interaction. If the occupancy of the cavity is kept low, the cavity mode a acts as an engineered reservoir which can be exploited to cool the hybrid modes β_1 and β_2, in an ideal case, into their ground states, to generate the stationary entanglement of modes b_1 and b_2. We will show in such state there is EPR steering and the steering can be manipulated and enhanced. To acquire large steady-state entanglement, one demands that the squeezing parameter $r = \text{arctanh}(G_+/G_-)$ should be large, i.e., requires the increase of G_+ (keep G_- constant). However, increasing G_+ will result in the suppression of this cooling ability due to $G_+/G_- \to 1$ and $\mathcal{G} = \sqrt{G_-^2 - G_+^2} \to 0$ in Eq. (6), which means the increase of the coupling ratio G_+/G_- has two competing effects, as previously studied in Refs. [55,56,58–60]. Thus the maximal steady entanglement is obtained by balancing of these two competing effects.

III. ENTANGLEMENT AND TWO-WAY EPR STEERING

In this section, we discuss the generation of entanglement and two-way EPR steering between mechanical modes by numerical simulation. In fact, the entanglement generation between two mechanical modes by two-tone driving can be intuitively drawn from the previous work [55], which has been demonstrated by different research groups [56,60]. The reason we discuss entanglement generation here is to compare the parameter region where entanglement exists with the region where EPR steering exists. When the dissipation and input noises induced by a Markovian environment are considered, following the standard technique [63], the quantum Langevin equations (QLEs) governing the dynamics of the linearized system can be written as

$$\begin{aligned} \dot{b}_j &= i[H_{\text{lin}}, b_j] - \frac{\gamma_j}{2} b_j + \sqrt{\gamma_j} b_{j,\text{in}}, \\ \dot{a} &= i[H_{\text{lin}}, a] - \frac{\kappa}{2} a + \sqrt{\kappa} a_{\text{in}}. \end{aligned} \tag{7}$$

$b_{j,\text{in}}$ and a_{in} are independent zero-mean vacuum input noise operators obeying the following correlation function:

$$\begin{aligned} \langle b_{j,\text{in}}(t) b_{j,\text{in}}^\dagger(t') \rangle &= (n_{j,\text{th}} + 1)\delta(t - t'), \\ \langle b_{j,\text{in}}^\dagger(t) b_{j,\text{in}}(t') \rangle &= n_{j,\text{th}} \delta(t - t'), \\ \langle a_{\text{in}}(t) a_{\text{in}}^\dagger(t') \rangle &= (n_a + 1)\delta(t - t'), \\ \langle a_{\text{in}}^\dagger(t) a_{\text{in}}(t') \rangle &= n_a \delta(t - t'), \end{aligned} \tag{8}$$

with $n_{j,\text{th}}$ and n_a being equilibrium mean thermal occupancies of the jth mechanical mode and cavity, respectively. Introduce the position and momentum quadratures corresponding to the bosonic annihilation operator o $(o = b_j, a, b_{j,\text{in}}, a_{\text{in}})$,

$$Q_o = \frac{o + o^\dagger}{\sqrt{2}}, \qquad P_o = \frac{o - o^\dagger}{i\sqrt{2}}, \tag{9}$$

and define the vectors of quadrature operators and input noises as

$$\begin{aligned} R &= [Q_{b_1}, P_{b_1}, Q_{b_2}, P_{b_2}, Q_a, P_a]^T, \\ N &= \big[\sqrt{\gamma_1} Q_{b_{1,\text{in}}}, \sqrt{\gamma_1} P_{b_{1,\text{in}}}, \sqrt{\gamma_2} Q_{b_{2,\text{in}}}, \\ &\times \sqrt{\gamma_2} P_{b_{2,\text{in}}}, \sqrt{\kappa} Q_{a_{\text{in}}}, \sqrt{\kappa} P_{a_{\text{in}}}\big]^T. \end{aligned} \tag{10}$$

Then the linearized QLEs of the quantum fluctuations can be rewritten as

$$\frac{dR}{dt} = MR + N, \tag{11}$$

where M is a 6×6 matrix:

$$M = \begin{pmatrix} -\frac{\gamma_1}{2} & \Delta_1 & 0 & 0 & 0 & G_- - G_+ \\ -\Delta_1 & -\frac{\gamma_1}{2} & 0 & 0 & -G_- - G_+ & 0 \\ 0 & 0 & -\frac{\gamma_2}{2} & \Delta_2 & 0 & G_- - G_+ \\ 0 & 0 & -\Delta_2 & -\frac{\gamma_2}{2} & -G_- - G_+ & 0 \\ 0 & G_- - G_+ & 0 & G_- - G_+ & -\frac{\kappa}{2} & 0 \\ -G_- - G_+ & 0 & -G_- - G_+ & 0 & 0 & -\frac{\kappa}{2} \end{pmatrix}. \tag{12}$$

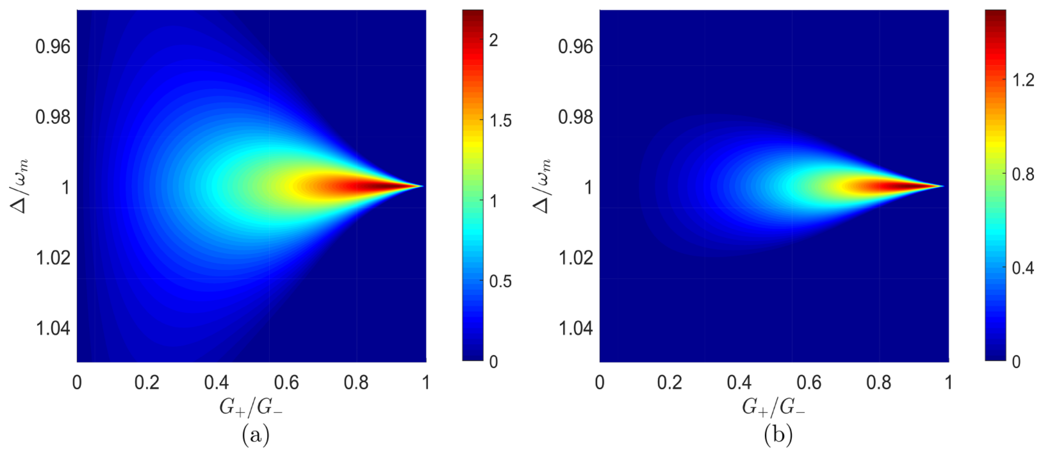

FIG. 2. Density plots of the mechanical entanglement E_N (a) and two-way EPR steering $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) (b) versus the ratio of coupling strengths G_+/G_- and the detuning Δ/ω_m. The chosen system parameters are (in units of ω_m) $\kappa = 0.1$, $\Omega = 0.1$, $\gamma_1 = \gamma_2 = 10^{-3}$, $G_- = 0.1$, and $n_{1,\text{th}} = n_{2,\text{th}} = 0$.

The system is stable only if all eigenvalues of the drift matrix M have negative real parts, which can be derived from the Routh-Hurwitz criterion [64]. Since the system is linearized, it remains in Gaussian state from an initial Gaussian state whose information-related properties [53–59] can be entirely characterized by a 6×6 covariance matrix (CM) σ with components $\sigma_{k,l} = \langle R_k R_l + R_l R_k\rangle/2$ $(k, l = 1, 2, \dots, 6)$. The steady-state CM fulfills the Lyapunov equation [65]

$$M\sigma + \sigma M^T = -D, \tag{13}$$

where the diffusion matrix $D = \text{diag}[\gamma_1(2n_{1,\text{th}} + 1)/2, \gamma_1(2n_{1,\text{th}} + 1)/2, \gamma_2(2n_{2,\text{th}} + 1)/2, \gamma_2(2n_{2,\text{th}} + 1)/2, \kappa(2n_a + 1)/2, \kappa(2n_a + 1)/2]$, whose components characterizing the stationary-noise correlations has been defined through $\delta(t - t')D_{k,l} = \langle N_k(t)N_l(t') + N_l(t')N_k(t)\rangle/2$. By utilizing Eq. (13), one can study the system's properties of entanglement and EPR steering. Here we focus on the entanglement and EPR steering between the two mechanical modes, so we should first extract the first four rows and columns of the full 6×6 CM σ to obtain the reduced 4×4 CM σ_m, which can be written as the block matrix form

$$\sigma_m = \begin{pmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{pmatrix}, \tag{14}$$

where V_1, V_2, and V_{12} are 2×2 matrices. We quantify the entanglement between the two oscillators by adopting the logarithmic negativity [66] which has been proposed as a reliable quantitative estimate of continuous-variable entanglement [67]. The definition of logarithmic negativity E_N is given by

$$E_N = \max[0, -\ln(2\eta^-)], \tag{15}$$

where $\eta^- \equiv 2^{-1/2}\{\Sigma - [\Sigma^2 - 4\det\sigma_m]^{1/2}\}^{1/2}$ is the smallest symplectic eigenvalue of the partially transposed CM, with $\Sigma \equiv \det V_1 + \det V_2 - 2\det V_{12}$. If $E_N > 0$, i.e., $\eta^- < 1/2$, the mechanical modes are entangled and the larger E_N the higher the degree of the entanglement. Moreover, the quantification of EPR steering has been introduced for arbitrary two-mode Gaussian states of a continuous-variable system [68]. The quantum steerability of Gaussian modes $b_1 \to b_2$ and $b_2 \to b_1$ is quantified, respectively, as [23]

$$\mathcal{G}_{1\to2} = \max[0, S(2V_1) - S(2\sigma_m)],$$
$$\mathcal{G}_{2\to1} = \max[0, S(2V_2) - S(2\sigma_m)], \tag{16}$$

with $S(\sigma) = [\ln\det(\sigma)]/2$. $\mathcal{G}_{1\to2} > 0$ ($\mathcal{G}_{2\to1} > 0$) means the presence of EPR steering from mode b_1 (b_2) to mode b_2 (b_1), and the value of $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) represents the strength of the steerability.

Now we numerically simulate the behavior of the entanglement and EPR steering in the parameter space by employing E_N and $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$). From Eq. (3), we can see the system dynamics depends on the coupling strength $G_\pm$ and the detuning Δ. Specifically, Eq. (6) clearly shows that the ratio G_+/G_- determines the coupling strength $\mathcal{G}$ between the cavity and Bogoliubov modes. Therefore, we investigate the dependence of E_N and $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) on the parameters G_+/G_- and Δ/ω_m, and the numerical results are shown in Figs. 2 and 3. Note that we here choose identical mechanical damping rates $\gamma_1 = \gamma_2$ and the ideal environment $n_{1,\text{th}} = n_{2,\text{th}} = 0$, combined with the simultaneous cooling of the two Bogoliubov modes, so the EPR steering obtained here is two-way symmetrical, i.e., $\mathcal{G}_{1\to2} = \mathcal{G}_{2\to1}$, as shown in Figs. 2(b) and 3. From Fig. 2, we can see that the EPR steering $\mathcal{G}_{1\to2}$ has similar variation trends as E_N and reaches the maximum value at the same position. The parameter region existing EPR steering is contained within the region existing entanglement, which is a sign that the nonclassical correlation of steering is stronger than entanglement.

Figure 3(a) shows the entanglement and steering first increase and then decrease sharply with the increase of G_+/G_-. That is because the larger G_+/G_- the larger squeezing parameter r=arctanh(G_+/G_-) that leads to the increase of entanglement. However, when $G_+/G_- \to 1$, the coupling strength $\mathcal{G}=\sqrt{G_-^2 - G_+^2} \to 0$ means the cooling ability is suppressed and the entanglement will be reduced. Therefore, E_N and $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) are nonmonotonic functions of G_+/G_- and take maximum values for a specific G_+/G_-, and there is no entanglement and EPR steering for G_+/G_-=1. From Fig. 3(b), it is clear that the entanglement and EPR

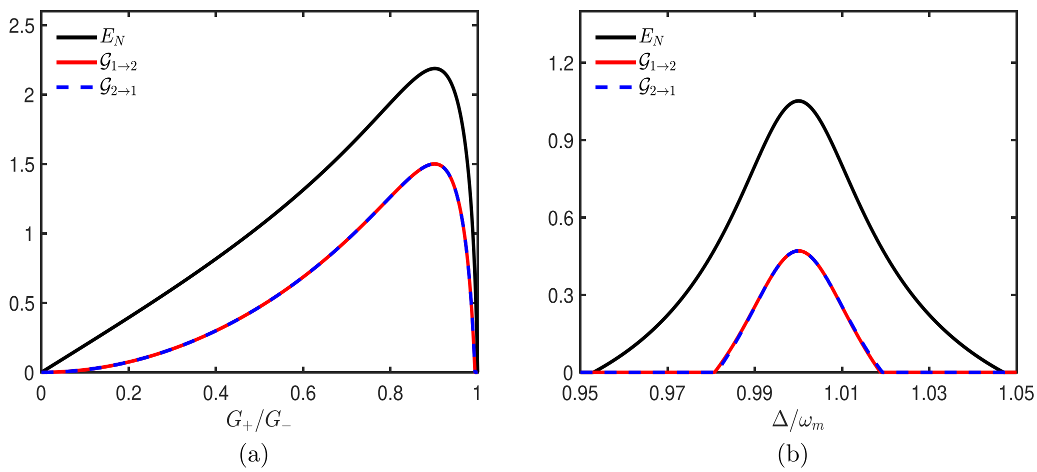

FIG. 3. The entanglement E_N and two-way EPR steering ($\mathcal{G}_{1\to2}$ and $\mathcal{G}_{2\to1}$) versus the ratio of coupling strengths G_+/G_- for $\Delta/\omega_m = 1$ (a) and the detuning Δ/ω_m for $G_+/G_- = 0.5$ (b), respectively. The chosen parameters are the same as those in Fig. 2.

steering reach their maximums when $\Delta/\omega_m = 1$, which can be explained from Eqs. (3) and (6). To get the desired Hamiltonian equation (6), we have chosen $\Delta_1 = -\Delta_2 = \Omega$, i.e., $\Delta = \omega_m$. If $\Delta \neq \omega_m$, there will be a parametric-amplification-like interaction between the two Bogoliubov modes in Eq. (6) that will affect the cooling and reduce the entanglement.

IV. MANIPULATING THE DIRECTION OF THE EPR STEERING

Due to the identical damping rates and the two Bogoliubov dissipation channels, the EPR steering obtained above is the symmetrical two-way steering. Here we investigate how to manipulate the steering direction via imbalanced mechanical dampings and thermal noises, whose physical mechanism is qualitatively clear, that is, the asymmetrical external environment of the two oscillators will cause asymmetrical quantum correlation that is beneficial for the generation of one-way EPR steering.

We firstly numerically simulate the manipulation of the EPR steering using imbalanced damping rates in Fig. 4, which displays the stationary entanglement and EPR steering of the mechanical modes as functions of the coupling ratio G_+/G_-, where different mechanical damping rate ratios γ_1/γ_2 are used in different subplots. From Figs. 4(a) to 4(d) the damping-rate ratio γ_1/γ_2 gradually increases, and we can see that the maximal values of entanglement and EPR steering decrease with the increase of γ_1, which implies that the damping is harmful for the degrees of the entanglement and EPR steering. The regions of two-way, one-way, and no-way EPR steering are respectively indicated by different colors in Fig. 4. With the increase of γ_1/γ_2, the region of one-way steering gradually becomes lager. When the ratio γ_1/γ_2 is large enough, only the steering from mode b_2 to b_1 $\mathcal{G}_{2\to1}$ is present, and $\mathcal{G}_{1\to2}$ disappears completely, as shown in Fig. 4(c) where $\gamma_1/\gamma_2 = 5$. That is because the large γ_1/γ_2 means the interaction between b_1 and its thermal bath is stronger than that between b_2 and its bath, resulting in the steering $\mathcal{G}_{1\to2}$ dropping faster than the steering $\mathcal{G}_{2\to1}$. That is to say, the mechanical mode with larger damping rate is more difficult to steer the other one. As γ_1/γ_2 continues to increase, both $\mathcal{G}_{1\to2}$ and $\mathcal{G}_{2\to1}$ disappear completely, but the mechanical entanglement still exists as shown in Fig. 4(d) where $\gamma_1/\gamma_2 = 13$, which is also attributed to the stricter nonclassical correlation of steering than of entanglement. From the analysis above, it can be seen that one can manipulate the EPR steering direction and obtain one-way steering by adjusting the damping difference between the two modes.

Now we begin to analyze the manipulation of the EPR steering via asymmetrical thermal noises of the two mechanical modes. The subplots in Fig. 5 display the mechanical entanglement E_N and EPR steering $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) as functions of G_+/G_-, for different thermal noises. The symmetrical two-way steering is present for symmetrical noises $n_{1,\text{th}} = n_{2,\text{th}}$ as shown in Fig. 5(a). From Figs. 5(b) and 5(c), it can be seen that the imbalanced thermal noises will induce asymmetrical EPR steering, and the larger the difference between thermal phonon numbers of the two thermal baths, the larger the one-way steering region. When the thermal noise is large enough, the EPR steering vanishes but the entanglement can still be obtained as shown in Fig. 5(d). From Fig. 5, we can also see that, with the increase of the thermal phonon numbers, the degrees of the entanglement and EPR steering decrease, but for $n_{1,\text{th}} > n_{2,\text{th}}$, the steerability $\mathcal{G}_{1\to2} > \mathcal{G}_{2\to1}$. That is to say, the thermal noise as a factor of decoherence will reduce the nonclassical correlation but have a positive effect on the generation of one-way steering. Therefore, it is worth noting that the larger damping rate γ_1 leads to one-way steering from mechanical mode b_2 to b_1, but the larger thermal noise $n_{1,\text{th}}$ gives one-way steering in the opposite direction. That is because, according to Refs. [22–24,35], the conditions $\mathcal{G}_{1\to2} > 0$ and $\mathcal{G}_{2\to1} > 0$ can be expressed as $|\langle b_1 b_2\rangle| > \sqrt{\langle b_2^\dagger b_2\rangle(\langle b_1^\dagger b_1\rangle + 1/2)}$ and $|\langle b_1 b_2\rangle| > \sqrt{\langle b_1^\dagger b_1\rangle(\langle b_2^\dagger b_2\rangle + 1/2)}$, respectively, in terms of correlation-based inequalities, which means that the mode with more occupancies is more likely to steer the mode with less occupancies. The lesser damping rate and larger thermal noise will be beneficial to the mode's occupancy and thus improve the ability of steering.

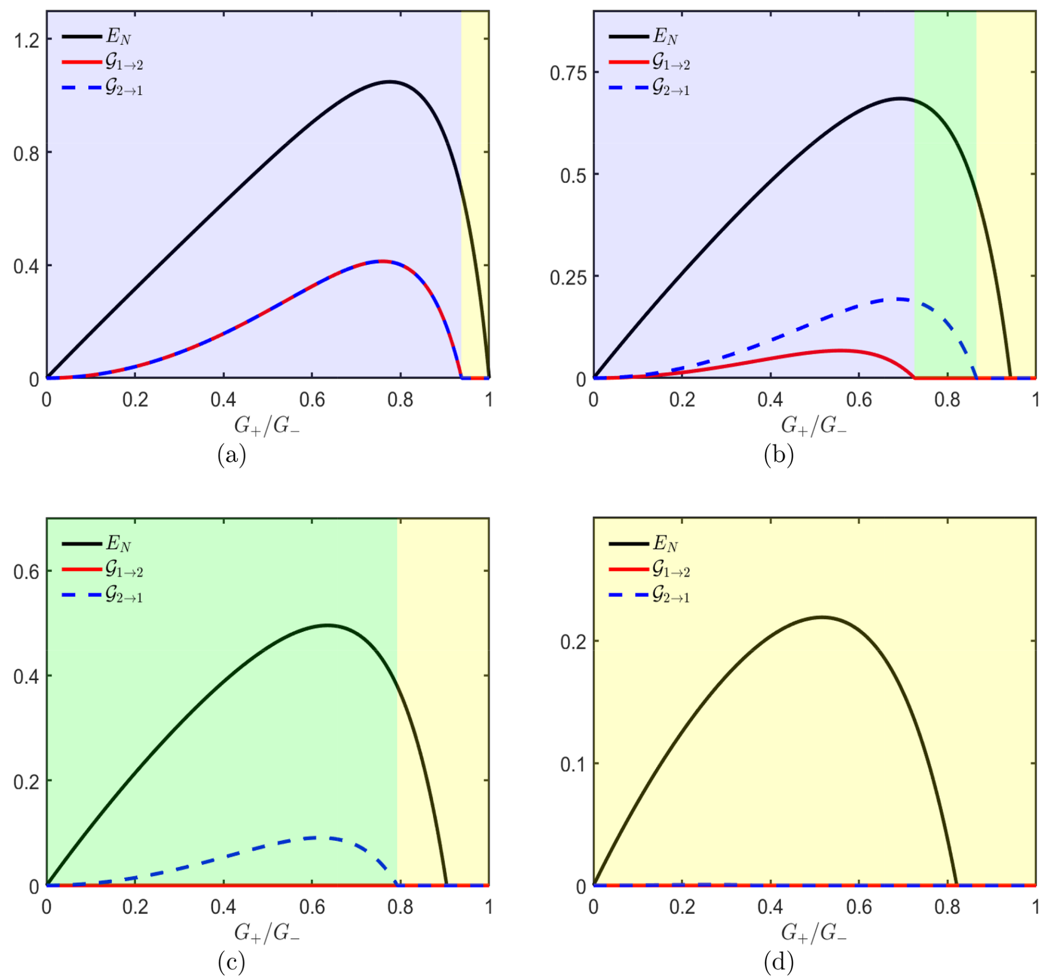

FIG. 4. Mechanical entanglement E_N and EPR steering $\mathcal{G}_{1\to2}, \mathcal{G}_{2\to1}$ as functions of the effective coupling ratio G_+/G_- for different damping rate ratios γ_1/γ_2: (a) $\gamma_1/\gamma_2 = 1$, (b) $\gamma_1/\gamma_2 = 3$, (c) $\gamma_1/\gamma_2 = 5$, and (d) $\gamma_1/\gamma_2 = 13$, where $\gamma_2 = 10^{-2}\omega_m$ and the other parameters are the same as those in Fig. 3(a).

V. ENHANCING ENTANGLEMENT AND EPR STEERING BY INTRODUCING PARAMETRIC AMPLIFIER

In this section, we study how to enhance the entanglement and EPR steering and increase the range of asymmetric steering between two mechanical modes in the system above. It has been demonstrated that a PA inside an optomechanical system can enhance the cooling of the mechanical oscillator [69,70]. Here, we will show that, by putting a PA in the optomechanical cavity in Fig. 1, the mechanical entanglement and the EPR steering discussed above can be enhanced. We assume that the gain of the PA is Λ depending on the power of the pump field driving the PA, and the frequency and the phase of the pump field are $2\omega_c$ and θ, respectively. The Hamiltonian of the system in the rotating frame with respect to H_0 is given by

$$H'_{\rm lin} = \Lambda(a^{\dagger 2}e^{i\theta} + a^2 e^{-i\theta}) + \Delta_1 b_1^\dagger b_1 + \Delta_2 b_2^\dagger b_2 + G_+(b_1 a + b_1^\dagger a^\dagger) + G_+(b_2 a + b_2^\dagger a^\dagger) + G_-(b_1^\dagger a + b_1 a^\dagger) + G_-(b_2^\dagger a + b_2 a^\dagger), \tag{17}$$

where $\Delta_1 = -\Delta_2 = \Omega$. The first term shows the parametric amplification process. By replacing the Hamiltonian in Eq. (3) with Eq. (17) and deriving again Eqs. (5)–(12), the new drift matrix reads as

$$M' = \begin{pmatrix} -\frac{\gamma_1}{2} & \Delta_1 & 0 & 0 & 0 & G_- - G_+ \\ -\Delta_1 & -\frac{\gamma_1}{2} & 0 & 0 & -G_- - G_+ & 0 \\ 0 & 0 & -\frac{\gamma_2}{2} & \Delta_2 & 0 & G_- - G_+ \\ 0 & 0 & -\Delta_2 & -\frac{\gamma_2}{2} & -G_- - G_+ & 0 \\ 0 & G_- - G_+ & 0 & G_- - G_+ & -\frac{\kappa}{2} + 2\Lambda\cos\theta & 2\Lambda\sin\theta \\ -G_- - G_+ & 0 & -G_- - G_+ & 0 & 2\Lambda\sin\theta & -\frac{\kappa}{2} - 2\Lambda\cos\theta \end{pmatrix}.$$

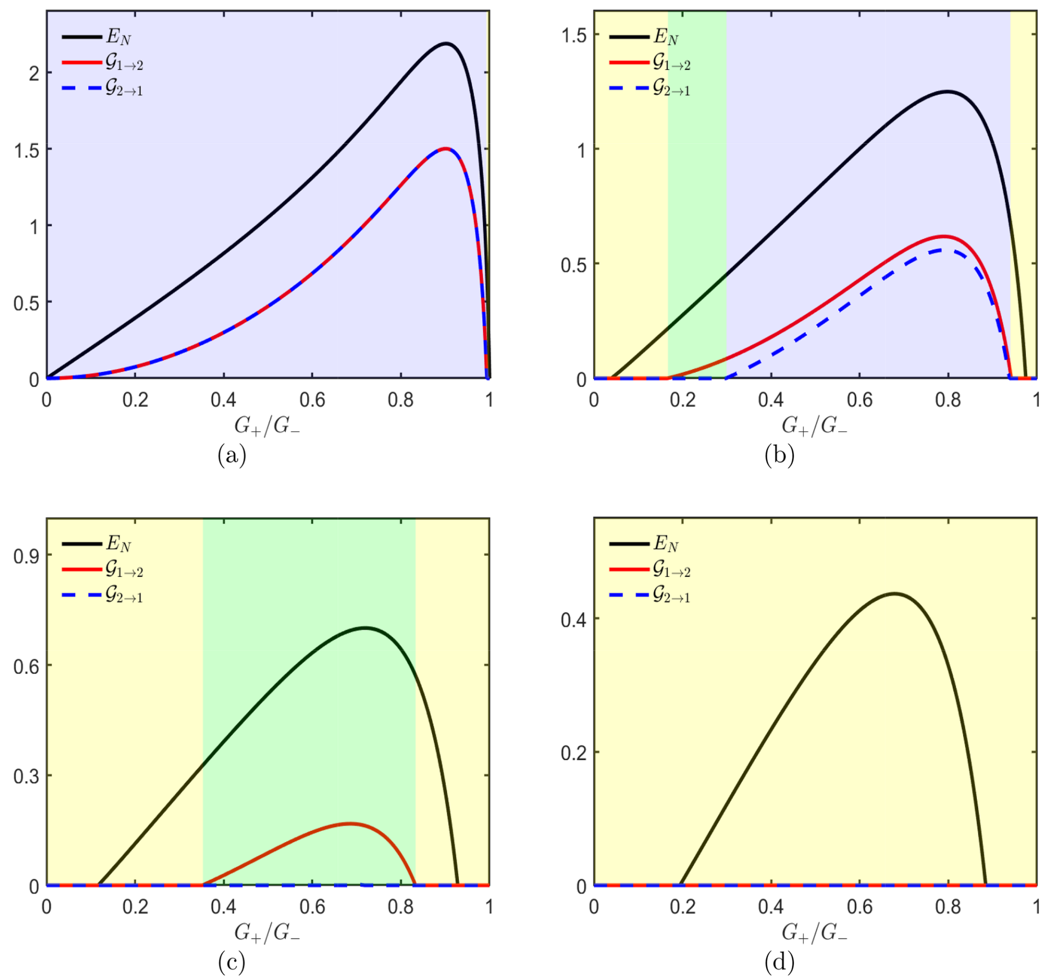

FIG. 5. Mechanical entanglement E_N and EPR steering $\mathcal{G}_{1\to2}$, $\mathcal{G}_{2\to1}$ as functions of the effective coupling ratio G_+/G_- for different thermal noises $n_{1,\text{th}}$ and given $n_{2,\text{th}}=0$: (a) $n_{1,\text{th}}=0$, (b) $n_{1,\text{th}}=4$, (c) $n_{1,\text{th}}=11$, and (d) $n_{1,\text{th}}=17$. The other parameters are the same as those in Fig. 3(a).

Then by solving the Lyapunov equation, we can obtain the new CM σ_m', which will be used to investigate the effect of the strength and phase of the PA on the mechanical entanglement and EPR steering.

We numerically simulate the mechanical entanglement and the two-way EPR steering $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) versus the ratio G_+/G_- for different Λ values in Figs. 6(a) and 6(c), respectively. It can be found that both the entanglement and the steering are significantly improved by introducing the PA, and the degrees of both the entanglement and the steering increase with the increase of the PA strength Λ. In particular, when $G_+/G_-=0$, the entanglement still exists when the PA is present, but the steering equals to 0 whether the PA is added or not, which means, without the two-tone driving, one can generate the entanglement but cannot generate the EPR steering by using the PA and a single driving. In Figs. 6(b) and 6(d), we evaluate the effect of the phase θ of the PA on the mechanical entanglement E_N and two-way EPR steering $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) corresponding to the optimal ratio G_+/G_-, which shows that E_N and $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) fluctuate periodically with θ. The amplitudes of the fluctuations increase with the increase of Λ, and the fluctuation period is 2π. Such dependence of E_N and $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) on the phase θ is physically intuitive due to the fact that the different phase θ of the PA corresponds to the different squeezing direction of the cavity field generated by the parametric amplification process, so the two-mode squeezing between the two mechanical oscillators originating from the cavity field naturally changes with θ.

In order to explore the effect of the PA on the asymmetry of the steering, in Fig. 7, we introduce imbalanced damping rates and compare the asymmetrical steering $\mathcal{G}_{1\to2}$ and $\mathcal{G}_{2\to1}$. It can be seen from Fig. 7, by adding the PA in the cavity, the asymmetry of the steering $|\mathcal{G}_{2\to1}-\mathcal{G}_{1\to2}|$ is significantly increased. We use the pink (or dark gray) to depict the region of one-way steering without PA, and the green (or light gray) to depict the expanded region of one-way steering after adding PA, respectively, which shows that the introduction of the PA not only increases the one-way steering, but also expands the region of the one-way steering.

Thermal noise, i.e., the environment temperature, is a key factor affecting the performance of the optomechanical system. Therefore, we now analyze and discuss its effects on entanglement generation and EPR steering. Without loss of generality, we assume the two mechanical modes are in the

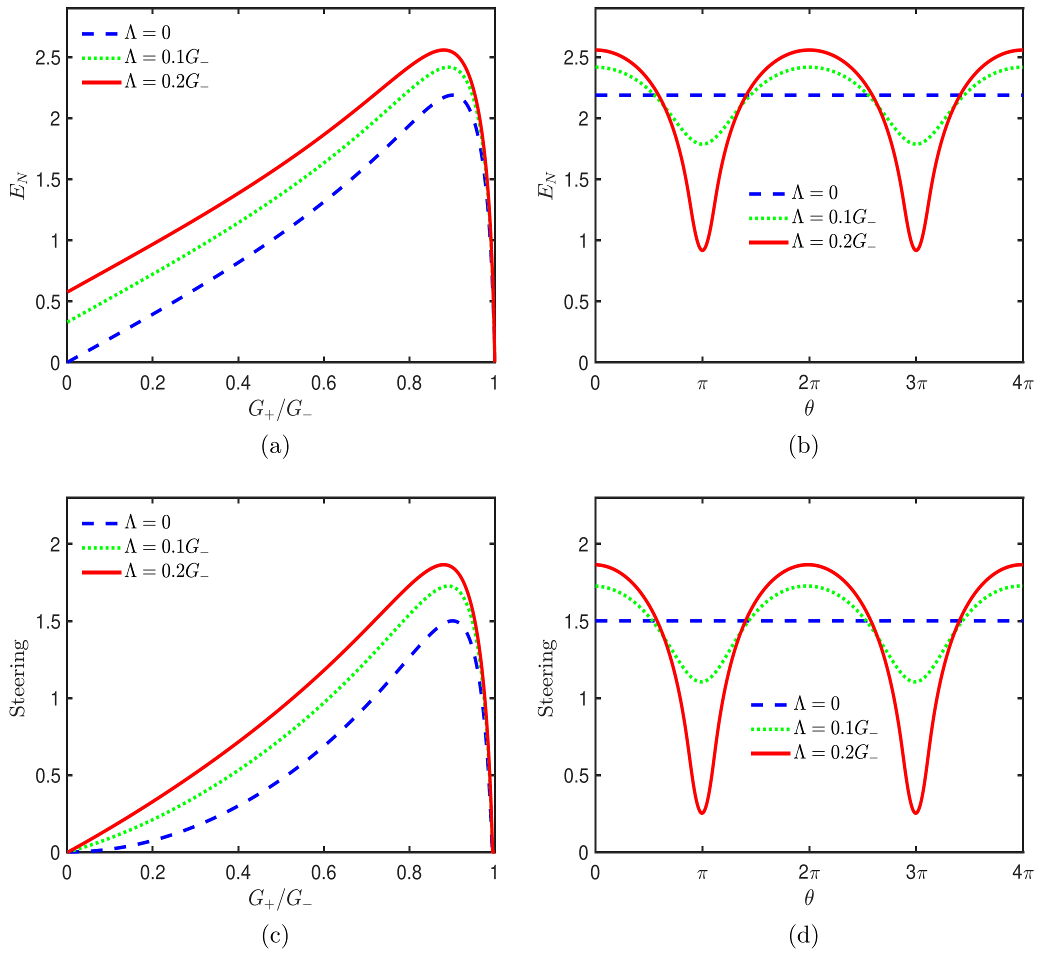

FIG. 6. Mechanical entanglement E_N (a) and two-way EPR steering $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) (c) as functions of the effective couplings ratio G_+/G_-, for $\Lambda = 0$, $\Lambda = 0.1G_-$, $\Lambda = 0.2G_-$, and $\theta = 0$. The maximized entanglement E_N (b) and maximized two-way EPR steering (d) for the optimal ratio G_+/G_- as functions of the phase θ. The other parameters are the same as those in Fig. 3(a).

same environment, i.e., $n_{1,\mathrm{th}} = n_{2,\mathrm{th}} = n_{\mathrm{th}}$. Figure 8 shows the degrees of entanglement and EPR steering for the optimal ratio G_+/G_- versus the mean thermal phonon number n_{th}, in which dashed and solid lines denote the case without and with

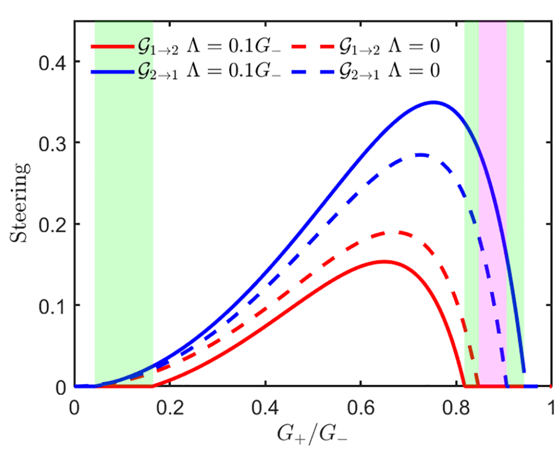

FIG. 7. Asymmetrical EPR steering as functions of the effective couplings ratio G_+/G_- for the cases without PA ($\Lambda = 0$) and with PA ($\Lambda = 0.1G_-$, $\theta = \pi/2$), where $\gamma_1 = 2\gamma_2 = 2 \times 10^{-2}\omega_m$; other parameters are the same as those in Fig. 3(a).

the PA, respectively. Obviously, both the robustness of the entanglement and the EPR steering against the environment temperature can be strengthened by introducing the PA. Comparing Fig. 8(a) with 8(b), we can find that EPR steering is more sensitive to thermal noise than entanglement, regardless of the presence or absence of PA.

VI. CONCLUSIONS

In summary, we have studied how to generate mechanical entanglement and EPR steering in the cavity optomechanical system via two Bogoliubov dissipation pathways. The numerical results showed that the entanglement and EPR steering can be obtained by controlling the two-tone driving. It has been shown that the direction of the EPR steering can be manipulated by imbalanced damping rates or asymmetrical thermal noises of the mechanical modes, which provides two effective ways to achieve one-way EPR steering. What is more, we numerically simulated the entanglement and EPR steering in the case adding a PA into the optomechanical cavity, which showed that, due to the introduction of the PA, the degrees of entanglement and EPR steering are enhanced, and the parameter region of the one-way steering is expanded. The entanglement and EPR steering for the presence of the PA

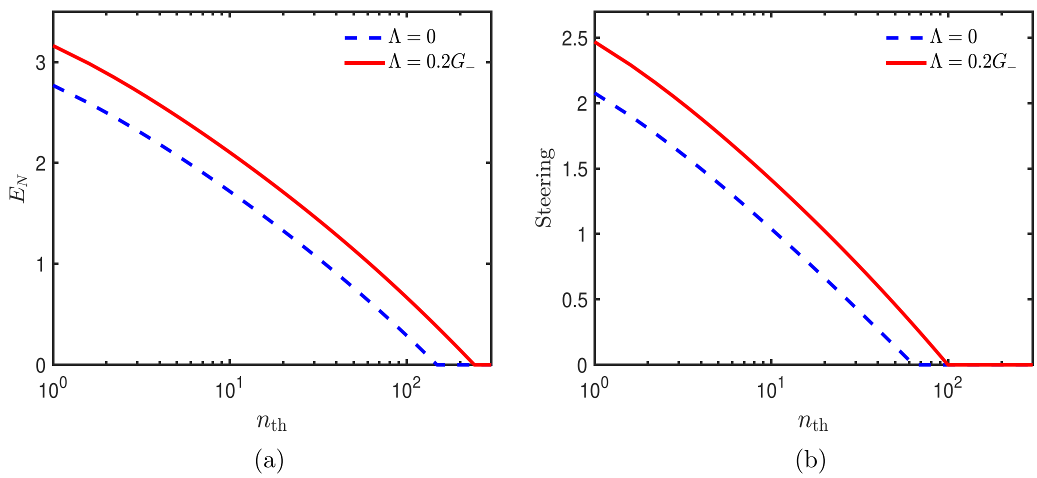

FIG. 8. Mechanical entanglement E_N (a) and two-way EPR steering $\mathcal{G}_{1\to2}$ ($\mathcal{G}_{2\to1}$) (b) for the optimal ratio G_+/G_- as functions of the mean thermal phonon number $n_{\rm th}$ of the mechanical modes for $\gamma_1 = \gamma_2 = 10^{-4}\omega_m$, $\theta = 0$. The other parameters are the same as those in Fig. 3(a).

have stronger robustness against the thermal noise compared with that without the PA. Given the potential application of the one-way steering in one-sided device-independent quantum key distribution, the generation, manipulation, and enhancement schemes for the EPR steering presented here may be meaningful for quantum cryptography and quantum information processing.

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