Letter

Non-Hermitian chiral anomalies

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The chiral anomaly underlies a broad number of phenomena, from enhanced electronic transport in topological metals to anomalous currents in the quark-gluon plasma. The discovery of topological states of matter in non-Hermitian systems raises the question of whether there are anomalous conservation laws that remain unaccounted for. To answer this question, we consider both two and four space-time dimensions, presenting a unified formulation to calculate anomalous responses in Hermitianized, anti-Hermitianized, and non-Hermitian systems of massless electrons with complex Fermi velocities coupled to non-Hermitian gauge fields. Our results indicate that the quantum conservation laws of chiral currents of non-Hermitian systems are not related to those in Hermitianized and anti-Hermitianized systems, as would be expected classically, due to different anomalous terms that we derive. We further present some physical consequences of our non-Hermitian anomaly that may have implications for a broad class of emerging experimental systems that realize non-Hermitian Hamiltonians.

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Introduction. Quantum anomalies explain transport phenomena in many branches of physics, including high-energy physics [1–4], astrophysics [5,6], and condensed matter physics [7–9]. Anomalies account for the fact that, upon quantization, the conservation law associated with a classical symmetry can be broken, resulting in observable anomalous transport currents [10]. While anomalous currents can be corrected by interactions among particles, the chiral anomaly [11], the imbalance between left and right movers due to quantum fluctuations [12], remains universal. The universal chiral anomaly coefficient appears in dissipationless transport currents [10] due to electromagnetic [13–18] and strain fields [19] in condensed matter systems, and in anomalous transport in the quark-gluon plasma [3,10].

The universality of the chiral anomaly, i.e., its robustness against local perturbations, is intimately related to topological properties of the Hamiltonian [20]. While most of these Hamiltonians respect the Hermiticity condition, non-Hermitian Hamiltonians, which are effective descriptions of systems coupled to an environment [21–25], introduce new classes of topological systems which do not have any Hermitian analog [26–30]. This motivates the question we address in this Letter: Are chiral quantum anomalies in non-Hermitian systems different in any way from those of (anti-)Hermitian systems? So far, one of the main links between anomalous field theories and non-Hermitian systems are the extension of the index theorem to non-Hermitian systems [31] and the reinterpretation of the non-Hermitian skin effect as a consequence of an anomaly [32,33]. The skin effect [34–39] is one of the central differences between topological non-Hermitian and Hermitian systems and results in a macroscopic number of states accumulating at the boundary of the system. The action describing a non-Hermitian system in $d+0$ dimensions is mathematically equivalent to that of a $[(d-1)+1]$-dimensional Hermitian system [33]. This establishes a link between the non-Hermitian skin effect of a d-dimensional system to a $[(d-1)+1]$-dimensional anomalous Hermitian theory. Additionally, Ref. [31] showed that a lattice version of a $\mathcal{PT}$-symmetric continuum model, i.e., non-Hermitian systems with a real spectrum, can display quantum anomalies similar to those in Hermitian systems. In $\mathcal{PT}$-symmetric systems the chiral magnetic effect [4], an anomalous transport current parallel to a magnetic field, can exist in equilibrium in non-Hermitian systems [40].

In this Letter, we present a generic $(d+1)$-dimensional non-Hermitian formulation that consolidates anomalous chiral responses in $(d+1)$-dimensional Hermitian, anti-Hermitian, and non-Hermitian systems. More specifically, we study the chiral anomaly in two and four space-time dimensions for non-Hermitian massless Weyl fermions, with complex Fermi velocities, coupled to complex gauge fields. We also (anti)symmetrize the non-Hermitian action and introduce an (anti-)Hermitianized action. All these models lack $\mathcal{PT}$ and Lorentz symmetry.

By establishing a unified notation for the Hermitianized, anti-Hermitianized, and non-Hermitian systems, we show that all of these systems classically conserve both the vector and chiral currents but exhibit anomalous responses when quantized (Fig. 1). Furthermore, the anomalous currents in non-Hermitian systems are not given by the simple addition of currents associated with the Hermitianized and

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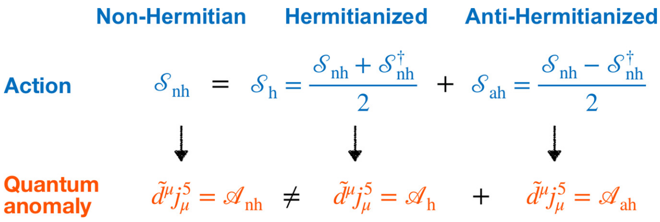

FIG. 1. Anomalies in Hermitianized, anti-Hermitianized, and non-Hermitian actions. For the non-Hermitian Hamiltonian $\mathcal{S}_{\rm nh}$, as well as its Hermitianized ($\mathcal{S}_{\rm h}$) and anti-Hermitianized ($\mathcal{S}_{\rm ah}$) forms, the conserved classical chiral current will be anomalous upon including quantum fluctuations. Note that the definition of current remains the same in all systems and $\tilde{d}_\mu = f^\nu_\mu \partial_\nu$ where f^ν_μ is introduced in Table I. While $\mathcal{S}_{\rm nh} = \mathcal{S}_{\rm h} + \mathcal{S}_{\rm ah}$, for the anomalous currents we get $\mathcal{A}_{\rm nh} \neq \mathcal{A}_{\rm h} + \mathcal{A}_{\rm ah}$.

anti-Hermitianized actions, as would be expected classically (Fig. 1).

In the main text, we follow Fujikawa's path-integral approach [41–44], which proves to be a controlled method to evaluate the chiral anomalies emerging from the non-Hermitian system. Based on these results, we present a Chern-Simons description of our non-Hermitian models and briefly discuss currents associated with the non-Hermitian anomalous Hall effect and non-Hermitian chiral magnetic effect. We also evaluate the $(1+1)$-dimensional anomalies using the diagrammatic method, for which we discuss some subtleties arising from the non-Hermicity of the action. Detailed calculations can be found in the Supplemental Material (SM) [45], which also includes an effective bosonic theory for the $(1+1)$-dimensional non-Hermitian systems.

Non-Hermitian chiral anomaly from Fujikawa's method. Within Fujikawa's method [41–44], the covariant form of the chiral anomaly [10] is evaluated by the change of the measure of the path integral after applying both vector and chiral transformations.

We apply Fujikawa's method to the three different actions presented in Fig. 1. These three systems, in the language of the path integral, correspond to a non-Hermitian action and its (anti)symmetrized form, generating the (anti-)Hermitian action.

We consider a non-Hermitian, non-$\mathcal{PT}$ symmetric system consisting of massless fermions, with complex Fermi velocities, coupled to non-Hermitian gauge fields (V, W) [46] described by the non-Hermitian Euclidean-space action $\mathcal{S}_{\rm nh}$ and the partition function $\mathcal{Z}_{\rm nh}$,

$$\mathcal{Z}_{\rm nh} \propto \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{\mathcal{S}_{\rm nh}}, \tag{1}$$

$$\mathcal{S}_{\rm nh} = i \int d^d x [\bar{\Psi}\gamma^\mu(\mathcal{D}_{{\rm nh},\mu}\Psi)], \tag{2}$$

$$\not{\mathcal{D}}_{\rm nh} = \gamma^\mu \mathcal{D}_{{\rm nh},\mu} = \gamma^\mu M^\nu_\mu \partial_\nu - i\gamma^\mu M^\nu_\mu (V_\nu + \gamma^5 W_\nu), \tag{3}$$

in units where $c = \hbar = 1$. The dimension d is even, and the greek indices take values between 1 and d, where repeated indices are summed over. The gamma matrices γ^μ obey $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, where $g^{\mu\nu} = -\delta^{\mu\nu}$ is the Euclidean metric, and the fifth gamma-matrix is $\gamma^5 = -\prod_\mu \gamma^\mu$. The M is a rank d diagonal matrix such that $M = {\rm diag}[v_1, \ldots, v_d]$ with $v_d = 1$ and spatial elements $v_{i\neq d}$ are complex-valued Fermi velocities. Here, $\not{\mathcal{D}}_{\rm nh}$ is the Dirac operator, and $\overline{\Psi} = \Psi^\dagger \gamma^0$ denotes the Dirac adjoint with $\gamma^0 = i\gamma^4$.

In the SM [45], we propose a microscopic model based on a nonreciprocal anisotropic heterostructure as a platform to experimentally realize the introduced linear band system with complex Fermi velocities.

TABLE I. $f, \tilde{A}$, and $\tilde{\mathcal{F}}$ for Hermitianized, $\mathcal{S}_{\rm h}$, anti-Hermitianized, $\mathcal{S}_{\rm ah}$, and non-Hermitian, $\mathcal{S}_{\rm nh}$, systems. A stands for gauge fields V and W. $\tilde{\mathcal{F}}$ is presented for two and four dimensions. The matrix B for non-Hermitian systems is introduced in Eq. (17).

	$\mathcal{S}_{\rm h}$	$\mathcal{S}_{\rm ah}$	$\mathcal{S}_{\rm nh}$
f^ν_μ	${\rm Re}[M^\nu_\mu]$	${\rm Im}[M^\nu_\mu]$	M^ν_μ
$\tilde{A}_\mu$	${\rm Re}[M^\nu_\mu A_\nu]$	${\rm Im}[M^\nu_\mu A_\nu]$	$M^\nu_\mu A_\nu$
$\tilde{\mathcal{F}}_2$	$4\pi\lvert{\rm Re}[v_{\rm f}]\rvert$	$4\pi\lvert{\rm Im}[v_{\rm f}]\rvert$	$4\pi\sqrt{\det[B]}$
$\tilde{\mathcal{F}}_4$	$32\pi^2\lvert\det[{\rm Re}[M]]\rvert$	$32\pi^2\lvert\det[{\rm Im}[M]]\rvert$	$32\pi^2\sqrt{\det[B]}$

To show the differences between non-Hermitian and (anti-)Hermitian systems, we also explore the chiral anomaly in the (anti-)Hermitianized form of Eq. (2). By symmetrizing/antisymmetrizing $\mathcal{S}_{\rm nh}$ in Eq. (2), the Hermitian/anti-Hermitian action $\mathcal{S}_{\rm h/ah}$ yields

$$\mathcal{S}_{\rm h/ah} = \frac{i}{2}\int d^d x[\bar{\Psi}\gamma^\mu(\mathcal{D}_{{\rm nh},\mu}\Psi) \mp \overline{(\mathcal{D}_{{\rm nh},\mu}\Psi)}\gamma^\mu\Psi], \tag{4}$$

$$= i\int d^d x \bar{\Psi}\gamma^\mu \mathcal{D}_{{\rm h/ah},\mu}\Psi, \tag{5}$$

where $\overline{(\mathcal{D}_{{\rm nh},\mu}\Psi)} = (\mathcal{D}_{{\rm nh},\mu}\Psi)^\dagger\gamma^0$. Here, $\mathcal{D}_{\rm h}$, $\mathcal{D}_{\rm ah}$ are the modified Dirac operators which are given by

$$\mathcal{D}_{{\rm h},\mu} = {\rm Re}[M^\nu_\mu]\partial_\nu - i{\rm Re}[M^\nu_\mu V_\nu] - i\gamma^5{\rm Re}[M^\nu_\mu W_\nu], \tag{6}$$

$$\mathcal{D}_{{\rm ah},\mu} = i{\rm Im}[M^\nu_\mu]\partial_\nu + {\rm Im}[M^\nu_\mu V_\nu] + \gamma^5{\rm Im}[M^\nu_\mu W_\nu]. \tag{7}$$

Note that our Hermitianized system with $M \in \mathbb{R}$ represents the Lorentz preserving Hermitian model when $M = \mathbb{1}_{d\times d}$ and $\{V, W\} \in \mathbb{R}$.

Our calculations are simplified by defining a unified representation that incorporates all cases

$$\tilde{\mathcal{Z}} \propto \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} e^{\tilde{\mathcal{S}}}, \text{ with } \tilde{\mathcal{S}} = i\int d^d x \bar{\Psi}\gamma^\mu \tilde{\mathcal{D}}_\mu \Psi, \tag{8}$$

$$\tilde{\mathcal{D}}_\mu = \tilde{d}_\mu - i\tilde{V}_\mu - i\gamma^5 \tilde{W}_\mu. \tag{9}$$

Here, $\tilde{d}_\mu = f^\nu_\mu \partial_\nu$. The matrix f and the gauge field $\tilde{A} \in \{\tilde{V}, \tilde{W}\}$, with their associated elements given in Table I, map our generic action $\tilde{S}$ into the Hermitianized, anti-Hermitianized, or non-Hermitian actions.

Classically, $\tilde{\mathcal{S}}$, or equivalently, $\mathcal{S}_{\rm h}$, $\mathcal{S}_{\rm ah}$, or $\mathcal{S}_{\rm nh}$, carries a local $U_{\rm A}(1) \times U_{\rm V}(1)$ symmetry, where $U_{\rm A(V)}(1)$ is the chiral (vector) $U(1)$ symmetry; see the SM [45]. Quantum mechanical fluctuations reduce this underlying classical symmetry in all cases, as we will show in the following.

While the quantum action in Eq. (8) remains invariant under a chiral transformation of the spinor,

$$\Psi_{\rm rot} = e^{-i\gamma^5\beta}\Psi, \tag{10}$$

the rotated partition function acquires a phase given by the Jacobian of its measure. Hence, we continue by calculating the Jacobian of the rotated path-integral measure. For this purpose, one may decompose the rotated spinors into the eigenbasis of the non-Hermitian Dirac operator ($\tilde{\not{D}} \neq \tilde{\not{D}}^{\dagger}$), which is biorthogonal with right and left eigenvectors. However, rather than working with the eigenbasis of $\tilde{\not{D}}$, which introduces unnecessary complications due to the biorthogonal basis of a non-Hermitian $\tilde{\not{D}}$, we employ an equally correct but more straightforward approach which is using the eigenbases of the Hermitian Laplacian operators $\not{D}\not{D}^{\dagger}$ and $\not{D}^{\dagger}\not{D}$, given by

$$\tilde{\not{D}}\tilde{\not{D}}^{\dagger}|\eta_n\rangle = |\lambda|_n^2|\eta_n\rangle, \quad \tilde{\not{D}}^{\dagger}\tilde{\not{D}}|\xi_n\rangle = |\lambda|_n^2|\xi_n\rangle, \tag{11}$$

$$\tilde{\not{D}}^{\dagger}|\eta_n\rangle = \lambda_n^*|\xi_n\rangle, \quad \tilde{\not{D}}|\xi_n\rangle = \lambda_n|\eta_n\rangle. \tag{12}$$

Here, $\{\lambda_n\}$ are complex eigenvalues and $\{|\xi\rangle\}$ and $\{|\eta\rangle\}$ are the corresponding eigenvectors (see the SM [45]). Using these eigenbases and following the standard Fujikawa method allows us to write

$$\mathcal{D}\Psi_{\text{rot}}\mathcal{D}\bar{\Psi}_{\text{rot}} = \tilde{\mathcal{J}}^5[\beta]\mathcal{D}\Psi\mathcal{D}\bar{\Psi} = e^{\tilde{\mathcal{S}}^5[\beta]}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}, \tag{13}$$

$$\tilde{\mathcal{S}}^5[\beta] = i\int d^dx\beta(x)\tilde{\mathcal{A}}^5(x), \tag{14}$$

where $\tilde{\mathcal{J}}^5$ is the path-integral Jacobian due to an infinitesimal chiral transformation. The exponent of this Jacobian is regularized by the heat-kernel regularization method [47,48] (see details in the SM [45]). Up to the first order in the fields, $\tilde{\mathcal{A}}^5$ in $d = 2$ dimensions reads

$$\tilde{\mathcal{A}}^5 = \frac{-\varepsilon^{\mu\nu}}{\tilde{\mathcal{F}}_2}[i(\tilde{F}_{\mu\nu}[\tilde{V}^{\dagger}] - \tilde{F}^{\dagger}_{\mu\nu}[\tilde{V}])], \tag{15}$$

and up to the second order in the fields, $\tilde{\mathcal{A}}^5$ in $d = 4$ dimensions casts

$$\tilde{\mathcal{A}}^5 = \frac{\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}[\tilde{F}_{\mu\nu}[\tilde{V}^{\dagger}]\tilde{F}_{\eta\zeta}[\tilde{V}^{\dagger}] + \tilde{F}^{\dagger}_{\mu\nu}[\tilde{V}]\tilde{F}^{\dagger}_{\eta\zeta}[\tilde{V}] + \tilde{F}^{\dagger}_{\mu\nu}[\tilde{W}]\tilde{F}^{\dagger}_{\eta\zeta}[\tilde{W}] + \tilde{F}_{\mu\nu}[\tilde{W}^{\dagger}]\tilde{F}_{\eta\zeta}[\tilde{W}^{\dagger}]]. \tag{16}$$

Here, $\tilde{F}_{\mu\nu}[\tilde{A}] = \tilde{d}_{\mu}\tilde{A}_{\nu} - \tilde{d}_{\nu}\tilde{A}_{\mu}$, and $\tilde{F}^{\dagger}_{\mu\nu}[\tilde{A}] = \tilde{d}^{\dagger}_{\mu}\tilde{A}_{\nu} - \tilde{d}^{\dagger}_{\nu}\tilde{A}_{\mu}$ with $\tilde{d}^{\dagger}_{\mu} = -f^{*\nu}_{\mu}\partial_{\nu}$. The explicit form of $\tilde{\mathcal{F}}_{d+1}$ for all cases is presented in Table I, where it is formulated in terms of the determinant of a matrix B, with matrix elements:

$$B^{\alpha\beta} = \delta^{\mu\nu}f^{*\alpha}_{\mu}f^{\beta}_{\nu} - \frac{1}{2}[\gamma^{\mu},\gamma^{\nu}]\frac{f^{*\alpha}_{\mu}f^{\beta}_{\nu} - f^{\alpha}_{\mu}f^{*\beta}_{\nu}}{2}. \tag{17}$$

For real gauge fields V and W, and for $M = \mathbb{1}_{d\times d}$, $\tilde{\mathcal{A}}^5$ reproduces the Lorentz invariant Hermitian results [12,45,49].

We evaluate the change of the path-integral measure under a local vector transformation using the same method as for the chiral rotations,

$$\Psi_{\text{rot}} = e^{-i\kappa(x)}\Psi, \tag{18}$$

where κ is the rotation angle. For an infinitesimal κ,

$$\mathcal{D}\Psi_{\text{rot}}\mathcal{D}\bar{\Psi}_{\text{rot}} = e^{\tilde{\mathcal{S}}[\kappa]}\mathcal{D}\Psi\mathcal{D}\bar{\Psi} = e^{i\int d^dx\kappa(x)\tilde{\mathcal{A}}}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}. \tag{19}$$

Up to the first order in the fields, $\tilde{\mathcal{A}}$ in $d = 2$ dimensions is

$$\tilde{\mathcal{A}} = \frac{-\varepsilon^{\mu\nu}}{\tilde{\mathcal{F}}_2}[i(\tilde{F}^{\dagger}_{\mu\nu}[\tilde{W}] - \tilde{F}_{\mu\nu}[\tilde{W}^{\dagger}])], \tag{20}$$

and up to the second order in the fields, $\tilde{\mathcal{A}}$ in $d = 4$ dimensions casts [50]

$$\tilde{\mathcal{A}} = \frac{-\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}[\tilde{F}_{\mu\nu}[\tilde{V}^{\dagger}]\tilde{F}_{\eta\zeta}[\tilde{W}^{\dagger}] + \tilde{F}^{\dagger}_{\mu\nu}[\tilde{V}]\tilde{F}^{\dagger}_{\eta\zeta}[\tilde{W}]], \tag{21}$$

In the limit where $W = 0$, $V \in \mathbb{R}$, and $M = \mathbb{1}_{d\times d}$, $\tilde{\mathcal{A}}$ reduces to the well-known Lorentz preserving Hermitian result $\mathcal{A} = 0$ [12,41].

The rotated action in Eq. (8) is modified under the chiral and vector transformations in Eqs. (10) and (18), such that

$$\tilde{\mathcal{S}}_{\text{rot}} - \tilde{\mathcal{S}} = -\int d^2x[\beta(x)\tilde{d}_{\mu}j^{5,\mu} - \kappa(x)\tilde{d}_{\mu}j^{\mu}], \tag{22}$$

where the chiral and vector currents are $j^{5,\mu} = \bar{\Psi}\gamma^{\mu}\gamma^5\Psi$ and $j^{\mu} = \bar{\Psi}\gamma^{\mu}\Psi$, respectively. To enforce the invariance of $\tilde{\mathcal{Z}}$ under the chiral and vector rotations, we differentiate the partition function with respect to β and κ and obtain the anomaly equations $\tilde{\mathcal{A}}^5 = i\tilde{d}_{\mu}j^{5,\mu}$ and $\tilde{\mathcal{A}} = -i\tilde{d}_{\mu}j^{\mu}$.

After analytical continuation $\tau \to it$, we obtain the divergence of currents in the Minkowski space. Using the elements of M and V, we rewrite the divergence of chiral currents as

$$\tilde{d}_{\mu}j^{5,\mu} \propto v_1\tilde{E}_1^{\dagger} + v_1^*\tilde{E}_1 \quad \text{in } d = 2, \tag{23}$$

$$\tilde{d}_{\mu}j^{5,\mu} \propto v_1v_2v_3(\tilde{E}^{\dagger}\cdot\tilde{B}^{\dagger} + \tilde{E}^{5\dagger}\cdot\tilde{B}^{5\dagger}) + v_1^*v_2^*v_3^*(\tilde{E}\cdot\tilde{B} + \tilde{E}^5\cdot\tilde{B}^5) \quad \text{in } d = 4. \tag{24}$$

Here, the complex electric fields read $\tilde{E}_j = (\exp[2i\phi_j]\partial_tV_j - \partial_jV_0)$ and $\tilde{E}^5_j = (\exp[2i\phi_j]\partial_tW_j - \partial_jW_0)$, where $\exp[i\phi_j] = v_j/|v_j|$ with $i, j, k \neq t$ being a spatial index. Similarly, the complex magnetic field casts $\tilde{B}^i = \varepsilon^{ijk}\tilde{B}_{jk}$ and $\tilde{B}^{5,i} = \varepsilon^{ijk}\tilde{B}^5_{jk}$ with $\tilde{B}_{jk} = \exp[2i\phi_k]\partial_jV_k - \exp[2i\phi_j]\partial_kV_j$ and $\tilde{B}^5_{jk} = \exp[2i\phi_k]\partial_jW_k - \exp[2i\phi_j]\partial_kW_j$. It is notable that $\tilde{d}_{\mu}j^{5,\mu}$ in Eqs. (23) and (24) obey the common form of the chiral anomaly reported in Hermitian systems in the absence [44] or presence [51] of interactions.

The non-Hermitian anomaly $[\tilde{d}_{\mu}j^{5,\mu}]_{\text{nh}}$ is different from the naive summation of $[\tilde{d}_{\mu}j^{5,\mu}]_{\text{h}} + [\tilde{d}_{\mu}j^{5,\mu}]_{\text{ah}}$. One can see this difference by noticing, for example, that the prefactor $\tilde{\mathcal{F}}_2$ in (15) [or $\tilde{\mathcal{F}}_4$ in Eq. (B73)] is different for all three cases. This is the main result of this Letter (see Fig. 1).

A Chern-Simons description of non-Hermitian Weyl semimetals. To explore some physical consequences of non-Hermitian anomalies, we explore a situation in which the chiral gauge field W is absent. In this case, the anomaly-induced action, given in Eq. (14), in real-time representation casts a Chern-Simons action as

$$\tilde{\mathcal{S}}^5[\beta] = \int dtd^3x\frac{4\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}\tilde{d}_{\mu}\beta(x)\tilde{V}^{\dagger}_{\nu}\tilde{d}_{\eta}\tilde{V}^{\dagger}_{\zeta} + \int dtd^3x\frac{4\varepsilon^{\mu\nu\eta\zeta}}{\tilde{\mathcal{F}}_4}\tilde{d}^{\dagger}_{\mu}\beta(x)\tilde{V}_{\nu}\tilde{d}^{\dagger}_{\eta}\tilde{V}_{\zeta}, \tag{25}$$

where we have performed an integration by parts and dropped a total derivative term. We then obtain the associated current as a sum of the functional derivatives of $\tilde{\mathcal{S}}^5[\beta]$ with respect to both V and $V^{\dagger}$, which results in

$$M^{\alpha}_{\nu}j^{\nu} = \frac{8\varepsilon^{\mu\nu\eta\zeta}\partial_{\delta}\beta}{\tilde{\mathcal{F}}_4}\text{Re}\big[M^{\alpha *}_{\nu}M^{\delta}_{\mu}M^{\rho}_{\zeta}\tilde{E}^{\dagger}_{\rho}\big], \quad \mu = 1, 2, 3, \tag{26}$$

$$M^{\alpha}_{\nu} j^{\nu} = \frac{8\varepsilon^{0\nu\eta\zeta}\partial_t\beta}{\tilde{\mathcal{F}}_4}\mathrm{Re}\big[M^{\alpha *}_{\nu} M^{\iota}_{\eta} M^{\rho}_{\zeta} \tilde{B}^{\dagger}_{\iota\rho}\big], \tag{27}$$

where the complex electric ($\tilde{E}$) and magnetic fields ($\tilde{B}$) are defined below Eq. (24). In the limit where $V \in \mathbb{R}$ and $M = \mathbb{1}_{4\times4}$, Eq. (26) represents the Hermitian anomalous Hall effect and Eq. (27) coincides with the Hermitian chiral magnetic effect [52]. $\partial_0\beta$ and $\partial_\delta\beta$ can be associated to the energy and spatial separation of Weyl nodes, respectively, which for a non-Hermitian Weyl semimetal can be complex valued. As a result, one can view the complex currents in Eqs. (26) and (27) as a representation of the non-Hermitian anomalous Hall effect and non-Hermitian chiral magnetic effect.

Assigning $\boldsymbol{j}$ as the polarization current as $\boldsymbol{j} = \partial_t \boldsymbol{P}$ where $\boldsymbol{P}$ denotes the electric polarization, Eq. (27) simplifies to

$$M^{\alpha}_{\nu} P^{\nu} = \frac{8\varepsilon^{0\nu\eta\zeta}\beta}{\mathcal{F}_4}\mathrm{Re}\big[M^{\alpha *}_{\nu} M^{\iota}_{\eta} M^{\rho}_{\zeta} \tilde{B}^{\dagger}_{\iota\rho}\big]. \tag{28}$$

When $\beta = \pi$, which can occur in time-reversal-invariant topological insulators [53], the quantized magnetoelectric effect of Hermitian systems, written as $\boldsymbol{P} = e^2\boldsymbol{B}/4\pi$, is modified to

$$M^{\alpha}_{\nu} P^{\nu} = \frac{\varepsilon^{0\nu\eta\zeta}}{4\pi\sqrt{\det[B]}}\mathrm{Re}\big[M^{\alpha *}_{\nu} M^{\iota}_{\eta} M^{\rho}_{\zeta} \tilde{B}^{\dagger}_{\iota\rho}\big], \tag{29}$$

in non-Hermitian systems. Such a modified magnetoelectric polarization will modify the electromagnetic responses of non-Hermitian topological insulators, including image monopole charges [54] or the Casimir effect [55]. In the SM, we study the Witten effect as an example. The polarization in Eq. (29) will change the induced electric charge created by a magnetic monopole from $e/2$ of the Hermitian system [56], to an arbitrary value (see SM) that depends on the phase of the complex Fermi velocity.

One-loop diagrammatic calculation of the $(1+1)$-*dimensional anomaly.* We also derive the chiral anomaly for the (anti)-Hermitianized, and non-Hermitian actions in $1+1$ dimensions using the diagrammatic method (see the SM [45]). Within this approach, we integrate out the fermionic degrees of freedom from the underlying partition function, which results in an effective action for the gauge fields. The functional integration of an Hermitian action, with a Lagrangian density $\mathcal{L} = \Psi^{\dagger}(i\gamma^0 \not{\mathcal{D}})\Psi$, requires an orthogonal eigenbasis of the self-adjoint operator $(i\gamma^0\not{\mathcal{D}})$ to yield the effective action $\Gamma[V, W] = -i\ln[\det(i\gamma^0\not{\mathcal{D}})]$. By using the product rules of determinants and logarithms, and omitting any constant vacuum contributions, $\Gamma[V, W]$ is rewritten as $\Gamma[V, W] = -i\ln[\det(i\not{\mathcal{D}})]$. When the operator $(i\gamma^0\not{\mathcal{D}})$ is not self-adjoint, one should instead use the corresponding biorthogonal eigenbasis to evaluate the functional integral. To take advantage of the well-developed Hermitian field theory and to avoid the complexities of finding functional determinants from a biorthogonal basis, we construct the self-adjoint operator $\not{\mathcal{D}}\not{\mathcal{D}}^{\dagger}_{\mathcal{L}}$, where $\not{\mathcal{D}}^{\dagger}_{\mathcal{L}} \equiv \gamma^{\mu}\mathcal{D}^{\dagger}_{\mu}$. Formally, this is done by considering the sum of the effective actions with respect to the operators $i\gamma^0\not{\mathcal{D}}$ and $i\gamma^0\not{\mathcal{D}}^{\dagger}_{\mathcal{L}}$ and using properties of the determinant and the logarithm to obtain $\Gamma[V, W] = -\frac{i}{2}\ln[\det(\not{\mathcal{D}}\not{\mathcal{D}}^{\dagger}_{\mathcal{L}})]$ (see SM [45]). For the various actions given in Eqs. (2) and (5), we expand this effective action up to the second order in the gauge fields. These calculations give rise to divergent momentum integrals which are treated with gauge-invariant regularization methods [57].

The vector (chiral) currents are defined as the sum of the functional derivatives of the effective action with respect to V and $V^{\dagger}$ (W and $W^{\dagger}$), from which the expressions for divergence of the vector and chiral currents (terms first order in the gauge fields), e.g., in Eq. (23), follow.

Summary. We have found non-Hermitian anomalies in massless Dirac fermions with complex velocities coupled to non-Hermitian gauge fields. We have presented a unified non-Hermitian formulation to bring the Hermitianized, anti-Hermitianized, and non-Hermitian cases under one umbrella. Our results show that non-Hermiticity allows different anomalous terms in the conservation laws for the chiral current in both in two and four dimensions. Interestingly, these anomalous terms could not be inferred by simply adding the Hermitianized and anti-Hermitianized results, as would be expected classically (see Fig. 1). In this sense, these anomalies are different and richer than those that occur in Hermitian systems. We further demonstrate this point by presenting the non-Hermitian anomalous Hall effect and non-Hermitian chiral magnetic effect.

Finally, we note that paritylike anomalies may exist in non-Hermitian systems. Exploring them might give a different interpretation to the nonuniversality of the Hall response in non-Hermitian Chern insulators [58].

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Sh.S. carried out the path-integral and bosonization calculations and proposed the microscopic model with input from all authors. J.D.H. carried out the diagrammatic calculations with input from all authors. Sh.S. drafted the manuscript, to which all authors contributed. A.G.G. conceived and supervised the project.

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[46] We emphasize that the gauge field W is not subject to a gauge principle, but this does not render it less physical. For example, if the Weyl separation is allowed to be a complex quantity, as in a non-Hermitian system, its variations over time and space will lead to the complex electric and magnetic fields.

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