Letter

Unifying the Anderson transitions in Hermitian and non-Hermitian systems

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Non-Hermiticity enriches the tenfold Altland-Zirnbauer symmetry class into the 38-fold symmetry class, where critical behavior of the Anderson transitions (ATs) has been extensively studied recently. Here, we propose a correspondence of the universality classes of the ATs between Hermitian and non-Hermitian systems. We illustrate that the critical exponents of the length scale in non-Hermitian systems coincide with the critical exponents in the corresponding Hermitian systems with additional chiral symmetry. A remarkable consequence of the correspondence is superuniversality, i.e., the ATs in some different symmetry classes of non-Hermitian systems are characterized by the same critical exponent. In addition to the comparisons between the known critical exponents for non-Hermitian systems and their Hermitian counterparts, we obtain the critical exponents in symmetry classes AI, AII, AII†, CII†, and DIII in two and three dimensions. Estimated critical exponents are consistent with the proposed correspondence. According to the correspondence, some of the exponents also give useful information of the unknown critical exponents in Hermitian systems, paving a way to study the ATs of Hermitian systems by the corresponding non-Hermitian systems.

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Introduction. Scattering, transmission, and interference of waves in dissipative media lead to a rich variety of physical phenomena. A prime example is localization, where a propagating wave and its counterpropagating wave caused by scattering form a standing wave. After Anderson's seminal work [1], which predicted delocalization-localization transitions of electron wave functions in disordered solids, a general scaling theory of localization was introduced [2,3]. Subsequent development of field theory descriptions, as well as renormalization-group analyses, clarified the universality classes of the Anderson transitions (ATs) in three fundamental symmetry classes of time-reversal symmetry: Wigner-Dyson classes [4,5]. Furthermore, chiral symmetry [6,7] and particle-hole symmetry enrich the universality classes into the tenfold symmetry classification [8].

Like other continuous phase transitions, a universality class of the ATs is characterized by scaling properties of an effective theory. Based on the single-parameter-scaling hypothesis, the critical exponents of the ATs in the ten symmetry classes have been numerically studied [9–24]. It is commonly believed that the universality classes are determined solely by spatial dimension and symmetry, being independent from details of Hamiltonians. In some cases, two distinct symmetry classes share the same scaling property, which is called superuniversality [25–31]. Superuniversality can be numerically observed by precisely determining critical exponents and other universal scaling properties. In this Research Letter, we show that superuniversality emerges also in non-Hermitian disordered systems.

Recently, the ATs in non-Hermitian disordered systems have attracted considerable research interest [32–45]. Non-Hermitian disordered systems describe random media with amplification or dissipation, which include open classical systems [46–50], as well as quantum systems of quasiparticles with finite lifetime [51–54]. In contrast to Hermitian systems, non-Hermitian systems are classified into 38 symmetry classes [55–57]. However, universality classes of the ATs in these 38 symmetry classes have yet to be understood clearly.

In this Research Letter, we propose a correspondence between the ATs in Hermitian systems and those in non-Hermitian systems and develop a unified understanding about the ATs. We argue that the critical behavior of the length scale in non-Hermitian systems is identical to the critical behavior in the corresponding Hermitian system with additional chiral symmetry. To examine the proposed correspondence, we carry out extensive numerical studies of critical exponents in non-Hermitian disordered systems. In particular, we study in this Research Letter the universal critical behavior of the ATs for

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TABLE I. Critical exponents ν and normalized localization lengths Λ_c at the Anderson transitions for non-Hermitian symmetry classes (NHSCs) in three dimensions (3D) and two dimensions (2D). Non-Hermitian Hamiltonians $\mathcal{H}$ are classified by time-reversal symmetry (TRS) $\mathcal{U}\mathcal{H}^*\mathcal{U}^\dagger = \mathcal{H}$, particle-hole symmetry (PHS) $\mathcal{U}\mathcal{H}^T\mathcal{U}^\dagger = -\mathcal{H}$, time-reversal symmetry† (TRS†) $\mathcal{U}\mathcal{H}^T\mathcal{U}^\dagger = \mathcal{H}$, particle-hole symmetry† (PHS†) $\mathcal{U}\mathcal{H}^*\mathcal{U}^\dagger = -\mathcal{H}$, chiral symmetry (CS) $\mathcal{U}\mathcal{H}^\dagger\mathcal{U}^\dagger = -\mathcal{H}$, and sublattice symmetry (SLS) $\mathcal{U}\mathcal{H}\mathcal{U}^\dagger = -\mathcal{H}$, with unitary matrices $\mathcal{U}$. In classes AI, AI†, AII, AII†, and AIII, $\mathcal{H}$ respects TRS with $\mathcal{U}\mathcal{U}^* = +1$, TRS† with $\mathcal{U}\mathcal{U}^* = +1$, TRS with $\mathcal{U}\mathcal{U}^* = -1$, TRS† with $\mathcal{U}\mathcal{U}^* = -1$, and CS, respectively. In classes CII†, $\mathcal{H}$ respects TRS† with $\mathcal{U}\mathcal{U}^* = -1$, PHS† with $\mathcal{U}\mathcal{U}^* = -1$, and CS. In class DIII, $\mathcal{H}$ respects TRS with $\mathcal{U}\mathcal{U}^* = -1$, PHS with $\mathcal{U}\mathcal{U}^* = 1$, and CS. The symmetry class depends not only on the symmetry of $\mathcal{H}$ but also on the eigenvalue E. For example, when the symmetry of $\mathcal{H}$ is that of the non-Hermitian symmetry class AI and $E \neq E^*$, the symmetry class is the non-Hermitian class A. For comparison, critical exponents for the corresponding Hermitian symmetry classes (HSCs) are also listed. The square brackets denote the 95% confidence intervals estimated by the Monte Carlo simulation. Note that we found a significant discrepancy in critical exponents between 3D Hermitian class BDI and 3D non-Hermitian class AI.

$\mathcal{H}$	Energy	NHSC	ν	Λ_c	HSC	ν
				3D		
A	E	A	1.00 ± 0.04[a] [45]	0.598 [0.593, 0.605] [45]	AIII	1.06 ± 0.02[b] [17]
AI†	E	AI†	1.19 ± 0.01[a] [45]	0.837 [0.835, 0.839] [45]	CI	1.17 ± 0.02[b] [17], 1.16 ± 0.02[a] [18]
AI	$E = E^*$	AI	0.933 [0.799, 1.041][c]	0.269 [0.259, 0.293][c]	BDI	1.12 ± 0.06[b] [17], 0.80 ± 0.02[a] [18]
AII	$E = E^*$	AII	0.8745 [0.8710, 0.8783][c]	0.936 [0.935, 0.937][c]	CII	unknown
AII†	E	AII†	0.903 [0.896, 0.908][c]	0.581 [0.576, 0.586][c]	DIII	0.85 ± 0.05 [15]
				2D		
AII	$E \neq E^*$	A	1.562 [1.524, 1.609][c]	1.290 [1.276, 1.303][c]	AIII	unknown
AII	$E = E^*$	AII	no AT found[c]	no AT found[c]	CII	unknown
AII†	E	AII†	1.377 [1.331, 1.439][c]	0.48 [0.29, 0.61][c]	DIII	1.5 ± 0.1 [58], ≈2.0 [24]
AIII	$E = -E^*$	AIII	2.7 ± 0.1[b] [37]	unknown	A	2.59 ± 0.01[a] [12][d]
CII†	$E = 0$	CII†	2.740 [2.706, 2.773][c]	1.852 [1.848, 1.855][c]	AII	2.75 ± 0.04[a] [14][d]
DIII	$E = 0$	DIII	2.757 [2.726, 2.788][c]	1.852 [1.847, 1.855][c]	AII	2.75 ± 0.04[a] [14][d]

[a]Double standard deviation 2σ.
[b]Standard deviation σ.
[c]This Research Letter.
[d]$\Lambda_c = 1.284$ [1.268, 1.305] for class A; $\Lambda_c = 1.844 \pm 0.004$ for class AII.

non-Hermitian models in classes AI, AII, AII†, CII†, and DIII in two dimensions (2D) and three dimensions (3D). We calculate the localization lengths of these models by the transfer matrix method, analyze them by the finite-size scaling [45], and determine values of the critical exponents of the ATs, as summarized in Table I. Combining with the critical exponents for classes A and AI† previously obtained in Refs. [44,45], we show that the critical exponents in these non-Hermitian symmetry classes are consistent with the known critical exponents in the corresponding Hermitian symmetry classes, supporting the correspondence of the ATs between Hermitian and non-Hermitian systems. Notably, estimated critical exponents in some non-Hermitian systems also provide useful information of critical behavior in Hermitian symmetry classes with chiral or particle-hole symmetry, where the critical exponents were previously difficult to estimate.

Unified universality classes. Our correspondence of the ATs between Hermitian and non-Hermitian systems is based on Hermitization [33,34,38,56,59]. A non-Hermitian Hamiltonian $\mathcal{H}$ with complex energy $E \in \mathbb{C}$ is mapped to the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ by

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & \mathcal{H} - E \\ \mathcal{H}^\dagger - E^* & 0 \end{pmatrix}. \quad (1)$$

By construction, the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ respects additional chiral symmetry $\tau_z \tilde{\mathcal{H}} \tau_z = -\tilde{\mathcal{H}}$. Let $|\phi_r\rangle$ and $|\phi_l\rangle$ be a right eigenmode and a left eigenmode of the non-Hermitian Hamiltonian $\mathcal{H}$ with eigenenergy E, respectively: $\mathcal{H}\,|\phi_r\rangle = E\,|\phi_r\rangle$ and $\mathcal{H}^\dagger\,|\phi_l\rangle = E^*\,|\phi_l\rangle$. Then, $(0 \;\; |\phi_r\rangle)^{\rm T}$ and $(|\phi_l\rangle \;\; 0)^{\rm T}$ comprise doubly degenerate zero modes of the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ [i.e., $\tilde{\mathcal{H}}(0 \;\; |\phi_r\rangle)^{\rm T} = \tilde{\mathcal{H}}(|\phi_l\rangle \;\; 0)^{\rm T} = 0$]. This is the Hermitization, which associates the non-Hermitian Hamiltonian $\mathcal{H}$ with the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ with chiral symmetry. Hermitization is relevant to non-Hermitian random matrices [33] and topological phases [38,56], as well as topological characterization [59,60] of the anomalous boundary physics due to non-Hermiticity (i.e., the non-Hermitian skin effect [61–63]). However, the significance of Hermitization has been unclear for the ATs.

We argue that Hermitization unifies the ATs in Hermitian and non-Hermitian systems. The ATs are continuous phase transitions that are characterized by the universal scaling properties of the localization lengths. As shown above, eigenmodes of $\mathcal{H}$ and the corresponding zero modes of $\tilde{\mathcal{H}}$ share the same spatial profiles, including the localization lengths. Therefore the universal scaling properties of the localization lengths in non-Hermitian systems, as well as the absence or presence of the ATs, are generally the same as those in their Hermitian counterparts. It is to be noted that the right eigenmode $|\phi_r\rangle$ and the corresponding left eigenmode $|\phi_l\rangle$ exhibit similar localization properties with the same localization length, since they correspond to zero modes in the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ with opposite chiralities. We also note that, although the Hermitization procedure always maps non-Hermitian Hamiltonians to Hermitian Hamiltonians with chiral symmetry, nonchiral symmetry classes can appear in the Hermitized Hamiltonians. Even if the Hermitized

Hamiltonians respect chiral symmetry, they can respect additional unitary symmetry and then be block diagonalized. In such a case, the relevant symmetry classes (or equivalently, classifying spaces) are not necessarily chiral classes [64].

For several non-Hermitian symmetry classes, we summarize the correspondence in Table I (see Ref. [56] and the Supplemental Material [64] for the correspondence of all the 38 symmetry classes). For these classes in 2D and 3D in Table I, we illustrate the correspondence by numerical evaluations of the critical exponents, as shown below.

Model and symmetry class. To study the AT in class AI, we introduce the following O(1) tight-binding model on a 3D cubic lattice:

$$\mathcal{H} = \sum_i \varepsilon_i c_i^\dagger c_i + \sum_{\langle i,j\rangle} V_{i,j} c_i^\dagger c_j, \tag{2}$$

where ε_i is the random potential characterized by the uniform distribution in $[-W/2, W/2]$ with the disorder strength W. Here, $\langle i, j\rangle$ denotes nearest-neighbor lattice sites. $V_{i,j}$ is set to either -1 or $+1$ randomly with equal probability, and $V_{i,j}$ and $V_{j,i}$ are treated as independent random numbers. Hermiticity is broken because of $V_{i,j}^* \neq V_{j,i}$, and reciprocity is absent in each disorder realization ($\mathcal{H}^{\rm T} \neq \mathcal{H}$). Still, $\mathcal{H}$ is statistically reciprocal in the sense that $\mathcal{H}$ and $\mathcal{H}^{\rm T}$ appear with equal probability in the ensemble. Eigenstates of $\mathcal{H}$ at real and complex energy E belong to non-Hermitian symmetry classes AI and A, respectively. For the real and complex E, the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ belongs to symmetry classes BDI and AIII, respectively.

To study the ATs in classes AII, AII†, CII†, and DIII, we introduce the following non-Hermitian extension of the SU(2) model [13,14,65] on 2D square and 3D cubic lattices:

$$\mathcal{H} = \sum_{i,\sigma} \varepsilon_{i,\sigma} c_{i,\sigma}^\dagger c_{i,\sigma} + \sum_{\langle i,j\rangle,\sigma,\sigma'} R(i,j)_{\sigma,\sigma'} c_{i,\sigma}^\dagger c_{j,\sigma'}, \tag{3}$$

with $\sigma = \uparrow, \downarrow$. The spin-dependent nearest-neighbor hoppings are parametrized by the SU(2) matrix

$$R(i,j) = \begin{pmatrix} e^{i\alpha_{i,j}}\cos(\beta_{i,j}) & e^{i\gamma_{i,j}}\sin(\beta_{i,j}) \\ -e^{-i\gamma_{i,j}}\sin(\beta_{i,j}) & e^{-i\alpha_{i,j}}\cos(\beta_{i,j}) \end{pmatrix}, \tag{4}$$

where i in front of $\alpha_{i,j}$ and $\gamma_{i,j}$ is the imaginary unit, $\alpha_{i,j}$ and $\gamma_{i,j}$ are uniformly distributed in $[0, 2\pi)$, and $\beta_{i,j}$ is distributed in $[0, \pi/2]$ according to the probability density $P(\beta)d\beta = \sin(2\beta)d\beta$. The hopping terms satisfy $R^\dagger(i,j) = R(j,i)$ for classes AII, AII†, and CII† ($\alpha_{i,j} = -\alpha_{j,i}$, $\gamma_{i,j} = \gamma_{j,i} + \pi$), while they satisfy $\sigma_z R^\dagger(i,j)\sigma_z = -R(j,i)$ for class DIII ($\alpha_{i,j} = -\alpha_{j,i} + \pi$, $\gamma_{i,j} = \gamma_{j,i} + \pi$). The on-site potentials $\varepsilon_{j,\sigma} = \omega_{j,\sigma}^r + i\omega_{j,\sigma}^i$ are complex valued, letting $\mathcal{H}$ be non-Hermitian. The complex-valued potentials are realized in classical optical systems with random amplification and dissipation [66–68]. $\omega_{j,\sigma}^r$ and $\omega_{j,\sigma}^i$ are independent for each site j and are uniformly distributed in $[-W_r/2, W_r/2]$ and $[-W_i/2, W_i/2]$, respectively. A relation between $\varepsilon_{j,\uparrow}$ and $\varepsilon_{j,\downarrow}$, as well as W_r and W_i, is chosen appropriately so that $\mathcal{H}$ will belong to the different symmetry classes among classes AII, AII†, CII†, and DIII [64]. The SU(2) models are reciprocal in classes AII†, CII†, and DIII; the SU(2) model in class AII is reciprocal only statistically, similarly to the O(1) model.

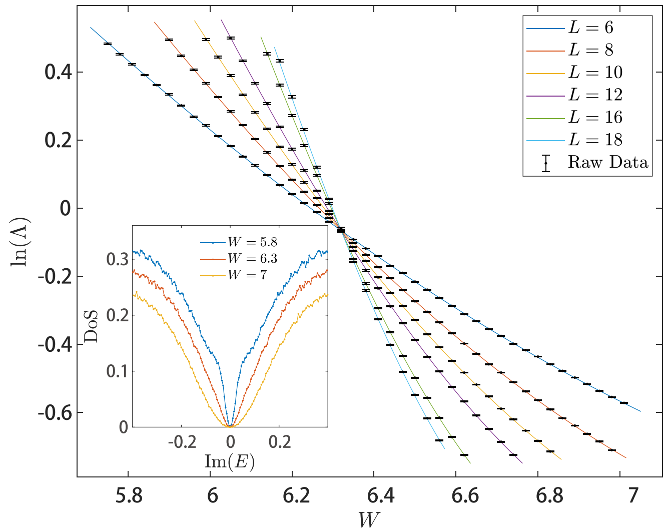

FIG. 1. Normalized localization lengths Λ as a function of the disorder strength $W \equiv W_r = W_i$ for 3D class AII at $E = 0$. The points with the error bars are the numerical data with the different system sizes L. The colored curves are the fitted curves. Inset: density of states (DoS) for the imaginary part of eigenenergies. Eigenenergies in the $16 \times 16 \times 16$ cubic system under periodic boundary conditions are calculated, and the average over the 640 samples is taken.

Transfer matrix study and polynomial fitting. The localization length and conductance of non-Hermitian systems were previously calculated by the transfer matrix method [45]. Thereby, the critical exponents of the ATs in classes A and AI† were determined precisely by the finite-size scaling analysis [9,69,70]. In this Research Letter, the localization lengths for the five symmetry classes are calculated for different complex-valued energies in a quasi-one-dimensional geometry ($L \times L_z$ in 2D and $L \times L \times L_z$ in 3D with $L_z \gg L$). The quasi-one-dimensional localization length $\xi(L)$ along the z direction is normalized by the system size L along the transverse direction. Being dimensionless, the normalized length $\Lambda = \xi(W, L)/L$ shows scale-invariant behavior at the AT as a function of L.

The single-parameter scaling [2,3] has been demonstrated to be successful in analyses of the quantum criticality of the ATs in Hermitian systems [9–24] and in non-Hermitian systems [41,44,45]. Apart from fine-tuned critical points such as multicritical points, critical properties of a generic continuous phase transition must be controlled by a saddle-point fixed point with only one relevant scaling variable. The scaling argument dictates that the dimensionless normalized localization length Λ follows a scaling function that depends on the relevant scaling variable and possibly many other irrelevant scaling variables. The universal critical exponent ν associated with the relevant scaling variable can be estimated based on a polynomial expansion of the scaling function in terms of the scaling variables [9].

Numerical results. The normalized quasi-one-dimensional localization lengths Λ for classes AI, AII, AII†, CII†, and DIII in 2D or 3D are calculated at different complex energies [64]. As an illustration, Fig. 1 shows Λ around the critical point at $E = 0$ for 3D class AII with different system sizes L and disorder strength W. As L increases, Λ increases below the critical point (delocalized phase) and decreases above the critical point (localized phase). In terms of numerical fitting based on the polynomial expansion [64], universal critical parameters of the ATs in the five non-Hermitian symmetry classes are obtained. The critical exponents ν as well as normalized localization lengths Λ_c at the critical point are summarized in Table I. Fitted critical parameters are confirmed to be stable against changing the system sizes and/or expansion orders [64].

Universal critical exponents of the ATs in the non-Hermitian symmetry classes are mostly consistent with the known exponents in the corresponding Hermitian symmetry classes (Table I). On the other hand, we also found discrepancies in the exponents between the 3D class AI model from Ref. [17,18] and the 2D class AII† model from Ref. [24]. Causes of these deviations are currently under investigation, which will be discussed in a separate paper.

Superuniversality. As a unique feature of non-Hermitian systems, our results show emergent superuniversality of the ATs: Two or more non-Hermitian disordered systems that belong to different symmetry classes in the 38-fold symmetry classification exhibit the same critical behavior of the length scale. In fact, the critical exponent of the 2D SU(2) model in class CII† is identical to that in class DIII (see Table I). In the Hermitian limit, for which parameters giving rise to non-Hermiticity vanish, these two different symmetry classes fall into two different universality classes with different critical exponents. Hence the superuniversality emerges as a consequence of non-Hermiticity.

Furthermore, the correspondence of the ATs in Hermitian and non-Hermitian systems can also be regarded as superuniversality. Hermitian and non-Hermitian systems exhibit distinct transport phenomena, which implies that the underlying effective theories are different. Nevertheless, our results in Table I illustrate that such different effective theories share the same scaling property of the length scale.

Implications as to unknown critical exponents in Hermitian chiral classes. As a by-product of our correspondence, we can provide useful information for critical exponents for unexplored Hermitian symmetry classes that are difficult to estimate. To our best knowledge, the critical exponents for 3D class CII and 2D class AIII are unknown in the Hermitian case. The critical exponents obtained in 3D class AII model with $E = E^*$ and 2D class AII model with $E \neq E^*$, respectively (see Table I), can be the critical exponents of the ATs in these Hermitian chiral symmetry classes, given that the universality classes of the ATs are determined only by spatial dimension and symmetry. Importantly, calculations of non-Hermitian systems are much easier than the Hermitian counterparts because degrees of freedom of minimal non-Hermitian models are often half. We note that the critical localization lengths Λ_c for Hermitian systems are also proposed by the non-Hermitian counterparts summarized in Table I.

The ATs of 2D Hermitian systems have remained elusive in chiral classes (AIII, BDI, and CII) [71–73] because of the vanishing β functions [6,7,74]. We fail to find ATs for our 2D O(1) models in class AI (not shown). Similarly, the 2D non-Hermitian models in Refs. [39,41] exhibit no ATs. On the other hand, for our 2D SU(2) model in class AII with $E \neq E^*$, which corresponds to class AIII in the Hermitian counterpart, we find the AT [64] and estimate the critical exponent (Table I). It could merit further study to investigate the 2D ATs on the basis of our correspondence.

Density of states. The density of states shows characteristic features and also contains relevant information about the ATs. In our non-Hermitian systems, the density of states around the real axis exhibits a sharp peak in class AI [64] and a soft gap in class AII (inset of Fig. 1). This behavior is consistent with the random-matrix behavior in classes AI and AII [34,64,75,76] and originates from the difference of time-reversal symmetry. In class AI, time-reversal symmetry imposes a constraint on each real eigenenergy. Because of this constraint, real eigenenergies remain real unless they are mixed with other real eigenenergies. Consequently, some of them are stable even against non-Hermitian perturbations, forming the sharp peak of the density of states. In class AII, by contrast, time-reversal symmetry creates Kramers pairs with real eigenenergies. In the presence of non-Hermitian perturbations, they are fragile and form complex-conjugate pairs [77]. Hence eigenenergies tend to be away from the real axis, which leads to the soft gap of the density of states. We give other heuristic discussions in the Supplemental Material [64].

Nonreciprocity. In the numerical studies, we focused on statistically reciprocal models as illustrative examples. Nonreciprocity can give rise to unique non-Hermitian topology [38,56] and further change the universal critical properties. Nevertheless, our correspondence of the ATs in Hermitian and non-Hermitian systems should remain valid even in the presence of nonreciprocity since it is based solely on Hermitization. We conjecture that even if nonreciprocity changes critical behavior of the ATs in non-Hermitian systems because of an additional mechanism such as topology, the critical behavior in the corresponding Hermitian systems should also change by the same mechanism and thus coincides with the non-Hermitian counterpart. As an example of this, the ATs in one-dimensional nonreciprocal systems are characterized by $\nu = 1$ [32,43], which coincides with the critical behavior in the corresponding Hermitized systems [78,79]. The conjecture can be argued for the case of symmetry-conserving energy [64]. It is worthwhile to further confirm our correspondence for higher-dimensional nonreciprocal systems.

Summary and concluding remarks. In this Research Letter, we propose a correspondence of the ATs between Hermitian and non-Hermitian systems. The 38-fold non-Hermitian symmetry class is mapped to the tenfold Hermitian symmetry class in terms of the universal scaling properties of the length scale. Consequently, superuniversality emerges in non-Hermitian systems: The ATs in several distinct symmetry classes share the same universal scaling properties around their critical points. To test this correspondence, we study the ATs in classes AI, AII, AII†, CII†, and DIII in 2D and 3D and estimate the critical exponents by the transfer matrix method. The estimated critical exponents are consistent with the correspondence and superuniversality in non-Hermitian disordered systems. From the correspondence, we also provide useful information of the unknown critical exponents for 2D class AIII and 3D class CII in Hermitian systems. Investigating non-Hermitian systems is a new efficient way to study critical behavior of the ATs in Hermitian systems since non-Hermitian matrices are often half the size of the corresponding Hermitian matrices. We note that conformal invariance [80–82] should emerge at the ATs in 2D non-Hermitian systems from the correspondence. The multifractal properties at the ATs [83] in 2D and 3D non-Hermitian systems should also be unified with the Hermitian counterparts.

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