Letter

Time scaling of entanglement in integrable scale-invariant theories

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In two-dimensional isotropic scale-invariant theories, the time scaling of the entanglement entropy of a segment is fixed via the conformal symmetry. We consider scale invariance in a more general sense and show that in integrable theories in which the scale invariance is anisotropic between time and space, parametrized by z, most of the entanglement is carried by the *slow modes*. At early times entanglement grows linearly due to the contribution of the fast modes, before smoothly entering a slow mode regime where it grows forever with $t^{\frac{1}{1-z}}$. The slow-mode regime admits a logarithmic enhancement in bosonic theories. We check our analytical results against numerical simulations in corresponding fermionic and bosonic lattice models and find extremely good agreement. We show that due to the dominance of the slow modes in these non-relativistic theories, local quantum information is scrambled independent of z in a stronger way, compared to their relativistic counterparts.

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I. INTRODUCTION

Understanding the dynamics of entanglement is a central problem on the interface of statistical physics, condensed-matter physics, quantum field theory (QFT), quantum information, and gravitational physics. Among a wide set of theoretical questions tied with this interdisciplinary topic are thermalization and relaxation of many-body systems, dynamics of quantum phase transitions, and evaporation of black holes (see, e.g., Ref. [1]). Besides these theoretical interests, in recent years the revolutionary experiments with cold atoms have made it possible to experimentally probe different features of closed quantum systems (see, e.g., Ref. [2]).

In the QFT context, the conformal symmetry is strong enough to fix the dynamics of entanglement for certain subregions in two-dimensional conformal field theories (CFT) [3,4]. A related question is: how about the dynamics of entanglement in nonrelativistic scale-invariant theories in two (and higher) dimensions? The symmetry groups admitting anisotropic scale-invariance are not powerful enough to fix the dynamics. Despite this fact, we show that it is possible to learn important lessons about entanglement dynamics in (anisotropic) scale-invariant integrable theories, utilizing the celebrated quasiparticle (QP) picture [4,5].

Our main focus is on two-dimensional theories with the dispersion relation

$$\omega = k^z, \tag{1}$$

where z is positive and $z \neq 1$ [6]. These theories are invariant under Lifshitz scaling, which is interesting partially due to the symmetry structure of quantum critical points [7]. Our main focus is on $z > 1$. We study the dynamics of entanglement entropy (EE) and mutual information (MI) followed by a quantum quench. The only relevant scale in this problem is the one in the prequench state, which we denote by m_0. This scale is basically identified with the parameter that we take it to vanishes after the quench. We consider m_0 to be finite in our analysis and all physical quantities are compared with this scale. In our analysis we denote the $k < m_0$ modes as *slow modes* and the $k > m_0$ modes as *fast modes*. The scope of these modes after the quantum quench is illustrated in Fig. 1.

II. DOMINANCE OF THE SLOW MODES

We consider integrable models with the dispersion relation (1) and use the QP picture, uplifted with the integrability knowledge of the final steady state [4,5] to understand the dynamics of entanglement. The EE of a connected interval of length ℓ is given by

$$S(t) = 2t \int_0^{k_\ell^*} dk\, s(k) v(k) + \ell \int_{k_\ell^*}^{\infty} dk\, s(k), \tag{2}$$

where $v(k)$ is the group velocity of the QPs given by $v(k) = \partial\omega/\partial k = z\, k^{z-1}$, $s(k)$ denotes the individual contribution of modes with momentum k to the entropy, and

$$k_\ell^* = \left(\frac{\ell}{2zt}\right)^{\frac{1}{z-1}} \tag{3}$$

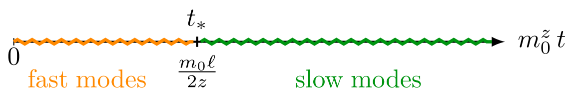

FIG. 1. After the quantum quench at $t = 0$, before and after $t_* \equiv m_0^{1-z}\ell/2z$, the fast and the slow modes carry the entanglement. For $z > 1$ the role of the slow modes is dominant and vice versa for $z < 1$.

is a characteristic momentum corresponding to time t, in which $k > k_\ell^*$ has been saturated before t while $k < k_\ell^*$ still contribute to the time evolution. This k_ℓ^* is a decreasing function of time and $k_\ell^*(t_*) = m_0$. For $t < t_*$ the fast modes only contribute to the dynamics and afterwards the slow modes take this role. The role of the slow modes stands until infinite time, though there is no (finite) saturation time in these theories as apposed to relativistic scale-invariant cases.

In integrable theories, the state of the system will finally relax to a generalized Gibbs ensemble. This fixes $s(k)$ in terms of the expectation value of the number operator in the prequench state as [8]

$$2\pi\, s(k) = -n_k \ln n_k \pm (1 \pm n_k)\ln(1 \pm n_k), \tag{4}$$

where the upper and the lower signs correspond to bosonic and fermionic theories.

Our analysis is quite general for theories with Eq. (1). In order to perform explicit calculations, we consider two family of bosonic and fermionic theories as the prototypes to study anisotropic scale-invariant fixed points. These theories are generalizations of Klein-Gordon and Dirac fermion theories defined as [9]

$$\mathcal{S}_b = \frac{1}{2}\int dt d\vec{x} \{\dot{\phi}^2 - \phi\big[(-\Delta)^z - m^{2z}\big]\phi\}, \tag{5}$$

$$\mathcal{S}_f = \frac{1}{2}\int dt d\vec{x}\, \bar{\Psi}\big(i\gamma^0\partial_t + i\Delta^{\frac{z-1}{2}}\gamma\cdot\partial - m^z\big)\Psi, \tag{6}$$

where the bosonic theory is defined for integer values of z, the fermionic theory is defined for odd values of z, and both theories are invariant under Lifshitz scaling $(t, x) \to (\lambda^z t, \lambda x)$ when $m \to 0$. The explicit forms of $s(k)$ for bosonic and fermionic theories in our case of interest are given by

$$n_{k,b} = \frac{1}{4}\left(\frac{\omega}{\omega_0} + \frac{\omega_0}{\omega}\right) - \frac{1}{2},$$
$$n_{k,f} = \frac{1}{2} - \frac{1}{2}\frac{\omega}{\omega_0}. \tag{7}$$

We explicitly study these theories in two-dimensional spacetime, though the analysis is generalizable to higher dimensions for spherically symmetric entangling regions [10].

III. DYNAMICS OF ENTANGLEMENT ENTROPY

As a warm up, $s(k)$ is depicted in these theories in Fig. 2. Although these nonrelativistic theories are conceptually different from their relativistic counterparts, in which an upper bound exists on the group velocity of the propagating modes [4,11], these density plots show that $s(k)$ enjoys an effective light-cone structure. The entropy is expected to be dominantly affected by the slow modes, populated close to the time axis

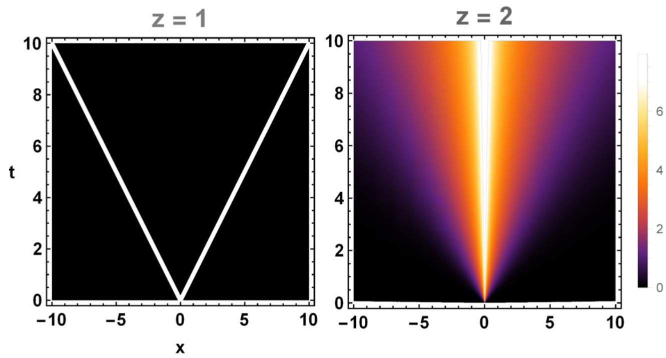

FIG. 2. Density plot of $s(k)$ for bosonic theory followed by a mass quench, on the t-x plane for $z = 1$ and 2. For $z = 1$ all modes propagate on the edge of the light cone. For $z = 2$, $s(k)$ is large for the slow modes while it almost vanishes for the fast modes. We set $m_0 = 1$ in these plots. The structure is similar for $z > 2$. Fermionic theories have the same behavior in a squeezed scale.

[12]. In the following we explicitly analyze the role of fast and slow modes separately and confirm our analysis with numerical checks.

A. Fast modes

The fast modes contribute to the entropy in the early steps of the evolution, namely, for $t < t_*$. The larger the value of z is, the more the velocity of all physical modes is increased; thus, they saturate earlier and this regime becomes shortened. In other words t_* decreases for larger values of z. In this regime we consider the expansion of $s(k)$ around small m_0/k. The crucial point is that for $t \ll t_*$ we have $k_* \gg m_0$, and *all* of the fast modes contribute to the linear growth of the EE through the lower bound of the first integral. Based on this, considering infinite order of this expansion leads to

$$S_{b/f}^{\text{fast}}(t) = \mathfrak{c}_{b/f}\, m_0^z\, t + \cdots, \tag{8}$$

where $\mathfrak{c}_b = (\pi - 2)/4$ and $\mathfrak{c}_f = 1/2$ for bosonic and fermionic theories, respectively. Note that these coefficients are *independent* of z for $z > 1$ and their values are equal to the corresponding relativistic counterparts (see, for instance, Ref. [14]). Such a behavior is expected since the structure of the fast modes in $s(k)$ does not very much depend on z, though one should be careful that the regime of validity of this linear scaling is *shortened* by a factor of $1/z$.

The structure of the ellipsis in Eq. (8) is given by $\mathfrak{m}_0^{2nz}\, \mathfrak{t}^{\frac{1-2nz}{1-z}}$, where $\mathfrak{t} \equiv t/\ell^z$, $\mathfrak{m}_0 \equiv m_0\ell$, and $n = 1, 2, \ldots$ stands for the expansion order. The scaling of these terms are suppressed with the factor of $\ell^{\frac{z-2nz}{z-1}}$. The first-order correction in fermionic theories is given by

$$S_{f,1}^{\text{fast}}(t) = \frac{2z^2\alpha_z \mathfrak{m}_0^{2z}}{(1-2z)}\, \mathfrak{t}^{\frac{1-2z}{1-z}}\left[\mathfrak{f}_1 - z\ln\left(\frac{2^{\frac{1}{z}} z\, \mathfrak{t}}{\mathfrak{m}_0^{1-z}}\right)\right], \tag{9}$$

where $\mathfrak{f}_1 = (1-z)(8z-3)/2(1-2z)$ and $\alpha_z = (2z)^{\frac{1}{1-z}}/(2\pi\, z)$. The same order of correction vanishes in bosonic theories due to the structure of $s(k)$. These subleading terms (mixed up with their counterparts among

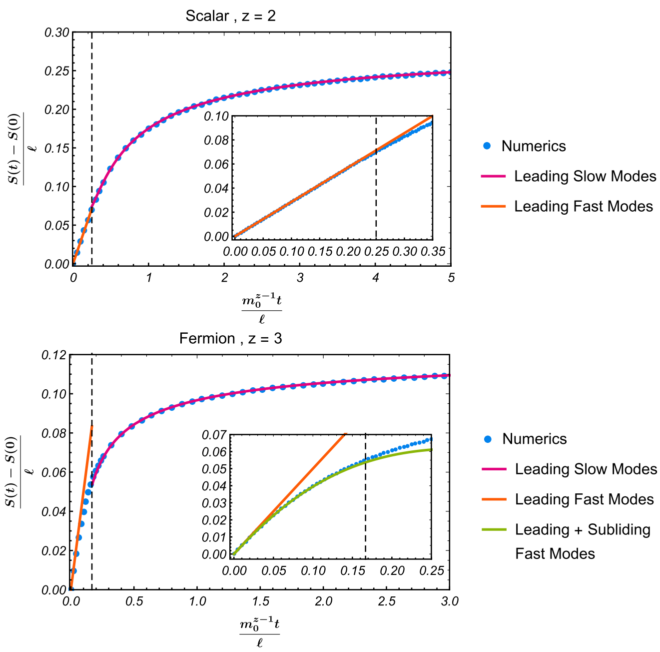

FIG. 3. QP prediction for the evolution of EE in scalar and fermionic theories. The dashed vertical line corresponds to $t = t_*$. The inner panels show the same graph with focus on $t < t_*$. We have set $m_0 = 1$ in both plots and consider $-\infty < k < +\infty$. We set $M_b = 0.7$ and $M_f = 0.95$. In the fast-mode regime of the scalar theory, the linear term well approximates almost all the $t < t_*$ region, though in the fermionic theory we need to add at least one higher-order correction. The structure of these plots is the same for any higher values of z.

the slow modes) provide a smooth transition between these two regimes around $t \sim t_*$.

B. Slow modes

The effect of the slow modes *drastically* changes the story of entanglement propagation in theories with $z > 1$ compared to relativistic scale-invariant theories, $z = 1$. Due to the comparatively large $s(k)$ of the slow modes, they carry most of the entanglement in these theories, starting to contribute from $t = t_*$ and standing until infinite time [15]. In this regime, the first-order expansion around small k/m_0 well approximates $s(k)$ for $k < M_{b/f} \lesssim m_0$. With this approximation we find

$$S_b^{\rm slow}(t) = \mathfrak{b}_1\, \ell\, M_{\rm b} + \alpha_z\, \mathfrak{t}^{\frac{1}{1-z}}\left[\mathfrak{b}_2 - z\ln\left(\frac{2z\,\mathfrak{t}}{\mathfrak{m}_0^{1-z}}\right)\right] + \cdots ,$$
$$S_f^{\rm slow}(t) = \frac{\ln 2}{2\pi}\,\ell\, M_{\rm f} + \ln 2\,\alpha_z(1-z)\,\mathfrak{t}^{\frac{1}{1-z}} + \cdots , \tag{10}$$

where $2\pi\,\mathfrak{b}_1 = \ln(m_0^z e^{z+1}/4M_{\rm b}^z)$ and $\mathfrak{b}_2 = 1 - z^2 + (z-1)(\ln 4 - 1)$. This $\mathfrak{t}^{\frac{1}{1-z}}$ function presents the scaling of EE during *most* of the evolution, namely, $t_* < t < \infty$. Note that the logarithmic term in the bosonic theories is originated from the logarithmic divergence of $s(k)$ of the very slow modes in these theories [16].

In Fig. 3 we show how our results in Eqs. (8), (9), and (10) lie on the top of the exact (numerically found) values of Eq. (2). The good approximation for the slow-mode regime in bosonic models is due to the blow up of $s(k)$ for the very slow modes, and in the fermionic models it is due to the slowly varying nature of $s(k)$ in the slow-mode regime. Though Eq. (10) works quite well, it is straightforward to work out the higher orders at which their structure is given as $\mathfrak{m}_0^{2nz}\,\mathfrak{t}^{\frac{1+2nz}{1-z}}$, with $n = 1, 2, \ldots$ The exact structure of the entropy around $t \approx t_*$ is complicated due to the mixture of the subleading effects of both regimes.

C. Numerical results

In this section we report how numerical results for the EE in the vacuum state of the bosonic and fermionic theories match with the aforementioned analytic predictions. To find these numerical results we use the correlation matrix method for Gaussian states [18] to study quantum quenches from a massive theory to a scale-invariant theory. We consider regularized versions of Eqs. (5) and (6) on an infinite lattice given by (see Ref. [19] and the Supplemental Material [20] for this Letter)

$$H_b = \frac{1}{2}\sum_n \left[\pi_n^2 + (\partial^z\phi_n)^2 + m^{2z}\phi_n^2\right], \tag{11}$$

$$H_f = \sum_n \left[-\frac{i}{2}\left(\Psi_n^\dagger\gamma^0\gamma^1\partial^z\Psi_n - \text{H.c.}\right) + m^z\Psi_n^\dagger\gamma^0\Psi_n\right], \tag{12}$$

where $\partial^z f_n = \sum_{k=0}^{z}(-1)^{z-k}{}_zC_k\, f_{n+k}$ and ${}_zC_k$ is the binomial coefficient [21]. Numerical results corresponding to a single interval are presented in Fig. 4. We find a very good agreement between numerical results and the QP predictions for a much wider family of parameters which we have not presented here. The time scaling of EE after boundary state quench and the entropy production in bosonic and fermionic Gaussian thermofield double states have been also found to be in a very good agreement with the QP predictions [10].

We would like to note that, besides our analytic results found from integrability conditions, the $\mathfrak{t}^{\frac{1}{1-z}}$ scaling can be justified semianalytically, via combining direct analytic calculations for small subsystems with numerical results, without using integrability conditions [10].

IV. AN IMPLICATION ON SCRAMBLING OF LOCAL QUANTUM INFORMATION

As an important implication of the dominance of slow modes, we focus on MI as an important correlation measure, the dynamics of which quantifies how local quantum information scrambles (spreads) over larger subregions [23,24]. In CFTs, although MI generally depends on the full spectrum of the theory, the pattern of its time evolution is well known. Putting aside the cases of very large central charges where the QP picture fails [26], in these theories (even more generally in any integrable theory in which most of the entanglement is carried by the fastest QP), MI exhibits a peak at some finite time. More precisely, when there is an upper bound on the QP velocities, MI starts to rise after a certain time and peaks at $t_p = (d + \ell)/(2v_m)$, where v_m is the velocity of the fastest mode, ℓ denotes the subregions' size, and d is the separation

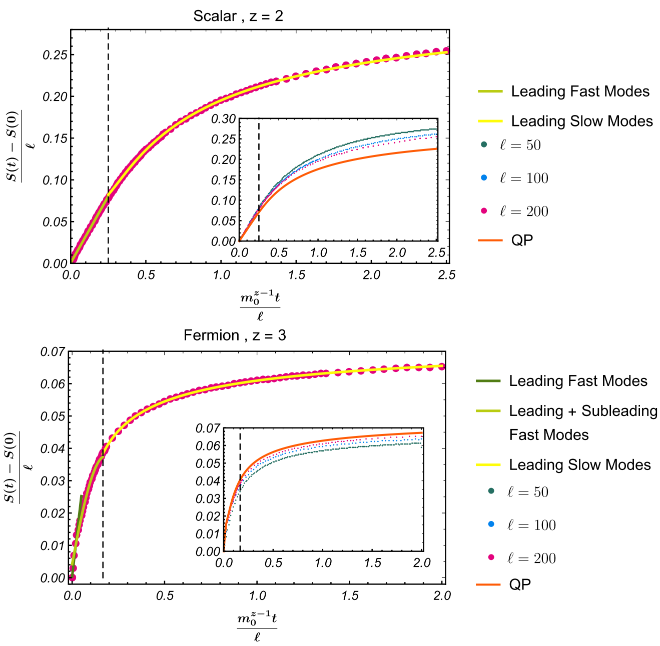

FIG. 4. Numerical results for the evolution of EE in scalar and fermionic theories. The inner panels show how numerics approach QP prediction as the system size is increased. We have set $m_0 = 1/2$ in the upper panel and $m_0 = 1/4$ in the lower panel. We have used an infrared (IR) cutoff of $m = 10^{-5}$ for the scalar theory. These numerics are found on a lattice where $-1 < k < +1$. The fitted functions for the scalar case are $0.32\,t$ (green) and $0.32 - t^{-1}(0.125 + 0.047\ln t)$ (yellow). The fitted functions for the fermionic case are $0.31\,t - 1.59\,t^{5/2}$ (green) and $0.076 - 0.016\,t^{-1/2}$ (yellow).

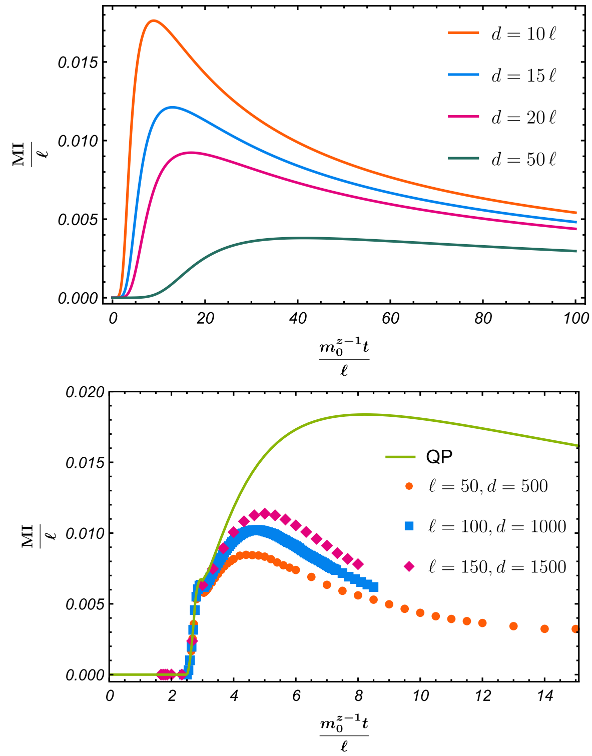

FIG. 5. MI for scalar theory (the structure is very similar to that for fermionic theories) for $z = 2$. The upper panel shows QP prediction for the existence of the peak and how it decays with d. In the lower panel we show the agreement between numerical calculations and QP analysis. We show how the numerics approach the QP prediction in $\ell \to \infty$ for $d/\ell = 10$. The tiny peak on the horizontal axis slightly before ~ 3 is a lattice effect.

between them. Moreover, MI starts to decay to zero after this peak.

It may seem that for $z > 1$, since there is no upper bound on the velocity of the propagating modes, MI should instantly peak and then start to decay. As we have shown in the previous section, the slow modes carry most of the entanglement, so the story is different with mostly known cases including CFTs where $z = 1$. In theories obeying Eq. (1), the MI starts to grow very slowly right after the quench (due to the very fast modes which carry a tiny amount of entanglement) and smoothly starts to rise significantly after the slower modes start to contribute. There is a peak due to the effect of the slow modes, and afterwards the peak decays *slowly*.

We analyze the dynamics of MI similar to the previous section. The QP expression for MI [5] can be expressed as

$$I(t) = \int_{k_d^*}^{k_{d+\ell}^*} dk\, s(k)\left(vt - \frac{d}{2}\right) + \int_{k_{d+\ell}^*}^{k_{d+2\ell}^*} dk\, s(k)\left(\frac{d}{2} + \ell - vt\right). \tag{13}$$

Since the whole resulting expression may not be informative, we only present it for far apart subregions, namely, $d \gg \ell$, where the peak of MI occurs at $m_0^z t_p \approx \mathfrak{r}_{b/f}^{z-1}\frac{m_0 d}{2z}$, where $\mathfrak{r}_b = (4e^{z-1})^{\frac{1}{z}}$ and $\mathfrak{r}_f = (\frac{2z+1}{\ln 4})^{\frac{1}{2z}}$. MI at the peak is given by

$$I(t_p) \approx \mathfrak{g}_{b/f}\,\frac{z}{(z-1)}\,\frac{\ell^2 m_0}{d}, \tag{14}$$

where $\mathfrak{g}_b = 1/\mathfrak{r}_b$ and $\mathfrak{g}_f = 1/\mathfrak{r}_f^{2z+1}$. The MI peak decays with d^{-1}, which is stronger than the relativistic case decaying with $d^{-\frac{1}{2}}$, due to the dominance of the fast modes [23]. Interestingly this scaling is *independent* of the dynamical exponent over all $z > 1$ theories. In Fig. 5 we have shown the evolution of MI predicted by the QP picture and confirmed numerically. The decay of the MI itself is also *independent* of the dynamical exponent in these theories given by

$$I_f(t > t_p) \approx \mathfrak{i}_{b/f}\,\frac{z\,\alpha_z}{(z-1)}\,\frac{\ell^2}{d^2}\left(\frac{t}{d^z}\right)^{\frac{1}{1-z}}, \tag{15}$$

where $\mathfrak{i}_b = 1$ and $\mathfrak{i}_f = \ln 2$. The bosonic case is again enhanced with a logarithmic correction as well.

We would like to also comment that entanglement revivals are also able to capture scrambling of local quantum information into global degrees of freedom [27]. By considering these theories on a compact spatial dimension, we find results very similar to those for MI for the shortening of the deep entanglement revival of a connected interval, which will be reported in future work [10].

V. CONCLUSIONS AND DISCUSSIONS

We analyzed the dynamics of entanglement in integrable scale-invariant theories where the slow modes play a crucial role. It is worth noting that, in long-range interacting models (see, e.g., Ref. [28]) which admit strict Lieb-Robinson bounds [29], certain experiments verify the existence of propagating modes which violate Lieb-Robinson bounds [30]. The scale-invariant theories studied here are similar to this family of models and our theoretical explanation is in complete agreement with the aforementioned experiments. The striking role of slow modes in these theories is tempting for the existence of Lieb-Robinson bounds. Proving such a bound would be a very interesting future direction. Another important direction is exploring the dynamics of EE near anisotropic quantum critical points (see Ref. [31] for a recent related study).

Finally we would like to compare our results with holographic studies in Lifshitz theories. Holographic studies result in linear growth and sharp saturation of EE, similar to relativistic theories [32], which is totally different from the results presented in this Letter corresponding to mass quench (as well as boundary state quench and entropy production in thermofield double states). It is worth noting that these dynamical results are not the first example of such a disagreement; holographic results for EE in Lifshitz theories also do not capture the z dependence of EE in static cases. An important question to pursue in future works would be the investigation of the origin of such disagreements between nonrelativistic field theories and the corresponding holographic models.

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