PHYSICAL REVIEW RESEARCH **4**, L012002 (2022)

Letter

Quantifying necessary quantum resources for nonlocality

Lucas Tendick,* Hermann Kampermann, and Dagmar Bruß

Institute for Theoretical Physics III, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany

(Received 22 February 2021; accepted 18 November 2021; published 14 January 2022)

Nonlocality is one of the most important resources for quantum information protocols. The observation of nonlocal correlations in a Bell experiment is the result of appropriately chosen measurements and quantum states. We quantify the minimal purity to achieve a certain Bell value for any Bell operator. Since purity is the most fundamental resource of a quantum state, this enables us also to quantify the necessary coherence, discord, and entanglement for a given violation of two-qubit correlation inequalities. Our results shine a light on the Clauser-Horne-Shimony-Holt inequality by showing that for a fixed Bell violation an increase in the measurement resources does not always lead to a decrease of the minimal state resources.

DOI: 10.1103/PhysRevResearch.4.L012002

It is arguably one of the most astonishing features of quantum theory that local measurements performed on certain quantum states can lead to the phenomenon of quantum nonlocality [1]. That is, the measurement statistics cannot be explained classically as they are not compatible with the principle of local realism. Mathematically this can be witnessed by the violation of a so-called Bell inequality [2]. Even though nonlocality [3] has been studied ever since the foundations of quantum theory [4], it is not yet completely understood.

Especially its connection to the properties of the used states and measurements remain challenging. On a qualitative level it is well understood that the resources entanglement and measurement incompatibility are necessary but not sufficient for nonlocality [5–7]; on a quantitative level things are much less clear. One particular example for open challenges is the anomaly of nonlocality [8,9] i.e., the effect that partially entangled states can lead to more nonlocality than the maximally entangled state. The situation becomes even more unclear when we include the influence of the state resources purity [10], coherence [11,12], and discord [13] which all found growing attention recently [14–19]. If one wants to analytically analyze the resources within quantum states and measurements and study their influence on nonlocal correlations, it is most natural to use the full description of the involved physical systems. Resources like purity, entanglement, and coherence are defined naturally in this so-called device-dependent (DD) formalism. An alternative approach to study nonlocality is the device-independent formalism which makes minimal assumptions on the involved systems and usually relies on numerical hierarchies [20,21]. We address in this Letter the following fundamental question in the DD scenario: What are the required properties of a quantum state and its measurements to exhibit nonlocality? In other words, we quantify the interplay between the resource of nonlocal correlations and other resources like purity, coherence, discord, and most famously entanglement on the state side and measurement incompatibility [22] on the measurement side. The physical situation we are going to consider is illustrated in Fig. 1. We derive from the spectrum of any given Bell operator an analytical expression for the minimal purity of a quantum state that is needed to achieve some fixed amount of nonlocality in terms of a Bell inequality violation. This result is general, i.e., it holds for any dimension, any number of parties, measurement settings, and outcomes. In a second step, we show that this criterion also provides the minimal amount of coherence, discord, and entanglement needed for the violation of an inequality with any Bell-diagonal Bell operator, which is of particular interest for the case of two-qubit systems. As an application of our results, we present a closed expression for the maximal possible violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [23] given some fixed amount of entanglement or purity and a given level of measurement incompatibility. This enables us to establish a surprising link between the incompatibility of quantum measurements and the minimal entanglement needed. More precisely, we show that highly incompatible projective measurements need, in some instances, a higher amount of entanglement in order to show some fixed CHSH nonlocality than less incompatible projective measurements. In other words, a smaller resource on the measurement side does *not* require a higher resource on the state side, which is counterintuitive. An analogous result follows for the case of the two-setting linear steering inequality [24].

Preliminaries. In general, we are considering Hermitian Bell operators of the form

$$I = \sum_{a,b,x,y} c_{ab|xy} M_{a|x} \otimes M_{b|y}, \tag{1}$$

*Lucas.Tendick@hhu.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

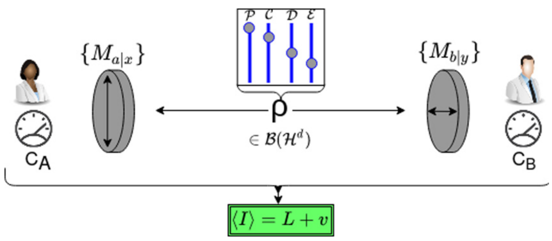

FIG. 1. Illustration of a Bell experiment. A (bounded) quantum state $\rho \in \mathcal{B}(\mathcal{H}^d)$ with adjustable resources purity $\mathcal{P}$, coherence $\mathcal{C}$, discord $\mathcal{D}$, and entanglement $\mathcal{E}$ is distributed to Alice and Bob who perform measurements $\{M_{a|x}\}$ and $\{M_{b|y}\}$ with also adjustable incompatibilities C_A and C_B. The interplay between the state resources and the measurement resources results in the observed Bell value $\langle I\rangle$. Minimal resource requirements for an observed Bell violation v beyond the local bound L are derived in the text.

where the real coefficients $c_{ab|xy}$ together with the local bound L (see below) describe the corresponding Bell inequality. The measurements are described by positive semidefinite operators $M_{a|x}, M_{b|y}$ with outcomes a, b and inputs x, y which form a positive operator-valued measure such that $\sum_a M_{a|x} = \mathbb{1}$ and $\sum_b M_{b|y} = \mathbb{1}$. A Bell inequality is given by

$$\sum_{a,b,x,y} c_{ab|xy}\, p(ab|xy)_{\mathrm{LHV}} \leqslant L, \tag{2}$$

with the (real) local bound L for all correlations obeying a so-called local hidden-variable model (LHV). This inequality may be violated by some entangled quantum states ρ, where the probability distribution is given by $p(ab|xy) = \mathrm{Tr}[(M_{a|x} \otimes M_{b|y})\rho]$. We call states which violate (at least) one Bell inequality nonlocal. The achieved Bell value is denoted by $\langle I\rangle = \mathrm{Tr}(I\rho) = L + v$ where $v > 0$ is the amount by which the bound L is violated. During the course of this Letter, we will often use the spectral decomposition of a quantum state $\rho = \sum_i^d \lambda_i |\phi_i\rangle\langle\phi_i|$ with $\lambda_i \geqslant 0$ and $\sum_i^d \lambda_i = 1$ and the Bell operator $I = \sum_j^d \mu_j |\Psi_j\rangle\langle\Psi_j|$ with real eigenvalues μ_j where d is the dimension of $\rho \in \mathcal{B}(\mathcal{H}^d)$ and $\mathcal{B}(\mathcal{H}^d)$ denotes the set of bounded operators. The sets $\{|\phi_i\rangle\}, \{|\Psi_j\rangle\}$ form orthonormal bases. We order (without loss of generality) the eigenvalues in descending order, i.e., $\lambda_i \geqslant \lambda_s$ for $i < s$ and $\mu_j \geqslant \mu_t$ for $j < t$.

Main task. We want to quantify the minimal quantum resources of a state ρ of dimension d in order to achieve some given violation v for a given Bell operator I (i.e., the measurements are fixed). Thus, we want to minimize a general resource quantifier $R(\rho)$ such that ρ is consistent with the observed data in terms of the Bell expectation value $\langle I\rangle$, i.e., we want to find

$$R^* = \min_{\rho}\{R(\rho) | \langle I\rangle = \mathrm{Tr}(\rho I) = L + v\}. \tag{3}$$

Optimizations of this form naturally occur in inference schemes based on entanglement witnesses [25–28]. The important difference to the task we consider here is that nonlocality itself is also a resource. In the context of nonlocality this problem has only been addressed for the CHSH inequality [29–34] with the main focus on entanglement. This approach based on the Bell operator makes use of the full information available and therefore allows us to study in a simple way how the required state resources depend on the chosen measurements.

Let us specify what we mean by the term quantum resource without going into detail. In any resource theory, one first defines the states which are no resource, the so-called void states (or free states), which constitute the set $\mathcal{V}$. Second, one defines the (maximal) set of operations Λ (free operations) that cannot turn a void state into a resource state. Finally, one has to find measures R which quantify the respective resource. The measures have to be faithful monotones, i.e., $R(\rho) = 0$ iff $\rho \in \mathcal{V}$ and $R[\Lambda(\rho)] \leqslant R(\rho)\ \forall\rho$ and free operations Λ. Additional properties of many measures are normalization for the maximal resource and additivity under tensor products. For more details, see [35]. For example, in the resource theory of entanglement, the free states are the separable ones, the free operations are local operations and classical communication, and a quantifier is the relative entropy of entanglement.

Purity. Our main result is an analytical result for the minimal purity of ρ needed to achieve the Bell value $\langle I\rangle = L + v$ for a general Bell operator I of any dimension d, any number of parties n, settings k, and outcomes m. We want to emphasize that the commonly used expression $\mathrm{Tr}(\rho^2)$ (known as linear purity) is not a proper purity measure [10] since it lacks additivity and normalization $[P(|\Psi\rangle) = \log_2(d)$ for any d-dimensional pure state $|\Psi\rangle]$ and does not vanish for the maximally mixed state. Instead one should use the Rényi 2-purity $\mathcal{P}_2(\rho) = \log_2[d\,\mathrm{Tr}(\rho^2)]$. Here, we will employ the generalized robustness of purity, which is easier to handle mathematically. It is defined via the general robustness quantifier

$$G_R(\rho) := \min_{\tau}\left\{x | x \geqslant 0, \exists \text{ a state } \tau, \frac{\rho + x\tau}{1+x} \in \mathcal{V}\right\}, \tag{4}$$

where the set $\mathcal{V}$ consists of the void states. $G_R(\rho)$ leads to the log robustness $\log_2[G_R(\rho) + 1]$, which is a proper measure of the resources considered in this Letter [16,36]. Because it is fully determined via $G_R(\rho)$, we focus the main discussions in this Letter on the generalized robustness for simplicity. Since τ can be any state, $G_R(\rho)$ can be seen as general noise robustness of ρ with respect to a void set $\mathcal{V}$ and can therefore be used to quantify a general resource G. In the case of purity, the void set $\mathcal{V}$ consists of only the maximally mixed state $\mathbb{1}/d$. It was shown in [36] that the generalized robustness of purity is given by

$$P_R(\rho) = d\lambda_1(\rho) - 1. \tag{5}$$

Thus, minimizing $P_R(\rho)$ reduces to minimizing $\lambda_1(\rho)$. In order to show our main result we first answer the (easier to solve) reverse question: Given $P_R(\rho)$, what is the maximal possible Bell value $\langle I\rangle_{\max} = L + v_{\max}$ the state ρ can achieve for a fixed Bell operator I?

Theorem 1. Given the Hermitian operator $I = \sum_{j=1}^d \mu_j |\Psi_j\rangle\langle\Psi_j|$ with $\mu_j \geqslant \mu_t$ for $j < t$ and a fixed robustness of purity $P_R(\rho)$ of a quantum state ρ. The maximal expectation value $\langle I\rangle_{\max}$ can be achieved by $\rho = \sum_{i=1}^r \lambda_i |\Psi_i\rangle\langle\Psi_i|$, where $\lambda_i \geqslant 0$, $\sum_{i=1}^r \lambda_i = 1$, $\lambda_i \geqslant \lambda_s$

for $i < s$, and is given by

$$\langle I\rangle_{\max} = \sum_{j=1}^{r} \mu_j \lambda_j, \tag{6}$$

where r is an integer such that $\frac{1}{r-1} > \lambda_1 \geqslant \frac{1}{r}$ and all eigenvalues λ_i for $i \in \{1, \ldots, r-1\}$ are equal to $\lambda_1 = (1 + P_R)/d$.

Proof. The theorem follows from the generalization of Ruhe's trace inequality [37] and the fact that it is optimal to choose all eigenvalues λ_i equal to λ_1 except the lowest nonzero one, which is given by normalization. The integer r defines the rank of the optimal ρ which we construct from the $\{\lambda_i\}$ and the eigenstates of I. This choice is unique for nondegenerate eigenvalues of I. See [38] for the specifics of the proof. ■

Theorem 1 can be used reversely (see Lemma 1 in the Supplemental Material [38]), which provides our first main result. Namely, for given $\langle I\rangle_{\max}$ we can use Eq. (6) to determine the minimal $P_R(\rho)$ or $\lambda_1(\rho)$ needed to achieve the Bell value $\langle I\rangle_{\max}$. In order to determine $\lambda_1(\rho)$ one only needs to find the integer r such that Theorem 1 is valid. The usefulness of Theorem 1 lies in its simplicity. Not only does it allow one to minimize the generalized robustness of purity $P_R(\rho)$ for a fixed expectation value of the most general Bell operator via an easily accessible criterion; also, one needs to check at most d linear equations. We also proved a more involved analogon to Theorem 1 with respect to the Rényi 2-purity $\mathcal{P}_2(\rho)$. See the Supplemental Material [38] for a detailed discussion.

Equality of quantum resources for two qubits. In the following we show which effect minimizing the purity has on the other state resources. In other words, we demonstrate the power of Theorem 1 by showing that for the subset of two-qubit correlation inequalities, i.e., inequalities without single party correlation terms, the states of minimal generalized robustness of purity for a fixed violation v also minimize the respective generalized robustnesses of coherence $C_R(\rho)$, discord $D_R(\rho)$, and entanglement $E_R(\rho)$, which in fact turn out to be equal. This is of particular interest since for every quantum state the hierarchy [10]

$$\mathcal{P}(\rho) \geqslant \mathcal{C}(\rho) \geqslant \mathcal{D}(\rho) \geqslant \mathcal{E}(\rho) \tag{7}$$

holds when quantified by the same distance-based [39] measure and coherence is quantified with respect to any product basis. We will in particular choose the product basis that minimizes the coherence of the state ρ. This notion of coherence coincides with the notion of symmetric quantum discord with respect to all subsystems [10,40]. Therefore, we will only summarize the concept of coherence [12] here; for more details about discord, see [41]. Coherence in general is a basis-dependent concept and is connected to the ability of a state to be in a superposition of some (fixed) basis states. The void states δ are called incoherent states. These are diagonal with respect to a fixed basis $|i\rangle$, i.e,

$$\delta = \sum_i p_i |i\rangle\langle i|, \quad p_i \geqslant 0, \quad \sum_i p_i = 1. \tag{8}$$

Note that our notion of coherence corresponds to a minimization over all states equivalent to ρ under local unitaries.

Our result is summarized in the following theorem.

Theorem 2. Given a Bell operator of the form

$$I = \sum_{x,y} g_{x,y} A_x \otimes B_y, \tag{9}$$

with real coefficients $g_{x,y}$ and local observables $A_x = \vec{a}_x \cdot \vec{\sigma}$, $B_y = \vec{b}_y \cdot \vec{\sigma}$ where $\vec{a}_x, \vec{b}_y$ are Bloch vectors and $\vec{\sigma}$ is the vector containing the Pauli matrices. For a fixed expectation value $\langle I\rangle = L + v$, where L is the local bound and $v > 0$, there exists a two-qubit quantum state ρ_{opt} which simultaneously minimizes the generalized robustness of purity P_R, coherence with respect to all product bases C_R, and entanglement E_R.

Proof. The proof relies on the fact that the states of minimal entanglement are Bell-diagonal states (BDS), which are entangled if and only if $\lambda_1 > 1/2$. The generalized robustness of entanglement [42] reduces for two-qubit BDS to $E_R(\rho_{\text{BDS}}) = 2\lambda_1(\rho_{\text{BDS}}) - 1$. Using this fact and Lemma 1 (see the Supplemental Material [38]) the optimal state ρ_{opt} can always be chosen to be of at most rank 2. This enables us to show that the closest separable state is always incoherent in some product basis. Therefore minimizing λ_1 minimizes all state resources. We relocated the specifics of the proof to the Supplemental Material [38]. ■

Note that an equivalence between coherence and entanglement for maximally correlated states has also been shown in different contexts [16,17]. We want to highlight that there is a straightforward generalization to genuine-multipartite entanglement (GME) quantification for N-qubit Greenberger-Horne-Zeilinger (GHZ) -diagonal Bell operators (e.g., two-setting full-correlation inequalities [43]) when we ask for a violation v which requires GME [44], since the optimal states will then be diagonal in the GHZ basis and the GME of these states is completely characterized by $\lambda_1 > 1/2$ [45], analogously to two-qubit BDS.

However, in general the hierarchy (7) will not be tight. Based on numerical optimization we find that there is indeed a (nontrivial) gap between purity, coherence, and entanglement for judiciously chosen observables in the $I3322$ inequality [46], an inequality with three settings and two outcomes for both parties including single party expectation values. One reason for this is the fact that the considered Bell operator is, in contrast to those for the previous discussed correlation inequalities, not diagonal in the Bell basis. That leads to different optimal states for the respective resources. See the Supplemental Material [38] for more details.

CHSH inequality. Remarkably, our results bring insights into the well-known CHSH inequality and systems of two qubits. The CHSH operator [23] is defined as

$$I = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2, \tag{10}$$

with $|\langle I\rangle| \leqslant 2$ for local-realistic models. The general form of Eq. (6) can for the case of two-qubit states be reduced to at most rank-2 solutions

$$\lambda_1 \mu_1 + (1 - \lambda_1)\mu_2 = L + v, \tag{11}$$

which recovers the finding made in [31]. Furthermore it is well known [47] that if the observables fulfill $A_i^2 = B_j^2 = \mathbb{1}$, it holds

$$I^2 = 4\mathbb{1} \otimes \mathbb{1} - [A_1, A_2] \otimes [B_1, B_2], \tag{12}$$

where $[X, Y]$ denotes the commutator between X and Y. The observables X and Y describing projective measurements are called incompatible, i.e., they cannot be measured jointly if and only if $[X, Y] \neq 0$. This quantum effect is the central aspect of the famous Heisenberg-Robertson uncertainty relation [48]. The use of incompatible measurements is necessary but not sufficient for Bell nonlocality [6,7]. There exists a resource theory [22] which allows the quantification of measurement incompatibility of one party. Let us introduce as a quantifier C for the (global) incompatibility the product of the single party measurement incompatibilities defined by the operator norm (largest absolute eigenvalue) of the commutators. Namely, $C = C_A C_B = \|[A_1, A_2]\|\|[B_1, B_2]\|$. This is well motivated since $C = 0$ if and only if one of the parties holds compatible measurements, i.e., the CHSH inequality cannot be violated and $C = 4$ is achieved with Pauli commutation relations only. Note that the single party incompatibility C_A is directly related to incompatibility quantifiers studied in [22]. After some algebra, we obtain the eigenvalues of I as a function of C, i.e. [49,50],

$$\mu_{1/4} = \pm\sqrt{4 + C}, \quad \mu_{2/3} = \pm\sqrt{4 - C}. \tag{13}$$

This shows that the quantity C quantifies the maximal nonlocality which can possibly be revealed by the given observables. By introducing the global measurement incompatibility we can study relations between the necessary resources contained in the states and those contained in the measurements, when wanting to achieve a certain nonlocality. The maximal possible violation given in Eq. (6) reduces to

$$\langle I\rangle_{\max} = \sqrt{4 + C}\lambda_1 + \sqrt{4 - C}(1 - \lambda_1). \tag{14}$$

Note that after inserting the optimal incompatibility $C_{\max} = \frac{4(2\lambda_1 - 1)}{2\lambda_1^2 - 2\lambda_1 + 1}$ to maximize the Bell value $\langle I\rangle_{\max}$ for fixed λ_1 one easily recovers the special case [30] and notably the result [34] where a formula for the maximal CHSH value of a two-qubit state in terms of its concurrence was found.

Intuitively one would expect now for a fixed violation of the CHSH inequality, that there is a trade-off between the necessary measurement resources and the necessary state resources in the sense that more of the resource in the measurements requires less resource in the state. This, however, is not always the case. As one can see in Fig. 2 there are parameter regions where less resources on the measurement side go together with less resources on the state side. Especially for very small violations, weakly incompatible measurements require much less entanglement for the same amount of nonlocality. We want to emphasize that the behavior of the other resources with respect to the quantifier C is qualitatively the same, since these are also monotonic functions of $\lambda_1(\rho)$. We further highlight that extensions of the considered Bell operators to higher dimensions, such as those in [49], can in the case of suboptimal extensions only increase the necessary purity while keeping the quantifier C constant. Is the surprising behavior discussed above a generic feature, or does it possibly depend on the chosen quantifiers for measurement incompatibility and/or the resources in the state? We discuss other possible quantifiers for state resources in the Supplemental Material [38] and conclude that the behavior is generic, by arguing that when other quantifiers are chosen

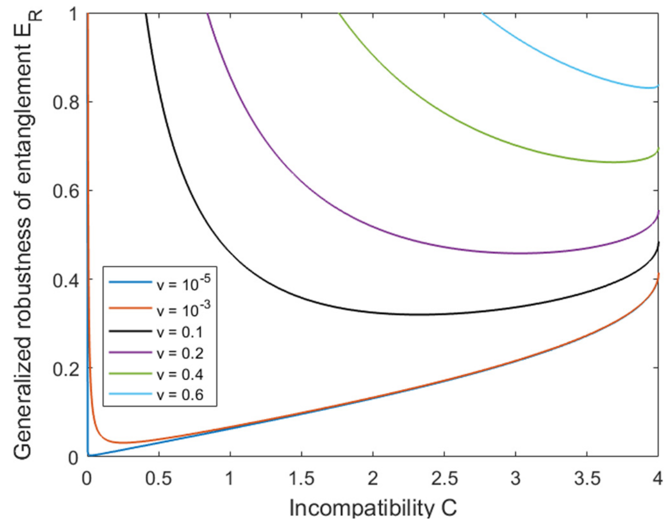

FIG. 2. The minimal generalized robustness of entanglement $E_R(\rho)$ for a given level of incompatibility C for different amounts of desired violation v. The curves diverge at some C because there is no state achieving the given violation. For low violations the effect that less entanglement for lower C is necessary becomes clearly visible, for a large regime of C.

only purity could possibly show a qualitatively different behavior. We show that this is indeed the case for the relative entropy of purity, while the Rényi 2-purity shows a similar behavior as the generalized robustness of purity as a function of C. For measurement incompatibility, we also show that the generalized robustness of incompatibility displays the same qualitative behavior. However, in general, it is still an open question whether these results are influenced by the particular choice of the incompatibility quantifier.

We strengthen this conclusion by highlighting that plots of the same qualitative behavior follow for the two-setting linear steering inequality [24] given by

$$F_2 = \left|\sum_{i=1}^{2}\langle A_i \otimes B_i\rangle\right| \leqslant \sqrt{2}, \tag{15}$$

where Bob's measurements have to be aligned orthonormally while Alice is free to choose any projective measurements. In this case, the eigenvalues of the steering operator of F_2 only depend on $C_A = \|[A_1, A_2]\|$ in an analogous way to the CHSH inequality, i.e.,

$$\tilde{\mu}_{1/4} = \pm\sqrt{2 + C_A}, \quad \tilde{\mu}_{2/3} = \pm\sqrt{2 - C_A}, \tag{16}$$

from which a behavior of the resources that is analogous to that for the CHSH inequality follows. This shows that the qualitative dependency of the state resources on the measurement incompatibility is not just due to our definition of the bipartite quantifier C, but a true physical phenomenon.

Discussion. In the present Letter we have analyzed the minimal resource requirements on the states and measurements for a given level of Bell nonlocality. We have shown that the minimal purity necessary to achieve a certain Bell value for the most general Bell operator can be found analytically via an easily accessible criterion. Since the purity of a state is its most fundamental resource which bounds *all* other resources of this state, this has major consequences for the inference of other necessary resources such as coherence and entanglement. We demonstrated this concretely by showing that the generalized robustness of all state resources can be minimized by the same state for two-qubit correlation inequalities. Finally, we have connected the nonlocality of quantum correlations, the incompatibility of quantum measurements, and the state's resources via the CHSH inequality. This revealed the counterintuitive effect, that sometimes more state resources are required to reach the same level of nonlocality, when the measurement resources are increased. While the CHSH inequality is by far the most studied Bell inequality, this behavior has, to the best of our knowledge, not been reported so far. The same effect is also prevalent for a steering inequality and thus excludes the existence of any possible conservation law for the necessary resources in states and measurements, regarding steering.

Several points are open for future research. First, one should investigate more general Bell scenarios, including the optimization over all Bell operators for a particular Bell inequality. Second, one could investigate further important resource measures. Finally, one should further investigate how the spectrum of Bell operators depends on the properties of the used measurement operators.

Acknowledgments. We thank Lennart Bittel, Federico Grasselli, Martin Kliesch, and Gláucia Murta for helpful discussions. This research was partially supported by the EU H2020 QuantERA ERA-NET Cofund in Quantum Technologies project QuICHE, and by the Federal Ministry of Education and Research (BMBF).

[1] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).
[2] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).
[3] J. I. de Vicente, On nonlocality as a resource theory and nonlocality measures, J. Phys. A: Math. Theor. **47**, 424017 (2014).
[4] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777 (1935).
[5] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).
[6] F. Hirsch, M. T. Quintino, and N. Brunner, Quantum measurement incompatibility does not imply Bell nonlocality, Phys. Rev. A **97**, 012129 (2018).
[7] E. Bene and T. Vértesi, Measurement incompatibility does not give rise to Bell violation in general, New J. Phys. **20**, 013021 (2018).
[8] A. A. Méthot and V. Scarani, An anomaly of non-locality, Quantum Inf. Comput. **7**, 157 (2007).
[9] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).
[10] A. Streltsov, H. Kampermann, S. Wölk, M. Gessner, and D. Bruß, Maximal coherence and the resource theory of purity, New J. Phys. **20**, 053058 (2018).
[11] J. Aberg, Quantifying superposition, arXiv:quant-ph/0612146.
[12] A. Streltsov, G. Adesso, and M. B. Plenio, *Colloquium*: Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).
[13] H. Ollivier and W. H. Zurek, Quantum Discord: A Measure of the Quantumness of Correlations, Phys. Rev. Lett. **88**, 017901 (2001).
[14] S. Luo, Quantum discord for two-qubit systems, Phys. Rev. A **77**, 042303 (2008).
[15] K. Modi, T. Paterek, W. Son, V. Vedral, and M. Williamson, Unified View of Quantum and Classical Correlations, Phys. Rev. Lett. **104**, 080501 (2010).
[16] H. Zhu, M. Hayashi, and L. Chen, Coherence and entanglement measures based on Rényi relative entropies, J. Phys. A: Math. Theor. **50**, 475303 (2017).
[17] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Measuring Quantum Coherence with Entanglement, Phys. Rev. Lett. **115**, 020403 (2015).
[18] T. R. Bromley, M. Cianciaruso, and G. Adesso, Frozen Quantum Coherence, Phys. Rev. Lett. **114**, 210401 (2015).
[19] C. Spee, Certifying the purity of quantum states with temporal correlations, Phys. Rev. A **102**, 012420 (2020).
[20] M. Navascués, S. Pironio, and A. Acín, A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations, New J. Phys. **10**, 073013 (2008).
[21] T. Moroder, J.-D. Bancal, Y.-C. Liang, M. Hofmann, and O. Gühne, Device-Independent Entanglement Quantification and Related Applications, Phys. Rev. Lett. **111**, 030501 (2013).
[22] T. Heinosaari, J. Kiukas, and D. Reitzner, Noise robustness of the incompatibility of quantum measurements, Phys. Rev. A **92**, 022115 (2015).
[23] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).
[24] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Phys. Rev. A **80**, 032112 (2009).
[25] K. M. R. Audenaert and M. B. Plenio, When are correlations quantum?—Verification and quantification of entanglement by simple measurements, New J. Phys. **8**, 266 (2006).
[26] J. Eisert, F. G. S. L. Brandão, and K. M. R. Audenaert, Quantitative entanglement witnesses, New J. Phys. **9**, 46 (2007).
[27] O. Gühne, M. Reimpell, and R. F. Werner, Estimating Entanglement Measures in Experiments, Phys. Rev. Lett. **98**, 110502 (2007).
[28] D. Cavalcanti and M. O. T. Cunha, Estimating entanglement of unknown states, Appl. Phys. Lett. **89**, 084102 (2006).
[29] R. Horodecki, M. Horodecki, and P. Horodecki, Entanglement processing and statistical inference: The Jaynes principle can produce fake entanglement, Phys. Rev. A **59**, 1799 (1999).
[30] J. Batle and M. Casas, Nonlocality and entanglement in qubit systems, J. Phys. A: Math. Theor. **44**, 445304 (2011).
[31] J. Batle, M. Casas, A. R. Plastino, and A. Plastino, Inference schemes and entanglement determination, Phys. Rev. A **65**, 024304 (2002).

[32] J. Batle, M. Casas, A. R. Plastino, and A. Plastino, On the fake inferred entanglement associated with the maximum entropy inference of quantum states, J. Phys. A: Math. Gen. **34**, 6443 (2001).

[33] K. Bartkiewicz, B. Horst, K. Lemr, and A. Miranowicz, Entanglement estimation from Bell inequality violation, Phys. Rev. A **88**, 052105 (2013).

[34] F. Verstraete and M. M. Wolf, Entanglement versus Bell Violations and their Behavior under Local Filtering Operations, Phys. Rev. Lett. **89**, 170401 (2002).

[35] E. Chitambar and G. Gour, Quantum resource theories, Rev. Mod. Phys. **91**, 025001 (2019).

[36] Y. Luo and Y. Li, Robustness of purity: An operational approach to resource theory of purity, Eur. Phys. J. D **73**, 89 (2019).

[37] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications* (Springer, New York, 2011).

[38] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevResearch.4.L012002 for more details on derivations, as well as for more details on the hierarchy of quantum resources in the $I3322$ scenario and a discussion about the incompatibility quantifier of C. This includes Refs. [51–54].

[39] Note that the hierarchy also holds for measures which are no distance in a strict mathematical sense, like the relative entropy, due to the inclusion relations of the void sets. It also holds for the generalized robustness.

[40] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Quantum coherence in multipartite systems, Phys. Rev. A **92**, 022112 (2015).

[41] A. Bera, T. Das, D. Sadhukhan, S. S. Roy, A. Sen(De), and U. Sen, Quantum discord and its allies: A review of recent progress, Rep. Prog. Phys. **81**, 024001 (2017).

[42] M. Steiner, Generalized robustness of entanglement, Phys. Rev. A **67**, 054305 (2003).

[43] V. Scarani and N. Gisin, Spectral decomposition of Bell's operators for qubits, J. Phys. A: Math. Gen. **34**, 6043 (2001).

[44] G. Svetlichny, Distinguishing three-body from two-body nonseparability by a Bell-type inequality, Phys. Rev. D **35**, 3066 (1987).

[45] M. Cianciaruso, T. R. Bromley, and G. Adesso, Accessible quantification of multiparticle entanglement, npj Quantum Inf. **2**, 16030 (2016).

[46] D. Collins and N. Gisin, A relevant two qubit Bell inequality inequivalent to the CHSH inequality, J. Phys. A: Math. Gen. **37**, 1775 (2004).

[47] L. J. Landau, On the violation of Bell's inequality in quantum theory, Phys. Lett. A **120**, 54 (1987).

[48] H. P. Robertson, The uncertainty principle, Phys. Rev. **34**, 163 (1929).

[49] S. L. Braunstein, A. Mann, and M. Revzen, Maximal Violation of Bell Inequalities for Mixed States, Phys. Rev. Lett. **68**, 3259 (1992).

[50] J. Kiukas and R. F. Werner, Maximal violation of Bell inequalities by position measurements, J. Math. Phys. **51**, 072105 (2010).

[51] S. Boyd and L. Vandenberghe, *Convex Optimization* (Cambridge University Press, Cambridge, UK, 2004).

[52] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Quantifying Entanglement, Phys. Rev. Lett. **78**, 2275 (1997).

[53] H. Zhu, L. Chen, and M. Hayashi, Additivity and non-additivity of multipartite entanglement measures, New J. Phys. **12**, 083002 (2010).

[54] R. Uola, C. Budroni, O. Gühne, and J.-P. Pellonpää, One-to-One Mapping between Steering and Joint Measurability Problems, Phys. Rev. Lett. **115**, 230402 (2015).