

Stochastic thermodynamics of fractional Brownian motion

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This paper is concerned with the stochastic thermodynamics of nonequilibrium Gaussian processes that can exhibit anomalous diffusion. In the systems considered, the noise correlation function is not necessarily related to friction. Thus there is no conventional fluctuation-dissipation relation (FDR) of the second kind and no unique way to define a temperature. We start from a Markovian process with time-dependent diffusivity (an example being scaled Brownian motion). It turns out that standard stochastic thermodynamic notions can be applied rather straightforwardly by introducing a time-dependent temperature, yielding the integral fluctuation relation. We then proceed to our focal system, that is, a particle undergoing fractional Brownian motion (FBM). In this case, the noise is still Gaussian, but the noise correlation function is nonlocal in time, defining a non-Markovian process. We analyze in detail the consequences when using the conventional notions of stochastic thermodynamics with a constant medium temperature. In particular, the heat calculated from dissipation into the medium differs from the log ratio of path probabilities of forward and backward motion, yielding a deviation from the standard integral fluctuation relation for the total entropy production if the latter is defined via system entropy and heat exchange. These apparent inconsistencies can be circumvented by formally defining a time-nonlocal temperature that fulfills a generalized FDR. To shed light on the rather abstract quantities resulting from the latter approach, we perform a perturbation expansion in terms of $\epsilon = H - 1/2$, where H is the Hurst parameter of FBM and $1/2$ corresponds to the Brownian case. This allows us to calculate analytically, up to linear order in ϵ , the generalized temperature and the corresponding heat exchange. By this, we provide explicit expressions and a physical interpretation for the leading corrections induced by non-Markovianity.

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I. INTRODUCTION

Within the last decades, the framework of stochastic thermodynamics (ST) [1–3] has been established as a powerful tool to analyze the dynamical and thermodynamic properties of small, mesoscopic systems out of equilibrium [4–7]. Paradigmatic examples whose thermodynamic fluctuation properties have been studied experimentally are driven colloidal particles [8], biopolymers [9], and molecular Szilard-type engines [10]. However, concepts of ST are nowadays also used for open quantum systems [11,12], nonlinear electronic circuits [13], electron shuttles [14], and open, coarse-grained systems [15]. In these mesoscopic systems, observables of interest such as the position of a particle typically fluctuate strongly due to interactions with an environment. The key step of ST is to define thermodynamic quantities such as heat, work, and entropy along single fluctuating trajectories [1,2], allowing one to investigate not only ensemble averages as in the (phenomenological) thermody-

namics of large, macroscopic systems, but also fluctuations of these quantities.

These fluctuations are constrained by fundamental symmetry relations, known as fluctuation relations (FRs) (see Refs. [3,6,7,16,17] for collections and reviews). Applied to the (total) entropy production, they allow for negative entropy production on the trajectory level but reduce to the conventional second law of thermodynamics (expressing positiveness of the entropy production) upon averaging. FRs were put forward by Evans *et al.* [18] in numerical simulations of shear-driven systems, but later they were mathematically proven for different dynamics [19–21] and also experimentally confirmed [4–10]. More generally, FRs relate the probability density functions of certain thermodynamic observables to those of conjugate (typically time-reversed) processes. An important example in the FR collection is the Jarzynski relation [22,23] involving the nonequilibrium work of driven systems, which is of great importance because of its applicability for measuring free-energy landscapes [4]. Subsequently, many other associated relations have been discovered, such as the Crooks fluctuation relation [24,25], the Hummer-Szabo relation [26], and integral FRs (IFRs) [8,27,28].

Within the realm of classical systems, most of the work on FRs and other aspects of ST (such as the recently discovered thermodynamic uncertainty relation (TUR) [29,30]) has been devoted to fluctuating systems exhibiting *normal* diffusion. Considering, for simplicity, one-dimensional (1D) motion of a

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Brownian particle in a suspension, “normal” diffusion implies that the mean-square displacement (MSD) $\langle x^2(t) \rangle$ (with x being the distance traveled at time t , averaged over an ensemble of particles) increases linearly in t at long times. Such processes are typically modeled by a conventional Langevin equation (LE) involving white noise, which is related to the friction via the (second) fluctuation-dissipation relation (FDR) [31].

In this paper we are interested in the ST of systems exhibiting *anomalous* diffusion, where $\langle x^2(t) \rangle \propto t^\alpha$ with $\alpha \neq 1$. Here, the case $\alpha < 1$ is referred to as subdiffusion, while $\alpha > 1$ corresponds to superdiffusion [32–34]. Anomalous dynamics occurs in a large variety of systems (see, e.g., Refs. [33,35–37]). Typically, subdiffusion is related to crowding phenomena (where the motion is hindered by obstacles) or spatial confinement [37–39]. In turn, superdiffusion occurs, e.g., in glassy material [40], for cell migration [41,42], and in the foraging of biological organisms [36]. From a theoretical point of view, various types of models have been proposed to describe anomalous dynamics [43]. One class of these models is Markovian in character, where the future of the observable, e.g., x , only depends on the current value. Examples of (semi-)Markovian models yielding anomalous diffusion include continuous-time random walks [44,45], heterogeneous diffusion processes [46], anomalous diffusion in disordered media [47,48], and scaled Brownian motion [49,50]. However, there are furthermore many non-Markovian models predicting anomalous diffusion. Prominent examples are generalized Langevin equations (GLEs) with friction (“memory”) kernels and colored noise [51–53], as well as the paradigmatic case of fractional Brownian motion (FBM) [54,55], where memory arises through power-law-correlated Gaussian noise. The FBM process, which has been widely observed in experiments (see, e.g., Ref. [56] for references), is of particular interest in this paper.

Despite the broad occurrence of anomalous diffusion in mesoscopic and biological systems, applications of concepts of ST to such systems are still rare, and many open questions remain. This concerns both anomalous models with Markovian character and anomalous models with non-Markovian character. Existing studies mainly focus on FRs. For example, a series of papers using (non-Markovian) GLEs has confirmed the validity of the Crooks and the Jarzynski FRs, as well as of transient and steady-state FRs [57–61]. More generally, the validity of (different) FRs for GLE-like dynamics has been shown in Ref. [62]. Notably, the above-mentioned results in the framework of GLEs have been obtained under the assumption that the noise correlation function and the memory kernel are related (in fact, proportional) to each other by the FDR of the second kind (FDR2) [31]. The latter should be distinguished from the FDR of the first kind (FDR1), which relates the response of a system with respect to an external perturbation to (equilibrium) correlation functions in the absence of that perturbation. In overdamped GLE models of driven systems without FDR2 [63,64], the conventional form of FRs may not be obtained for thermodynamic observables. This problem is also explored in very recent works modeling fluctuations of a Brownian particle in an active bath, for which GLEs with two different kinds of noises have been used, typically Gaussian white and (exponentially) colored noise

[65]. For the latter, representing the active bath, FDR2 is broken, and deviations from conventional FRs have been reported, arising in such models [65–70]. Beyond GLE models, forms different from conventional (steady-state and transient) FRs, dubbed *anomalous fluctuation relations* [6], were also obtained for systems with non-Gaussian noises [71–78], in glassy systems [79,80], and in continuous-time random walks for certain exponents of the (power-law) waiting time distribution [81]. More recently, studies of FRs and further concepts of ST have been extended to other nontrivial systems of current interest, particularly to active particles [65,82–84] and systems with time delay [85–88]. We also mention recent studies of TURs in systems displaying anomalous dynamics [89] and time delay [90]. All these developments highlight the ongoing strong interest in understanding the ST of systems beyond standard Brownian motion. However, to the best of our knowledge, most studies have focused on specific aspects (such as FRs and TURs), while the general framework of ST for anomalous processes seems still underdeveloped.

In this paper, we aim at filling this gap by a systematic study of two paradigmatic stochastic processes that can exhibit anomalous diffusion, one being Markovian and the other being non-Markovian. Both of these processes involve Gaussian noise, yet nontrivial (in one case, non-Markovian) noise correlation functions. For these two exemplary processes, we systematically apply the framework of “standard” ST focusing, in particular, on definitions of heat, medium, and total entropy production, and the IFR. We do not impose *a priori* the presence of an FDR (of any kind), thereby considering systems which have been called “athermal” [3]. Indeed, breaking FDRs of any kind was found to be characteristic for active biological systems driven out of equilibrium [91]. Experimental examples concerning FDR1 include hair bundles [92], active cytoskeletal networks [93], and neutrophil biological cells [94]. In nonliving systems a violation of FDR1 has been demonstrated as well, for example, in glassy systems based on both numerical [95] and experimental [96] evidence. A breaking of FDR2 has been reported for numerous nonequilibrium systems including heated Brownian particles [97], a probe particle in a nonequilibrium fluid [98], particle-bath systems in external oscillating fields [99], and systems with nonstationary noise [100], among many others. In what follows we refer to FDR2 when mentioning FDR.

Throughout this paper we focus on the overdamped limit (although mass effects can clearly influence the dynamics; see, e.g., Ref. [101]). Including inertia in our investigation would imply significantly expanding the formalism of ST. For example, already for simple Brownian systems it is well known that adding inertia yields a modification of detailed fluctuation relations for the housekeeping heat [102], the creation of an additional source for entropy production [103], and a violation of the thermodynamic uncertainty relation [104]. As the main objective of this work is to investigate the effect of non-Markovian dynamics and anomalous diffusion, we here chose to focus on particles with negligible mass, where the overdamped limit seems justified.

To start with, we discuss in Sec. II a model that involves a time-dependent noise intensity (diffusivity). A prominent example of such a process (which was originally proposed by Batchelor in the context of turbulence [105]) is scaled

Brownian motion [49,50,106–112]. In this paper, we utilize this rather simple, and still Markovian, generalization of standard Brownian motion to review some core concepts of ST definitions. In particular, we discuss the role of the FDR and, related to that, the definition of an (effective) temperature [6,60,61,63,64,66–70,82,113,114] for definitions of heat production and the validity of the standard IFR for the entropy production.

In Sec. III of this paper we turn to our major topic, that is, the ST of FBM. FBM is a non-Markovian process that can generate all modes of anomalous diffusion, from subdiffusion to normal diffusion to superdiffusion. This property of FBM makes it a versatile and nowadays widely used model for numerous experimental observations of anomalous diffusion in nature and the laboratory [56]. Examples include the motion of tracers in viscoelastic media [53], crowded *in vitro* environments [115–117], living cells [118,119], and intracellular media [120]. Given its quite universal applicability, the investigation of ST concepts for FBM systems is both timely and relevant. Our goal here is to unravel the challenges implied by the non-Markovianity and the absence of the FDR for the definition of heat production, entropy production, and the related IFR. To this end, we employ a fractional differential approach and a perturbation expansion. As a main result, we provide explicit expressions and an interpretation for the leading corrections induced by non-Markovianity to the usual temperature and heat.

II. BROWNIAN MOTION WITH TIME-DEPENDENT NOISE STRENGTH

In this section we revisit some key concepts of ST considering, specifically, a Langevin equation (LE) with a time-dependent noise intensity. After introducing relevant thermodynamic quantities (Sec. II A), we proceed in Sec. II B by (re)deriving a standard IFR following essentially corresponding arguments for standard Brownian motion [27]. In this way, we lay the foundation of our later treatment of the more complex case of (non-Markovian) FBM.

A. Langevin equation and energetics

Let us consider an overdamped particle (henceforth called the “system”) which diffuses in one dimension through a medium acting as a heat bath. As in the standard Brownian picture, the bath interacts with the particle through a stochastic force $\xi(t)$ whose correlations are specified below, as well as by friction. The dynamics of the system is governed by the LE

$$\dot{x}(t) = \mu F(x(t), \lambda(t)) + \xi(t), \quad (1)$$

where $\mu = 1/\gamma$ denotes the mobility (with γ being the friction constant) and $F(x(t), \lambda(t))$ describes a force acting on the particle. As usual, F can consist of a conservative part arising from a potential V and/or a nonconservative part directly applied to the system, that is,

$$F(x(t), \lambda(t)) = -\partial_x V(x, \lambda(t)) + f(t, \lambda(t)). \quad (2)$$

Here, $\lambda(t)$ is a control parameter which can be tuned in order to manipulate the trajectory of the particle. An example of such a nonconservative force is an optical tweezer [121] that

drags the system with a time-(in)dependent velocity and (or) in response to the state of the system in order to control it. In what follows, we assume that the stochastic force ξ is described by a Gaussian process with zero mean, i.e., $\langle \xi(t) \rangle = 0$ (with $\langle \dots \rangle$ being an average over noise realizations) and a time-dependent correlation function

$$\langle \xi(t)\xi(t') \rangle = 2K(t)\delta(t-t'), \quad (3)$$

where $K(t)$ is the time-dependent noise strength (sometimes called “time-dependent diffusivity”). By this time dependency, our model contrasts with the LE of standard Brownian motion, where K is constant and equals the diffusion constant D . We note, however, that despite the time dependence of $K(t)$, the model considered here is still Markovian in the sense that the stochastic forces $\xi(t)$ at different times are uncorrelated [as indicated by the delta function in Eq. (3)].

A prominent example of $K(t)$ which indeed generates anomalous diffusion is *scaled Brownian motion* (SBM) [49,50]. In SBM, $K(t)$ has a power-law dependence on time, that is,

$$K(t) = \alpha K_\alpha t^{\alpha-1}. \quad (4)$$

With this choice, the MSD [for one-dimensional motion in the absence of $F(x(t), \lambda(t))$ and $x(t=0) = 0$] is given as $\langle x^2(t) \rangle = 2K_\alpha t^\alpha$ [49], indicating the possibility of generating sub- or superdiffusive processes when choosing α smaller or greater than unity, respectively. For $\alpha = 1$, one recovers standard Brownian motion with $K(t) = K_1 = D$.

So far, Eqs. (3) and (4) have been introduced as a simple generalization of standard Brownian motion. Importantly, however, here we do not impose any relation between the noise strength, $K(t)$, and the particle’s mobility, μ , or equivalently, the friction γ . This is in contrast to the ordinary Brownian case, where the noise strength identified with the diffusion coefficient obeys $D = \mu T$, with T being the temperature of the bath (and we have set the Boltzmann constant $k_B = 1$). We recall in this context that the relation $D = \mu T$ is just another formulation of FDR2, which formally follows when setting the noise correlation of standard Brownian motion, $\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t')$, proportional to the delta-like friction kernel $\gamma(t-t')$ that appears when rewriting the left-hand side of Eq. (1) in a GLE-like manner (see, e.g., Ref. [64]). Having this in mind, it becomes clear that for a system with time-dependent noise strength (such as SBM with $\alpha \neq 1$), FDR2 is broken if the mobility or (inverse) friction is assumed to be constant [we come back to this point below Eq. (12)]. Models with a time-local dissipation term [see Eq. (1)] and noise with a time-dependent correlation function [see Eq. (4)], as well as with a time-nonlocal one (see Sec. III), have been widely used to describe anomalous diffusion observed in *in vitro* and *in vivo* experiments; see, e.g., Refs. [37,94,122]. Therefore we believe those models are relevant to consider in the context of ST. As we will proceed to show, the resulting absence of the (conventional) FDR does not impose any problems for several definitions and relations in standard ST [27]. However, complications appear when considering the so-called medium entropy production.

To start with, we consider the heat exchange between the particle and the bath due to the friction and thermal fluctuations. For an infinitesimal displacement $dx(t)$ of the particle

during the time interval dt , the fluctuating heat dissipated into the medium is given by

$$\delta Q(t) = (\gamma \dot{x}(t) - \gamma \xi(t)) \odot dx(t), \quad (5)$$

where the symbol \odot in Eq. (5) denotes a Stratonovich product [123]. Henceforth, we will drop this symbol for the sake of brevity. Combining Eq. (5) with Eqs. (1) and (2), and integrating over time, one obtains the total heat flowing from the particle into the medium during the time t , that is,

$$Q_{[x]}(t) = \int_0^t F(x(t'), \lambda(t')) \dot{x}(t') dt', \quad (6)$$

stochastic trajectory considered. Equation (6) has exactly the same form as in the standard case [1–3]. Similarly, the fluctuating work done on the particle is given (as in the standard case) by

$$\delta W(t) = \partial_\lambda V(x, \lambda(t)) d\lambda + f(t, \lambda(t)) dx(t), \quad (7)$$

yielding the first law on a trajectory level [1],

$$dU(t) = \delta W(t) - \delta Q(t), \quad (8)$$

with dU being an increment of the system's total energy.

We now consider contributions to the entropy production. For overdamped motion involving only the particle's position, the so-called system entropy is defined by [27]

$$S_{[x]}(t) = -\ln P(x(t), t), \quad (9)$$

where $P(x, t)$ denotes the probability distribution function (PDF) of the particle displacement evaluated along the trajectory considered. For a Markovian system, $P(x, t)$ is the solution of the Fokker-Plank equation (FPE) corresponding to the LE. With the initial distribution $P(x_0, 0)$ with $x_0 = x(t = 0)$, the change in the system entropy along the stochastic trajectory during time t follows as

$$\Delta S_{[x]}(t) = -\ln P(x, t) + \ln P(x_0, 0) = \ln \frac{P(x_0, 0)}{P(x, t)}. \quad (10)$$

From here, one usually proceeds by defining the so-called medium entropy $S_{[x]}^m$, either by comparing path probabilities of forward and backward processes or by directly starting from the fluctuating heat exchange with the environment. For standard Brownian motion these two routes yield the same results [3]. This, however, is not automatically the case for the model at hand.

To show this, we start by defining $S_{[x]}^m$ via the heat exchange (for a discussion of path probabilities, see Sec. II B). In standard Brownian motion, the (trajectory-dependent) change in medium entropy is given as $\Delta S_{[x]}^m = Q_{[x]}/T$, where the heat exchange during time t , $Q_{[x]}$, is given by Eq. (6), and the bath temperature T is determined by the FDR. In the present model, however, the noise strength depends on time, such that the very definition of a temperature is not obvious. To proceed, we consider two different scenarios.

(i) We first assume that the medium temperature is a constant, T_0 , whose value is, however, undetermined. In particular, T_0 is not related to the noise. Defining now the (fluctuating) medium entropy as in standard Brownian motion

and using Eq. (6), we obtain

$$\Delta S_{[x]}^m(t, T_0) = \frac{Q_{[x]}(t)}{T_0} = \frac{1}{T_0} \int_0^t dt' F(x(t'), \lambda(t')) \dot{x}(t'). \quad (11)$$

(ii) Our second choice is motivated by the time dependence of the noise strength. Specifically, we introduce a time-dependent temperature via

$$T(t) = \frac{K(t)}{\mu}. \quad (12)$$

Equation (12) may be understood as an *ad hoc* generalization of the FDR2 of standard Brownian motion. This can be seen when we formally multiply both sides by $2\delta(t - t')/\mu$. Then the right-hand side of Eq. (12) equals the correlation function of the renormalized noise $\langle \xi'(t)\xi'(t') \rangle = \langle \xi(t)\xi(t') \rangle / \mu^2$ [see Eq. (3)], while the left-hand side contains the delta-like friction kernel [i.e., $\gamma(t - t') = \gamma\delta(t - t')$] implicitly assumed in Eq. (1). Thus one obtains $\langle \xi'(t)\xi'(t') \rangle = \gamma(t - t')T(t)$, that is, the FDR2 with time-dependent temperature.

Having these considerations in mind, the change in the medium entropy along the trajectory may be defined as

$$\Delta S_{[x]}^m(t) = \int_0^t dt' \frac{1}{T(t')} F(x(t'), \lambda(t')) \dot{x}(t'). \quad (13)$$

As we will see in Sec. II B, only the second choice [scenario (ii)] is consistent with the definition of $S_{[x]}^m$ via path probabilities, as well as with the usual IFR for the total entropy production. It seems worthwhile to note that the introduction of an effective, in our case time-dependent temperature is not a new concept at all. Indeed, generalized temperatures have been used, e.g., in weak turbulence, granular matter, and glassy material [113] and, more recently, for active matter [66–70,82]. We remark, however, that its straightforward definition based on FDRs has been criticized [114].

B. Integral fluctuation relation and total entropy production

We now discuss consequences of SBM dynamics or, more generally, a time-dependent noise strength, for FRs, particularly the IFR. To this end, we recall [2,3,27] that the key ingredient for the derivation of FRs from the LE is the probability of observing a certain path of the particle. For an arbitrary Gaussian process $\xi(t)$, such as the one in Eq. (1), the conditional path probability that the particle is at position $x(t)$ at time t , given that it was at $x(0)$ at $t = 0$, is given by [124,125]

$$P[x(t)|x(0)] = \exp \left[-\frac{1}{2} \int_0^t dt_2 \int_0^t dt_1 \xi(t_1) G(t_1, t_2) \xi(t_2) \right], \quad (14)$$

where the kernel $G(t_1, t_2)$ is the functional inverse of the noise correlation function, i.e.,

$$\int_0^\infty dt_3 G(t_1, t_3) \langle \xi(t_3)\xi(t_2) \rangle = \delta(t_1 - t_2). \quad (15)$$

For the present system with time-dependent noise strength, it follows from Eq. (3) that

$$G(t_1, t_2) = \frac{\delta(t_1 - t_2)}{2K(t_1)}. \quad (16)$$

Inserting Eq. (16) into (14) and substituting $\xi(t)$ via Eq. (1), we obtain

$$P[x(t)|x(0)] \propto \exp \left[- \int_0^t dt_1 \frac{[\dot{x}(t_1) - \mu F(x(t_1), \lambda(t_1))]^2}{4K(t_1)} \right], \tag{17}$$

where the (negative of the) exponent corresponds to the action of the present model and the proportionality sign signals the (missing) Jacobian arising from the substitution of ξ . In fact, Eq. (17) is formally identical to the path probability of standard Brownian motion (in the presence of a force F), the only difference being the appearance of the time-dependent noise strength in the denominator rather than the diffusion constant D .

As a next step, we calculate the ratio of the probabilities of the forward and backward paths, the latter involving the system’s dynamics under time reversal. The forward path $[x]$, whose probability is denoted by $P[x(t)|x(0)]$, starts from an initial point $x(0)$ chosen from the distribution $P_0(x(0))$ and ends at $x(t)$ under the control protocol $\lambda(t)$. The corresponding reversed path $[\bar{x}]$ starts from the final position of the forward path, with the distribution $P_1(x(t))$, and ends at the initial position of the forward path, i.e., $\bar{x}(0) = x(t)$ and $\bar{x}(t) = x(0)$, under the reversed protocol, $\tilde{\lambda}(t)$. Note that in the present model the noise strength $K(t)$ is time dependent; see Eq. (3). However, since the resulting noise correlation function is symmetric in time (as in the normal case), the time dependence of $K(t)$ does not impose any complication. With these considerations, we find that the logarithm of (conditional) path probabilities in the forward and the backward direction, which is a key ingredient for defining the total entropy production (and the IFR), is given by

$$\ln \frac{P[x(t)|x(0)]}{P[\bar{x}(t)|\bar{x}(0)]} = \int_0^t dt_1 \frac{\mu}{K(t_1)} F(x, \lambda(t_1)) \dot{x}(t_1). \tag{18}$$

We now compare the right-hand side of Eq. (18) with our previously stated expressions for the change in medium entropy defined via the heat exchange; see Eqs. (11) and (13). One immediately observes consistency with the second expression [scenario (ii)], that is,

$$\ln \frac{P[x(t)|x(0)]}{P[\bar{x}(t)|\bar{x}(0)]} = \Delta S_{[x]}^m(t, T(t)). \tag{19}$$

Thus, by introducing a time-dependent temperature via a generalized FDR [see Eq. (12)], the previously defined medium entropy production becomes consistent with the logarithm of the path probability ratio, in complete analogy to the case of standard Brownian motion. Clearly, this is not the case if we define *ad hoc* a constant temperature T_0 [scenario (i)]. In that case, where an FDR is lacking, the medium entropy production defined via Eq. (11) obviously differs from Eq. (18).

To proceed towards an IFR, we consider the quantity $R_{[x]}$ defined as

$$R_{[x]} = \ln \frac{P[x(t)|x(0)]P_0(x(0))}{P[\bar{x}(t)|\bar{x}(0)]P_1(\bar{x}(0))}, \tag{20}$$

which fulfills the exact relation [28]

$$\langle e^{-R_{[x]}} \rangle = 1. \tag{21}$$

We stress that Eq. (21) is entirely a mathematical expression that does not rely on any physical interpretation of $R_{[x]}$. Following the usual approach [27], we decompose $R_{[x]}$ into a “bulk” term determined by the log ratio of conditional probabilities for forward and backward dynamics and a “boundary” term governed by the log ratio of the distributions of the initial and final values, i.e., $P_1(\bar{x}(0)) = P_1(x(t))$ and $P_0(x(0))$. Setting $P_1(x(t)) = P(x, t)$, the latter being the PDF of the particle displacement with the distribution of initial condition $P_0(x(0))$, the boundary term becomes equal to the change in system entropy ΔS considered in Eq. (10) [27]. In this case, we therefore have

$$R_{[x]} = \ln \frac{P[x(t)|x(0)]}{P[\bar{x}(t)|\bar{x}(0)]} + \Delta S_{[x]} = \int dt_1 \frac{\mu}{K(t_1)} F(x, \lambda(t_1)) \dot{x}(t_1) + \Delta S_{[x]}, \tag{22}$$

where we have used Eq. (18) in the second line.

Comparing Eq. (22) with Eq. (13), we see that the first term in Eq. (22) becomes indeed equal to the medium entropy production if we define the latter based on a time-dependent temperature fulfilling a generalized FDR, Eq. (12). In this case [scenario (ii)] we thus obtain the usual relations

$$R_{[x]} = \Delta S_{[x]}^m(t, T(t)) + \Delta S_{[x]} = \Delta S_{[x]}^{\text{tot}} \tag{23}$$

for the definition of the total entropy production [2,3,27] via the quantity $R_{[x]}$. Combining Eqs. (23) and (21), we immediately find

$$\langle e^{-\Delta S_{[x]}^{\text{tot}}} \rangle = 1. \tag{24}$$

In contrast, if we assume a constant medium temperature $[T_0]$; see scenario (i)] and define the medium entropy production via Eq. (11), an inconsistency arises: In this case, the quantity $R_{[x]}$ is obviously different from the sum of medium and system entropy production. Rather, we have from Eqs. (22) and (11)

$$R_{[x]} = \Delta S_{[x]} + \Delta S_{[x]}^m(t, T_0) + \int dt_1 \left(\frac{\mu}{K(t_1)} - \frac{1}{T_0} \right) F(x, \lambda(t_1)) \dot{x}(t_1). \tag{25}$$

If we still define the total entropy production $\Delta S_{[x]}^{\text{tot}}$ as the sum of system and medium entropy production [the latter being defined by Eq. (11)], we have from Eqs. (25) and (21)

$$\langle e^{-[\Delta S_{[x]}^{\text{tot}} + \int dt (\frac{\mu}{K(t)} - \frac{1}{T_0}) F(x, \lambda(t)) \dot{x}(t)]} \rangle = 1. \tag{26}$$

Clearly, the exponent deviates from the total entropy production alone. This suggests that we interpret the term involving $\mu/K(t) - 1/T_0$ as an indicator [114] of how far the IFR for the total entropy production deviates from the standard one. Note, however, that this all depends on how we define the term “total entropy production”: One could also argue that, in the case of a constant medium temperature (not related to noise correlations), the “total” entropy production includes an additional term, namely, just the integral term appearing on the right-hand side of Eq. (26).

III. FRACTIONAL BROWNIAN MOTION

We now extend our discussion towards a more complex, non-Markovian diffusion process, namely, fractional Brownian motion (FBM). Physically, we could think, for example, of a colloidal particle diffusing through a homogeneous, yet viscoelastic medium (a situation which may be mapped onto FBM; see, e.g., Ref. [115]). The homogeneity of the medium allows one to consider the friction coefficient γ , and thus the mobility $\mu = \gamma^{-1}$, as independent of space and time. The medium's viscoelasticity then enters only through the properties of the noise. Specifically, we consider the LE

$$\dot{x}(t) = \mu F(x(t), \lambda(t)) + \xi_{\text{fGn}}^H(t), \quad (27)$$

where we have assumed, in analogy to the previous model equation (1), that the particle is also subject to a force F . Furthermore, $\xi_{\text{fGn}}^H(t)$ denotes the fractional Gaussian noise (FGN) with zero mean, i.e., $\langle \xi_{\text{fGn}}^H(t) \rangle = 0$, and correlation function [126–128]

$$\begin{aligned} \langle \xi_{\text{fGn}}^H(t_1) \xi_{\text{fGn}}^H(t_2) \rangle &= 2K_H(2H - 1)|t_1 - t_2|^{2H-2} \\ &+ 4K_H H |t_1 - t_2|^{2H-1} \delta(t_1 - t_2). \end{aligned} \quad (28)$$

In Eq. (28), H is the so-called Hurst parameter, whose range is given by $0 < H < 1$. The Hurst parameter is related to the exponent α governing the long-time behavior of the MSD as $2H = \alpha$. Thus the motion of the particle is subdiffusive for $H < 1/2$, diffusive for $H = 1/2$, and superdiffusive for $H > 1/2$. Furthermore, the prefactor K_H plays the role of the noise strength. For later purposes, we note that the noise correlation function of FGN, Eq. (28), depends (only) on the time difference $t_1 - t_2$ rather than separately on both times.

The process referred to as FBM emerges via an integration over time. Specifically, in the absence of a force F , the trajectory of the particle follows from Eq. (27) as

$$x(t) = \xi_{\text{fBm}}^H(t) = \int_0^t dt_1 \xi_{\text{fGn}}^H(t_1), \quad (29)$$

where $\xi_{\text{fBm}}^H(t)$ is the characteristic noise of an FBM process, with zero mean and correlation function

$$\langle \xi_{\text{fBm}}^H(t_1) \xi_{\text{fBm}}^H(t_2) \rangle = K_H (t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}). \quad (30)$$

Based on this connection, we henceforth refer to the system at hand as an ‘‘FBM-driven’’ particle. We stress that due to the time nonlocality of the FGN and FBM noise correlation functions in Eqs. (28) and (30), respectively, the dynamics of the FBM-driven particle is indeed non-Markovian, that is, the motion of the particle depends on its past. This is different from the case of delta-correlated noise with time-dependent strength considered in Sec. II [see Eq. (3)]. A common feature of both models is that the noise is not related to the mobility of the particles, which is, in both cases, a constant, μ . In other words, there is no FDR2. We now discuss consequences for the thermodynamic properties for the (non-Markovian) FBM model.

As recalled in Sec. II A, the definitions of the (trajectory-dependent) work done on the system and the heat dissipated into the medium do not involve the statistical properties of the noise appearing in the LE (as long as this noise originates from the medium). In particular, these definitions do not rely

on the Markovianity or non-Markovianity of the noise correlation functions. We can therefore employ Eq. (6) as the definition of the total heat dissipated into the medium also for the FBM-driven model. Furthermore, since we are still considering $x(t)$ as the relevant dynamical variable, we can also apply the expressions for the system entropy and system entropy production given in Eqs. (9) and (10). However, as expected, complications arise when determining the medium entropy production, since the latter requires a definition of the temperature.

Following essentially our approach in Sec. II A, we consider two scenarios for the definition of temperature. Within the first scenario [scenario (i)], the temperature is considered to be a constant throughout the medium, T_0 , whose value is yet to be quantified. In this case, the medium entropy production defined via the heat exchange is given by Eq. (11). Secondly [scenario (ii)], we introduce a generalized, time-dependent temperature defined in such a way that the resulting medium entropy production equals the corresponding expression arising from the log ratio of path probabilities. Since this is more involved than in the Markovian case discussed before, we postpone the definition of the generalized temperature to the next section.

A. Path probability ratio of the FBM-driven system

In what follows, we aim at calculating the log ratio of forward and backward path probabilities for the FBM-driven system using two distinct approaches, resulting in two representations. This twofold strategy will later facilitate the interpretation and analysis of the expressions needed in the IFR.

First, we start directly with the expression for the (conditional) path probability given in Eq. (14). This is possible, since for the FBM-driven system given in Eq. (27), the noise term is still Gaussian, i.e., we can set $\xi = \xi_{\text{fGn}}^H$. As before, the kernel G appearing in Eq. (14) is defined by the functional inverse of the noise correlation function. In the present case, we have $\int dt_3 G(t_1, t_3) \langle \xi_{\text{fGn}}^H(t_3) \xi_{\text{fGn}}^H(t_2) \rangle = \delta(t_1 - t_2)$ involving the correlation function of fractional Gaussian noise [see Eq. (28)]. For simplicity, we henceforth write the inverse of G as $\langle \xi_{\text{fGn}}^H(t_1) \xi_{\text{fGn}}^H(t_2) \rangle^{-1}$. By substituting ξ_{fGn}^H from Eq. (27) one obtains

$$\begin{aligned} P[x(t)|x(0)] &\propto \exp \left\{ -\frac{1}{2} \int_0^t dt_1 \int_0^t dt_2 (\dot{x}(t_1) - \mu F(t_1)) \right. \\ &\quad \left. \times \langle \xi_{\text{fGn}}^H(t_1) \xi_{\text{fGn}}^H(t_2) \rangle^{-1} (\dot{x}(t_2) - \mu F(t_2)) \right\}. \end{aligned} \quad (31)$$

Note that we neglect [as in (17)] the Jacobian of the transformation, due to its irrelevance in calculating the forward and backward path probability ratio. Considering the reversed trajectory obtained by $\dot{x} \rightarrow -\dot{x}$, and taking into account that the noise correlation function equation (28) is symmetric with respect to time, the logarithm of the forward and backward path ratio follows as

$$\begin{aligned} \ln \frac{P[x(t)|x(0)]}{\tilde{P}[\tilde{x}(t)|\tilde{x}(0)]} &= 2\mu \int_0^t dt_1 \int_0^t dt_2 F(t_1) \\ &\quad \times \langle \xi_{\text{fGn}}^H(t_1) \xi_{\text{fGn}}^H(t_2) \rangle^{-1} \dot{x}(t_2), \end{aligned} \quad (32)$$

which is only a function of the forward path. The above expression can be rewritten in a more familiar form by introducing (similar to Sec. II B for the SBM case) a generalized, time-dependent temperature that is proportional to the functional inverse of the noise correlation function. For FGN, this function involves two times, t_1 and t_2 , with the simplification that it only depends on the time difference; see Eq. (28). We therefore introduce the “temperature”

$$T^{-1}(t_1 - t_2) := 2\mu(\xi_{\text{FGn}}^H(t_1)\xi_{\text{FGn}}^H(t'))^{-1}, \quad (33)$$

such that $\int dt_3(2\mu)^{-1}T(t_1 - t_2)(\xi_{\text{FGn}}^H(t_3)\xi_{\text{FGn}}^H(t_2)) = \delta(t_1 - t_2)$.

Equation (33) may be considered as a generalized FDR (of the second kind), since it relates the generalized temperature to the mobility μ and the noise autocorrelation function, in analogy to our argument below Eq. (12), for the case of SBM. With this, Eq. (32) becomes

$$\ln \frac{P[x(t)|x(0)]}{\bar{P}[\bar{x}(t)|\bar{x}(0)]} = \int_0^t dt_1 \int_0^{t_1} dt_2 F(t_1) \times T^{-1}(t_1 - t_2)\dot{x}(t_2). \quad (34)$$

Combining the left-hand side of Eq. (34) with the boundary term involving the distribution of initial and final values of P as described before [see Eq. (22)], we obtain for the quantity $R_{[x]}$ in Eq. (20)

$$R_{[x]} = \Delta S_{[x]} + \int_0^t dt_1 \int_0^{t_1} dt_2 F(t_1)T^{-1}(t_1 - t_2)\dot{x}(t_2). \quad (35)$$

By definition, the so-obtained $R_{[x]}$ fulfills the IFR equation (21). We also see, however, that in order to view $R_{[x]}$ as a “total entropy production” (which appears in the IFR of standard Brownian motion), we have to introduce an unusual form of medium entropy production, that is, $\Delta S_{[x]}^m = \int_0^t dt_1 \int_0^{t_1} dt_2 F(t_1)T^{-1}(t_1 - t_2)\dot{x}(t_2)$. Clearly, the price to pay is the introduction of the time-nonlocal temperature according to Eq. (33). This strategy corresponds to scenario (ii) referred to at the beginning of Sec. III, i.e., it is analogous to the introduction of a time-dependent temperature in the SBM case [see Eq. (12)]. Furthermore, from the preceding expressions it is obvious that if we defined the medium entropy production with a constant temperature [scenario (i); see Eq. (11)], then the sum of this quantity and the system entropy would be different from $R_{[x]}$ and therefore would not fulfill the IFR, just as in the SBM system.

So far, we have evaluated the log ratio of path probabilities following the standard approach. As an alternative, we now employ a *fractional differential approach* [32–34]. To start with, we integrate Eq. (27) over time, yielding

$$x(t) - x_0 = \mu\bar{F}(x(t), \lambda(t)) + \xi_{\text{FBm}}^H(t), \quad (36)$$

where $\bar{F}(x(t), \lambda(t)) = \int_0^t dt' F(x(t'), \lambda(t'))$, $x_0 = x(0)$, and we have used Eq. (29) relating ξ_{FGn}^H to ξ_{FBm}^H . Equation (36) can be formally solved in terms of the Riemann-Liouville fractional differential operator ${}_0D_t^\beta$ [32,33], yielding

$${}_0D_t^{(H+\frac{1}{2})}(x(t) - x_0) - \mu {}_0D_t^{(H+\frac{1}{2})}\bar{F}(x(t), \lambda(t)) = \xi(t). \quad (37)$$

On the right-hand side of Eq. (37), $\xi(t)$ is a standard, Gaussian *white* noise with zero mean and autocorrelation function

$\langle \xi(t)\xi(t') \rangle = 2D\delta(t - t')$ (with D being the diffusion constant). For the special case $H = 1/2$, the fractional differential operator reduces to a normal time derivative, i.e., ${}_0D_t^\beta = d/dt$. This ensures that Eq. (37) reduces to the standard Brownian equation of motion for $H = 1/2$.

The path probability corresponding to Eq. (37) follows from Eq. (14) where, in the present case, $G(t_1, t_2) = \delta(t_1 - t_2)/(2D)$. We thus obtain

$$P[x(t)|x(0)] \propto \exp \left\{ -\frac{1}{4D} \int_0^t [{}_0D_{t'}^{(H+\frac{1}{2})}(x(t') - x_0) - \mu {}_0D_{t'}^{(H+\frac{1}{2})}\bar{F}(x(t'), \lambda(t'))]^2 dt' \right\}. \quad (38)$$

We note in passing that the use of Eq. (38) for actual calculations of quantities, such as the PDF of the particle displacement, is quite involved when $H > 1/2$. This is because additional boundary conditions involving fractional derivatives at $t = 0$ are required. Here we are rather interested in the log ratio of the forward and backward paths. To calculate the conjugate trajectory, we use a protocol that is slightly different from the conventional time-reversal protocol, defined as ${}_0D_t^{(H+\frac{1}{2})}(x(t) - x_0) \rightarrow -{}_0D_t^{(H+\frac{1}{2})}(x(t) - x_0)$. This prescription provides a backward trajectory in time with fractal dimension $H + 1/2$. With this we find

$$\ln \frac{P[x(t)|x(0)]}{\bar{P}[\bar{x}(t)|\bar{x}(0)]} = \frac{\mu}{D} \int_0^t [{}_0D_{t'}^{(H+\frac{1}{2})}(x(t') - x_0) \times {}_0D_{t'}^{(H+\frac{1}{2})}\bar{F}(x(t'), \lambda(t'))] dt'. \quad (39)$$

Before proceeding, some consistency checks are in order. First, for $H = 1/2$ and using ${}_0D_t^{(1)} = d/dt$, we recover, as we should, the expression for a normal Brownian particle in a heat bath of temperature $T_0 = D/\mu$ (according to Einstein’s relation). Second, Eq. (39) becomes equivalent to Eq. (18), that is, the log ratio of path probabilities for time-dependent noise strength, if the fractional derivatives are replaced by the ordinary time derivatives (i.e., by formally setting $H = 1/2$), and $K(t)$ is set to the constant $K_{(1/2)} = D$.

In the more interesting, non-Markovian case ($H \neq 1/2$), Eq. (39) may be considered as an alternative expression to Eq. (34) for the path probability ratio of an FBM-driven particle. In Eq. (34), the non-Markovianity enters via the nontrivial time dependence in the “temperature” defined via the functional inverse of the noise correlation function. In contrast, Eq. (39) involves a constant prefactor μ/D , suggesting that we define a constant temperature $T_0 = D/\mu$. The non-Markovian character here rather appears through the presence of fractional derivatives.

Despite these differences, similar problems of interpretation occur when we try to make the connection to thermodynamics, particularly to the medium entropy production. Especially, the integral term in Eq. (39) is equal to the conventional dissipated heat equation (6) only if $H = 1/2$. In this case the log ratio is equivalent to the medium entropy production defined in Eq. (11). For any other value of the Hurst parameter ($H \neq 1/2$) that yields a non-Markovian anomalous dynamics, no immediate conclusion about the physical meaning of the log ratio of the forward and backward path probabilities can be made. Thus the total entropy production

is not straightforwardly defined either. To proceed, we can formally introduce [based on the right-hand side of Eq. (39)] a generalized heat function

$$\mathbb{Q}(t) = \int_0^t {}_0D_{t'}^{(H+\frac{1}{2})} (x(t') - x_0) {}_0D_{t'}^{(H+\frac{1}{2})} \bar{F}(x(t'), \lambda(t')) dt', \tag{40}$$

from which we define the medium entropy production as $\Delta S_{[x]}^m = \mathbb{Q}(t)/T_0$, with $T_0 = D/\mu$. We then find from the full path probability ratio [including the bulk term given in Eq. (39), which leads to the system entropy] a relation (formally) resembling the standard IFR:

$$\langle e^{-\Delta S_{[x]} - \frac{\mathbb{Q}}{T_0}} \rangle = \langle e^{-\Delta S_{[x]}^{\text{tot}}} \rangle = 1, \tag{41}$$

where, in this case, $\Delta S_{[x]}^{\text{tot}} = \Delta S_{[x]} + \mathbb{Q}/T_0$.

So far, we have studied the ST of FBM-driven systems by introducing and exploiting different definitions of temperatures, medium entropy productions, and heat functions. These definitions were motivated by the desire to formulate, consistent with standard ST for Brownian motion, an IFR based on path probability ratios involving the total entropy production. We have shown that in order to achieve such consistency, one has to introduce either a time-nonlocal temperature $T(t_1 - t_2)$ or a generalized heat function \mathbb{Q} . Both quantities seem rather artificial. In the following section, we will shed some light on these quantities by utilizing a perturbation method [129–133].

B. Perturbation theory

In this section, we use perturbation theory to further investigate the ST of the FBM-driven system. Our main focus is to better understand the definitions of the generalized temperature and generalized heat function introduced in Eqs. (33) and (40), respectively.

As a starting point, we rewrite the Hurst parameter characterizing the FBM process [see Eqs. (28) and (30)] as

$$H = 1/2 + \epsilon, \tag{42}$$

where ϵ is now considered as a small (perturbation) parameter. Equation (42) reflects the special role of the case $H = 1/2$, for which the noise correlation function reduces to a delta function, and the (Markovian) LE (27) describes the normal diffusion of a particle under the influence of a force. By setting $\epsilon \neq 0$, the noise correlation function becomes nonlocal in time (i.e., non-Markovian), accompanied by an anomalous behavior of the particle’s MSD. Thus increasing ϵ from zero to some positive or negative value in the range $[-1/2, 1/2]$ corresponds to a smooth transition from Markovian behavior (with diffusive dynamics) to non-Markovian behavior and anomalous dynamics.

Instead of applying the perturbation method directly to the kernel, as was done in Refs. [129–133] for calculating the path probability, here we perform our perturbation analysis on the level of the LE. This will allow us not only to calculate the log ratio of the forward and backward path probabilities, but also to study the ST of the system for small values of ϵ .

We start from the integrated LE (36). The FBM noise appearing on the right-hand side of the equation can be represented by the Riemann-Liouville fractional integral ${}_0I_t^\beta$ of

Gaussian white noise [134], that is,

$$\begin{aligned} \xi_{\text{FBM}}^H(t) &= \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t dt' (t - t')^{H-\frac{1}{2}} \xi(t') \\ &= {}_0I_t^{(H+\frac{1}{2})} \xi. \end{aligned} \tag{43}$$

Substituting Eq. (43) into the LE (36), one obtains

$$x(t) - x_0 = \mu \bar{F}(x(t), \lambda(t)) + {}_0I_t^{(H+\frac{1}{2})} \xi. \tag{44}$$

We note that for $H = 1/2$, the conventional LE for normal Brownian motion is recovered by differentiating both sides with respect to time (recall that \bar{F} corresponds to the time-integrated force). Our goal is now to expand the H -dependent terms in Eq. (44), where H is given in Eq. (42), up to the first order in ϵ . To this end, we perform a Taylor expansion of Eq. (43) around $\epsilon = 0$, yielding

$$\begin{aligned} {}_0I_t^{(H+\frac{1}{2})} \xi(t) &= \int_0^t dt' \xi(t') + \epsilon \left[\zeta \int_0^t dt' \xi(t') \right. \\ &\quad \left. + \int_0^t dt' \ln(t - t') \xi(t') \right] + O(\epsilon^2), \end{aligned} \tag{45}$$

where ζ is the Euler-Mascheroni constant given by the negative sign of the first derivative of the gamma function with respect to ϵ at $\epsilon = 0$, $\zeta = -\Gamma'(1) \simeq 0.577$. Substituting Eq. (45) into Eq. (44) and differentiating both sides with respect to time yields

$$\begin{aligned} \dot{x}(t) - \mu F(x(t), \lambda(t)) &= \xi(t) + \epsilon \left[(\zeta + \ln \tau) \xi(t) + \int_0^t dt' |t - t'|^{-1} \xi(t') \right]. \end{aligned} \tag{46}$$

Here, the parameter τ is chosen to separate the two coinciding times and is considered to be a small cutoff time. It is introduced in order to avoid the divergence of the log term for the two coinciding times by using a regularization technique. As we will proceed to show, this parameter appears only as a constant in the (renormalized) diffusion coefficient. This correction can later be removed by choosing a particular value for τ .

Inspecting Eq. (46), we see that to zeroth order of ϵ (i.e., $\sim \epsilon^0$), it reduces to the LE for normal diffusion,

$$\dot{\xi}(t) = \dot{x}(t) - \mu F(x(t), \lambda(t)), \tag{47}$$

as it should. We now insert this zeroth-order result to replace $\xi(t)$ in the first-order equation (46). Solving with respect to $\xi(t)$, we obtain

$$\begin{aligned} \xi(t) &= K_\epsilon^{-1} [\dot{x}(t) - \mu F(x(t), \lambda(t))] \\ &\quad - \epsilon \int_0^t dt' |t - t'|^{-1} [\dot{x}(t') - \mu F(x(t'), \lambda(t'))], \end{aligned} \tag{48}$$

where we have introduced $K_\epsilon^{-1} = 1 - \epsilon(\zeta + \ln \tau)$.

We now proceed towards the path probability. To this end, we recall that $\xi(t)$ is a Gaussian process, such that the path probability can be readily found from Eq. (14), with $G(t_1, t_2) = \delta(t_1 - t_2)/2D$. Substituting $\xi(t)$ from Eq. (48), we

find

$$P[x(t)|x(0)] \propto \exp \left\{ -\frac{1}{4D} \int_0^t dt' \left[K_\epsilon^{-1}(\dot{x}_{t'} - \mu F_{t'}) - \frac{\epsilon}{4D} \int_0^{t'} dt'' |t' - t''|^{-1} (\dot{x}_{t''} - \mu F_{t''}) \right]^2 \right\}. \tag{49}$$

To better see the impact of ϵ , we expand Eq. (49) up to the first order in this parameter, yielding

$$P[x(t)|x(0)] \propto \exp \left\{ \left(-\frac{1}{4D} + \frac{\epsilon(\zeta + \ln \tau)}{2D} \right) \times \int_0^t dt' (\dot{x}_{t'} - \mu F_{t'})^2 \right\} \times \exp \left\{ \frac{\epsilon}{2D} \int_0^t dt' (\dot{x}_{t'} - \mu F_{t'}) \times \int_0^{t'} dt'' \frac{(\dot{x}_{t''} - \mu F_{t''})}{|t' - t''|} \right\}. \tag{50}$$

On the right-hand side of Eq. (50), the first exponential already resembles the path probability of a normal diffusive process, with a correction in the prefactor of the integral. This correction, which can be interpreted as a renormalization of the diffusion constant, can be set to zero by choosing $\tau = e^{-\zeta}$ (recall that τ is a free parameter). In this way the first exponential becomes equivalent to the Brownian case. The second exponential in Eq. (50), however, reflects the non-Markovian character of the noise correlation function, as seen from the double time integral in the exponent (and the prefactor ϵ of the integral). In this sense, the second exponential represents the signature of non-Markovianity within our first-order expansion. We note that the result, Eq. (50), matches the perturbative path probability of the FBM calculated in Refs. [129–133].

We are now in the position to calculate the log ratio of the forward and the backward path probabilities (with the final goal of investigating the IFR). Following the same protocol for time reversal as before in the standard approach [see Eq. (32)], we find

$$\ln \frac{P[x(t)|x(0)]}{\tilde{P}[\tilde{x}(t)|\tilde{x}(0)]} = \frac{\mu}{D} \int_0^t dt' \dot{x}_{t'} F_{t'} - \epsilon \frac{\mu}{D} \times \int_0^t dt' \int_0^{t'} dt'' \frac{\dot{x}_{t''} F_{t'}}{|t' - t''|} + O(\epsilon^2). \tag{51}$$

Equation (51) provides a useful starting point for a *physical* interpretation of the log ratio of the forward and backward probabilities for the FBM-driven system. To this end, we compare Eq. (51) with the corresponding (exact) results obtained via the standard and fractional differential approaches leading to Eqs. (32) and (39), respectively. Within the standard approach we have defined a time-dependent temperature $T(t_1 - t_2)$ [see Eq. (33)] in order to identify the log ratio given in Eq. (32) as a medium entropy production [see second term in Eq. (35)]. We can now specify this temperature up to first order in ϵ . Specifically, we compare Eqs. (51) and (34), after plugging into the latter the ansatz $T^{-1}(t_1 - t_2) =$

$T_{(0)}^{-1}(t_1 - t_2) + \epsilon T_{(1)}^{-1}(t_1 - t_2) + O(\epsilon^2)$. By this we identify

$$T_{(0)}^{-1}(t_1 - t_2) = \frac{\mu}{D} \delta(t_1 - t_2), \tag{52}$$

$$T_{(1)}^{-1}(t_1 - t_2) = \frac{\mu}{D} |t_1 - t_2|^{-1},$$

where the superscript -1 is now meant as an ordinary inverse (not a functional inverse anymore). By inverting the zeroth-order term to get $T_{(0)} = T_0 \delta(t_1 - t_2)$ with $T_0 = D/\mu$, we see that this term is related to the classical definition of the temperature in normal Brownian motion. In contrast to $T_{(0)}$, the first-order term $T_{(1)}^{-1}$ is nonlocal in time and thereby introduces the impact of the non-Markovianity of the dynamics.

With the definitions in Eq. (52), we can now rewrite Eq. (34) [or, equivalently, Eq. (51)] in terms of the standard medium entropy production of a system at fixed temperature plus correction terms, i.e.,

$$\ln \frac{P[x(t)|x(0)]}{\tilde{P}[\tilde{x}(t)|\tilde{x}(0)]} = \Delta S_{[x]}^{m,(0)}(t) + \epsilon \Delta S_{[x]}^{m,(1)}(t) + \dots, \tag{53}$$

where

$$\Delta S_{[x]}^{m,(0)} = \frac{\mu}{D} \int_0^t dt_1 \dot{x}(t_1) F(t_1),$$

$$\Delta S_{[x]}^{m,(1)} = \frac{\mu}{D} \int_0^t dt_1 \int_0^{t_1} dt_2 \frac{\dot{x}(t_2) F(t_2)}{|t_1 - t_2|}. \tag{54}$$

Thus the zeroth order matches the conventional definition of the medium entropy production, while the first order includes the effect of the non-Markovianity.

Another important quantity, which we have introduced within the fractional differential approach for the path probability ratio [see Eq. (39)], is the generalized heat function given in Eq. (40). To shed light on the physical meaning of this function, we first rewrite Eq. (51) as

$$\ln \frac{P[x(t)|x(0)]}{\tilde{P}[\tilde{x}(t)|\tilde{x}(0)]} = \frac{1}{T_0} \Delta Q(t) - \frac{\epsilon}{T_0} (\dot{x} \Sigma_F(t) + F \Sigma_{\dot{x}}(t)) + O(\epsilon^2), \tag{55}$$

where $\Delta Q(t)/T_0$ corresponds to the first term on the right-hand side of Eq. (51) [which equals $\Delta S_{[x]}^{m,(0)}(t)$ introduced in Eq. (54)] and

$$\dot{x} \Sigma_F(t) = \int_0^t dt'' \dot{x}_{t''} \int_0^{t''} dt' \frac{F_{t'}}{(t'' - t')} = \int_0^t dt'' \dot{x}_{t''} \tilde{F}_{t''},$$

$$F \Sigma_{\dot{x}}(t) = \int_0^t dt'' F_{t''} \int_0^{t''} dt' \frac{\dot{x}_{t'}}{(t'' - t')} = \int_0^t dt'' F_{t''} \tilde{x}_{t''}. \tag{56}$$

Here, we have introduced a retarded velocity \tilde{x} and retarded force \tilde{F} . The two terms arise from a splitting of the double time integral in the second term in Eq. (51).

Equation (55) reveals that, upon deviating from the normal-diffusion regime ($\epsilon = 0$), an additional heat exchange between the system and the (viscoelastic) medium takes place. This is due to the memory imposed by the environment, which is then translated into a retardation of the force and the velocity. We note that the two terms in Eq. (56) arise through the perturbation expansion around $\epsilon = 0$; as such, they are

independent of ϵ . Having this in mind, we can conclude that positive values of ϵ , which correspond to superdiffusion, lead to a *reduction* in the heat exchange, whereas negative values corresponding to subdiffusion lead to an *increase* in heat exchange.

Furthermore, it is now evident that $\Delta Q(t)$ and the sum ${}_x\Sigma_F(t) + {}_F\Sigma_x(t)$ are the zeroth order and first order of the generalized heat exchange function, respectively, i.e.,

$$\mathbb{Q}(t) = \Delta Q(t) - \epsilon({}_x\Sigma_F(t) + {}_F\Sigma_x(t)) + O(\epsilon^2). \quad (57)$$

We note that this conclusion could also be obtained by directly expanding the generalized heat function, Eq. (40). However, the singularities in the case $H = 1/2$ are handled more conveniently and systematically with the current approach.

We finally turn back to the IFR. Combining Eq. (55) with the expression for the boundary term of the (full) path probability ratio and using Eq. (21), we obtain a ‘‘perturbative form’’ of an IFR for the entropy production, which up to the first order in ϵ reads

$$\langle e^{-\Delta S_{[x]}^{\text{tot}} + \frac{\epsilon}{T_0}({}_x\Sigma_F + {}_F\Sigma_x) + O(\epsilon^2)} \rangle = 1. \quad (58)$$

Here, $\Delta S_{[x]}^{\text{tot}} = \Delta S_{[x]}^{m,(0)}(t) + \Delta S_{[x]}$. Equation (58) nicely demonstrates how the additional heat exchange defined in Eq. (57) enters into the IFR of the entropy production.

We finally remark that the appearance of additional terms supplementing the conventional total entropy production in the IFR is in line with other studies for diffusion in complex environments such as active baths [65] or systems with time-delayed feedback [88], although the underlying processes are very different. Interestingly, in Ref. [65] these additional contributions were interpreted in terms of a mutual information production between particle and bath dynamics. For our FBM-driven system, if we define $\Delta S_{[x]}^I = \mathbb{Q} - \Delta Q$, by using Eq. (57) we can trivially rewrite Eq. (58) as

$$\langle e^{-\Delta S_{[x]}^{\text{tot}} + \Delta S_{[x]}^I} \rangle = 1. \quad (59)$$

Whether one could express $\Delta S_{[x]}^I$ in terms of mutual information production remains an interesting open question.

IV. CONCLUSIONS

In this paper we have explored the applicability of ST to systems displaying anomalous diffusion. We have studied two important cases, namely, Markovian systems with time-dependent noise strength (such as SBM) and FBM. The latter provides a paradigmatic example of a *non-Markovian* process yielding anomalous diffusion, where the non-Markovianity stems from the noise correlation function. Methodologically, we have essentially followed the definitions and derivations of ST quantities and the IFR for standard Brownian systems [3]. Not surprisingly, the treatment of FBM turned out to be challenging.

One of the major results concerns the role of a (generalized) FDR of the second kind, connected with the definition of a (generalized) temperature. For conventional Brownian dynamics, these issues are straightforward: The FDR relating the (delta-like) noise correlation function with constant diffusion coefficient D to the constant mobility μ (which implies a delta-like friction kernel) leads directly to the definition

of a (constant) temperature $T_0 = D/\mu$. This immediately allows one to define the heat exchange with the medium, as well as the medium entropy production consistent with the corresponding expression from the log ratio of (forward and backward) path probabilities. Furthermore, consideration of the full log ratio (i.e., the quantity $R_{[x]}$) directly leads to the total entropy production ΔS^{tot} (as the sum of system and medium entropy) and the IFR related to this quantity.

As we showed in Sec. II, these well-established concepts have to be handled with care already for the relatively simple (Markovian) case of a time-dependent noise strength. In that case, the noise correlation function is not related to mobility, i.e., there is no FDR from the LE. Therefore the definition of temperature is not obvious. If we define the temperature as a time-dependent function $T(t)$ related to the noise strength, thereby introducing a ‘‘generalized FDR’’ (of the second kind), and define the heat exchange accordingly, then the medium entropy production defined through the heat becomes consistent with the corresponding path probability expression. The IFR for ΔS^{tot} then follows automatically. In contrast, if we set the temperature to a constant, we can still define heat exchange, but the two routes towards the medium entropy production now yield different results. As a consequence, we observe deviations from the IFR for ΔS^{tot} if the latter is defined in a physical way as ‘‘system entropy plus heat exchange.’’ We stress that, regardless of any definitions, the IFR for the quantity $R_{[x]}$, that is, the log ratio of path probabilities, is always true by definition. The question, rather, is whether $R_{[x]}$ corresponds to the physical total entropy production or to a somewhat modified quantity. This is what we mean by ‘‘deviation’’ here.

Similar conceptual issues arise in the FBM case. However, here the analysis becomes more demanding due to the non-Markovian character of the noise correlation function. This leads (when requiring consistency between different routes to the medium entropy production) to a temperature depending on a finite time difference, which clearly reveals the presence of memory effects. In other words, one can introduce some form of an FDR, but the price to pay is a temperature with memory. An alternative view comes up when treating the problem via functional differentiation. Along these lines, consideration of the log ratio of path probabilities suggests a *constant* temperature (due to the white noise appearing in the fractional LE) but a highly nontrivial heat function whose physical interpretation remains obscure. So again, there is a price to pay. We then have shown that these quantities, the time-nonlocal temperature and the generalized heat function, can be interpreted to some extent via a perturbation expansion of the Hurst parameter H around the Brownian case ($H = 1/2$). The zeroth-order expressions recover the standard results for Brownian motion. A major result consists of our *explicit* first-order expressions for the generalized temperature and heat dissipation, both reflecting clearly the presence of memory. For example, the first-order correction to the heat dissipation can be physically interpreted as extra heat exchanges between the system and the medium that include the memory of the environment through either a retarded force or a retarded velocity.

We close with some more general remarks on the embedding of our work in the field of ST. The starting point of

this paper was the wealth of literature concerning the ST of Brownian and Markovian systems. Within this framework, it has been shown that FRs provide a universal relationship that is valid even very far from equilibrium thus generalizing conventional linear response theory. Starting from FRs, expressions for a nonlinear response have also been obtained going beyond Onsager reciprocity relations [21,135,136]. Accordingly, it would be very interesting to calculate nonlinear response relations for non-Markovian systems from FRs as well, both with and without FDR, in order to learn more about the importance of Markovianity and FDR in nonequilibrium situations. Furthermore, as pointed out in the Introduction, there have been several recent efforts to generalize aspects of ST, particularly FRs, towards non-Markovian systems described by GLEs. In this paper we have asked, more generally, what can be learned when we apply the “standard” ST scheme with notions such as entropy, heat, and temperature to systems exhibiting anomalous dynamics, which are “athermal” [3] in the sense that there is no direct relation between temperature and noise. We emphasize (again) that such processes are by no means “exotic” mathematical artifacts, as they are widely observed in physical and biological experiments. In particular, they may have important applications to better understand

active matter, such as the motion of a tracer particle in an active bath [66–70] and the dynamics of a single active particle [65], if the persistence of the active particle(s) reflecting the self-propulsion was anomalous, in the sense of being stronger than exponentially correlated. We remark that anomalous features of dynamics in active baths have already been observed experimentally and modeled theoretically [137]. Our analysis shows that ST for strongly correlated processes indeed suggests new definitions for thermodynamic quantities such as temperature or heat and a respective interpretation of the physical contributions. We thus view our present analysis as an important contribution to the development of ST for anomalous dynamics.

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