

Entanglement and separability in continuum Rokhsar-Kivelson states

Christian Boudreault^{1,2,*}, Clément Berthière^{2,3,†} and William Witczak-Krempa^{2,3,4,‡}

¹Département des sciences de la nature, Collège militaire royal de Saint-Jean,
15 Jacques-Cartier Nord, Saint-Jean-sur-Richelieu, Québec, Canada J3B 8R8

²Université de Montréal, C. P. 6128, Succursale Centre-ville, Montréal, Québec, Canada H3C 3J7

³Centre de Recherches Mathématiques, Université de Montréal, Montréal, Québec, Canada H3C 3J7

⁴Institut Courtois, Université de Montréal, Montréal (Québec), Canada H2V 0B3



(Received 26 November 2021; revised 17 June 2022; accepted 24 August 2022; published 30 September 2022)

We study a vast family of continuum Rokhsar-Kivelson (RK) states, which have their ground state encoded by a local quantum field theory. These describe certain quantum magnets, and are also important in quantum information. We prove the separability of the reduced density matrix of two disconnected subsystems, implying the absence of entanglement between the two subsystems—a stronger statement than the vanishing of logarithmic negativity. As a particular instance, we investigate the case where the ground state is described by a relativistic boson, which is relevant for certain magnets or Lifshitz critical points with dynamical exponent $z = 2$, and we propose nontrivial deformations that preserve their RK structure. Specializing to 1D systems, we study a deformation that maps the ground state to the quantum harmonic oscillator, leading to a gap for the boson. We study the resulting correlation functions, and find that cluster decomposition is restored. We analytically compute the c function for the entanglement entropy along a renormalization group flow for the wavefunction, which is found to be strictly decreasing as in CFTs. Finally, we comment on the relations to certain stoquastic quantum spin chains. We show that the Motzkin and Fredkin chains possess unusual entanglement properties not properly captured by previous studies.

DOI: [10.1103/PhysRevResearch.4.033251](https://doi.org/10.1103/PhysRevResearch.4.033251)

I. INTRODUCTION

Rokhsar-Kivelson (RK) states [1,2] are ground states that quantum mechanically encode the partition function of a classical system. They describe numerous quantum magnets at low temperature [3,4], quantum critical phase transitions [5], and novel spin chains [6,7]. One can in principle construct a RK wavefunction for a quantum theory in $(d + 1)$ space-time dimensions from an arbitrary d -dimensional classical model (e.g., statistical mechanics). This quantum-classical correspondence can actually be made more precise, e.g., for lattice systems [2,6,8]. The normalization factor of the (unnormalized) RK ground state $|\Psi_0^{(d+1)}\rangle$ is interpreted as the partition function of the lower-dimensional classical system, $\langle \Psi_0^{(d+1)} | \Psi_0^{(d+1)} \rangle = Z_{cl}^{(d)}$. Because of their simple form and relation to lower-dimensional classical theories, RK states offer a useful and controlled framework to study, e.g., correlation functions and entanglement measures such as entanglement entropy, which is notoriously hard to achieve in quantum many-body systems. Systems poised at a RK point have been

extensively studied, in particular their entanglement properties [3,4,9–24].

In this paper, we study the entanglement and correlations properties of a vast family of *continuum* RK states for which the dual classical models are local quantum field theories (QFTs). We prove the separability of the reduced density matrix of two disconnected subsystems for continuum RK states, implying the absence of entanglement between the two subsystems—a stronger statement than the vanishing of logarithmic negativity [25–28].

As a particular instance, we investigate Lifshitz ground states with dynamical exponent $z = 2$. We propose nontrivial deformations that preserve their RK structure, making these theories rare examples of nonrelativistic theories, which admit analytic treatment. By examining critical Lifshitz theories, comparing entanglement entropy and capacity of entanglement we argue that entanglement is effectively carried by maximally entangled EPR pairs, unlike systems with emergent Lorentz invariance (e.g., gapless Dirac fermions).

We also apply our findings to certain stoquastic spin chains introduced by Shor, Bravyi, *et al.* [29]. We show that Motzkin and Fredkin spin chains possess unusual entanglement properties not properly captured by continuum descriptions proposed in previous studies. Our paper raises the urgent question regarding the correct field theoretical description of those spin chains, and we conjecture possibilities.

This paper is organized as follows. We begin in Sec. II with some background on (continuum) RK states, and study their reduced density matrices. We then prove the separability of the reduced density matrix of two disconnected subsystems

*christian.boudreault@cmrsj-rmcjsj.ca

†clement.berthiere@umontreal.ca

‡w.witczak-krempa@umontreal.ca

for continuum RK wavefunctional. In Sec. III, we review key concepts for Lifshitz theories, and in particular the RK structure of their ground states. We introduce in Sec. IV a massive deformation preserving the RK property of the Lifshitz ground state, and then proceed to study this model in $1+1$ dimensions. We present our results on correlations and entanglement properties of the (massive) theory. We find that cluster decomposition, violated in the massless case, is restored by the regulating mass. Along the way, aside from Rényi entropies and mutual information, we compute other entanglement-related quantities: the capacity of entanglement, the entropic c function, and the sharp limit coefficient of corner entanglement. Our results on the positive-field version of the massless Lifshitz theory are presented in Sec. V, where we discuss its relation to Motzkin and Fredkin spin chains and results in the literature. We conclude in Sec. VI with a summary of our main results, and give an outlook on future study. Appendix A is devoted to a non-Gaussian deformation of Lifshitz theory for which the ground state is described by $SL(2, \mathbb{R})$ -conformal quantum mechanics. This deformation preserves the conformal spatial symmetry of the ground state, and constrains the form of the correlators and Rényi entropies. Finally, Appendix B contains our results on the capacity of entanglement.

II. ROKHSAR-KIVELSON STATES, REDUCED DENSITY MATRICES, AND SEPARABILITY

In this paper, we study the vast family of quantum RK states, that is, ground states of the form

$$\Psi_0[\phi] \propto e^{-\frac{1}{2}S_{\text{cl}}[\phi]}, \quad (1)$$

for some *local* Euclidean action $S_{\text{cl}}[\phi]$ involving a finite number of derivatives of the field. Such a state is the ground state of a parent Hamiltonian

$$H = \frac{1}{2} \int d^d x \{A^\dagger(x), A(x)\}, \quad (2)$$

quadratic in the operators

$$A(x) = \frac{1}{\sqrt{2}} \left(i\Pi(x) + \frac{1}{2} \frac{\delta S_{\text{cl}}[\phi]}{\delta \phi(x)} \right), \quad (3)$$

where, in the field-eigenstate Schrödinger picture, canonical quantization demands $\Pi(x) = -i\delta/\delta\phi(x)$. Clearly, positive semidefiniteness is preserved for any deformation of the type

$$\frac{1}{2}S_{\text{cl}} \mapsto \frac{1}{2}S_{\text{cl}} + \Lambda, \quad (4)$$

where the deformations Λ is real and local to ensure reality and locality of the action. With $A(x)$ mapped to

$$A'(x) = A(x) + \frac{\delta}{\delta\phi(x)} \Lambda = e^{-\Lambda} A(x) e^{\Lambda}, \quad (5)$$

we see that (normalizable) states annihilated either by the $A(x)$'s or the $A'(x)$'s are related under the transformation $\Psi'_0 = e^{-\Lambda} \Psi_0$:

$$A(x)\Psi_0[\phi] = 0 \iff A'(x)\Psi'_0[\phi] = 0. \quad (6)$$

Such states, if any, must be ground states of their respective Hamiltonians. We emphasize that the new Hamiltonian H'

is generally *not* the result of a simple frame transformation, as $H' \neq e^{-\Lambda} H e^{\Lambda}$. (H' and $e^{-\Lambda} H e^{\Lambda}$ agree if and only if Λ is imaginary valued.)

In this paper, we shall study a particular instance of RK states, that is Lifshitz ground states and perform deformations thereof. In the following section, we first derive general results for continuum RK wavefunctional, such as a formula for Rényi entanglement entropies, and show the separability of the reduced density matrix of disconnected subsystems.

A. Reduced density matrix and replica trick

We derive formulas for the Rényi entanglement entropies of continuum RK wavefunctional. For simplicity, we work in one spatial dimension, though our results generalize naturally to higher dimensions. The Rényi entanglement entropies for a finite bipartition $\{A, B\}$ are defined as

$$S_n(A) = \frac{1}{1-n} \log \text{Tr} \rho_A^n, \quad (7)$$

where ρ_A is the reduced density matrix on subsystem A . The entanglement entropy is obtained from (7) by taking the replica limit $n \rightarrow 1$.

We begin with normalized states formally written as $|\Psi\rangle = \int \mathcal{D}\phi \Psi[\phi] |\phi\rangle$, in one-to-one correspondence with Schrödinger wavefunctionals Ψ , and we make two assumptions:

Assumption 1. The ϕ are (dimensionless) real fields, almost everywhere C^1 on $[0, L]$, and $\mathcal{D}\phi$ is a measure for such fields.

Field eigenstates

$$\chi_\phi[\phi'] = \begin{cases} 1, & \phi' = \phi, \\ 0, & \text{otherwise,} \end{cases} \quad (8)$$

have no spatial entanglement: for almost any open partition $\{A, B\}$ of the physical space, $\chi_\phi = \chi_{\phi|_A} \otimes \chi_{\phi|_B}$. Equivalently,

$$|\phi\rangle = |\alpha\rangle_\phi \otimes |\beta\rangle_\phi, \quad (9)$$

where $\alpha = \phi|_A$ and $\beta = \phi|_B$ are C^1 almost everywhere on their respective domain, and $\phi = \{\phi_1, \dots, \phi_M\}$ is an *anchor* for these fields: $\alpha(x_i) = \phi_i = \beta(x_i)$ for each x_i at the common boundary of A and B . We now make an assumption specific to a distinguished class of states $|\Psi\rangle$:

Assumption 2. Ψ is real-valued, and for α, β as above, $\Psi[\phi] = a[\alpha] \otimes b[\beta]$ for some real functionals a, b . This means that the state Ψ is *local*.

Then

$$\begin{aligned} |\Psi\rangle &= \int d\phi \int (\mathcal{D}\alpha)_\phi a[\alpha] |\alpha\rangle_\phi \otimes \int (\mathcal{D}\beta)_\phi b[\beta] |\beta\rangle_\phi \\ &= \int d\phi |\Psi_A\rangle_\phi \otimes |\Psi_B\rangle_\phi, \end{aligned} \quad (10)$$

which formally resembles a Schmidt decomposition for $|\Psi\rangle$ with anchors as Schmidt index, and can be cast in actual Schmidt form $|\Psi\rangle = \sum_{i=1}^{\infty} \sigma_i |u_i\rangle \otimes |v_i\rangle$, if necessary, by expressing the ϕ integral as a convergent sum over increasingly finer mesh elements such that the i th mesh element contains point ϕ_i , and where $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ are respectively obtained by Gram-Schmidt orthonormalization of the sets $\{|\Psi_A\rangle_{\phi_i} - \sum_{j<i} |\Psi_A\rangle_{\phi_j}\}$, and $\{|\Psi_B\rangle_{\phi_i} - \sum_{j<i} |\Psi_B\rangle_{\phi_j}\}$. (Here,

$j < i$ means that j is a coarser mesh element containing i .) We will not require the Schmidt decomposition, and will be content with (10). Actually, we will use Assumption 2 only at the end of the argument, so for now we simply write the state in terms of anchored fields, $|\Psi\rangle = \int d\phi \int (\mathcal{D}\alpha\beta)_\phi \Psi[\alpha\beta]|\alpha\rangle_\phi \otimes |\beta\rangle_\phi = \int d\phi |\Psi\rangle_\phi$. Note that in higher dimensions, integrals over fields become path integrals. The corresponding density matrix is

$$\begin{aligned} \rho &= |\Psi\rangle\langle\Psi| \\ &= \int (d\phi\phi')(\mathcal{D}\alpha\beta)_\phi(\mathcal{D}\alpha'\beta')_{\phi'} |\alpha\rangle_\phi |\beta\rangle_\phi \Psi[\alpha\beta] \\ &\quad \otimes \Psi^*[\alpha'\beta']_{\phi'} \langle\alpha'|_{\phi'} \langle\beta'|, \end{aligned} \quad (11)$$

and the reduced density is

$$\begin{aligned} \rho_A &= \text{Tr}_B \rho \\ &= \int d\phi(\mathcal{D}\alpha\alpha'\beta)_\phi |\alpha\rangle_\phi (\Psi[\alpha\beta]\Psi^*[\alpha'\beta])_\phi \langle\alpha'|. \end{aligned} \quad (12)$$

Note that all fields now have the same anchor due to the partial trace $\text{Tr}_B(\cdot) = \int d\phi(\mathcal{D}\beta)_\phi \langle\beta| \cdot |\beta\rangle_\phi$. When $\text{Tr}_A \rho_A^n$ is explicitly written out, we find a cyclic product of projectors of the form

$$\begin{aligned} \text{Tr}_A(\cdots |\alpha_i\rangle_{\phi_i} \langle\alpha'_i|_{\alpha_{i+1}} \langle\alpha'_{i+1}|_{\phi_{i+1}} \langle\alpha'_{i+1}| \cdots) \\ = \prod_i \delta_{\alpha'_i \alpha_{i+1}} \delta_{\phi_i \phi_{i+1}}, \end{aligned} \quad (13)$$

forcing all anchors to agree: $\phi_1 = \cdots = \phi_n = \phi$. Thus

$$\begin{aligned} \text{Tr}_A \rho_A^n &= \int d\phi \prod_i \int (\mathcal{D}\alpha_i\beta_i)_\phi |\Psi[\alpha_i\beta_i]|^2 \\ &= \int d\phi \left[\int (\mathcal{D}\alpha\beta)_\phi |\Psi[\alpha\beta]|^2 \right]^n \\ &= \int d\phi \left[\langle\Psi|\Psi\rangle_\phi \right]^n. \end{aligned} \quad (14)$$

Manifestly, $\text{Tr}_A \rho_A^n = \text{Tr}_B \rho_B^n$. Up to this point we have only used Assumption 1. If we now use the second assumption, we obtain

$$\text{Tr}_A \rho_A^n = \int d\phi \left[\int (\mathcal{D}\alpha)_\phi a^2[\alpha] \int (\mathcal{D}\beta)_\phi b^2[\beta] \right]^n. \quad (15)$$

Furthermore, on any interval $U = (x_{i_1}, x_{i_2})$ for x_i at the common boundary of A and B , the functional Ψ factors as $\Psi|_U \times \cdots$, and we may consistently define a local ‘‘action’’ by adjoining pieces of the form

$$\frac{1}{2} S_U = -\log \Psi|_U. \quad (16)$$

Then (15) splits into factors $\langle\phi_{i_2}, x_{i_2}|\phi_{i_1}, x_{i_1}\rangle \stackrel{\text{def}}{=} \int (\mathcal{D}\alpha)_\phi (\Psi|_U)^2[\alpha] = \int (\mathcal{D}\alpha)_\phi e^{-S_U[\alpha]}$, which we identify as the propagators of the dynamics associated to Ψ . It follows that

$$\text{Tr}_A \rho_A^n = \int d\phi [f_M(\phi, \mathbf{x})]^n, \quad (17)$$

with $f_M(\phi, \mathbf{x}) = \langle h_2, L|\phi_M, x_M\rangle \cdots \langle\phi_1, x_1|h_1, 0\rangle$ for boundary conditions $\phi(0) = h_1$, $\phi(L) = h_2$. If the system is

periodic, the (variable) boundary value $\phi_0 = \phi(0) = \phi(L)$ is an additional anchor to be integrated on in (17).

For the dimensional fields, as in (34), we have $\dim f_M = (\text{length})^{-M/2}$. Under the rescaling $\phi \rightarrow \sqrt{\epsilon}\phi$, $f_M \rightarrow \epsilon^{-M/2} f_M$, where ϵ is a local length scale (e.g., a lattice constant or UV cutoff), we have

$$\int d\phi [f_M]^n \rightarrow \epsilon^{(n-1)M/2} \int d\phi [f_M]^n, \quad (18)$$

which, using (17) in (7), yields the Rényi entropies

$$S_n(A) = \frac{1}{1-n} \log \left[\epsilon^{(n-1)M/2} \int d\phi [f_M(\phi, \mathbf{x})]^n \right]. \quad (19)$$

To eliminate any concerns about normalization, one can construct the entropy out of the explicitly normalized object $\rho/\text{Tr} \rho$, which amounts to making the change $f_M(\phi, \mathbf{x}) \rightarrow f_M(\phi, \mathbf{x})/\int d\phi' f_M(\phi', \mathbf{x})$ in the above formula.

B. Separability of $\rho_{A \cup B}$ for disjoint subsystems

We show here that for general continuum states satisfying the RK property (which include the models considered next in the present paper), the reduced density $\rho_{A \cup B}$ is (mixed) separable,

$$\rho_{A \cup B} = \sum_{j \in J} p_j (\rho_{A,j} \otimes \rho_{B,j}), \quad (20)$$

for any disconnected subsystems A and B . In the above expression, J is some index set, and p is a probability distribution on J . Thus, tracing out the complement of $A \cup B$ completely disentangles $|\Psi_0\rangle$. However, as long as p is not the trivial distribution, A and B will share mutual information.

Consider a general tripartition $\{A, C, B\}$, where C is to be traced out. From (12), a state satisfying both assumptions of Sec. II A has reduced density

$$\langle\alpha\beta|\rho_{A \cup B}|\alpha'\beta'\rangle = \int (\mathcal{D}\gamma)_\phi \Psi[\alpha\beta\gamma] \Psi[\alpha'\beta'\gamma], \quad (21)$$

where α, β are as in (9), ϕ is the anchor for these fields, and γ is restricted to the complement of $A \cup B$. By the second assumption, $\Psi[\alpha\beta\gamma] = a[\alpha]b[\beta]\Gamma[\gamma]$, where Γ is the restriction of Ψ to the complement of $A \cup B$, so

$$\langle\alpha\beta|\rho_{A \cup B}|\alpha'\beta'\rangle = \int (\mathcal{D}\gamma)_\phi a[\alpha]b[\beta]\Gamma^2[\gamma]a[\alpha']b[\beta']. \quad (22)$$

Since A and B do not have a boundary in common, they do not share any anchor. Therefore, the reduced density is a separable mixed state as it can be written as

$$\rho_{A \cup B} = \int d\phi p(\phi) \rho_A(\phi) \otimes \rho_B(\phi), \quad (23)$$

with probability distribution

$$p(\phi) = \int (\mathcal{D}\alpha\beta\gamma)_\phi a^2[\alpha]b^2[\beta]\Gamma^2[\gamma], \quad (24)$$

and subsystem densities

$$\begin{aligned}\rho_A(\phi) &= \frac{\int (\mathcal{D}\alpha\alpha')_\phi |\alpha\rangle_\phi a[\alpha] a[\alpha']_\phi \langle\alpha'|}{\int (\mathcal{D}\alpha)_\phi a^2[\alpha]} = |\Psi_A\rangle_{\phi\phi} \langle\Psi_A|, \\ \rho_B(\phi) &= \frac{\int (\mathcal{D}\beta\beta')_\phi |\beta\rangle_\phi b[\beta] b[\beta']_\phi \langle\beta'|}{\int (\mathcal{D}\beta)_\phi b^2[\beta]} = |\Psi_B\rangle_{\phi\phi} \langle\Psi_B|.\end{aligned}\quad (25)$$

Note that (24) is a probability distribution by virtue of the normalization of $|\Psi\rangle$, and that the vectors $|\Psi_A\rangle_\phi$ and $|\Psi_B\rangle_\phi$ in (25) are normalized by construction. If necessary, the integral in (23) can be converted to a convergent countable sum over increasingly fine mesh elements, in the same way the (discrete) Schmidt form was obtained in Sec. II A. As obvious from the subsystem densities (25), ρ_{AUB} is invariant under partial transpositions

$$\rho_{AUB}^{T_A} = \rho_{AUB} = \rho_{AUB}^{T_B}. \quad (26)$$

This is consistent with the known fact that, for Gaussian states like the massive deformation ground state considered in the main text, invariance under partial transposition implies separability [30]. That observation was used in [23] to prove mixed separability in the massless Lifshitz theory. Our treatment is quite general, valid for real-valued RK wavefunctionals, independently of the dimension of the underlying classical theory. It thus provides the explicit separability of ρ_{AUB} for quantum states satisfying the aforementioned properties, which includes both the massive deformation (Gaussian) and the singular deformation (non-Gaussian) of the Lifshitz theory considered in this paper. Separability implies the vanishing of logarithmic negativity.

Our result on the separability of continuum RK states for disconnected subsystems A and B should carry through to the case where the fields are compact, i.e., for $\phi \sim \phi + 2\pi R$, with $R > 0$ being the compactification radius. Indeed, since A and B do not have a common boundary, they do not share any anchor so that the compactness of the field, which manifests itself in the boundary conditions, should not affect our argument to obtain a separable reduced density (23). We emphasize that the crucial point is the wavefunctional being a local function of the field such that the fields on A , B , C can only talk through the boundaries. It can also be shown that the logarithmic negativity vanishes in the compact case. Consider its replica formulation [31], $\mathcal{E}(A : B) = \lim_{n_e \rightarrow 1} \log \text{Tr}(\rho_{AUB}^{T_A})^{n_e}$, where the analytic continuation is taken over the even integers n_e . Taking n_e copies of $\rho_{AUB}^{T_A}$, the sewing conditions between the copies make all anchors agree, i.e., all the replica fields agree at the boundary between C and A , B . Furthermore, the partial transposition $(\rho_{AUB}^{T_A})^n$ is not sensitive to the parity of n , and one can show that $\text{Tr}(\rho_{AUB}^{T_A})^n = \text{Tr}(\rho_{AUB})^n$. Taking the limit $n \rightarrow 1$, it then follows from the unit normalization of the density matrix ($\text{Tr} \rho_{AUB}^{T_A} = \text{Tr} \rho_{AUB} = 1$) that the topological (winding) sector contribution coming from the compact nature of the field is trivial, yielding a zero logarithmic negativity (see [23] for an explicit calculation for Gaussian ground states of certain Lifshitz theories in one and two spatial dimensions).

III. LIFSHITZ CRITICAL POINT

Scale invariance plays a central role in the study of dynamical critical phenomena, far-from-equilibrium statistical dynamics, and quantum criticality [32–35]. Taken isotropic, scale invariance is often enhanced to conformal symmetry, with profound consequences. However, many systems at criticality exhibit anisotropic scaling between space and time, called Lifshitz scaling,

$$t \rightarrow \lambda^z t, \quad \mathbf{x} \rightarrow \lambda \mathbf{x}, \quad \lambda > 0, \quad (27)$$

with characteristic dynamical critical exponent $z \neq 1$. Lifshitz scaling is encountered in a variety of contexts, from nonrelativistic mechanics [36–39] and critical systems [5,40], to nonrelativistic holographic duality [41–45] and quantum gravity [46,47]. Here, we are interested in a certain class of nonrelativistic quantum field theories admitting Lifshitz symmetry. Originally introduced as the “quantum Lifshitz model” in $2+1$ dimensions for $z = 2$ [5], general $(d+1)$ -dimensional Lifshitz theories with (even) positive integer z possess the remarkable feature that their ground-state wavefunctional takes a local form, given in terms of the action of a d -dimensional classical model—a RK wavefunctional.

The real, noncompact $(d+1)$ -dimensional $z = 2$ Lifshitz quantum critical boson is the QFT with Hamiltonian

$$H = \frac{1}{2} \int d^d x (\Pi^2 + \kappa^2 (\nabla^2 \phi)^2), \quad (28)$$

with canonical commutation relations $[\phi(x), \Pi(x')] = i\delta(x - x')$. The parameter κ is dimensionless, and $\Pi(x) = -i\delta/\delta\phi(x)$ in the Schrödinger picture. In addition to invariance under Lifshitz scaling (27), and the obvious \mathbb{Z}_2 symmetry, this theory is invariant under affine shifts of the field $\phi(x) \rightarrow \phi(x) + ax + b$. This is also called polynomial shift symmetry.

The Hamiltonian (28) is quadratic in the operators

$$A(x) = \frac{1}{\sqrt{2}} (i\Pi(x) - \kappa \nabla^2 \phi), \quad (29)$$

as it is easily verified that $H = \frac{1}{2} \int d^d x \{A^\dagger(x), A(x)\}$. Note that because $\Pi(x)$ is Hermitian, $A^\dagger(x)$ is obtained from (29) by the single replacement $i \rightarrow -i$. Alternatively, it will be convenient to diagonalize the Hamiltonian to the normal-ordered form

$$H = \int d^d x A^\dagger(x) A(x) + E_{\text{vac}}, \quad (30)$$

where $E_{\text{vac}} = \frac{1}{2} \int d^d x [A(x), A^\dagger(x)] = -\frac{\kappa}{2} \int d^d x \nabla_x^2 \delta^d(x-y)|_{x=y}$ is a positive, UV-divergent multiple of the identity. As such, it can be considered as a vacuum-energy shift relating the otherwise identical eigensystems of H and $\int d^d x A^\dagger(x) A(x)$ [5]. A ground state is now found satisfying

$$A(x)|\Psi_0\rangle = 0, \quad \forall x. \quad (31)$$

Positive semidefiniteness of $\int d^d x A^\dagger(x) A(x)$ ensures that such a state is indeed a ground state. The corresponding functional-differential equation, $(\frac{\delta}{\delta\phi} - \kappa \nabla^2 \phi)\Psi_0[\phi] = 0$, has nontrivial solution

$$\Psi_0[\phi] = \frac{1}{\sqrt{\mathcal{Z}}} e^{-\frac{1}{2} \int d^d x \kappa (\nabla\phi)^2}, \quad (32)$$

with normalization factor

$$\mathcal{Z} = \int \mathcal{D}\phi(x) e^{-\int d^d x \kappa(\nabla\phi)^2}. \quad (33)$$

One recognizes \mathcal{Z} as the partition function of a d -dimensional free Euclidean scalar field with classical action $S_{\text{cl}}[\phi] = \int d^d x \kappa(\nabla\phi)^2$. This local action appearing in Ψ_0 is conformally invariant in d (spatial) dimensions. We thus have an emergent *spatial* conformal symmetry in the ground state of the parent Hamiltonian H . We emphasize that S_{cl} is only inherent to Ψ_0 , and does not coincide with the action of the parent Hamiltonian H . (A relationship between these actions can be established via stochastic quantization [48].) Dimensionlessness of S_{cl} requires that

$$\dim \phi = (\text{length})^{-\frac{d}{2}+1}, \quad (34)$$

and Lifshitz scaling in turn implies

$$\dim \Pi = (\text{length})^{-\frac{d}{2}-1}. \quad (35)$$

Equivalently, the dimension of H is $(\text{length})^{-2}$ as expected from the Lifshitz scaling $t \sim x^2$. For simplicity, we will mostly consider the case $d = 1$, for which S_{cl} is the Euclidean action of a free nonrelativistic particle. We will derive certain properties of the entanglement entropy of the $d = 2$ theory that follow from our $d = 1$ results in Sec. IV C. Using the Gaussian propagators of this underlying theory,

$$\langle \phi', x' | \phi, x \rangle = \sqrt{\frac{\kappa}{\pi(x' - x)}} e^{-\frac{\kappa(\phi' - \phi)^2}{x' - x}}, \quad (36)$$

with $x' > x$, one can compute ground-state correlation functions, and Rényi entanglement entropies [20].

In this paper, we perform two distinct deformations of the Lifshitz point preserving the Rokhsar-Kivelson structure. In Sec. IV we consider a deformation that breaks the emergent conformal symmetry of $S_{\text{cl}}[\phi]$ with a (mass) scale, whereas in Appendix A we consider a nontrivial deformation, which preserves this symmetry when $d = 1$. (The affine field-shift symmetry is lost in both cases, and will play no further role.)

IV. DEFORMATION BY A MASS TERM

We now explicitly break the Lifshitz scaling symmetry of (28) with a length scale $1/m$, setting $\Lambda[\phi] = \frac{1}{2} \int d^d x m^2 \phi(x)^2$, so that the Euclidean action defining the ground state becomes

$$S_{\text{cl}}[\phi] = \int d^d x (\kappa(\nabla\phi)^2 + m^2 \phi^2), \quad (37)$$

and

$$A(x) = \frac{1}{\sqrt{2}} \left(\frac{\delta}{\delta\phi(x)} - \kappa \nabla^2 \phi + m^2 \phi \right). \quad (38)$$

Expression (37) is seen to correspond to the Euclidean action of a massive relativistic scalar (Klein-Gordon). The original Lifshitz Hamiltonian (28) is thereby deformed to

$$H_{\text{mass}} = H_{\text{Lif}} + \frac{1}{2} \int d^d x (2\kappa m^2 (\nabla\phi)^2 + m^4 \phi^2). \quad (39)$$

The Hamiltonians $H_{\text{mass}} = \frac{1}{2} \int d^d x \{A^\dagger(x), A(x)\}$ and $H_{\text{mass}}^{\text{normal}} = \int d^d x A^\dagger(x)A(x)$ differ by $\frac{1}{2} \int d^d x [A^\dagger(x), A(x)] =$

$\frac{1}{2} \text{tr}(\kappa \nabla^2 - m^2)$, a UV-divergent multiple of the identity. Their eigensystems are thus identical up to an infinite zero-point energy shift, with common ground state

$$\Psi_0[\phi] = \frac{1}{\sqrt{\mathcal{Z}}} e^{-\frac{1}{2} \int d^d x (\kappa(\nabla\phi)^2 + m^2 \phi^2)}. \quad (40)$$

Indeed it is clear that $A(x)|\Psi_0\rangle = 0$. Due to the scale m , the new ground state has lost conformal symmetry. H_{mass} has positive-infinite ground-state energy $E_{\text{vac}} = \frac{1}{2} \text{tr}(-\kappa \nabla^2 + m^2)$, whereas the normal-ordered form has ground-state energy zero. The new terms in (39) are both relevant under renormalization group (RG), and the theory flows to the massive relativistic scalar in the IR. Let us be more precise: the usual RG transformation on the theory sends $x_i \rightarrow bx_i$ ($1 \leq i \leq d$) and $t \rightarrow b^z t$, with $z = 2$. The scaling dimensions of $(\nabla\phi)^2$ and ϕ^2 are distinct, meaning that the fine tuning between the mass-dependent terms in (39) would be lost under RG. The key to preserve the RK structure is to consider an alternative RG transformation that acts on the ground-state wavefunction (40), not directly on the Hamiltonian. This wavefunction RG transformation now acts on the $(0 + d)\text{D}$ relativistic massive Euclidean scalar. All coordinates are dilated by the same factor, $x_i \rightarrow bx_i$. The mass is relevant, and leads to the usual infinite mass trivial IR fixed point in $(0 + d)\text{D}$. The wavefunction RG transformation corresponds to the trajectory in theory space where m increases in (39). In other words, the trajectory in the space of Hamiltonians is obtained by constructing the parent Hamiltonian for the ground state (40).

The normalization factor for Ψ_0 coincides with the Euclidean partition function of the d -dimensional massive scalar,

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-\int d^d x (\kappa(\nabla\phi)^2 + m^2 \phi^2)}. \quad (41)$$

The density matrix operator corresponding to Ψ_0 is

$$\rho = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \mathcal{D}\phi' e^{-(S_{\text{cl}}[\phi] + S_{\text{cl}}[\phi'])/2} |\phi\rangle \langle \phi'|. \quad (42)$$

We now specialize to $d = 1$ with Dirichlet boundary conditions (BC): $\phi(0) = 0 = \phi(L)$, but many of our results will also apply to other choices of boundary conditions. Apart from simplicity, one motivation for this choice is the search for the elusive effective field theories of the Motzkin and Fredkin spin chains [29,49] whose unique, highly entangled, frustration-free ground states reproduce the logarithmic scaling of entanglement entropy found in critical spin chains. When ϕ is used to represent a ‘‘height field’’ for the spin variables, i.e., $\partial_x \phi(x) = S_z(x)$, the ground-state property $S_z^{\text{tot}} = 0$ translates into Dirichlet boundary conditions (up to a constant) on ϕ . The *positive-field* version of (28) is a parent Hamiltonian for the *positive-field* version of (32), which captures many spin and entanglement features of the Motzkin and Fredkin ground states [19,20,50], while having a markedly different excitation spectrum [19,20]. We note that bulk properties are expected not to depend on the field positivity constraint [19,20].

In one dimension, \mathcal{Z} is the partition function of a single particle with Euclidean Lagrangian $\mathcal{L}_m = \kappa(\partial_x \phi)^2 + m^2 \phi^2$, i.e., a quantum harmonic oscillator of ‘‘mass’’ $M = 2\kappa$ and

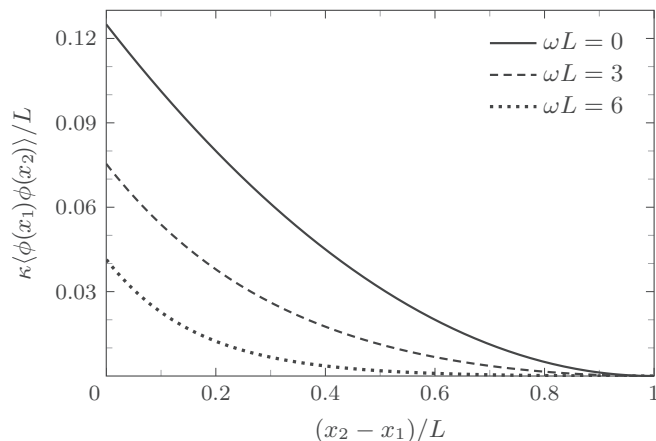


FIG. 1. Correlator $\frac{\kappa}{L} \langle \phi(x_1) \phi(x_2) \rangle$ from (50), for a central interval of length $x_2 - x_1$, and for different values of ωL . Correlators vanish when $x_2 - x_1 \rightarrow L$ due to the boundary condition $\phi(0) = 0 = \phi(L)$.

“frequency”

$$\omega = m/\sqrt{\kappa}. \quad (43)$$

The standard propagator of the Euclidean oscillator, $\langle \phi', x' | \phi, x \rangle = \int_{\phi(x)=\phi}^{\phi(x')=\phi'} \mathcal{D}\phi e^{-\int_x^{x'} dx \mathcal{L}_m}$, is [51]

$$\langle \phi', x' | \phi, x \rangle = \sqrt{\frac{\kappa \omega}{\pi \sinh \omega(x' - x)}} \times \exp \left[\frac{-\kappa \omega ((\phi^2 + \phi'^2) \cosh \omega(x' - x) - 2\phi\phi')}{\sinh \omega(x' - x)} \right], \quad (44)$$

which reduces to (36) in the limit $m = \omega = 0$. Vacuum expectation values of local operators $\hat{O}_x | \phi \rangle$ can be expressed in terms of the propagator associated to Ψ_0 via the mapping $\mathcal{Z}(\Psi_0 | \Psi_0) = \int_{\text{BC}} \mathcal{D}\phi e^{-\int d^d x S_{\text{cl}}[\phi]}$. In particular we obtain from (42)

$$\langle \phi(x) \rangle = \text{Tr}[\rho \hat{\phi}(x)] = \int_{-\infty}^{\infty} d\phi \phi f_1(\phi, x), \quad (45)$$

with single-point probability distribution $f_1(\phi, x) = \mathcal{Z}^{-1} \langle 0, L | \phi, x \rangle \langle \phi, x | 0, 0 \rangle$. Since f_1 is symmetric under $\phi \rightarrow -\phi$, field and gradient have vanishing vacuum expectation values,

$$\langle \phi(x) \rangle = \int_{-\infty}^{\infty} d\phi \phi f_1(\phi, x) = 0, \quad (46)$$

$$\langle \partial_x \phi(x) \rangle = \partial_x \langle \phi(x) \rangle = 0.$$

Other boundary conditions not breaking \mathbb{Z}_2 symmetry, periodic conditions for instance, will have the same vanishing expectations.

We may formally define the reduced density matrix on the single-point set $\{x\}$ as

$$\rho(x) = \int_{-\infty}^{\infty} d\phi f_1(\phi, x) | \phi, x \rangle \langle \phi, x |. \quad (47)$$

Then for a local operator \hat{O}_x as above, $\langle \hat{O}_x \rangle = \text{Tr}_{\{x\}}[\rho(x) \hat{O}_x] = \int_{-\infty}^{\infty} d\phi f_1(\phi, x) \langle \phi, x | \hat{O}_x | \phi, x \rangle$. The argument is readily generalized to any product of local operators.

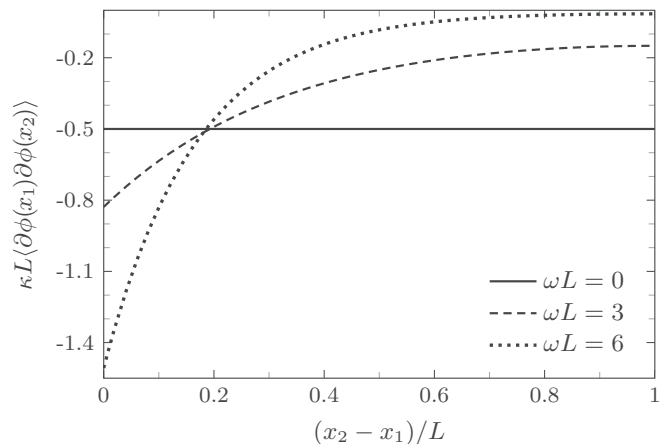


FIG. 2. Correlator $\kappa L \langle \partial \phi(x_1) \partial \phi(x_2) \rangle$ from (51), for a central interval of length $x_2 - x_1$, and for different values of ωL .

A. Correlations in the ground state

We have shown in Sec. II B that the reduced density matrix ρ_{AUB} is a separable mixed state for any disconnected subsystems A, B . Therefore, A and B are not entangled, and the correlations left in ρ_{AUB} do not arise from entanglement [52–55]. As A, B come into contact, the result does not hold anymore and quantum-driven contact terms are expected. For operators \hat{O}_1, \hat{O}_2 local at x_1, x_2 , respectively, we define

$$\rho(x_1, x_2) = \int_{-\infty}^{\infty} d\phi_1 d\phi_2 f_2 \bigotimes_{i=1,2} | \phi_i, x_i \rangle \langle \phi_i, x_i |, \quad (48)$$

with two-point probability distribution

$$f_2(\phi_1, \phi_2, x_1, x_2) = \mathcal{Z}^{-1} \langle 0, L | \phi_2, x_2 \rangle \langle \phi_2, x_2 | \phi_1, x_1 \rangle \langle \phi_1, x_1 | 0, 0 \rangle. \quad (49)$$

Then $\langle \hat{O}_1 \hat{O}_2 \rangle = \text{Tr}_{\{x_1, x_2\}}[\rho(x_1, x_2) \hat{O}_1 \hat{O}_2]$. The function f_2 is not symmetric under the *individual* operations $\phi_i \rightarrow -\phi_i$. The field correlator and gradient correlator at positions $x_1 < x_2$ are found to be

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{\sinh \omega x_1 \sinh \omega(L - x_2)}{2\kappa \omega \sinh \omega L}, \quad (50)$$

$$\langle \partial_{x_1} \phi(x_1) \partial_{x_2} \phi(x_2) \rangle = -\frac{\omega \cosh \omega x_1 \cosh \omega(L - x_2)}{2\kappa \sinh \omega L}. \quad (51)$$

The correlators $\frac{\kappa}{L} \langle \phi(x_1) \phi(x_2) \rangle$ and $\kappa L \langle \partial \phi(x_1) \partial \phi(x_2) \rangle$ for a centered interval of length $x_2 - x_1$ are displayed in Figs. 1 and 2 in terms of the dimensionless variable $(x_2 - x_1)/L$. Note that these functions depend on ω and L only through the combination ωL . Because single-point field expectations vanish, we can identify two-point functions $\langle \phi_1 \phi_2 \rangle$ and connected correlators $C_{\phi_1, \phi_2} = \langle \phi_1 \phi_2 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle$ for this model. By the same token, we identify $\langle \partial \phi_1 \partial \phi_2 \rangle$ and $C_{\partial \phi_1, \partial \phi_2} = \langle \partial \phi_1 \partial \phi_2 \rangle - \langle \partial \phi_1 \rangle \langle \partial \phi_2 \rangle$.

In the massless limit, $\omega L \rightarrow 0$, we recover the results

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{x_1(L - x_2)}{2\kappa L}, \quad (52)$$

$$\langle \partial_{x_1} \phi(x_1) \partial_{x_2} \phi(x_2) \rangle = -\frac{1}{2\kappa L}. \quad (53)$$

To obtain the thermodynamic limit, the field correlator is best written in the following form:

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{L}{8\kappa} - \frac{x_2 - x_1}{4\kappa} - \frac{(x_1 - \frac{L}{2})(x_2 - \frac{L}{2})}{2\kappa L}. \quad (54)$$

The first term is a divergent contact term or bulk second moment $\langle \phi(L/2)^2 \rangle$. The universal, second term is translation-invariant and linearly increasing (in absolute value) with separation, as should be since ϕ scales as $(\text{length})^{1/2}$. The final term vanishes when deep enough in the bulk. We thus conclude that the critical Lifshitz boson ($\omega = 0$) violates cluster decomposition. Note that $\langle \partial_{x_1}\phi(x_1)\partial_{x_2}\phi(x_2) \rangle_{\text{massless}}$ is constant, and vanishes in the thermodynamic limit,

$$\lim_{L \rightarrow \infty} \langle \partial_{x_1}\phi(x_1)\partial_{x_2}\phi(x_2) \rangle_{\text{massless}} = 0, \quad (55)$$

as observed in [19,20] where it meant that bulk S_z spins are uncorrelated in the continuous limit of the ground state of the Motzkin and Fredkin chains, a result also consistent with the discrete case [50]. Taking the spatial derivatives of (52) to get the $\partial_x\phi$ correlator yields a delta-function correlation function (we have assumed non-coincidental points above). The reason for the vanishing of $\langle \partial_{x_1}\phi(x_1)\partial_{x_2}\phi(x_2) \rangle$ deep in the bulk is thus the scaling dimension of ϕ , which is $-1/2$. As such, one can say that the operator product expansion of $\partial\phi$ with itself only contains an ultralocal contact term.

Expression (52) confirms that cluster decomposition is not satisfied in the massless case. On the other hand, we see from (50) that cluster decomposition is satisfied when $\omega > 0$ (or $\omega \gg 1/L$ for a finite system). Looking deep into the bulk where $\omega x_1 \rightarrow \infty$ and $\omega(L - x_2) \rightarrow \infty$, we obtain for all values of $\omega(x_2 - x_1)$,

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{4\kappa\omega} e^{-\omega(x_2-x_1)}, \quad (56)$$

and

$$\langle \partial_{x_1}\phi(x_1)\partial_{x_2}\phi(x_2) \rangle = -\frac{\omega}{4\kappa} e^{-\omega(x_2-x_1)}, \quad (57)$$

satisfying cluster decomposition at large separations. Connected correlations with an exponential bound $Ce^{-m(x_2-x_1)}$ are characteristic of a gapped phase with mass m . We identify $\xi = \omega^{-1} = \sqrt{\kappa}/m$ as the correlation length, and recover the quantum critical Lifshitz boson in the massless limit. Expressions (56) and (57) do not depend on the choice of boundary conditions. We are surprised to find that when the correlation length is infinite (requiring both $\omega \rightarrow 0$ and $L \rightarrow \infty$ to be effective), the theory develops an IR divergence in bulk correlations. The corresponding limit is singular,

$$\lim_{L \rightarrow \infty} \lim_{\omega \rightarrow 0} \langle \phi(x_1)\phi(x_2) \rangle \sim \frac{L}{8\kappa} - \frac{x_2 - x_1}{4\kappa}, \quad (58)$$

and

$$\lim_{\omega \rightarrow 0} \lim_{L \rightarrow \infty} \langle \phi(x_1)\phi(x_2) \rangle \sim \frac{1}{4\kappa\omega} - \frac{x_2 - x_1}{4\kappa}, \quad (59)$$

from which we may single out the nonsingular subleading behavior $\langle \phi(x_1)\phi(x_2) \rangle_{\text{regular}} = -(x_2 - x_1)/4\kappa$. A similar phenomenon is observed in the mutual information [see (80) and (81)].

We emphasize that for $x_1 < x_2$ deep within the bulk,

$$\langle \partial_{x_1}\phi(x_1)\partial_{x_2}\phi(x_2) \rangle = \begin{cases} 0, & m = 0, \\ -\frac{\omega}{4\kappa} e^{-\omega(x_2-x_1)}, & m > 0. \end{cases} \quad (60)$$

It is a remarkable fact that, contrary to intuition, this ground-state correlator becomes trivial only for the *massless* theory. Indeed, a mass for ϕ corresponds to a gap in the theory, revealed by an exponential decay of correlations in the observables. In the massless limit of large systems, the exponential decay is usually replaced by power-law on separation as the correlation length diverges, resulting in *enhanced* correlations. The vanishing of the self-correlations of an operator as the gap closes is unorthodox.

B. Entanglement in the ground state

We consider a finite bipartition $\{A, B\}$ of a one-dimensional system. By analogy with the “entangling surface” in higher dimensions, we call “surface” the multiple-point boundary between A and B , that is $\partial A = \{x_1, x_2, \dots, x_M\}$. Rényi entanglement entropies are defined in (7), and we derived in Sec. II A an expression in terms of the $(0 + 1)$ -dimensional partition function $\int_{\text{BC}} \mathcal{D}\phi e^{-\int d^d x S_{\text{cl}}[\phi]}$, namely,

$$S_n(A) = \frac{1}{1-n} \log \left[\epsilon^{(n-1)M/2} \int d\phi [f_M(\phi, \mathbf{x})]^n \right], \quad (61)$$

where $f_M(\phi, \mathbf{x}) = \mathcal{Z}^{-1} \langle 0, L | \phi_M, x_M \rangle \cdots \langle \phi_1, x_1 | 0, 0 \rangle$, and ϵ is a length scale readily identified as a UV cutoff. In a lattice regularization, for instance, ϵ is naturally present as the lattice constant. For the massive deformation with Dirichlet conditions, the Gaussian propagators (44) give $f_M(\phi, \mathbf{x}) = C e^{-\phi^T K \phi}$ with normalization factor $C = f_M(\mathbf{0}, \mathbf{x}) = \sqrt{\det K / \pi^M}$. Integrating (61) over \mathbb{R}^M , we find

$$S_n = -\log \frac{\mathcal{Z}_D^{AB}}{\mathcal{Z}} + \frac{1}{2} \log \frac{1}{\epsilon^M} - \frac{M}{2(1-n)} \log n, \quad (62)$$

where

$$\frac{\mathcal{Z}_D^{AB}}{\mathcal{Z}} = \frac{1}{\mathcal{Z}} \langle 0, L | 0, x_M \rangle \cdots \langle 0, x_1 | 0, 0 \rangle = f_M(\mathbf{0}, \mathbf{x}) \quad (63)$$

is a product of propagators with Dirichlet conditions on ∂A , normalized by the free propagator $\mathcal{Z} = \langle 0, L | 0, 0 \rangle$. Note that $\mathcal{Z}_D^{AB} = \mathcal{Z}_D^A \mathcal{Z}_D^B$. We see that S_n is independent of the Rényi index, up to a constant term. We have computed the bipartite Rényi entanglement entropy from first principles, and obtained (62). This is a generalization to our $(1 + 1)$ -dimensional case of the celebrated Fradkin-Moore formula $S = -\log(\mathcal{Z}_D^{AB}/\mathcal{Z}_F)$ for the bipartite entanglement entropy of $(2 + 1)$ -dimensional conformal quantum critical theories [9], which have the RK property and whose ground states correspond to a lower-dimensional CFT_2 . Then $\mathcal{Z}_D^{AB} = \mathcal{Z}_D^A \mathcal{Z}_D^B$ is the lower-dimensional CFT_2 partition function of configurations with Dirichlet conditions on ∂A , and \mathcal{Z}_F is the partition function of free configurations. For the massive deformation of the $(1 + 1)$ -dimensional Lifshitz theory, however, the lower-dimensional theory is not conformal invariant, and is only $(0 + 1)$ -dimensional, so that partition functions are simple quantum mechanical propagators. We note that expression (62) at finite mass generalizes naturally to other dimensions.

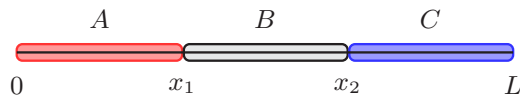


FIG. 3. Some system partitions. The single-point surface $\{x_1\}$ generates the bipartition $\{A, B \cup C\}$ with Rényi entropies $S_n(A)$ given in (64), while $\{x_1, x_2\}$ generates the bipartition $\{A \cup C, B\}$ with Rényi entropies $S_n(B)$ found in (65).

1. Rényi entanglement entropies

For a single-point surface separating the “boundary interval” A from the rest of the system as in Fig. 3, we find the Rényi entropies

$$S_n(A) = \frac{1}{2} \log \frac{\sinh \omega L_A \sinh \omega(L - L_A)}{\omega \epsilon \sinh \omega L} + b_1(n), \quad (64)$$

while for a two-point surface separating the bulk interval B from the rest of the system as in Fig. 3, we obtain

$$S_n(B) = \frac{1}{2} \log \frac{\sinh \omega L_A \sinh \omega L_B \sinh \omega L_C}{(\omega \epsilon)^2 \sinh \omega L} + b_2(n), \quad (65)$$

where the constant $b_M(n) = \frac{M}{2} \log \frac{\pi}{\kappa} - \frac{M}{2(1-n)} \log n$, with $M = 1$ and $M = 2$ in (64) and (65), respectively. The leading universal terms are indeed independent of the Rényi index n . These expressions are in perfect agreement with the exact calculation over the discrete versions of the Hamiltonian (39) and ground state (40). Once all interaction terms between a discrete subsystem and its complement have been singled out, one can apply the replica trick and compute a finite number of Gaussian integrals, yielding (62) in the continuous limit, up to constant terms. We apply these results to compute the capacity of entanglement in Appendix B.

Let us now consider different limiting regimes of the Rényi entropies. First, in the massless case, expressions (64) and (65) reduce to

$$S_n(A) = \frac{1}{2} \log \frac{L_A(L - L_A)}{\epsilon L} + b_1(n), \quad (66)$$

and

$$S_n(B) = \frac{1}{2} \log \frac{L_A L_B L_C}{\epsilon^2 L} + b_2(n), \quad (67)$$

respectively. The entropy (67) of a bulk interval shows an IR divergence, while that of a boundary interval (66) does not. Formula (66) agrees with that obtained in [23] for the discrete massless theory, and matches (B13) of reference [19]. The massless limit of (65), given by (67), however, does not agree with the results of [19]. In fact, our entropy formula (61) is different from the one used in [19]. We will say more about this below (79).

For large systems we find

$$S_n(A) \xrightarrow{L \rightarrow \infty} \frac{1}{2} \log \frac{\sinh \omega L_A}{\omega \epsilon} - \frac{1}{2} \omega L_A + \text{const}, \quad (68)$$

and

$$S_n(B) \xrightarrow{L_A, L_C \rightarrow \infty} \frac{1}{2} \log \frac{\sinh \omega L_B}{(\omega \epsilon)^2} - \frac{1}{2} \omega L_B + \text{const}. \quad (69)$$

Here, the constant has the general form $b_M - \frac{Q-1}{2} \log 2$ when Q subintervals have infinite length. At low momenta, quan-

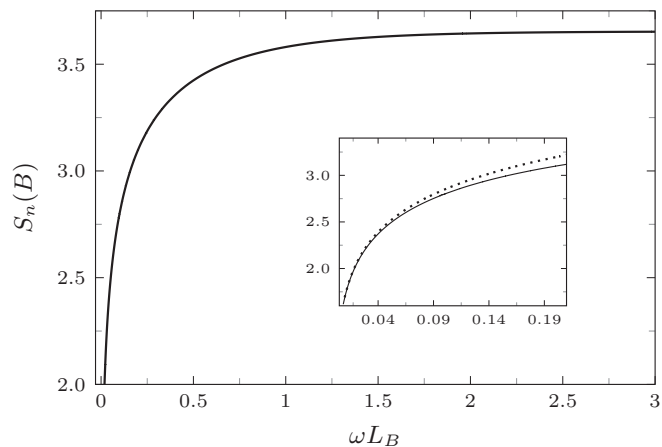


FIG. 4. Bulk interval Rényi entropy $S_n(B)$ in the large system limit (69), as a function of ωL_B . The order n and mass-scale ω are unspecified, contributing only an overall constant. An area law $S_n \sim C_{\text{upper bound}}$ is observed at large values of ωL_B , suppressing dependence on L_B and maintaining only a mass-scale dependence $C_{\text{upper bound}} = \log(\xi/\epsilon) + \text{const.}$, see (71). Inset: Small ωL_B behavior of S_n compared with right-hand side of (70) (dashed) showing logarithmic deviation from an area law in the massless case.

tum fluctuations are suppressed by the effective mass and the ground state disentangles, meaning that entanglement is localized in the smallest length scales, and only the fine-tuning of mass to zero will entangle the largest scales. A plot of $S_n(B)$ as a function of ωL_B is given in Fig. 4. In the low mass limit, $\omega L_B \rightarrow 0$, the entropy behaves like

$$S_n(B) = \frac{1}{2} \log \omega L_B - \frac{1}{2} \omega L_B - \log \epsilon \omega + \text{const}, \quad (70)$$

departing from the area-law behavior by an unbounded logarithmic dependence on subsystem length, characteristic of a gapless system. The prefactor is independent of n . In the opposite $\omega L_B \rightarrow \infty$ limit (large mass or large subinterval),

$$S_n(B) = -\log \epsilon \omega + \text{const} = \log \frac{\xi}{\epsilon} + \text{const}. \quad (71)$$

This is an area law $S_n \sim C_{\text{upper bound}}$ for the gapped system, the general expression for the upper bound being $C_{\text{upper bound}} = \frac{M}{2} \log \frac{\xi}{\epsilon} + \text{const}$, from (62), where the number of surface points M is the one-dimensional analog of a surface area. The entropy is independent of L_B in this limit, as expected for massive excitations. Recalling that the correlation length $\xi = \omega^{-1} = \sqrt{\kappa}/m$, the upper bound is seen to be a decreasing function of the mass.

Unexpectedly, we observe a close connection between (69) and the entanglement entropy of a single bulk interval of length L_B (and surface area $M = 2$) in a finite temperature CFT₂ [56],

$$S_1^{\text{CFT}}(L_B, T) = \frac{c}{3} \log \frac{\sinh \pi L_B T}{\pi \epsilon T} + \text{const}. \quad (72)$$

With the identification $c = 3/2$ and $\pi T = \omega = m/\sqrt{\kappa}$, we can rewrite (69) as

$$S_n^{\text{Lif}}(L_B, m) = S_1^{\text{CFT}}(L_B, T) - S_1^{\text{CFT}}(L_B, \infty) + \dots, \quad (73)$$

where $S_1^{\text{CFT}}(L_B, \infty) = \frac{c}{3}\pi L_B T$ stands for the high-temperature asymptotic behavior of S_1^{CFT} , that is, the thermal entropy. The ellipsis stands for nonuniversal terms independent of L_B . The moderately massive Lifshitz theory thus has the entanglement entropy of a moderately hot CFT₂, $S_n^{\text{Lif}}(L_B, m) \sim S_1^{\text{CFT}}(L_B, T)$ for small T, m . To our knowledge, only the correspondence between the massless Lifshitz theory and zero temperature CFT₂ had been observed so far.

2. Mutual information

Expression (62) enables us to give the analytic form of the mutual information $I_n(A : B) = S_n(A) + S_n(B) - S_n(A \cup B)$ between disconnected subsystems A and B ,

$$I_n(A : B) = \log \frac{f_{M_A+M_B}(\mathbf{0}, \mathbf{x}_A \cup \mathbf{x}_B)}{f_{M_A}(\mathbf{0}, \mathbf{x}_A) f_{M_B}(\mathbf{0}, \mathbf{x}_B)}, \quad (74)$$

where the M_A surface points of A are positioned at \mathbf{x}_A , the M_B surface points of B are positioned at \mathbf{x}_B , and $\mathbf{x}_A \cap \mathbf{x}_B = \emptyset$. A graphical representation of formula (74) for two bulk intervals A, B is given in Fig. 5(a). A graphical example pertaining to another partition is provided in Fig. 5(b), which serves to illustrate a very peculiar property of the (massive) Lifshitz ground state, namely that the mutual information $I(A : B)$ is insensitive to any component of A with no neighbor in B . [In Fig. 5(b), one such component is A_1 , with no propagators leaving or reaching its surface points.] The mutual information is universal, as expected, and independent of the Rényi index n , which will be omitted from now on. [Note that formula (74) holds whenever a wavefunctional can be mapped to a local action on noncompact fields with propagators that are quadratic forms of the fields, i.e., when $f_M(\phi, \mathbf{x}) = f_M(\mathbf{0}, \mathbf{x})e^{-\phi^T K \phi}$ for multiple-point surface $\partial = \{x_1, \dots, x_M\}$, dimensionless field values $\phi_i \in \mathbb{R}$, and normalization factor $f_M(\mathbf{0}, \mathbf{x}) = \sqrt{\det K/\pi^M}$.] For two disconnected bulk intervals as in Fig. 5(a) we find

$$I(A : B) = \frac{1}{2} \log \frac{\sinh \omega L_{C_1 \cup A \cup C_2} \sinh \omega L_{C_2 \cup B \cup C_3}}{\sinh \omega L_{C_2} \sinh \omega L}, \quad (75)$$

which is increasing with the lengths of A, B , and decreasing with the separation L_{C_2} . In particular, $I(A : B) \rightarrow 0$ as $\omega L_{C_2} \rightarrow \infty$. In the limit $L_{C_1}, L_{C_3} \rightarrow \infty$ it becomes

$$I(A : B) = -\frac{1}{2} \log \sinh \omega L_{C_2} + \frac{1}{2} \omega L_{C_2} - \frac{1}{2} \log 2, \quad (76)$$

showing that the dependence on L_A and L_B disappears deep in the bulk, regardless of separation. In this limit, we observe a logarithmic divergence at vanishing separations

$$I(A : B) \sim \frac{1}{2} \log \frac{1}{\omega L_{C_2}}, \quad \omega L_{C_2} \rightarrow 0, \quad (77)$$

and an exponential decay at large separations

$$I(A : B) \sim \frac{1}{2} e^{-2\omega L_{C_2}}, \quad \omega L_{C_2} \rightarrow \infty. \quad (78)$$

However, the mutual information does not vanish when the size of A and/or B vanishes. This is in stark contrast with most common theories. In a free fermion CFT₂, for instance, the mutual information between two intervals of length ℓ separated by a distance r is $I(\ell, r) = \frac{1}{3} \log \frac{(r+\ell)^2}{r(2\ell+r)}$ [57,58],

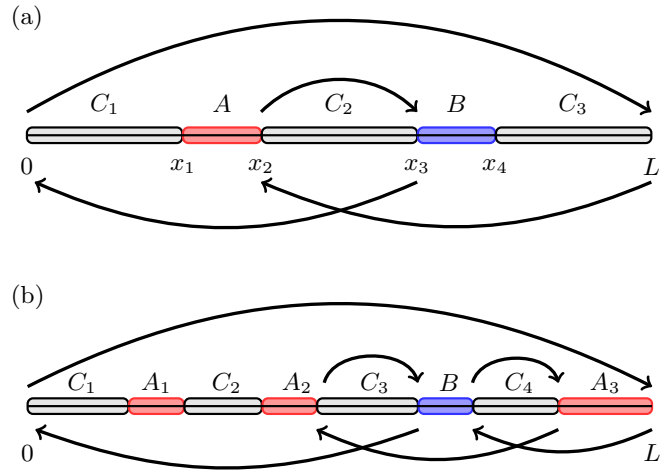


FIG. 5. Graphical representations of the mutual information between two disconnected bulk intervals A and B . In the general expression (74), the numerator contains all simple-interval propagators, including one propagator over the full system. Only those not present in the denominator survive, and are represented as arrows above the axis. The remaining denominator propagators are represented as arrows below the axis. The result is a cycle visiting the system’s boundary, as well as surface points of one subsystem that are adjacent to a surface point of the other subsystem.

which clearly vanishes as ℓ tends to zero. We note that in the massless limit of (75), we obtain

$$I(A : B) = \frac{1}{2} \log \frac{(L_{C_1} + L_A + L_{C_2})(L_{C_2} + L_B + L_{C_3})}{L_{C_2} L}, \quad (79)$$

which does not agree with the result found in [19] where mutual information seems to vanish. In this reference, the reduced density $\rho_{A \cup B}$ is assumed, wrongly, to depend only on the difference fields $\phi_A \equiv \phi_2 - \phi_1$ and $\phi_B \equiv \phi_4 - \phi_3$, where A, B are as in Fig. 5(a) and $\phi_i = \phi(x_i)$. The resulting expression is of the form $\rho_{A \cup B} \sim \int \mathcal{D}\phi_A \mathcal{D}\phi_B h(\phi_A, \phi_B) |\phi_A \phi_B\rangle \langle \phi_A \phi_B|$, with h a Gaussian function of ϕ_A, ϕ_B . This expression, however, neglects the crucial dependence of $\rho_{A \cup B}$ on the “anchor” $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, as derived in (23). A similar comment holds for other reduced densities as well. We have already mentioned that the expression obtained in [19] for the entanglement entropy of a single bulk interval is at odds with (the massless limit of) our expression (65). We find that far from vanishing, the mutual information diverges like $\frac{1}{2} \log L/L_{C_2}$ deep in the bulk. This is not entirely unexpected since correlators of the massless theory do not possess the cluster decomposition property. It is another manifestation of the bulk IR divergence that develops when the correlation length goes infinite (requiring both $\omega \rightarrow 0$ and $L \rightarrow \infty$ to be effective). The corresponding limit is singular,

$$\lim_{L_{C_1}, L_{C_3} \rightarrow \infty} \lim_{\omega \rightarrow 0} I(A : B) \sim \frac{1}{2} \log L - \frac{1}{2} \log L_{C_2}, \quad (80)$$

and

$$\lim_{\omega \rightarrow 0} \lim_{L_{C_1}, L_{C_3} \rightarrow \infty} I(A : B) \sim \frac{1}{2} \log \frac{1}{\omega} - \frac{1}{2} \log L_{C_2}, \quad (81)$$

which to be compare with the divergence in field correlations in (58) and (59).

We note that the mutual information shared by two boundary intervals, i.e., taking $L_{C_1} = L_{C_3} = 0$ in (79), reduces to [59]

$$I(A : B) = \frac{1}{2} \log \frac{(L - L_A)(L - L_B)}{L(L - L_A - L_B)}, \quad (82)$$

since $L = L_A + L_B + L_{C_2}$. Then, in the large L limit (or equivalently large separation), the mutual information does vanish as

$$I(A : B) \xrightarrow{L \rightarrow \infty} \frac{L_A L_B}{2L^2}. \quad (83)$$

The vanishing of the mutual information between two far apart boundary intervals can be understood by looking at the fields correlator (52). For this configuration, the two surface points separating $A \cup B$ from the rest of the system are located at $x_1 = L_A$ and $x_2 = L - L_B$, and the corresponding correlator vanishes in the regime $L \gg L_A, L_B$, i.e., $\langle \phi(x_1)\phi(x_2) \rangle = L_A L_B / (2\kappa L) \rightarrow 0$, hence recovering cluster decomposition.

Finally, we may also compute the mutual information between two adjacent subsystems A, B [see Fig. 5(a) setting L_{C_2} empty] using the entropy (65) for an interval in the bulk. One finds

$$I(A : B) = \frac{1}{2} \log \frac{\sinh \omega L_A \sinh \omega L_B \sinh \omega L_{C_1 \cup A} \sinh \omega L_{B \cup C_3}}{(\omega \epsilon)^2 \sinh \omega L \sinh \omega L_{A \cup B}}, \quad (84)$$

where we omitted an unimportant constant. It presents, as expected, a UV divergence. Deep in the bulk, i.e., in the limit $L_{C_1}, L_{C_3} \rightarrow \infty$, the mutual information shared by two adjacent subsystems behaves as

$$I(A : B) \sim \frac{1}{2} \log \frac{\sinh \omega L_A \sinh \omega L_B}{(\omega \epsilon)^2 \sinh \omega L_{A \cup B}}, \quad (85)$$

dependent only on L_A and L_B . The massless limit of (84) reads

$$I(A : B) = \frac{1}{2} \log \frac{(L_{C_1} + L_A)L_A L_B (L_B + L_{C_3})}{\epsilon^2 (L_A + L_B)L}, \quad (86)$$

which, deep in the bulk, yields

$$I(A : B) \sim \frac{1}{2} \log \frac{L_A L_B}{\epsilon(L_A + L_B)} + \frac{1}{2} \log \frac{L}{\epsilon}. \quad (87)$$

Taking the massless limit $\omega L_{A/B} \rightarrow 0$ from (85) instead, one gets

$$I(A : B) \sim \frac{1}{2} \log \frac{L_A L_B}{\epsilon(L_A + L_B)} - \frac{1}{2} \log \omega \epsilon. \quad (88)$$

We thus observe both UV and IR divergences in the mutual information between adjacent subsystems. The first term is of the same form as the mutual information and logarithmic negativity between two adjacent intervals in an infinite system for a CFT₂ [31].

3. Entropic c function

Being universal, i.e., cutoff independent, c functions are important tools in the study of (unitary) Lorentz-invariant field theories and their RG fixed points, and are known to

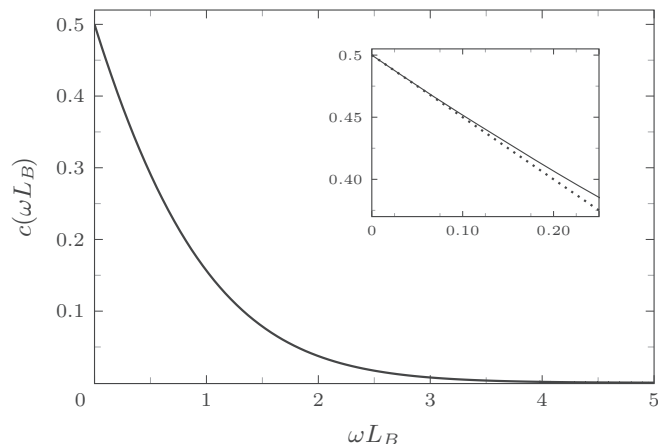


FIG. 6. Bulk interval c function, see (90), as a function of dimensionless subsystem mass-length ωL_B . Inset: Small ωL_B behavior of c compared with right-hand side of (91), first line (dashed). The curve tends to the critical value $c = 1/2$.

be monotone decreasing for such theories in two dimensions [60,61]. The decreasing of the entanglement entropy under the RG flow is a property demonstrably true for relativistic theories [61], but known to admit nonrelativistic exceptions [62]. The entropic c_n function is defined as

$$c_n(\ell) = \ell \frac{dS_n}{d\ell}, \quad (89)$$

where S_n is the (Rényi) entanglement entropy of an interval of length ℓ . Regrettably, c functions are difficult to compute for general theories, and analytical answers are scarce and most explicit results are obtained numerically. Here, however, we are able to analytically compute the entropic c -function of the massive Lifshitz theory.

The entropic c_n function corresponding to (69) is then

$$c(\omega L_B) = \frac{1}{2} \omega L_B (\coth \omega L_B - 1), \quad (90)$$

with asymptotics

$$c(\omega L_B) \sim \begin{cases} \frac{1}{2} - \frac{1}{2} \omega L_B, & \omega L_B \ll 1, \\ \omega L_B e^{-2\omega L_B}, & \omega L_B \gg 1. \end{cases} \quad (91)$$

Note that we have dropped the n dependence since it plays no role in this theory. A plot is given in Fig. 6. Interestingly, it is monotone decreasing under wavefunction RG flow, even though the theory is *not* Lorentz invariant. (In fact, not even Lifshitz invariant.) Lorentz invariance is thus not necessary for the monotone decreasing of the c function. Identifying the necessary conditions is still an open question.

C. 2D corner entanglement by dimensional reduction

With a few important exceptions, quantum critical states in $2 + 1$ spacetime dimensions have Rényi entropy that scales as

$$S(A) = \mathcal{B} \frac{\ell}{\epsilon} - \sum_i a(\theta_i) \log \frac{\ell}{\epsilon} + \text{const}, \quad (92)$$

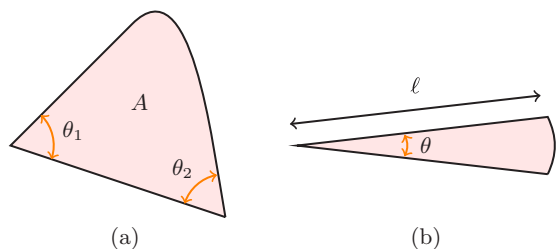


FIG. 7. (a) Subsystem A with two corners in the one-dimensional boundary, with opening angles θ_1 and θ_2 . (b) Sharp corner characterized by the linear dimension ℓ .

where ℓ is a linear dimension characterizing subregion A , $\epsilon \ll \ell$ is a UV cutoff, and i ranges over corners in the one-dimensional boundary of A , each with opening angle θ_i [see Fig. 7(a)]. The subleading logarithmically divergent terms measure the so-called *corner entanglement*, and the corner function $a(\theta)$ is universal and expected to depend only on scale-invariant geometric features of A [63–65].

In the sharp corner limit $\theta \rightarrow 0$ of noninteracting CFTs, the corner function may be analytically computed through dimensional reduction, by first considering a rectangular strip A with length ℓ and width $r \ll \ell$. The corresponding entropy is [66]

$$S(A) = \mathcal{B} \frac{\ell}{\epsilon} - \kappa_c \frac{\ell}{r}, \quad (93)$$

where κ_c is a universal coefficient also characterizing the mutual information shared by two regions A, B of length ℓ and separation $r \ll \ell$:

$$I(A : B) = \kappa_c \frac{\ell}{r}. \quad (94)$$

Expressions (93) and (94) are examples of relations that hold when a region of interest is characterized by m large linear dimensions ℓ_1, \dots, ℓ_m and n small linear dimensions r_1, \dots, r_n . Indeed, the entropy of such a region should be extensive in the large dimensions ℓ_i , which may then be considered periodic and Fourier analyzed, effectively reducing the dimension from $m + n$ to n . In the case of the thin rectangular strip A , the dimensional reduction leaves only one space dimension, and recasts κ_c into an integral of the entropic c function of the lower-dimensional QFT [66],

$$\kappa_c = \frac{1}{\pi} \int_0^\infty dx c(x). \quad (95)$$

A sharp corner of length ℓ and opening angle $\theta \ll 1$, as in Fig. 7(b), can then be built out of thin strips of length $d\ell'$ and width $\theta\ell'$, such that from (93) one obtains

$$S(A) = \int_\epsilon^\ell d\ell' \left(\frac{\mathcal{B}}{\epsilon} - \frac{\kappa_c}{\theta\ell'} \right) = \mathcal{B} \frac{\ell}{\epsilon} - \frac{\kappa_c}{\theta} \log \frac{\ell}{\epsilon}. \quad (96)$$

Comparison with (92) gives

$$a(\theta \rightarrow 0) = \frac{\kappa_c}{\theta}. \quad (97)$$

As it characterizes the (regulator-independent) leading divergence of $a(\theta \rightarrow 0)$, κ_c is also called the *sharp limit coefficient*.

We now consider a *nonconformal* theory, namely the $(2 + 1)$ -dimensional $z = 2$ real Lifshitz theory, with Lagrangian density

$$\mathcal{L}_2 = \frac{1}{2} [\dot{\phi}^2 - \kappa^2 (\nabla^2 \phi)^2]. \quad (98)$$

Compacting y to an S_1 of circumference ℓ_y , we decompose the field and the Lagrangian in Fourier modes of momenta $k = 2\pi n/\ell_y$, $n \in \mathbb{Z}$,

$$\begin{aligned} \phi(x, y) &= \sum_k e^{iky} \phi_k(x), \\ \mathcal{L}_2(x, y) &= \sum_{k, k'} e^{i(k'+k)y} \mathcal{L}_{2, k, k'}(x), \end{aligned} \quad (99)$$

with

$$\mathcal{L}_{2, k, k'} = \frac{1}{2} [\dot{\phi}_k \dot{\phi}_{k'} - \kappa^2 \phi_k (\partial_x^2 - k^2) (\partial_x^2 - k'^2) \phi_{k'}]. \quad (100)$$

Integrating y out, we obtain $\mathcal{L}_1(x) = \int_{S_1} dy \mathcal{L}_2 = \ell_y \sum_k \mathcal{L}_{1, k}$ where

$$\mathcal{L}_{1, k} = \frac{1}{2} \left[|\dot{\phi}_k|^2 - \kappa^2 |\partial_x^2 \phi_k|^2 - 2\kappa m_k^2 |\partial_x \phi_k|^2 - m_k^4 |\phi_k|^2 \right], \quad (101)$$

and $m_k^2 = \kappa k^2$. Since $\mathcal{L}_{1, k}(\phi_k) = \mathcal{L}_{1, k}(\text{Re } \phi_k) + \mathcal{L}_{1, k}(\text{Im } \phi_k)$, each decoupled mode ϕ_k is a double copy of the real $(1 + 1)$ -dimensional massive deformation with Hamiltonian (39). Notice the fine-tuning of the $|\partial_x \phi_k|^2$ term, just as in (39), a direct consequence of the quadraticity of the Hamiltonian $\int d^d x A^\dagger A$. We find from the corresponding lower-dimensional c function (90), and identity (95),

$$\kappa_c = \frac{1}{\pi} \int_0^\infty dx \frac{x}{2} (\coth x - 1) = \frac{\pi}{24}. \quad (102)$$

This result is in agreement with previous results pertaining to the $(2 + 1)$ -dimensional $z = 2$ Lifshitz theory where, owing to the *spatial* conformal invariance of the ground-state wave-functional, the corner function may be given in closed form [9]. It is remarkable that relationship (95) between κ_c and the entropic c function still holds for the $z = 2$ Lifshitz theory, although its scale invariance is Lifshitz (anisotropic) instead of conformal. Interestingly, there is compelling evidence that the corner function can serve as a measure of the number of low-lying degrees of freedom in theories near criticality [9, 67, 68], generalizing the role played by the central charge of $(1 + 1)$ -dimensional CFTs.

V. POSITIVE BOSON AND MOTZKIN AND FREDKIN CHAINS

In this section, we consider the positive-field version of the (massless) Lifshitz ground state (32). Deep inside the bulk, the constraint $\phi \geq 0$ can be ignored such that correlation and entanglement properties are identical to those of the unconstrained theory. We compute the Rényi entropies of a boundary interval and of a bulk interval as an illustration. Finally, we comment on the relation between the positive Lifshitz boson and the Motzkin and Fredkin ground states, and discuss results in the literature.

A. Positive Lifshitz boson

The positive-field version of the ground state (32) reads

$$\Psi_0[\phi] = \frac{1}{\sqrt{\mathcal{Z}}} e^{-\frac{1}{2} \int d^d x \kappa (\nabla \phi)^2} \prod_x \theta(\phi(x)), \quad (103)$$

where $\theta(z)$ is the Heaviside function that enforces ϕ to be non-negative. This wavefunctional was conjectured in [19,20] to be a continuum version of the ground state of both Motzkin and Fredkin spin chains. The $\phi \geq 0$ constraint is then applied to match that of the lattice Motzkin/Dyck paths that appear in the ground state.

The corresponding propagator can be expressed in terms of the free one (36) as

$$G_{\mathbb{R}^+}(\phi', x'; \phi, x) = G_{\mathbb{R}}(\phi', x'; \phi, x) - G_{\mathbb{R}}(\phi', x'; -\phi, x), \quad (104)$$

which explicitly reads

$$G_{\mathbb{R}^+}(\phi', x'; \phi, x) = \sqrt{\frac{\kappa}{\pi(x' - x)}} \left(e^{-\frac{\kappa(\phi' - \phi)^2}{x' - x}} - e^{-\frac{\kappa(\phi' + \phi)^2}{x' - x}} \right). \quad (105)$$

By construction, the propagator $G_{\mathbb{R}^+}(\phi', x'; \phi, x)$ vanishes for $\phi = 0$ or $\phi' = 0$. Upon setting Dirichlet boundary conditions at $x = 0, L$, one thus needs to introduce a regulator h , i.e., $\phi(0, t) = \phi(L, t) = h$, which will be sent to zero at the end of the calculations. Furthermore, as the $M(> 1)$ -point functions for the positive boson are not Gaussian (they are sums of Gaussians), one cannot apply the (generalized) Fradkin-Moore formula (62). Instead, we compute the Rényi entropies directly from (61).

For a boundary interval A , as in Fig. 3, it is straightforward to find

$$S_n(A) = \frac{1}{2} \log \frac{L_A(L - L_A)}{\epsilon L} + b(n), \quad (106)$$

where $b(n) = \frac{2n+1}{2(n-1)} \log n + \frac{1}{1-n} \log \frac{\Gamma(n+1/2)}{\Gamma(3/2)^n} - \frac{1}{2} \log 4\kappa$.

For an interval B in the bulk (see Fig. 3), the moments of the associated reduced density matrix are given by the following complicated expression in terms of the hypergeometric function:

$$\begin{aligned} \text{Tr } \rho_B^n &= \frac{1}{\pi} \left(\frac{n\pi}{4\kappa} \right)^{1-n} \left(\frac{L_A L_B L_C}{\epsilon L} \right)^{(1-n)/2} w^{n/2} (w-1)^{n+1/2} \sum_{k=0}^n (-1)^k \\ &\times \binom{n}{k} \left[\frac{1}{n^2} \Gamma\left(\frac{n+1}{2}\right)^2 {}_2F_1\left(\frac{n+1}{2}, \frac{n+1}{2}, \frac{1}{2}, \frac{a_k^2}{w}\right) \right. \\ &+ \frac{\sqrt{w} \Gamma(n/2)^2}{2(n+1)^2 a_k} \left((w^{-1} a_k^2 - 1) {}_2F_1\left(\frac{n+2}{2}, \frac{n+2}{2}, \frac{-1}{2}, \frac{a_k^2}{w}\right) \right. \\ &\left. \left. + (1 - 2(n+2) a_k^2 w^{-1}) {}_2F_1\left(\frac{n+2}{2}, \frac{n+2}{2}, \frac{1}{2}, \frac{a_k^2}{w}\right) \right) \right], \quad (107) \end{aligned}$$

where $w = (L_A + L_B)(L_B + L_C)/(L_A L_C)$, and $a_k = 1 - 2k/n$. However, deep in the bulk, i.e., for $w \rightarrow 1$, only the $k = 0$, n

terms contribute to leading order in the sum above. We find

$$\begin{aligned} \text{Tr } \rho_B^n &\simeq \frac{1}{\pi} \left(\frac{n\pi}{4\kappa} \right)^{1-n} \left(\frac{L_A L_B L_C}{\epsilon L} \right)^{(1-n)/2} w^{n/2} (w-1)^{n+1/2} \\ &\times \left[n^{-2} 2\sqrt{\pi} \Gamma\left(n + \frac{1}{2}\right) \frac{1}{(w-1)^{n+1/2}} + \dots \right] \\ &= \left(\frac{L_A L_B L_C}{\epsilon L} \right)^{(1-n)/2} \left(\frac{n\pi}{4\kappa} \right)^{1-n} \frac{2}{n^2 \sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) + \dots, \quad (108) \end{aligned}$$

and one can check that this expression is properly normalized for $n = 1$. Finally, the Rényi entropy reads

$$\begin{aligned} S_n(B) &= \frac{1}{2} \log \frac{L_A L_B L_C}{\epsilon^2 L} + \text{const}, \\ &\sim \frac{1}{2} \log \frac{L_B}{\epsilon} + \frac{1}{2} \log \frac{L}{\epsilon} \quad (109) \end{aligned}$$

valid for $L_A, L_C \sim L \gg L_B$. We thus retrieve the entropy of the free massless theory in the regime where B is deep enough in the bulk. This is actually a general feature; the constraint $\phi \geq 0$ is inconsequential deep inside the bulk due to the exponentially small probability of ϕ approaching zero. Thus, in that regime, we expect correlation functions [69] and Rényi entropies to be given by those for the unconstrained boson (see Sec. IV). In particular, the mutual information of two disconnected intervals deep in the bulk is given by (80), i.e.,

$$I(A : B) \sim \frac{1}{2} \log \frac{L}{L_{C_2}}, \quad (110)$$

and that of two adjacent intervals by (87), that is

$$I(A : B) \sim \frac{1}{2} \log \frac{L_A L_B}{\epsilon(L_A + L_B)} + \frac{1}{2} \log \frac{L}{\epsilon}. \quad (111)$$

The former is only IR divergent while the latter is both UV and IR divergent.

B. Relation to Motzkin and Fredkin chains

The Motzkin and Fredkin models are spin-1 and spin-1/2 chains, respectively. The Hamiltonians describing these spin chains are local, frustration free, and with nearest-neighbor interactions. Their ground states are equal-weight superpositions of all states corresponding to Motzkin paths for the Motzkin model, and to Dyck paths for the Fredkin model. A Motzkin path is any path connecting $(x, y) = (0, 0)$ and $(x, y) = (L, 0)$ in the upper-half plane, formed by three types of moves (moving from left to right): diagonal-up, diagonal-down, and flat, (which correspond to the three state in the S_z basis for spin-1). Similarly, a Dyck path is a Motzkin path with only the diagonal-up/down moves allowed. The spin variables can thus be represented as a ‘‘height field’’ ϕ , i.e., $\partial_x \phi(x) = S_z(x)$. The ground-state property $S_z^{\text{tot}} = 0$ translates into Dirichlet boundary conditions on ϕ : $\phi(L) - \phi(0) = 0$. Mapping the problem to a random walk, the wavefunctional (103) of the positive Lifshitz boson was thus conjectured in [19,20] to capture the ground states of both Motzkin and Fredkin spin chains in the continuum limit. The only difference between the two being encoded in the dimensionless

parameter κ . Indeed, expression (106) for the Rényi entropy of a boundary interval matches exactly the results for the Motzkin model [19,50] and for the Fredkin spin chain [20,49], provided $\kappa = 3/4, 2$ in the Motzkin and Fredkin ground states, respectively. Correlation functions can be shown to match as well. However, our field theory result (109) for the (Rényi) entropy of a bulk interval does not agree with the spin chains calculations [50,70]. Our formula does reproduce the “geometric” part (i.e., the $1/2 \log L_B$ term) of the entropy of Motzkin and Fredkin ground states, but we also observe an additional IR divergent term that does not appear for the spin chains. We are thus lead to conclude that, as it stands, the continuum limit of Motzkin and Fredkin ground states is not adequately described by the positive Lifshitz boson ground state. In the Discussion, we discuss one modification to the wavefunctional that could potentially cure the discrepancies, namely making the field compact.

Let us close this section with a remark on the mutual information for the Motzkin and Fredkin ground states, which was studied in [70]. There, the author finds that the mutual information of two disconnected subsystems inside the bulk does not depend on their separation, and takes the form

$$I(A : B) \sim \frac{1}{2} \log \frac{L_A L_B}{L_A + L_B} + \text{const}, \quad (112)$$

in the limit $L_A, L_B \gg 1$, where L_X is the number of spins in the subsystem X . Surprisingly, this formula coincides with the geometric term (first term) in (111), which gives the mutual information shared by two *adjacent* intervals for the Lifshitz ground state. Furthermore, a length scale seems to be missing in (112) if one wishes to take the continuum limit. Indeed, expression (112) would then appear to be UV divergent, which is in contradiction with the fact that mutual information between disconnected subsystems must be UV finite.

VI. DISCUSSION AND OUTLOOK

In this paper, we considered an important and vast class of quantum states, namely Rokhsar-Kivelson states that quantum mechanically encode the partition function of a classical system. We studied entanglement and correlations properties of *continuum* RK states for which the dual classical models are local QFTs. As instances of such states, Lifshitz ground states and their deformations were analyzed.

We proved in Sec. IIB that for real-valued RK wavefunctionals of $(d + 1)$ -dimensional QFTs, the reduced density matrix ρ_{AUB} of two disconnected subsystems A and B is (mixed) separable, meaning that the partial trace over C (the complement of $A \cup B$) disentangles A from B . Accordingly, the mutual information results entirely from classical and quantum nonentangling correlations [52–55]. Furthermore, the separability of ρ_{AUB} for two disjoint subsystems implies the vanishing of the logarithmic negativity. This is in agreement with previous results where the logarithmic negativity was shown to vanish; in [20] for the ground state of the noncompact free massless $z = 2$ Lifshitz boson, and in [23] for local ground states of free massless Lifshitz theories with even positive integer z , both for compact and noncompact fields.

We have introduced nontrivial deformations of the non-compact $z = 2$ Lifshitz theory that preserve the RK structure of the original theory. Accordingly, the correlators of the ground-state wavefunctional are encoded in the partition function of a lower-dimensional, local action. For the massive deformation studied in Sec. IV, which explicitly breaks Lifshitz scaling, ground-state correlations are given by the Euclidean harmonic oscillator. As expected, we found an exponential decay of correlations with correlation length $\xi \sim m^{-1}$, and observed that cluster decomposition, violated in the massless case, is restored by the regulating mass.

We further computed, from first principles, the Rényi entanglement entropy over a general bipartition $\{A, B\}$ with finitely many surface points, $\partial A = \{x_1, \dots, x_M\}$, and the corresponding c function (independent of the Rényi index n). Expression (62) for $S_n(A)$ generalizes to our $(1 + 1)$ -dimensional non-Lifshitz-invariant case the celebrated Fradkin-Moore formula [9] pertaining to the $(2 + 1)$ -dimensional $z = 2$ Lifshitz theory. Application of this formula yields a general expression for the UV-finite mutual information between disjoint subsystems A_1 and A_2 . When A_1, A_2 are two intervals, the mutual information decays exponentially with separation, unless the mass is zero. In the massless case, the mutual information is IR divergent and persists at all separations, which can be seen as a consequence of the cluster decomposition being violated.

We also observed a relationship between corner entanglement in the $(2 + 1)$ -dimensional $z = 2$ Lifshitz theory and the entropic c function of the $(1 + 1)$ -dimensional massive deformation. Specifically, the sharp-limit coefficient κ_c of the corner entanglement is found to be the integrated c -function, see (102), for our non-Lorentz-invariant theory. This is a generalization of the corresponding relation between the sharp-limit coefficient of a CFT₃ and the entropic c function of a lower-dimensional QFT [66]. Interestingly, the entropic c -function is found to decrease under wavefunction RG flow even though the theory is not Lorentz-invariant. (In fact, not even Lifshitz-invariant.) Whether a similar result may hold for higher-dimensional versions of the massive deformation presented in this paper is worth investigating, especially in dimension $2 + 1$, where the F -theorem for CFTs guarantees the monotone decreasing of the entanglement entropy of a disk under RG flow [71]. We leave this for future study.

As a byproduct of our results on Rényi entropies, we computed the capacity of entanglement C_A in Appendix B. This quantity, analog of the heat capacity for thermal states, characterizes the width of the entanglement spectrum. We find that for Lifshitz ground states, the capacity of entanglement is finite and follows an area law. Comparison of capacity of entanglement with entanglement entropy tells us something about the entanglement structure of the quantum state under study. We found $C_A \ll S_A$ for Lifshitz ground states, which could be interpreted as entanglement being effectively carried by maximally entangled EPR pairs in such systems.

One motivation for starting this paper was in relation to the Motzkin and Fredkin spin chains [29,49]. The positive-field version of the Lifshitz ground state (32) was believed to be the continuum limit of the Motzkin and Fredkin ground states, and, indeed, captures many of their spin and entanglement features [19,20,50]. In comparing our field theory predictions to the spin chains results in the literature, the entanglement

entropy of a bulk interval (67) obtained here does not agree with that computed in [50]. An extra IR piece, absent for the spin chains, is at the origin of the discrepancy. This also implies that the form of the mutual information (110) cannot be that for Motzkin and Fredkin ground states. Indeed, the mutual information (110) for disconnected intervals obtained here does not agree with that of [70] computed on the lattice. However, as argued in Sec. V, the expression found in [70] (see (112)) does not possess a sensible continuum limit. Still, as it stands, the positive Lifshitz boson cannot be taken as the continuum limit of Motzkin and Fredkin ground states. It thus raises the question of which field theory does? A possible direction would be to look at the *compact* Lifshitz boson ground state. As may be observed in [18] in $(2+1)$ dimensions, potential IR divergent terms in the mutual information get canceled against an additional contribution coming from the winding modes present in the compact case. It would thus be interesting to generalize our paper for the one-dimensional theory to compact fields.

ACKNOWLEDGMENTS

It is a pleasure to thank B. Doyon for interesting discussions. Ch.B. thanks the Department of National Defense of Canada for financial support to facilitate completion of his Ph.D.. C.I.B. is supported by a CRM-Simons Postdoctoral Fellowship at the Université de Montréal. W.W.-K. was funded by a Discovery Grant from NSERC, a Canada Research Chair, a Grant from the Fondation Courtois, and a “Établissement de nouveaux chercheurs et de nouvelles chercheuses universitaires” Grant from FRQNT.

APPENDIX A: DEFORMATION BY A SINGULAR POTENTIAL

In this Appendix, we return to the original Lifshitz theory (28) and perform a nontrivial deformation (for $d \neq 2$) that preserves the Lifshitz scale invariance in all dimensions, as well as the spatial conformal symmetry of the ground state when $d = 1$. Set $\Lambda[\phi] = \int d^d x \frac{g}{2} \phi(x)^{2d/(d-2)}$, so that the classical action becomes $S_{\text{cl}}[\phi] = \int d^d x (\kappa(\nabla\phi)^2 + \frac{g}{2} \phi^{2d/(d-2)})$, and

$$A(x) = \frac{1}{\sqrt{2}} \left(\frac{\delta}{\delta\phi(x)} - \kappa \nabla^2 \phi + \frac{gd}{2(d-2)} \phi^{\frac{d+2}{d-2}} \right). \quad (\text{A1})$$

The corresponding ground state of $H_{\text{sing}}^{\text{normal}} = \int d^d x A^\dagger(x)A(x)$ is

$$\Psi_0[\phi] = \frac{1}{\sqrt{\mathcal{Z}}} e^{-\frac{1}{2} \int d^d x (\kappa(\nabla\phi)^2 + \frac{g}{2} \phi^{2d/(d-2)})}, \quad (\text{A2})$$

with normalization factor given in (A13). The parameter g is dimensionless for all d , and the operators $A(x)$ are invariant under Lifshitz rescaling $(\mathbf{x}, t) \rightarrow (\lambda\mathbf{x}, \lambda^2 t)$, as per (34) and (35). (The affine field-shift symmetry is lost however.) As a consequence, the ground-state wavefunctional $\Psi_0[\phi]$ is invariant under spatial scaling $\mathbf{x} \rightarrow \lambda\mathbf{x}$. Yet, $H_{\text{sing}}^{\text{normal}}$ hardly qualifies as a *bona fide* parent Hamiltonian for $|\Psi_0\rangle$, because it contains the awkward UV-divergent term

$$\frac{1}{2} \text{tr} \left(\kappa \nabla^2 - g \frac{d(d+2)}{2(d-2)^2} \phi^{\frac{4}{d-2}} \right), \quad (\text{A3})$$

whose physical meaning is rather opaque. Note that this term is nothing but the commutator $\frac{1}{2} \int d^d x [A^\dagger(x), A(x)]$. But we can no longer rely on the deformed Lifshitz theory $H_{\text{sing}}^+ = \frac{1}{2} \int d^d x \{A^\dagger(x), A(x)\}$, even though it is without divergence, since it differs from $H_{\text{sing}}^{\text{normal}}$ by precisely (A3), which is *not* a multiple of the identity, so this time $H_{\text{sing}}^{\text{normal}}$ and H_{sing}^+ do not share their eigenstates.

1. Supersymmetric deformation

The way around this difficulty is to deform H_{sing}^+ (and its Hilbert space) to include fermionic degrees of freedom [48]. Define the operators

$$Q = \int d^d x \bar{\psi}(x)A(x), \quad \bar{Q} = \int d^d x \psi(x)A(x)^\dagger, \quad (\text{A4})$$

with $\{\psi(x), \bar{\psi}(y)\} = \delta(x-y)$. In the Schrödinger picture, where $\psi(x)$ is a multiplication operator and $\bar{\psi}(x) = \delta/\delta\psi(x)$, consider the normalized Grassmann-valued functional $\Psi_F[\psi] = \prod_x \psi(x)$. Then $\psi(x)|\Psi_F\rangle = 0$ for all x . The Hamiltonian

$$H_{\text{sing}}^{\text{SUSY}} = \frac{1}{2} \{Q, \bar{Q}\} \quad (\text{A5})$$

is part of a supersymmetric structure

$$\{Q, Q\} = 0 = \{\bar{Q}, \bar{Q}\}, \quad [Q, H_{\text{sing}}^{\text{SUSY}}] = 0 = [\bar{Q}, H_{\text{sing}}^{\text{SUSY}}] \quad (\text{A6})$$

with unbroken supersymmetry: for $|\Omega\rangle = |\Psi_0\rangle \otimes |\Psi_F\rangle$, where $\langle\phi|\Psi_0\rangle = \exp(-S_{\text{cl}}[\phi])$ as before, we have

$$Q|\Omega\rangle = 0 = \bar{Q}|\Omega\rangle. \quad (\text{A7})$$

As in the paper of Dijkgraaf *et al.* [48], the presence of the fermionic sector enables both Q and \bar{Q} to annihilate $|\Omega\rangle$, making it a ground state of the *divergenceless* anticommutator $\{Q, \bar{Q}\}$. Since the product $\Psi_F[\psi] = \prod_x \psi(x)$ has no spatial entanglement, the supersymmetric ground state $|\Omega\rangle$ and the bosonic ground state $|\Psi_0\rangle$ possess the same structure of spatial entanglement. Moreover, $H_{\text{sing}}^{\text{SUSY}}$ is an unambiguous parent Hamiltonian for $|\Omega\rangle$. Interestingly, this type of supersymmetric structure naturally emerges when stochastically quantizing a classical action S_{cl} [72]. The parent Hamiltonian H^{SUSY} is then physically realized as the stochastic field theory whose $t \rightarrow \infty$ equilibrium correlations yield the correlations of the quantized action. Expanding (A5), we get

$$\begin{aligned} H_{\text{sing}}^{\text{SUSY}} &= \frac{1}{2} \int d^d x \left[\Pi^2 + \frac{1}{2} (S'_{\text{cl}})^2 + \frac{1}{2} (\bar{\psi} S'_{\text{cl}} \psi - \psi S'_{\text{cl}} \bar{\psi}) \right] \\ &= H_{\text{sing}}^+ + \frac{1}{4} \int d^d x [\bar{\psi} S'_{\text{cl}} \psi - \psi S'_{\text{cl}} \bar{\psi}]. \end{aligned} \quad (\text{A8})$$

In the first line, we have written $S_{\text{cl}}^{(n)} = (\delta^n/\delta\phi^n)S_{\text{cl}}$, for short. We emphasize that $|\Psi_0\rangle \otimes |\Psi_F\rangle$ is a ground state of the above Hamiltonian, despite the fact that $|\Psi_0\rangle$ is not a ground state of H_{sing}^+ . The corresponding Euclidean theory has partition function

$$\mathcal{Z}_E = \int \mathcal{D}\phi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\int d\tau d^d x \mathcal{L}_E}, \quad (\text{A9})$$

where, up to boundary terms,

$$\mathcal{L}_E = \frac{1}{2}(\partial_\tau \phi + \frac{1}{2}S'_{cl})^2 - \bar{\psi}(\partial_\tau + \frac{1}{2}S''_{cl})\psi. \quad (A10)$$

Specializing again to $d = 1$, we find

$$H_{sing}^+ = H_{Lif} + \frac{1}{2} \int dx \left(\frac{g\kappa}{\phi^3} \nabla^2 \phi + \frac{g^2}{4\phi^6} \right), \quad (A11)$$

and

$$\mathcal{L}_E = \frac{1}{2}(\partial_\tau \phi)^2 + \frac{\kappa^2}{2}(\nabla^2 \phi)^2 + \frac{g\kappa}{2}\phi^{-3}\nabla^2 \phi + \frac{g^2}{8}\phi^{-6} - \bar{\psi} \left(\partial_\tau - \kappa \nabla^2 + \frac{3g}{2}\phi^{-4} \right) \psi. \quad (A12)$$

We will mostly consider the stable case $g > 0$, for which the singular potential in (A11) prevents the field from vanishing anywhere. Without loss of generality, we can restrict ϕ to strictly positive real values $\phi(x) \in \mathbb{R}^{>0}$. The boundary conditions are chosen to be $\phi(0, t) = \phi(L, t) = h$, for an arbitrary but strictly positive regulator $h = 0^+$, preventing the divergence of the potential energy term. (Our analysis also holds for negative values of g as long as the regulating condition $\phi \geq h > 0$ is maintained.) The normalization factor of $|\Omega\rangle = |\Psi_0\rangle \otimes |\Psi_F\rangle$ is

$$\mathcal{Z} = \int_{\phi(0)=h=\phi(L)} \mathcal{D}\phi e^{-\int dx \left(\kappa(\partial_x \phi)^2 + \frac{g}{2\phi^2} \right)}. \quad (A13)$$

One can recognize \mathcal{Z} as the partition function of a single quantum mechanical particle with Euclidean Lagrangian $\mathcal{L}_g = \kappa(\partial_x \phi)^2 + \frac{g}{2\phi^2}$. The theory in spatial dimension $d = 1$ stands out as having full conformal spatial symmetry. The group of dilations $x \rightarrow \lambda x$, $\lambda > 0$, may be extended to the $SL(2, \mathbb{R})$ group of transformations

$$x \rightarrow x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \alpha\delta - \beta\gamma = 1, \quad (A14)$$

with Jacobian determinant

$$\frac{\partial x'}{\partial x} = \frac{1}{(\gamma x + \delta)^2} > 0. \quad (A15)$$

Fields transform as tensor densities of weight 1/2 corresponding to their length dimension

$$\phi'(x') = \sqrt{\frac{\partial x'}{\partial x}} \phi(x) = \frac{\phi(x)}{\gamma x + \delta}. \quad (A16)$$

(The sign of $\gamma x + \delta$ is immaterial, the action being quadratic in the field, so we drop the absolute value bars.) The action $\int dx \mathcal{L}_g(\phi, \partial_x \phi)$ is called $SL(2, \mathbb{R})$ -conformal because it is invariant, up to a boundary term, under the joint transformations (A14) and (A16). The quantized theory, conformal quantum mechanics (CQM), has been known for a long time [36,38]. Stochastic quantization of \mathcal{L}_g generates the $(1 + 1)$ -dimensional field theory H_{sing}^{SUSY} , whose supersymmetric ground state $|\Psi_0\rangle \otimes |\Psi_F\rangle$ contains the propagators of CQM. Conversely, knowledge of the propagators of CQM is analytic knowledge about $|\Psi_0\rangle \otimes |\Psi_F\rangle$ and its parent theory. Because $|\Psi_F\rangle$ is a spatial product, the bosonic ground state $|\Psi_0\rangle$ and the supersymmetric ground state $|\Psi_0\rangle \otimes |\Psi_F\rangle$ have the same spatial entanglements. The (normalized) eigenstates of the CQM

Hamiltonian, $H_g = -\frac{1}{4\kappa} \partial_\phi^2 + \frac{g}{2\phi^2}$, were identified in [36,38],

$$(H_g - E)\psi_E = 0 \iff \psi_E(\phi) = \sqrt{2\kappa\phi} J_\nu(\phi\sqrt{4\kappa E}), \quad (A17)$$

where $\nu = \sqrt{2g\kappa + \frac{1}{4}}$ and J_ν is a Bessel function of the first kind of order ν . We will only consider nonnegative values of ν (i.e., $g \geq -1/8\kappa$). The propagators are readily computed,

$$\langle \phi', x' | \phi, x \rangle = \frac{2\kappa\sqrt{\phi'\phi}}{x' - x} \exp\left[-\kappa\left(\frac{\phi'^2 + \phi^2}{x' - x}\right)\right] I_\nu\left(\frac{2\kappa\phi'\phi}{x' - x}\right), \quad (A18)$$

with I_ν the modified Bessel function of the first kind of order ν . Deep in the bulk, where $\langle \phi' \phi \rangle$ is large compared to $x' - x$ (see below), the singular potential is ineffective and we recover the free massless boson propagator $G_{\mathbb{R}}(\phi', x'; \phi, x) = \sqrt{\frac{\kappa}{\pi(x' - x)}} \exp[-\frac{\kappa(\phi' - \phi)^2}{x' - x}]$. On the other hand, in the limit $g \rightarrow 0$ (i.e., $\nu \rightarrow 1/2$), and using the fact that $I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh z$, we find the propagator of the positive-valued free field

$$G_{\mathbb{R}^{>0}}(\phi', x'; \phi, x) = \sqrt{\frac{\kappa}{\pi(x' - x)}} \left(e^{-\frac{\kappa(\phi' - \phi)^2}{x' - x}} - e^{-\frac{\kappa(\phi' + \phi)^2}{x' - x}} \right), \quad (A19)$$

because nothing is left of the singular potential but the constraint $\phi > 0$. We emphasize that the singular deformation is not a perturbation of the original Lifshitz point, but of the positive-valued free field. However, the propagators $G_{\mathbb{R}}$ and $G_{\mathbb{R}^{>0}}$ are indistinguishable deep in the interior.

2. Correlations in the ground state

Obviously, the field has nonvanishing vacuum expectation values

$$\langle \phi(x) \rangle_\nu = \int_h^\infty d\phi \phi f_1^{(\nu)}(\phi, x) = \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 1)} \sqrt{\frac{x(L - x)}{\kappa L}}, \quad (A20)$$

where $f_1^{(\nu)}(\phi, x) = \langle h, L | h, 0 \rangle^{-1} \langle h, L | \phi, x \rangle \langle \phi, x | h, 0 \rangle$ and $h = 0^+$ is the field regulator introduced below (A12) to prevent the divergence of the potential energy term. The regulator h is also necessary to prevent the individual vanishing of the propagators, given by (A18). However, the integral in (A20), and similar integrals, have a well-defined limit as h is sent to zero. We will work in that limit from now on. Interestingly, the only difference between first moments with different values of ν (or g) is a real-number prefactor. An even stronger relationship holds between all higher moments (see Appendix A 4),

$$\langle \phi^\alpha(x) \rangle_\mu = R(x, x, \mu, \nu) \langle \phi^{\alpha+2(\mu-\nu)}(x) \rangle_\nu, \quad (A21)$$

with

$$R(x_1, x_2, \mu, \nu) = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \mu)} \left(\frac{\kappa L}{x_1(L - x_2)} \right)^{\mu - \nu}. \quad (A22)$$

In particular, higher moments can be obtained from the *first* moment of the (positive-valued) free boson via the relations $\langle \phi^{1+2\mu}(x) \rangle_\nu \propto \langle \phi(x) \rangle_{\nu+\mu} \propto \langle \phi(x) \rangle_{1/2}$. The general

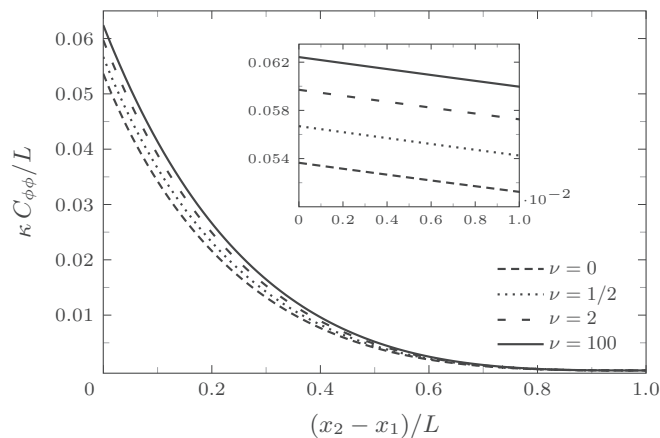


FIG. 8. Connected correlator $\frac{\kappa}{L} C_{\phi_1, \phi_2}(x_1, x_2)$ for a central interval of length $x_2 - x_1$ for $\nu = 0, 1/2, 2, 20$, obtained from (A20) and (A25). Inset: Small central interval regime. The linear behavior is in perfect agreement with (A29).

expression is

$$\langle \phi^{1+2\mu}(x) \rangle_\nu = \frac{\Gamma(\mu + \nu + \frac{3}{2})}{\Gamma(\nu + 1)} \left(\frac{x(L-x)}{\kappa L} \right)^{\frac{1}{2} + \mu}. \quad (\text{A23})$$

Deep in the bulk, $\langle \phi^{1+2\mu}(x) \rangle_\nu \sim (\sqrt{L})^{1+2\mu}$ for any ν .

Once again, the reduced density $\rho_{A \cup B}$ is a separable mixed state for any disconnected subsystems A, B (see Sec. II B). Therefore, A and B are not entangled, and their correlations are nonentangling [52–55], coming from the statistical mixture left in $\rho_{A \cup B}$, if any. As A, B come into contact, the result does not hold anymore and quantum-driven contact terms are expected. To compute the two-point function $\langle \phi(x_1)\phi(x_2) \rangle_\nu$, we find it best to express it in terms of the variable

$$w = \frac{(L_A + L_B)(L_B + L_C)}{L_A L_C}, \quad (\text{A24})$$

with subinterval lengths L_A, L_B, L_C as in Fig. 3. We find

$$\langle \phi(x_1)\phi(x_2) \rangle_\nu = \frac{L_B}{\kappa} \left[\frac{\Gamma(\frac{3}{2} + \nu)}{\Gamma(1 + \nu)} \right]^2 \frac{(w-1)^{1+\nu}}{w^{\frac{3}{2} + \nu}} \times {}_2F_1 \left[\frac{3}{2} + \nu, \frac{3}{2} + \nu; 1 + \nu; \frac{1}{w} \right], \quad (\text{A25})$$

where ${}_2F_1(a, b; c; x)$ is the Gaussian hypergeometric function. From (A20) and (A25), we can obtain analytic expressions for the connected correlator $C_{\phi_1, \phi_2} = \langle \phi_1 \phi_2 \rangle_\nu - \langle \phi_1 \rangle_\nu \langle \phi_2 \rangle_\nu$ and, by repeated differentiation, for the connected correlator $C_{\partial \phi_1, \partial \phi_2} = \langle \partial \phi_1 \partial \phi_2 \rangle_\nu - \langle \partial \phi_1 \rangle_\nu \langle \partial \phi_2 \rangle_\nu$. Here we omit the full expressions, but in Fig. 8 we plot $C_{\phi_1, \phi_2}(x_1, x_2)$ (in units of κ/L) for a central interval of length $x_2 - x_1$, and different values of ν . In Fig. 9 we plot $C_{\partial \phi_1, \partial \phi_2}(x_1, x_2)$ (in units of κL) for the same interval and values. Deep in the bulk, where $w \rightarrow 1$, we may expand (A25) in powers of

$$w - 1 = \frac{L L_B}{L_A L_C} = \frac{L(x_2 - x_1)}{x_1(L - x_2)} \sim \frac{1}{L}, \quad (\text{A26})$$

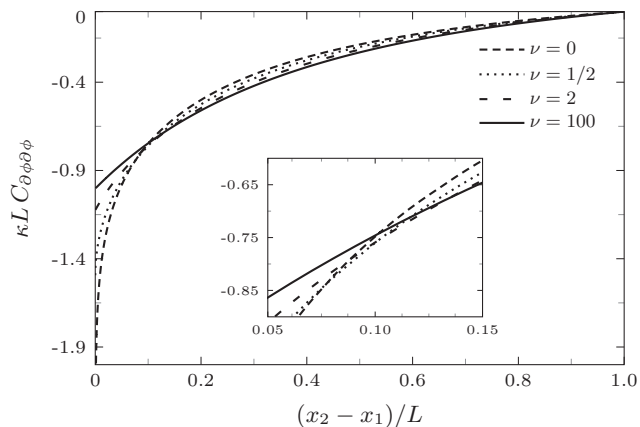


FIG. 9. Connected correlator $\kappa L C_{\partial \phi_1, \partial \phi_2}(x_1, x_2)$ for a central interval of length $x_2 - x_1$, and for $\nu = 0, 1/2, 2, 20$, obtained analytically from differentiating (A20) and (A25). Inset: Correlations are almost independent of ν when $x_2 - x_1 \sim 0.1L$.

leading to

$$\langle \phi(x_1)\phi(x_2) \rangle_\nu = \frac{(1+\nu)L}{4\kappa} - \frac{x_2 - x_1}{4\kappa} + O(1/L). \quad (\text{A27})$$

This expression is not restricted to central intervals. For the special case $\nu = 1/2$, we recover

$$\langle \phi(x_1)\phi(x_2) \rangle_{1/2} = \frac{3L}{8\kappa} - \frac{x_2 - x_1}{4\kappa} + O(1/L), \quad (\text{A28})$$

in agreement with the positive-valued boson. The field connected correlator deep in the bulk is found to be

$$C_{\phi, \phi}(x_1, x_2) = \langle \phi(x_1)\phi(x_2) \rangle_\nu - \langle \phi(x_1) \rangle_\nu \langle \phi(x_2) \rangle_\nu = \frac{\alpha_\nu L}{\kappa} - \frac{x_2 - x_1}{4\kappa} + O(1/L), \quad (\text{A29})$$

with α_ν a finite, positive constant independent of L ,

$$\alpha_\nu = \frac{1}{4} \left[1 + \nu - \left(\frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 1)} \right)^2 \right]. \quad (\text{A30})$$

The bulk two-point function (A27), and the bulk connected correlator (A29) consist in a constant leading term, which diverges like L in the thermodynamic limit, followed by a universal, translation-invariant bulk term linear in $x_2 - x_1$, and independent of ν . The universal behavior $\langle \phi(x_1)\phi(x_2) \rangle_\nu \sim (x_2 - x_1)^\Delta$, with exponent $\Delta = 1$, is a joint consequence of the dimension of the fields, $\dim \phi = \sqrt{\text{length}}$, and of the absence of a length scale other than L . Nondependence of this term on ν is yet another indication that the singular potential is ineffective in the bulk.

From (A29) and dimensionality, we expect $\partial \phi \partial \phi$ correlations to vanish in the limit $L \rightarrow \infty$. To obtain an analytic bulk expansion for $C_{\partial \phi_1, \partial \phi_2}$ beyond the leading order we find it necessary to restrict ν to chosen intervals or values. We provide four cases, beginning with $\nu = 0$, and the interval $\nu \in (0, 1)$.

Case $\nu = 0$:

$$C_{\partial\phi,\partial\phi} = \frac{\beta_0}{\kappa L} + \frac{1}{4\kappa L} \log \frac{x_2 - x_1}{L} - \frac{3(x_2 - x_1)}{4\kappa L^2} \log \frac{x_2 - x_1}{L} + \frac{\delta_0}{\kappa L^2} (x_2 - x_1) + O(1/L^3), \tag{A31}$$

with $\beta_0 = -\frac{1}{2} \log 2$, and $\delta_0 = \frac{3}{2} \log 2 - 1$.

Case $0 < \nu < 1$:

$$C_{\partial\phi,\partial\phi} = \frac{\beta_\nu}{\kappa L} + \frac{\gamma_\nu}{\kappa L^{1+\nu}} (x_2 - x_1)^\nu + O(1/L^2) \tag{A32}$$

with

$$\beta_\nu = -1 - \frac{1}{4\nu}, \quad \gamma_\nu = 4^\nu \left[\frac{\Gamma(\nu + 3/2)}{\Gamma(\nu + 1)} \right]^2 \csc \pi \nu. \tag{A33}$$

The leading term $\beta_\nu/\kappa L$, with coefficient given in (A33), is actually valid for all nonzero values of ν . The coefficient of fractional order γ_ν diverges near $\nu = 0$ and $\nu = 1$. As $\nu \rightarrow 1$, term $O(1/L^2)$ in (A32) can no longer be neglected. We also provide the expansions for the integer values $\nu = 1$ and $\nu = 2$.

Case $\nu = 1$:

$$C_{\partial\phi,\partial\phi} = \frac{\beta_1}{\kappa L} - \frac{9}{4\kappa L^2} (x_2 - x_1) \log(x_2 - x_1) + \frac{\delta_1}{\kappa L^2} (x_2 - x_1) + O(1/L^3). \tag{A34}$$

Case $\nu = 2$:

$$C_{\partial\phi,\partial\phi} = \frac{\beta_2}{\kappa L} + \frac{\delta_2}{\kappa L^2} (x_2 - x_1) + O(1/L^3), \tag{A35}$$

where $\delta_1 = \frac{9}{2} \log 2 - \frac{15}{4}$, and $\delta_2 = \frac{45}{8}$. In contrast to the $\phi\phi$ correlator, the non-constant leading term of the $\partial\phi\partial\phi$ correlator is strongly sensitive to the value of ν , not only in its coefficient but even in its functional dependence to the subinterval length $x_2 - x_1$. Case $\nu = 0$ presents a quasiconstant term $O(\log(x_2 - x_1))$, while case $\nu = 1$ has a quasilinear term $O((x_2 - x_1) \log(x_2 - x_1))$. For general integer ν , it is tempting to expect a quasipolynomial term $O((x_2 - x_1)^\nu \log(x_2 - x_1))$, subleading for $\nu \geq 2$.

Finally, the correspondence (A21) between higher moments can be generalized to multipoint functions deep in the bulk. For the points $0 < x_1 < x_2 < \dots < x_N < L$ with $x_1 \sim L/2 \sim L - x_N$, and writing $\phi(x_i) = \phi_i$, we find

$$\langle \phi_1^{\alpha_1} \dots \phi_N^{\alpha_N} \rangle_\mu = R(x_1, x_N, \mu, \nu) \langle \phi_1^{\alpha_1 + \mu - \nu} \phi_2^{\alpha_2} \dots \phi_{N-1}^{\alpha_{N-1}} \phi_N^{\alpha_N + \mu - \nu} \rangle_\nu. \tag{A36}$$

These relationships are a manifestation of the fact that bulk propagators are free, see section 4 of this Appendix. The effect of the singular potential on (A36) arises solely from the surface propagators $\langle \phi_1, x_1 | h, 0 \rangle$ and $\langle h, L | \phi_N, x_N \rangle$ between the system's boundaries and ϕ_1, ϕ_N . All other propagators are free. More generally, when some points are close to the boundary, more bulk propagators become boundary propagators. If, say, x_1 is close to the boundary, both $\langle \phi_1, x_1 | h, 0 \rangle$ and $\langle \phi_2, x_2 | \phi_1, x_1 \rangle$ will be boundary propagators, and α_2 will be modified accordingly in (A36).

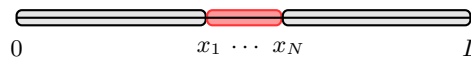


FIG. 10. N ordered points $x_1 < \dots < x_N$ deep in the bulk.

3. Entanglement in the ground state

We now compute the Rényi entanglement entropies of a boundary interval A as in Fig. 3, which are given by

$$S_n(A) = \frac{1}{1-n} \log \int_h^\infty d\phi [f_1^{(\nu)}(\phi, L_A)]^n + \frac{1}{2} \log \frac{1}{\epsilon}, \tag{A37}$$

where $f_1^{(\nu)} = \mathcal{Z}^{-1} \langle h, L | \phi, L_A \rangle \langle \phi, L_A | h, 0 \rangle$ is the single-point probability distribution obtained from the propagator (A18), $h = 0^+$ is the field regulator introduced below (A12), and \mathcal{Z} is a normalization factor given in (A13). Note that the (generalized) Fradkin-Moore formula (62) is not applicable here because its derivation requires that M -point functions be Gaussian. In the limit of vanishing regulator, we obtain

$$S_n(A) = \frac{1}{2} \log \frac{L_A(L - L_A)}{\epsilon L} + b(n, \nu), \tag{A38}$$

with

$$b(n, \nu) = \frac{n(1 + 2\nu) + 1}{2(n-1)} \log n + \frac{1}{1-n} \times \log \frac{\Gamma((1+n+2\nu)/2)}{\Gamma(1+\nu)^n} - \frac{1}{2} \log 4\kappa. \tag{A39}$$

Comparing (A38) with the entropy (66) in the (undeformed) massless case then reveals the same entanglement behavior for all values of ν . In particular, all c_n functions are identically equal to $1/2$, consistent with the unbroken spatial conformal symmetry of the ground-state wavefunctional deep in the bulk.

4. Multipoint moments

Consider N ordered points $x_1 < x_2 < \dots < x_N$ deep in the bulk, as shown in Fig. 10, and let us write $\phi(x_i) = \phi_i$. Based on the fact that propagators become free in the bulk, we expect a relationship between the functions

$$f_N^{(\nu)}(\phi, \mathbf{x}) = \frac{\langle h, L | \phi_N, x_N \rangle}{\langle h, L | h, 0 \rangle} \prod_{i=1}^{N-1} \langle \phi_{i+1}, x_{i+1} | \phi_i, x_i \rangle \langle \phi_1, x_1 | h, 0 \rangle \tag{A40}$$

with different values of ν . As in the previous section, $h = 0^+$ is a field regulator preventing the potential energy term from diverging. The propagators are given in (A18), which is repeated here for convenience:

$$\langle \phi', x' | \phi, x \rangle = \frac{2\kappa \sqrt{\phi' \phi}}{x' - x} \exp \left[-\kappa \left(\frac{\phi'^2 + \phi^2}{x' - x} \right) \right] I_\nu \left(\frac{2\kappa \phi' \phi}{x' - x} \right). \tag{A41}$$

The only dependence in ν is in the Bessel function. Assume that $x_1 \sim L/2 \sim x_N$. From (A27), we know that $\langle \phi_{i+1} \phi_i \rangle \sim L$, and therefore $\langle \phi_{i+1} \phi_i \rangle / (x_{i+1} - x_i) \gg 1$ for each $1 \leq i \leq N-2$. Neglecting fluctuations, and with $I_\nu(z) \sim e^z / \sqrt{2\pi z}$ for $z \gg 1$, we find that bulk propagators are asymptotic to those of the

positive-valued free boson

$$\langle \phi_{i+1}, x_{i+1} | \phi_i, x_i \rangle_\nu \sim G_{\mathbb{R}^{>0}}(\phi_{i+1}, x_{i+1}; \phi_i, x_i), \quad (\text{A42})$$

as shown in the text in (A19). Thus, $f_N^{(\nu)}(\phi, \mathbf{x})$ will depend on ν only through the boundary propagators

$$\frac{\langle h, L | \phi_N, x_N \rangle \langle \phi_1, x_1 | h, 0 \rangle}{\langle h, L | h, 0 \rangle}, \quad (\text{A43})$$

and ratios $f_N^{(\mu)}(\phi, \mathbf{x})/f_N^{(\nu)}(\phi, \mathbf{x})$ will be independent of bulk propagators. From (A20) we know that $\langle \phi_1 \rangle_{x_1} \sim L^{-1/2} \sim \langle \phi_N \rangle / (L - x_N)$. Therefore, both $\langle h \phi_1 \rangle / x_1$ and $\langle h \phi_N \rangle / (L - x_N)$ will be much smaller than 1 for small values of h . With $I_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(1+\nu)}$ for $z \ll 1$, we find

$$\frac{f_N^{(\mu)}(\phi, \mathbf{x})}{f_N^{(\nu)}(\phi, \mathbf{x})} = R(x_1, x_N, \mu, \nu) (\phi_1 \phi_N)^{\mu-\nu}, \quad (\text{A44})$$

with

$$R(x_1, x_N, \mu, \nu) = \frac{\Gamma(1+\nu)}{\Gamma(1+\mu)} \left(\frac{\kappa L}{x_1(L-x_N)} \right)^{\mu-\nu}. \quad (\text{A45})$$

Thus,

$$\begin{aligned} & \langle \phi_1^{\alpha_1} \cdots \phi_N^{\alpha_N} \rangle_\mu \\ &= \int d\phi \phi_1^{\alpha_1} \cdots \phi_N^{\alpha_N} f_N^{(\mu)}(\phi, \mathbf{x}) \\ &= R(x_1, x_N, \mu, \nu) \int d\phi \phi_1^{\alpha_1} \cdots \phi_N^{\alpha_N} (\phi_1 \phi_N)^{\mu-\nu} f_N^{(\nu)}(\phi, \mathbf{x}) \\ &= R(x_1, x_N, \mu, \nu) \langle \phi_1^{\alpha_1+\mu-\nu} \phi_2^{\alpha_2} \cdots \phi_{N-1}^{\alpha_{N-1}} \phi_N^{\alpha_N+\mu-\nu} \rangle_\nu, \end{aligned} \quad (\text{A46})$$

which is identical to (A36). The single-point relation (A21) is a special case, but its validity is actually slightly more general because it contains no bulk propagator. Thus the above result carries through even if x is close to the boundary, as long as the regulator h is 0^+ . Relationship (A46) is easily generalized if some points are close to the boundary. Then (A43) has to be modified to account for the additional boundary propagators.

APPENDIX B: CAPACITY OF ENTANGLEMENT

We investigate another information theoretic quantity, namely the capacity of entanglement (see, e.g., [73–75] for recent developments). It was introduced in [76] as the quantum information analog of the heat capacity for thermal systems. The capacity of entanglement for the reduced density matrix

ρ_A may be conveniently defined as

$$C_A = \lim_{n \rightarrow 1} n^2 \partial_n^2 \log \text{Tr} \rho_A^n = \lim_{n \rightarrow 1} n^2 \partial_n^2 (1-n) S_n(A). \quad (\text{B1})$$

It is seen to characterize the width of the eigenvalue spectrum of the reduced density matrix.

Using expression (65) for the Rényi entropy, the capacity of entanglement for the (massive) Lifshitz theory is simply given by

$$C_A = \frac{M}{2}. \quad (\text{B2})$$

We find that the capacity of entanglement is finite, independent of the mass m , follows an area law (the number of surface points M is the one-dimensional analog of a surface area), and is thus much smaller than the entanglement entropy. The latter observation is quite different from CFT results [73] where usually $C_A \sim S_A$.

An interesting take on [73] is that a capacity of entanglement much smaller than entanglement entropy could be interpreted as entanglement being effectively carried by maximally entangled EPR pairs. Conversely, quantum states for which $C_A \sim S_A$ would be better described by randomly entangled pairs of qubits. The capacity of entanglement thus provides insights into the entanglement structure of the ground states of Lifshitz theories.

For the sake of completeness, let us apply the results of Sec. V and Appendix A where we study the positive-valued Lifshitz boson and a singular deformation of this theory. For the positive boson, using (106) and (108), we find the capacity of entanglement to be

$$C_A^{(\text{pos})} = \frac{M}{2} - \left(5 - \frac{\pi^2}{2} \right), \quad (\text{B3})$$

where $M = 1, 2$ corresponds to the number of surface points of A . We conjecture this expression to be valid for $M \geq 1$. For the singular deformation we obtain

$$C_A^{(\nu)} = (\nu + 1/2)^2 \psi^{(1)}(\nu + 1) - \nu, \quad (\text{B4})$$

where $\psi^{(n)}(x)$ is the polygamma function of order n , and ν is related to the coupling parameter to the singular potential. This formula was derived for a boundary interval A (i.e., $M = 1$). We conjecture that for general subsystems with M surface points

$$C_A^{(\nu)} = \frac{M}{2} - (\nu + 1/2) \left(1 - (\nu + 1/2) \psi^{(1)}(\nu + 1) \right), \quad (\text{B5})$$

such that we recover the positive boson result for $\nu = 1/2$ (see Appendix A). We note that $C_A^{(\nu)} \leq C_A$, $\nu \geq 0$.

- [1] D. S. Rokhsar and S. A. Kivelson, Superconductivity and the Quantum Hard-Core Dimer Gas, *Phys. Rev. Lett.* **61**, 2376 (1988).
- [2] C. L. Henley, From classical to quantum dynamics at Rokhsar Kivelson points, *J. Phys.: Condens. Matter* **16**, S891 (2004).
- [3] M. P. Zaletel, J. H. Bardarson, and J. E. Moore, Logarithmic Terms in Entanglement Entropies of 2D Quantum Critical Points and Shannon Entropies of Spin Chains, *Phys. Rev. Lett.* **107**, 020402 (2011).

- [4] J.-M. Stéphan, G. Misguich, and V. Pasquier, Phase transition in the Rényi-Shannon entropy of Luttinger liquids, *Phys. Rev. B* **84**, 195128 (2011).
- [5] E. Ardonne, P. Fendley, and E. Fradkin, Topological order and conformal quantum critical points, *Ann. Phys.* **310**, 493 (2004).
- [6] L. H. Santos, Rokhsar-Kivelson models of bosonic symmetry-protected topological states, *Phys. Rev. B* **91**, 155150 (2015).

- [7] Y. Hirose, A. Oguchi, and Y. Fukumoto, Quantum dimer model containing Rokhsar-Kivelson point expressed by spin-1/2 Heisenberg antiferromagnets, *Phys. Rev. B* **101**, 174440 (2020).
- [8] C. Castelnuovo, C. Chamon, C. Mudry, and P. Pujol, From quantum mechanics to classical statistical physics: Generalized Rokhsar-Kivelson Hamiltonians and the “Stochastic Matrix Form” decomposition, *Ann. Phys.* **318**, 316 (2005).
- [9] E. Fradkin and J. E. Moore, Entanglement Entropy of 2D Conformal Quantum Critical Points: Hearing the Shape of a Quantum Drum, *Phys. Rev. Lett.* **97**, 050404 (2006).
- [10] S. Furukawa and G. Misguich, Topological entanglement entropy in the quantum dimer model on the triangular lattice, *Phys. Rev. B* **75**, 214407 (2007).
- [11] J.-M. Stéphan, S. Furukawa, G. Misguich, and V. Pasquier, Shannon and entanglement entropies of one- and two-dimensional critical wave functions, *Phys. Rev. B* **80**, 184421 (2009).
- [12] B. Hsu, M. Mulligan, E. Fradkin, and E.-A. Kim, Universal entanglement entropy in 2D conformal quantum critical points, *Phys. Rev. B* **79**, 115421 (2009).
- [13] J.-M. Stéphan, G. Misguich, and V. Pasquier, Rényi entropy of a line in two-dimensional Ising models, *Phys. Rev. B* **82**, 125455 (2010).
- [14] B. Hsu and E. Fradkin, Universal behavior of entanglement in 2D quantum critical dimer models, *J. Stat. Mech.* (2010) P09004.
- [15] M. Oshikawa, Boundary conformal field theory and entanglement entropy in two-dimensional quantum Lifshitz critical point, [arXiv:1007.3739](https://arxiv.org/abs/1007.3739).
- [16] J.-M. Stéphan, H. Ju, P. Fendley, and R. G. Melko, Entanglement in gapless resonating-valence-bond states, *New J. Phys.* **15**, 015004 (2013).
- [17] X. Chen, W. Witczak-Krempa, T. Faulkner, and E. Fradkin, Two-cylinder entanglement entropy under a twist, *J. Stat. Mech.* (2017) 043104.
- [18] T. Zhou, X. Chen, T. Faulkner, and E. Fradkin, Entanglement entropy and mutual information of circular entangling surfaces in the 2 + 1-dimensional quantum Lifshitz model, *J. Stat. Mech.* (2016) 093101.
- [19] X. Chen, E. Fradkin, and W. Witczak-Krempa, Quantum spin chains with multiple dynamics, *Phys. Rev. B* **96**, 180402(R) (2017).
- [20] X. Chen, E. Fradkin, and W. Witczak-Krempa, Gapless quantum spin chains: multiple dynamics and conformal wavefunctions, *J. Phys. A: Math. Theor.* **50**, 464002 (2017).
- [21] M. R. Mohammadi Mozaffar and A. Mollabashi, Entanglement in Lifshitz-type quantum field theories, *J. High Energy Phys.* **07** (2017) 120.
- [22] J. Angel-Ramelli, V. G. M. Puletti, and L. Thorlacius, Entanglement entropy in generalised quantum Lifshitz models, *J. High Energy Phys.* **08** (2019) 072.
- [23] J. Angel-Ramelli, C. Berthiere, V. G. M. Puletti, and L. Thorlacius, Logarithmic negativity in quantum Lifshitz theories, *J. High Energy Phys.* **09** (2020) 011.
- [24] J. Angel-Ramelli, Entanglement entropy of excited states in the quantum Lifshitz model, *J. Stat. Mech.* (2021) 013102.
- [25] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, *Phys. Lett. A* **223**, 1, (1996).
- [26] A. Peres, Separability Criterion for Density Matrices, *Phys. Rev. Lett.* **77**, 1413 (1996).
- [27] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65**, 032314 (2002).
- [28] M. B. Plenio, Logarithmic Negativity: A Full Entanglement Monotone That is not Convex, *Phys. Rev. Lett.* **95**, 090503 (2005).
- [29] S. Bravyi, L. Caha, R. Movassagh, D. Nagaj, and P. W. Shor, Criticality without Frustration for Quantum Spin-1 Chains, *Phys. Rev. Lett.* **109**, 207202 (2012).
- [30] L. Lami, A. Serafini, and G. Adesso, Gaussian entanglement revisited, *New J. Phys.* **20**, 023030 (2018).
- [31] P. Calabrese, J. Cardy, and E. Tonni, Entanglement Negativity in Quantum Field Theory, *Phys. Rev. Lett.* **109**, 130502 (2012).
- [32] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena, *Rev. Mod. Phys.* **49**, 435 (1977).
- [33] J. Cardy, *Scaling and Renormalization in Statistical Physics*, Cambridge Lecture Notes in Physics (Cambridge University Press, Cambridge, 1996).
- [34] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models*, Collection Alea-Saclay: Monographs and Texts in Statistical Physics (Cambridge University Press, Cambridge, 1999).
- [35] S. Sachdev, *Quantum Phase Transitions*, 2 ed. (Cambridge University Press, Cambridge, 2011).
- [36] V. de Alfaro, S. Fubini, and G. Faldsurlan, Conformal invariance in quantum mechanics, *Il Nuovo Cimento A* **34**, 569 (1976).
- [37] C. R. Hagen, Scale and conformal transformations in galilean-covariant field theory, *Phys. Rev. D* **5**, 377 (1972).
- [38] R. Jackiw, Introducing scale symmetry, *Phys. Today* **25**(1), 23 (1972).
- [39] J. M. Romero, V. Cuestab, J. A. Garcia, and J. D. Vergara, Conformal anisotropic mechanics and hořava gravity, in *XII Mexican Workshop on Particles and Fields*, edited by A. Ayala, G. Contreras, I. Leon, and P. Podesta, AIP Conf. Proc. No. 1361 (AIP, New York, 2011), p. 344.
- [40] M. Henkel, Phenomenology of local scale invariance: From conformal invariance to dynamical scaling, *Nucl. Phys. B* **641**, 405 (2002).
- [41] D. T. Son, Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry, *Phys. Rev. D* **78**, 046003 (2008).
- [42] K. Balasubramanian and J. McGreevy, Gravity Duals for Non-relativistic Conformal Field Theories, *Phys. Rev. Lett.* **101**, 061601 (2008).
- [43] J. L. F. Barbón and C. A. Fuertes, On the spectrum of nonrelativistic AdS/CFT, *J. High Energy Phys.* **09** (2008) 030.
- [44] G. Bertoldi, B. A. Burrington, and A. W. Peet, Thermodynamics of black branes in asymptotically Lifshitz spacetimes, *Phys. Rev. D* **80**, 126004 (2009).
- [45] V. Keränen, W. Sybesma, P. Szepietowski, and L. Thorlacius, Correlation functions in theories with Lifshitz scaling, *J. High Energy Phys.* **05** (2017) 033.
- [46] P. Hořava, Quantum gravity at a lifshitz point, *Phys. Rev. D* **79**, 084008 (2009).
- [47] P. Hořava, Membranes at quantum criticality, *J. High Energy Phys.* **03** (2009) 020.
- [48] R. Dijkgraaf, D. Orlando, and S. Reffert, Relating field theories via stochastic quantization, *Nucl. Phys. B* **824**, 365 (2010).

- [49] L. Dell'Anna, O. Salberger, L. Barbiero, A. Trombettoni, and V. E. Korepin, Violation of cluster decomposition and absence of light cones in local integer and half-integer spin chains, *Phys. Rev. B* **94**, 155140 (2016).
- [50] R. Movassagh, Entanglement and correlation functions of the quantum Motzkin spin-chain, *J. Math. Phys.* **58**, 031901 (2017).
- [51] G.-L. Ingold, Path integrals and their application to dissipative quantum systems, *Lect. Notes Phys.* **611**, 1 (2002).
- [52] H. Ollivier and W. H. Żurek, Quantum Discord: A Measure of the Quantumness of Correlations, *Phys. Rev. Lett.* **88**, 017901 (2001).
- [53] P. Giorda and M. G. A. Paris, Gaussian Quantum Discord, *Phys. Rev. Lett.* **105**, 020503 (2010).
- [54] G. Adesso, T. R. Bromley, and M. Cianciaruso, Measures and applications of quantum correlations, *J. Phys. A: Math. Theor.* **49**, 473001 (2016).
- [55] G. Adesso, M. Cianciaruso, and T. R. Bromley, An introduction to quantum discord and non-classical correlations beyond entanglement, [arXiv:1611.01959](https://arxiv.org/abs/1611.01959).
- [56] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, *J. Stat. Mech.* (2004) P06002.
- [57] H. Casini, C. D. Fosco, and M. Huerta, Entanglement and alpha entropies for a massive Dirac field in two dimensions, *J. Stat. Mech.* (2005) P07007.
- [58] B. Swingle, Mutual information and the structure of entanglement in quantum field theory, [arXiv:1010.4038](https://arxiv.org/abs/1010.4038).
- [59] Coincidentally, the mutual information (82) shared by two boundary intervals for the massless Lifshitz theory takes the exact same form (with a prefactor of 3/2) as the mutual information between two intervals separated by a distance r in an infinite system for free fermion CFT₂ [57,58],
- $$I_{\text{free fermions}}(A : B) = \frac{1}{3} \log \frac{(L_A + r)(L_B + r)}{r(L_A + L_B + r)},$$
- noticing that $L = L_A + L_B + r$, with $r \equiv L_{C_2}$ in (82).
- [60] A. B. Zamolodchikov, Irreversibility of the flux of the renormalization group in a 2D field theory, *JETP Lett.* **43**, 730 (1986).
- [61] H. Casini and M. Huerta, A Finite entanglement entropy and the c-theorem, *Phys. Lett. B* **600**, 142 (2004).
- [62] N. Laflorencie, Quantum entanglement in condensed matter systems, *Phys. Rep.* **646**, 1 (2016).
- [63] H. Casini and M. Huerta, Universal terms for the entanglement entropy in 2+1 dimensions, *Nucl. Phys. B* **764**, 183 (2007).
- [64] H. Casini, M. Huerta, and L. Leitaó, Entanglement entropy for a Dirac fermion in three dimensions: Vertex contribution, *Nucl. Phys. B* **814**, 594 (2009).
- [65] E. Fradkin, *Field Theories of Condensed Matter Physics*, 2 ed. (Cambridge University Press, Cambridge, 2013).
- [66] H. Casini and M. Huerta, Entanglement entropy in free quantum field theory, *J. Phys. A: Math. Theor.* **42**, 504007 (2009).
- [67] A. B. Kallin, E. M. Stoudenmire, P. Fendley, R. R. P. Singh, and R. G. Melko, Corner contribution to the entanglement entropy of an $O(3)$ quantum critical point in 2 + 1 dimensions, *J. Stat. Mech.* (2014) P06009.
- [68] P. Bueno and W. Witczak-Krempa, Bounds on corner entanglement in quantum critical states, *Phys. Rev. B* **93**, 045131 (2016).
- [69] The (connected) correlation functions for the positive Lifshitz boson may be found in Appendix A, setting $\nu = 1/2$ in the corresponding expressions.
- [70] L. Dell'Anna, Long-distance entanglement in Motzkin and Fredkin spin chains, *SciPost Phys.* **7**, 053 (2019).
- [71] H. Casini and M. Huerta, Renormalization group running of the entanglement entropy of a circle, *Phys. Rev. D* **85**, 125016 (2012).
- [72] G. Parisi and Y.-s. Wu, Perturbation Theory Without Gauge Fixing, *Sci. Sin.* **24**, 483 (1981).
- [73] J. de Boer, J. Järvelä, and E. Keski-Vakkuri, Aspects of capacity of entanglement, *Phys. Rev. D* **99**, 066012 (2019).
- [74] K. Kawabata, T. Nishioka, Y. Okuyama, and K. Watanabe, Probing Hawking radiation through capacity of entanglement, *J. High Energy Phys.* **05** (2021) 062.
- [75] K. Okuyama, Capacity of entanglement in random pure state, *Phys. Lett. B* **820**, 136600 (2021).
- [76] H. Yao and X.-L. Qi, Entanglement Entropy and Entanglement Spectrum of the Kitaev Model, *Phys. Rev. Lett.* **105**, 080501 (2010).