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Embedding the Yang-Lee quantum criticality in open quantum systems

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The Yang-Lee edge singularity is a quintessential nonunitary critical phenomenon characterized by anomalous scaling. However, an imaginary magnetic field involved in this phenomenon makes its physical implementation nontrivial. By embedding the original classical system which exhibits the Yang-Lee edge singularity in a quantum system with an ancilla qubit and by invoking the quantum-classical correspondence, we demonstrate a physical realization of the nonunitary quantum criticality in an open quantum system, where the nonunitary criticality is identified with the singularity at an exceptional point caused by postselection of quantum measurement.

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I. INTRODUCTION

The Yang-Lee zero [1,2] is a zero point of the partition function of the canonical ensemble and provides a mathematical origin of singularities of thermodynamic quantities at phase transitions. In the ferromagnetic Ising model, the Yang-Lee zeros distribute on the imaginary axis in the complex plane of an external magnetic field, and the distribution becomes dense as we approach the thermodynamic limit. In the paramagnetic phase, there is a nonzero lower bound on the absolute values of the Yang-Lee zeros, and the distribution of the zeros does not touch the real axis. In the vicinity of the lower bound, i.e., at the edge of the distribution, a critical phenomenon called the Yang-Lee edge singularity [3–9] arises. This critical phenomenon exhibits anomalous scaling. For example, a correlation function can increase with increasing the distance due to the negative scaling dimension of a field [4,6]. Also, it has been reported in related models that the entanglement entropy of a subsystem can decrease with increasing the size of the subsystem due to the negative central charge [10,11]. Such anomalous scaling with no counterpart in unitary critical systems is a unique feature of nonunitary critical phenomena [3–14].

While the concepts of Yang-Lee zeros and the Yang-Lee edge singularity were originally introduced as mathematical foundations of phase transitions, their physical realization has been of both theoretical and experimental interest. For example, schemes to determine the critical points and the critical exponents from Yang-Lee zeros have been proposed [15–17], and dynamical quantum phase transitions [18–21] may be interpreted as the real-time counterparts of the Yang-Lee zeros. Furthermore, experimental observations of the Yang-Lee zeros [22–26] were reported, and some features of the Yang-Lee edge singularity were extracted from experimental data [22,23,27,28]. The critical exponent of the density of Yang-Lee zeros has been obtained from the dependence of magnetization of an Ising ferromagnet on a real magnetic field combined with analytic continuation [22,23]. It can also be obtained from a finite-size scaling [29] of quantum coherence of a probe spin coupled to a many-body spin system [27]. Such scaling analysis also provides an effective central charge [28] of the conformal field theory [12–14,30,31] which describes the Yang-Lee edge singularity. However, a direct observation scheme and the physical meaning of the anomalous scaling accompanied by the negative scaling dimension in the Yang-Lee edge singularity have remained elusive due to an imaginary magnetic field involved in this critical phenomenon.

In this paper, we demonstrate that the Yang-Lee edge singularity can be implemented in quantum systems on the basis of the quantum-classical correspondence [32,33], where a classical system is mapped to a quantum system via the equivalent canonical partition function. The corresponding quantum system exhibits the Yang-Lee zeros and the Yang-Lee edge singularity of the classical ferromagnetic Ising model. In the Yang-Lee edge singularity of a classical system, the critical behavior of physical observables and that of the distribution of Yang-Lee zeros are discussed in a seminal work by Fisher [4]. In this work, however, an imaginary magnetic field is treated formally, which makes it highly nontrivial to physically interpret this critical phenomenon. By contrast, in our quantum counterpart, which is described by a non-Hermitian Hamiltonian [34–36], we find that both an imaginary magnetic field and the Yang-Lee edge singularity can be realized in an open system. To realize the Yang-Lee edge singularity in a quantum system, we embed the original classical system which exhibits the Yang-Lee edge singularity in a Hermitian quantum system with an ancilla so that a physical observable of the original system can be found as an expectation value conditioned on the measurement outcome

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of the ancilla. We demonstrate that such nonunitary operations of measurement and postselection extract the criticality in the form of a dynamical singularity at an exceptional point. We find unconventional scaling laws for finite-temperature dynamics that have not been discussed in classical systems. Such scaling laws are unique to quantum systems and of experimental relevance.

The rest of this paper is organized as follows. In Sec. II, we discuss how to implement the Yang-Lee edge singularity in open quantum systems. In Sec. III, we investigate the Yang-Lee quantum critical phenomena in finite-temperature systems. In Sec. IV, we present a possible experimental situation of the proposed open quantum system. In Sec. V, we summarize the paper and discuss future prospects. In Appendix A, the Yang-Lee edge singularity in the classical one-dimensional Ising model is reviewed to make this paper self-contained. In Appendix B, the quantum-classical correspondence between the classical one-dimensional ferromagnetic Ising model and a parity-time symmetric non-Hermitian Hamiltonian is discussed. In Appendix C, detailed derivations of some results in the main text are given. In Appendix D, expectation values of observables for finite-temperature quantum systems are calculated.

II. YANG-LEE EDGE SINGULARITY IN OPEN QUANTUM SYSTEMS

A. Yang-Lee edge singularity in quantum systems

A prototypical example that exhibits the Yang-Lee edge singularity is the classical one-dimensional Ising model with a pure-imaginary external magnetic field [6]:

$$H = -J\sum_j \sigma_j\sigma_{j+1} - ih\sum_j \sigma_j, \tag{1}$$

where $J > 0$, $h \in \mathbb{R}$, and $\sigma_j = \pm 1$ (see Appendix A for details of the classical Yang-Lee edge singularity). A quantum system to which this classical model is mapped via the quantum-classical correspondence is described by a parity-time ($\mathcal{PT}$) symmetric non-Hermitian Hamiltonian [34–36]

$$H_{\rm PT} = R(\cos\phi)\sigma^x + iR(\sin\phi)\sigma^z \tag{2}$$

with real parameters $R > 0$ and $\phi \in (-\pi/2, \pi/2)$. The canonical partition function of a classical system is obtained via the path-integral representation [37,38] of the quantum counterpart up to an error that scales as $\mathcal{O}((\Delta\beta_0)^2)$ with a segment width $\Delta\beta_0$ of the inverse temperature [39,40] (see Appendix B for details). Here, σ^x, σ^y, and σ^z denote the Pauli matrices, and the $\mathcal{PT}$ symmetry is described by $[H_{\rm PT}, \mathcal{PT}] = 0$ with $\mathcal{P} = \sigma^x$ and $\mathcal{T} = \mathcal{K}$, where $\mathcal{K}$ represents complex conjugation. This Hamiltonian has eigenenergies $E_\pm = \pm R\sqrt{\cos 2\phi}$. The corresponding right eigenvectors are given by

$$|E^R_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i\tan\phi \pm \frac{\sqrt{\cos 2\phi}}{\cos\phi} \\ 1 \end{pmatrix}, \tag{3}$$

and the left eigenvectors are given by $\langle E^L_\pm| = \langle E^R_\pm|\eta$ ($\langle E^L_\pm| = \pm\langle E^R_\mp|\sigma^x\rangle$ for $|\phi| < \pi/4$ ($|\phi| \geqslant \pi/4$). These eigenstates are normalized so as to satisfy the following normalization conditions: $\langle E^R_\pm|E^R_\pm\rangle = \langle E^L_\pm|E^L_\pm\rangle = 1$ for $|\phi| < \pi/4$, and $\langle E^L_p|E^R_q\rangle = \delta_{pq}\sqrt{\cos 2\phi}/\cos\phi$ for $p, q \in \{+, -\}$ [41]. Here,

$$\eta := \frac{1}{\sqrt{\cos 2\phi}}(\cos\phi + \sin\phi\,\sigma^y) \tag{4}$$

characterizes the pseudo-Hermiticity and satisfies $\eta H_{\rm PT} = H^\dagger_{\rm PT}\eta$ [42,43]. The parameter points $\phi = \pm\pi/4$ are exceptional points [44–46], at which the right (left) eigenvectors as well as the eigenenergies coalesce.

The quantum-classical correspondence shows that the Yang-Lee edge singularity manifests itself as the distribution of zeros of the partition function

$$Z = {\rm Tr}[e^{-\beta H_{\rm PT}}] = \sum_{p\in\{+,-\}} e^{-\beta E_p}, \tag{5}$$

and the associated critical phenomena appear in the expectation value of O given by [47–50]

$$\langle O\rangle_{\rm PT} := \frac{{\rm Tr}[Oe^{-\beta H_{\rm PT}}]}{Z} = \frac{1}{Z}\sum_p \frac{\langle E^L_p|O|E^R_p\rangle}{\langle E^L_p|E^R_p\rangle}e^{-\beta E_p}. \tag{6}$$

We note that the partition function Z takes a real value because the eigenenergies are either real or form a complex conjugate pair due to $\mathcal{PT}$ symmetry.

B. Embedding the non-Hermitian system in an extended Hermitian system

The dynamics governed by $H_{\rm PT}$ is realized in open quantum systems. In the following, we focus on the $\mathcal{PT}$-unbroken phase (i.e., $|\phi| < \pi/4$), and construct a model following Ref. [51]. By introducing an ancilla, we embed the non-Hermitian single-qubit system in an extended Hermitian two-qubit system described by the Hilbert space $\mathcal{H}_{\rm tot} = \mathcal{H}_{\rm A} \otimes \mathcal{H}_{\rm S}$, where $\mathcal{H}_{\rm A}$ and $\mathcal{H}_{\rm S}$ represent the Hilbert space of the ancilla and that of the system [51,52]. The total Hamiltonian of the two-qubit system is given by

$$H_{\rm tot} = r\sin\theta\, I_{\rm A}\otimes\sigma^x + r\cos\theta\,\sigma^y_{\rm A}\otimes\sigma^z, \tag{7}$$

where $r > 0$ and $\theta \in [0, \pi]$. We note that the total system has a conserved quantity

$$\tilde{H} := \sin\theta\,\sigma^x_{\rm A}\otimes I + \cos\theta\,\sigma^z_{\rm A}\otimes\sigma^y \tag{8}$$

since $[H_{\rm tot}, \tilde{H}] = 0$. We focus on the eigenspace $\mathcal{H}^{\rm PT}_{\rm tot}$ of the conserved quantity $\tilde{H}$ with eigenvalue $+1$. We assume an initial state $|\psi\rangle^{\rm PT}_{\rm tot} = |\uparrow\rangle_{\rm A}\otimes|\psi\rangle + |\downarrow\rangle_{\rm A}\otimes(\eta|\psi\rangle)$ ($\in \mathcal{H}^{\rm PT}_{\rm tot}$), which evolves in time as

$$e^{-itH_{\rm tot}}|\psi\rangle^{\rm PT}_{\rm tot} = |\uparrow\rangle_{\rm A}\otimes e^{-itH_{\rm PT}}|\psi\rangle + |\downarrow\rangle_{\rm A}\otimes\eta e^{-itH_{\rm PT}}|\psi\rangle, \tag{9}$$

where the parameters of $H_{\rm PT}$ in Eq. (2) are given by $R = r\sqrt{1+\cos^2\theta}/\sin\theta$ and $\phi = -\arctan(\cos\theta)$ (see Appendix C). By measuring the ancilla qubit after this time evolution and postselecting the event that projects the ancilla onto $|\uparrow\rangle_{\rm A}$, we obtain the time evolution of the system qubit generated by $H_{\rm PT}$. Such embedding in a Hermitian two-qubit system has been realized experimentally [53,54].

C. Observables in the extended Hermitian system

In the following, we show how to relate physical quantities for the canonical ensemble of $H_{\rm PT}$ to those of the extended

Hermitian system. The partition function of the system qubit with the non-Hermitian Hamiltonian $H_{\rm PT}$ is obtained from the partition function of the total system with the Hermitian Hamiltonian $H_{\rm tot}$ whose Hilbert space is constrained to $\mathcal{H}_{\rm tot}^{\rm PT}$:

$$\mathrm{Tr}_{\rm tot}\big[P_{\rm tot}^{\rm PT} e^{-\beta H_{\rm tot}}\big] = \mathrm{Tr}_{\rm S}[e^{-\beta H_{\rm PT}}] = Z, \tag{10}$$

where $P_{\rm tot}^{\rm PT} := (1/2)(I + \tilde{H})$ is the projection operator onto $\mathcal{H}_{\rm tot}^{\rm PT}$ [see Appendix C for the derivations of this result and Eqs. (11)–(16) below]. Then, the four formal expectation values for the canonical ensemble with respect to $H_{\rm PT}$ are given by

$$\frac{\langle P_{\rm tot}^{\rm PT}(|\uparrow\rangle_{\rm A\,A}\langle\uparrow| \otimes O)\rangle_{\rm tot}}{\langle P_{\rm tot}^{\rm PT}(|\uparrow\rangle_{\rm A\,A}\langle\uparrow| \otimes I)\rangle_{\rm tot}} = \sum_p \frac{e^{-\beta E_p}}{Z} \frac{\langle E_p^R|O|E_p^R\rangle}{\langle E_p^R|E_p^R\rangle}, \tag{11}$$

$$\frac{\langle P_{\rm tot}^{\rm PT}(|\downarrow\rangle_{\rm A\,A}\langle\downarrow| \otimes O)\rangle_{\rm tot}}{\langle P_{\rm tot}^{\rm PT}(|\downarrow\rangle_{\rm A\,A}\langle\downarrow| \otimes I)\rangle_{\rm tot}} = \sum_p \frac{e^{-\beta E_p}}{Z} \frac{\langle E_p^L|O|E_p^L\rangle}{\langle E_p^L|E_p^L\rangle}, \tag{12}$$

$$\frac{\langle P_{\rm tot}^{\rm PT}(|\downarrow\rangle_{\rm A\,A}\langle\uparrow| \otimes O)\rangle_{\rm tot}}{\langle P_{\rm tot}^{\rm PT}(|\downarrow\rangle_{\rm A\,A}\langle\uparrow| \otimes I)\rangle_{\rm tot}} = \sum_p \frac{e^{-\beta E_p}}{Z} \frac{\langle E_p^L|O|E_p^R\rangle}{\langle E_p^L|E_p^R\rangle}, \tag{13}$$

$$\frac{\langle P_{\rm tot}^{\rm PT}(|\uparrow\rangle_{\rm A\,A}\langle\downarrow| \otimes O)\rangle_{\rm tot}}{\langle P_{\rm tot}^{\rm PT}(|\uparrow\rangle_{\rm A\,A}\langle\downarrow| \otimes I)\rangle_{\rm tot}} = \sum_p \frac{e^{-\beta E_p}}{Z} \frac{\langle E_p^R|O|E_p^L\rangle}{\langle E_p^R|E_p^L\rangle}, \tag{14}$$

where $\langle\cdots\rangle_{\rm tot} := \mathrm{Tr}_{\rm tot}[\cdots e^{-\beta H_{\rm tot}}]/\mathrm{Tr}_{\rm tot}[e^{-\beta H_{\rm tot}}]$. In particular, the expectation value in Eq. (6), which exhibits the Yang-Lee edge singularity, is obtained as

$$\langle O\rangle_{\rm tot}^{\downarrow\uparrow} := \frac{\langle P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^- \otimes O)\rangle_{\rm tot}}{\langle P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^- \otimes I)\rangle_{\rm tot}} = \sum_p \frac{e^{-\beta E_p}}{Z} \frac{\langle E_p^L|O|E_p^R\rangle}{\langle E_p^L|E_p^R\rangle}, \tag{15}$$

where $\sigma_{\rm A}^- = \frac{1}{2}(\sigma_{\rm A}^x - i\sigma_{\rm A}^y) = |\downarrow\rangle_{\rm AA}\langle\uparrow|$. In the vicinity of the critical points $\theta_c = 0, \pi$, the quantity $\langle P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^- \otimes I)\rangle_{\rm tot} = (\sin\theta)/2$ in the denominator of $\langle O\rangle_{\rm tot}^{\downarrow\uparrow}$ in Eq. (15) approaches zero, leading to the singularity. The two-time correlation function $G(O(t_2), O(t_1)) = \langle O(t_2)O(t_1)\rangle_{\rm PT} - \langle O(t_2)\rangle_{\rm PT}\langle O(t_1)\rangle_{\rm PT}$ can be obtained in a similar manner. In particular, $\langle O(t_2)O(t_1)\rangle_{\rm PT}$ is obtained as

$$\frac{\langle e^{i\Delta t H_{\rm tot}}(\sigma_{\rm A}^- \otimes O)e^{-i\Delta t H_{\rm tot}} P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^- \otimes O)P_{\rm tot}^{\rm PT}\rangle_{\rm tot}}{\langle e^{i\Delta t H_{\rm tot}}(\sigma_{\rm A}^- \otimes I)e^{-i\Delta t H_{\rm tot}} P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^- \otimes I)P_{\rm tot}^{\rm PT}\rangle_{\rm tot}}, \tag{16}$$

where $\Delta t := t_2 - t_1$.

The quantities in Eqs. (11)–(16) can be interpreted as the expectation values for the subensembles conditioned on the measurement outcomes of $\sigma_{\rm A}^z$ for each bra and ket under the imaginary-time evolution. The denominator of $\langle O\rangle_{\rm tot}^{\downarrow\uparrow}$ in Eq. (15) is proportional to the probability amplitude of the measurement outcome corresponding to this type of the expectation value, and the vanishment of this probability amplitude is the physical origin of the Yang-Lee edge singularity. Here, a nontrivial equivalence with the classical many-body system with an imaginary field emerges as a consequence of nonunitary operations of measurement and postselection, which extract the criticality in the form of singularities at exceptional points $\theta = 0, \pi$. We note that the criticality in observables cannot arise from the canonical ensemble for $H_{\rm tot}$ alone. For example, the magnetization remains zero and does not exhibit any critical behavior when evaluated without measurement and postselection on the ancilla:

$$\frac{\mathrm{Tr}_{\rm tot}[(I_{\rm A} \otimes \sigma^z)\exp(-\beta H_{\rm tot})]}{\mathrm{Tr}_{\rm tot}[\exp(-\beta H_{\rm tot})]} = 0. \tag{17}$$

Practically, the expectation values in Eqs. (11)–(14) can be obtained from simultaneous measurements of an observable O of the system and another observable of the ancilla. For example, the quantity $\langle P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^- \otimes O)\rangle_{\rm tot}$ appearing in the numerator of $\langle O\rangle_{\rm tot}^{\downarrow\uparrow}$ in Eq. (15) is given by the following linear combination:

$$\tfrac{1}{2}\big[\langle P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^x \otimes O)\rangle_{\rm tot} - i\langle P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^y \otimes O)\rangle_{\rm tot}\big]. \tag{18}$$

Here, the first (second) term can be obtained by the simultaneous measurement of O and $\sigma_{\rm A}^x$ ($\sigma_{\rm A}^y$), and is proportional to the real (imaginary) part of $\mathrm{Tr}_{\rm S}[O\exp(-\beta H_{\rm PT})]$. The denominator of $\langle O\rangle_{\rm tot}^{\downarrow\uparrow}$ is given as a special case of $O = I$ in Eq. (18). The linear combination of the observables $\sigma_{\rm A}^x$ and $\sigma_{\rm A}^y$ in Eq. (18) reconstructs the off-diagonal matrix element of the reduced density matrix of the ancilla. Therefore, the present scheme for obtaining the expectation value (18) can be regarded as measurements of a system observable O combined with quantum state tomography [55–59] of the ancilla. The two-time correlation function can also be evaluated in a similar manner.

III. YANG-LEE QUANTUM CRITICAL PHENOMENA IN FINITE-TEMPERATURE SYSTEMS

A. Observables

Here we discuss scaling laws of physical quantities for a finite-temperature quantum system. In particular, finite-temperature scaling of a two-time correlation function is unique to quantum critical phenomena [60]. The quantum critical points are located at $\beta^{-1} = 0$ and $\phi = \pm\pi/4$. We here focus on the critical point at $\phi = \pi/4$. The magnetization $m = \langle\sigma^z\rangle_{\rm PT}$, the magnetic susceptibility $\chi = \frac{dm}{da}$ with $a := (R\sin\phi)/(R\cos\phi) = \tan\phi$ representing a normalized magnetic field, and the two-time correlation function $G(t_2, t_1) = \langle\sigma^z(t_2)\sigma^z(t_1)\rangle_{\rm PT} - \langle\sigma^z(t_2)\rangle_{\rm PT}\langle\sigma^z(t_1)\rangle_{\rm PT}$ are given by (see Appendix D for derivations)

$$\begin{aligned} m &= \frac{\cos\phi}{\sqrt{\cos 2\phi}} \frac{e^{-\beta E_-}\langle E_-^L|\sigma^z|E_-^R\rangle + e^{-\beta E_+}\langle E_+^L|\sigma^z|E_+^R\rangle}{e^{-\beta E_-} + e^{-\beta E_+}} \\ &= -i\frac{\sin\phi}{\sqrt{\cos 2\phi}} \tanh(\beta R\sqrt{\cos 2\phi}), \end{aligned} \tag{19}$$

$$\begin{aligned} \chi &= \frac{dm}{da} = (\cos^2\phi)\frac{\partial m}{\partial\phi} = \frac{-i\cos^3\phi}{(\cos 2\phi)^{\frac{3}{2}}} \\ &\quad\times \left[\tanh(\beta R\sqrt{\cos 2\phi}) - \frac{2\beta R(\sin^2\phi)\sqrt{\cos 2\phi}}{\cosh^2(\beta R\sqrt{\cos 2\phi})}\right], \end{aligned} \tag{20}$$

$$\begin{aligned} G(t_2, t_1) = \frac{\cos^2\phi}{\cos 2\phi}\Bigg[&(\tan^2\phi)\big(\tanh^2[\beta R\sqrt{\cos 2\phi}] - 1\big) \\ &+ \frac{\cosh\big[(\beta - 2i\Delta t)R\sqrt{\cos 2\phi}\big]}{\cosh[\beta R\sqrt{\cos 2\phi}]}\Bigg]. \end{aligned} \tag{21}$$

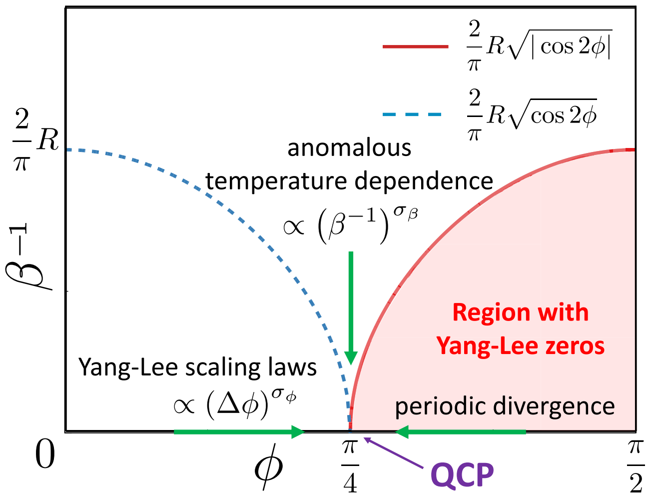

FIG. 1. Phase diagram of the Yang-Lee quantum critical system. The quantum critical point (QCP) is located at $\phi = \pi/4$ and $\beta^{-1} = 0$. In the $\mathcal{PT}$-unbroken phase (i.e., $|\phi| < \pi/4$), the $\Delta\phi$ dependencies of physical quantities obey the conventional scaling laws for the Yang-Lee edge singularity [6] (see Appendix A) in two successive limits, i.e., $\beta^{-1} \to 0$ followed by $\phi \to \pi/4 - 0$. In the $\mathcal{PT}$-broken phase (i.e., $|\phi| > \pi/4$), physical quantities diverge periodically and the corresponding limits cannot be defined by two sequential limits of $\beta^{-1} \to 0$ followed by $\phi \to \pi/4 + 0$. Unconventional scaling laws about the dependence on the temperature β^{-1} are obtained if the limit $\beta^{-1} \to 0$ is taken after $\phi \to \pi/4$. In the $\mathcal{PT}$-unbroken phase, a crossover between the two limiting behaviors occurs near the dashed curve given by $\beta^{-1} = (2/\pi)R\sqrt{\cos 2\phi}$.

Here, the pure-imaginary nature of the magnetization originates from $\mathcal{PT}$ symmetry. Indeed, because of this symmetry, we have

$$m^* = \frac{\mathrm{Tr}[\sigma^z e^{-\beta H_{\mathrm{PT}}^\dagger}]}{Z} = \frac{\mathrm{Tr}[\sigma^z(\sigma^x e^{-\beta H_{\mathrm{PT}}}\sigma^x)]}{Z} = -m. \tag{22}$$

Physically, this result arises from the projection $\sigma_{\mathrm{A}}^- = |{\downarrow}\rangle_{\mathrm{AA}}\langle{\uparrow}|$ in Eq. (15) onto the off-diagonal element of the reduced density matrix of the ancilla, which is complex-valued in general.

B. $\mathcal{PT}$-unbroken phase

First, we consider the $\mathcal{PT}$-unbroken phase (i.e., $|\phi| < \pi/4$) and examine the dependence of physical quantities on $\Delta\phi := \pi/4 - \phi$ by taking the limit $\phi \to \pi/4 - 0$ after $\beta^{-1} \to 0$, the latter of which corresponds to the thermodynamic limit for the classical counterpart in the quantum-classical correspondence. This order of the limits leads to the scaling laws in the classical system [6] (see Appendix A), where $\Delta\phi$ corresponds to a normalized magnetic field $\Delta a := 1 - a \propto \Delta\phi$ (see Fig. 1). By taking the limit of $\beta^{-1} \to 0$, we obtain

$$m \to \frac{-i\sin\phi}{\sqrt{\cos 2\phi}} \propto \Delta\phi^{-\frac{1}{2}}, \tag{23}$$

$$\chi \to \frac{-i\cos^3\phi}{(\cos 2\phi)^{3/2}} \propto \Delta\phi^{-\frac{3}{2}}, \tag{24}$$

$$\begin{aligned} G(t_2, t_1) &\to \frac{\cos^2\phi}{\cos 2\phi}\frac{\exp[(\beta - 2i\Delta t)R\sqrt{\cos 2\phi}]}{\exp[\beta R\sqrt{\cos 2\phi}]} \\ &= \frac{\cos^2\phi}{\cos 2\phi}\exp\left[-2\pi i\frac{\Delta t}{\pi/(R\sqrt{\cos 2\phi})}\right], \end{aligned} \tag{25}$$

where we have used the fact that hyperbolic functions behave as $\tanh(\beta R\sqrt{\cos 2\phi}) \to 1$ and $\cosh[(\beta - 2i\Delta t)R\sqrt{\cos 2\phi}] \to \frac{1}{2}\exp[(\beta - 2i\Delta t)R\sqrt{\cos 2\phi}]$ in the above limit. In particular, if Δt is replaced by an imaginary-time interval $-i\Delta\beta$, the two-time correlation function corresponds to the spatial correlation function $G_{\mathrm{cl}}(x)$ with the distance $x = \Delta\beta$ for the classical system

$$G_{\mathrm{cl}}(x) \propto \frac{e^{-x/\xi}}{(x/\xi)^2}x^{-(d-2+\eta)}, \tag{26}$$

where $d = 1$, $\eta = -1$, and $\xi \propto (h_c - h)^{-1/2}$ is the correlation length with the critical magnetic field h_c. Here, the singularities in Eqs. (23)–(25) originate from vanishing of the overlap $\langle E_p^L | E_p^R\rangle = \sqrt{\cos 2\phi}/\cos\phi$ between the left and right eigenstates with the same eigenenergy in the denominator of the resulting expressions [see also Eq. (6)].

C. Scaling laws in the extended Hermitian system

The results in Eqs. (23)–(25) are also obtained from the extended Hermitian system discussed in the previous section. In fact, from Eqs. (15) and (16), we obtain (see Appendix D for derivations)

$$\begin{aligned} m &= \frac{\mathrm{Tr}_{\mathrm{tot}}\left[P_{\mathrm{tot}}^{\mathrm{PT}}(\sigma_{\mathrm{A}}^- \otimes \sigma^z)e^{-\beta H_{\mathrm{tot}}}\right]}{\mathrm{Tr}_{\mathrm{tot}}\left[P_{\mathrm{tot}}^{\mathrm{PT}}(\sigma_{\mathrm{A}}^- \otimes I)e^{-\beta H_{\mathrm{tot}}}\right]} = \frac{i}{\tan\theta}\tanh(\beta r) \\ &\to \frac{i}{\tan\theta} \propto |\theta - \theta_c|^{-1}, \end{aligned} \tag{27}$$

$$\begin{aligned} \chi &= -\frac{i}{\sin^3\theta}\left[\tanh(\beta r) - \frac{2\beta r}{\cosh^2(\beta r)\,[1 + (\cos\theta)^{-2}]}\right] \\ &\to -\frac{i}{\sin^3\theta} \propto |\theta - \theta_c|^{-3}, \end{aligned} \tag{28}$$

$$\begin{aligned} G(t_2, t_1) &= -\frac{1}{\tan^2\theta}[1 - \tanh^2(\beta r)] + \frac{\cosh[(\beta - 2i\Delta t)r]}{(\sin^2\theta)(\cosh\beta r)} \\ &\to \frac{1}{\sin^2\theta}\exp\left(-2\pi i\frac{\Delta t}{\pi/r}\right) \end{aligned} \tag{29}$$

in the vicinity of the critical points $\theta_c = 0, \pi$.

D. $\mathcal{PT}$-broken phase

Next, we consider the $\mathcal{PT}$-broken phase (i.e., $|\phi| > \pi/4$) and evaluate the dependence of physical quantities on $\Delta\phi$ by taking the limit $\phi \to \pi/4 + 0$ after $\beta^{-1} \to 0$. In this phase, the magnetization, the magnetic susceptibility, and the two-time correlation function are given as follows:

$$m = -i\frac{\sin\phi}{\sqrt{|\cos 2\phi|}}\tan(\beta R\sqrt{|\cos 2\phi|}), \tag{30}$$

$$\begin{aligned} \chi &= \frac{i\cos^3\phi}{|\cos 2\phi|^{\frac{3}{2}}} \\ &\quad\times\left[\tan(\beta R\sqrt{|\cos 2\phi|}) - \frac{2\beta R(\sin^2\phi)\sqrt{|\cos 2\phi|}}{\cos^2(\beta R\sqrt{|\cos 2\phi|})}\right], \end{aligned} \tag{31}$$

$$G(t_2,t_1) = -\frac{\cos^2\phi}{|\cos 2\phi|}\Bigg[-(\tan^2\phi)\big(1+\tan^2[\beta R\sqrt{|\cos 2\phi|}]\big) + \frac{\cosh\left[(i\beta+2\Delta t)R\sqrt{|\cos 2\phi|}\right]}{\cos[\beta R\sqrt{|\cos 2\phi|}]}\Bigg]. \tag{32}$$

These quantities diverge periodically when the limit $\beta^{-1}\to 0$ is taken for some fixed $\phi>\pi/4$, which makes it impossible to define the above-mentioned double limits of these quantities (see Fig. 1). The condition for the divergence is given by

$$\beta R\sqrt{|\cos 2\phi|} = \left(n+\frac{1}{2}\right)\pi \tag{33}$$

for some integer n, which corresponds to the zeros of the partition function

$$Z = 2\cos(\beta R\sqrt{|\cos 2\phi|}), \tag{34}$$

i.e., the Yang-Lee zeros. Here, the real-valuedness of ϕ which satisfies this condition (33) is in accordance with the Lee-Yang circle theorem [2,61–63] in the classical system, which states that the fugacity $z=\exp(-2\beta h)$ corresponding to Yang-Lee zeros distributes on a unit circle on the complex z plane. These zeros appear only in the region defined by (see Fig. 1)

$$\beta^{-1}\leqslant \frac{2}{\pi}R\sqrt{|\cos 2\phi|}. \tag{35}$$

E. Anomalous temperature dependence

Finally, we consider the case in which the limit $\beta^{-1}\to 0$ is taken after the limit $\phi\to\pi/4$. This order of these two limits leads to unconventional scaling laws that have not been discussed in classical systems. By taking the limit $\phi\to\pi/4$, we obtain the following unconventional scaling laws:

$$m = -i\frac{\sin\phi}{\sqrt{\cos 2\phi}}\left[\beta R\sqrt{\cos 2\phi}+\mathcal{O}((\beta R\sqrt{\cos 2\phi})^3)\right] \to -\frac{i}{\sqrt{2}}\beta R, \tag{36}$$

$$\begin{aligned}\chi =& -i\left(\frac{1+\cos 2\phi}{2\cos 2\phi}\right)^{\frac{3}{2}}\left[\beta R\sqrt{\cos 2\phi}-\frac{1}{3}(\beta R\sqrt{\cos 2\phi})^3+\mathcal{O}((\beta R\sqrt{\cos 2\phi})^5)-\frac{(1-\cos 2\phi)\beta R\sqrt{\cos 2\phi}}{1+(\beta R\sqrt{\cos 2\phi})^2+\mathcal{O}((\beta R\sqrt{\cos 2\phi})^4)}\right]\\ &\to -\frac{i}{3\sqrt{2}}\left(\beta^3R^3+\frac{3}{2}\beta R\right),\end{aligned} \tag{37}$$

$$\begin{aligned}G(t_2,t_1) =& \frac{1+\cos 2\phi}{2\cos 2\phi}\left(\frac{\cos 2\phi-1}{\cos 2\phi+1}\left[1-\beta^2R^2\cos 2\phi+\mathcal{O}((\beta R\sqrt{\cos 2\phi})^4)\right]\right.\\ &\left.+\frac{1+\frac{1}{2}(\beta-2i\Delta t)^2R^2\cos 2\phi+\mathcal{O}((R\sqrt{\cos 2\phi})^4)}{1+\frac{1}{2}\beta^2R^2\cos 2\phi+\mathcal{O}((\beta R\sqrt{\cos 2\phi})^4)}\right)\to R^2\left(\frac{1}{2}\beta^2-i\beta\Delta t-(\Delta t)^2\right)+1,\end{aligned} \tag{38}$$

from which we obtain critical exponents $\sigma_\beta=-1,-3,-2$ for the power-law dependence on the temperature β^{-1} (see Fig. 1). In particular, the two-time correlation function behaves as $|G(t_2,t_1)|\propto(\Delta t)^2$ in the limit of $\Delta t\to\infty$, which is consistent with the anomalous divergent behavior of the spatial correlation function $G_{\rm cl}(x)\propto x^2$ at the critical point of the corresponding classical system [6] (see Appendix A). To understand the physical origin of the divergent behavior in the quantum system, we note that the factor $\cosh[(\beta-2i\Delta t)R\sqrt{\cos 2\phi}]$, which appears in the two-time correlation function in Eq. (21), becomes $\cosh[(i\beta+2\Delta t)R\sqrt{|\cos 2\phi|}]$ in the $\mathcal{PT}$-broken phase [see Eq. (32)] and exponentially diverges on a timescale $T\propto|\cos 2\phi|^{-1/2}$ as Δt increases, indicating an exponential amplification in this phase [64–68]. At the critical point (i.e., $\phi=\pm\pi/4$), the timescale T diverges and the divergent behavior of the two-time correlation function becomes a power law. We note that we can observe the criticality in Eqs. (36)–(38) in the extended Hermitian system by examining the temperature dependence of physical quantities with the parameters r and θ held fixed near the critical point.

In the $\mathcal{PT}$-unbroken phase (i.e., $|\phi|<\pi/4$), a crossover between the two limiting behaviors given in Eqs. (23)–(25) and in Eqs. (36)–(38) occurs around the temperature

$$\beta^{-1}\simeq\frac{2}{\pi}R\sqrt{\cos 2\phi}, \tag{39}$$

where the temperature is comparable to the energy gap (see Fig. 1).

IV. POSSIBLE EXPERIMENTAL SITUATION

In this section, we discuss a possible experimental situation of the open quantum system discussed in the previous sections. The dynamics governed by the non-Hermitian Hamiltonian $H_{\rm PT}$ has been experimentally realized in open quantum systems [53,54,69,70], and the scheme for embedding this non-Hermitian Hamiltonian in the Hermitian two-qubit system discussed in this paper has been implemented [53,54]. Among various quantum simulators, trapped ions [71–91] offer an ideal playground to explore the long-time dynamics at finite temperatures due to a long coherence time [71,90].

A. Magnetization

The Yang-Lee edge singularity in the magnetization m of the system can be found from Eq. (15) as

$$m = \frac{\mathrm{Tr}_{\mathrm{tot}}\left[P_{\mathrm{tot}}^{\mathrm{PT}}\left(\sigma_{\mathrm{A}}^{-} \otimes \sigma^{z}\right) e^{-\beta H_{\mathrm{tot}}}\right]}{\mathrm{Tr}_{\mathrm{tot}}\left[P_{\mathrm{tot}}^{\mathrm{PT}}\left(\sigma_{\mathrm{A}}^{-} \otimes I\right) e^{-\beta H_{\mathrm{tot}}}\right]} = \frac{\mathrm{Tr}_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{x} \otimes \sigma^{z}\right)\tilde{\rho}_{\mathrm{TFI}}\right] - i\mathrm{Tr}_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{z} \otimes \sigma^{z}\right)\tilde{\rho}_{\mathrm{TFI}}\right]}{\mathrm{Tr}_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{x} \otimes I\right)\tilde{\rho}_{\mathrm{TFI}}\right] - i\mathrm{Tr}_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{z} \otimes I\right)\tilde{\rho}_{\mathrm{TFI}}\right]}, \quad (40)$$

where $\tilde{\rho}_{\mathrm{TFI}} = P'_{\mathrm{tot}}\rho_{\mathrm{TFI}}P'_{\mathrm{tot}}$. Here $\rho_{\mathrm{TFI}} = e^{-\beta H_{\mathrm{TFI}}}/\mathrm{Tr}_{\mathrm{tot}}[e^{-\beta H_{\mathrm{TFI}}}]$ is a thermal equilibrium state of the total system for the Ising Hamiltonian with a transverse field

$$H_{\mathrm{TFI}} = r\sin\theta\, I_{\mathrm{A}} \otimes \sigma^{x} + r\cos\theta\, \sigma_{\mathrm{A}}^{z} \otimes \sigma^{z}, \quad (41)$$

which is related to H_{tot} as $H_{\mathrm{tot}} = e^{\frac{\pi}{4}i\sigma_{\mathrm{A}}^{x}} H_{\mathrm{TFI}} e^{-\frac{\pi}{4}i\sigma_{\mathrm{A}}^{x}}$. The transverse-field Ising Hamiltonian has been implemented in trapped ions [78,79,81–89], superconducting-circuit QED systems [92–97], and Rydberg atoms [98–105]. The projection operator P'_{tot} is given by

$$P'_{\mathrm{tot}} := e^{-\frac{\pi}{4}i\sigma_{\mathrm{A}}^{x}} P_{\mathrm{tot}}^{\mathrm{PT}} e^{\frac{\pi}{4}i\sigma_{\mathrm{A}}^{x}} = \tfrac{1}{2}(I + \tilde{H}') \quad (42)$$

with $\tilde{H}' := \sin\theta\ \sigma_{\mathrm{A}}^{x} \otimes I - \cos\theta\ \sigma_{\mathrm{A}}^{y} \otimes \sigma^{y}$, and it can be implemented by projection onto the eigenspace of $\tilde{H}'$ with eigenvalue $+1$ using, for example, the scheme proposed in Ref. [106], in which the center of mass of trapped ions is coupled to atomic states and plays a role of the meter in an indirect measurement of the Hamiltonian.

B. Two-time correlation function

The two-time correlation function $G(\sigma^{z}(t_2), \sigma^{z}(t_1))$ can be measured in a similar manner. Specifically, from Eq. (16), $\langle\sigma^{z}(t_2)\sigma^{z}(t_1)\rangle_{\mathrm{PT}}$ is obtained as

$$\begin{aligned}\langle\sigma^{z}(t_2)\sigma^{z}(t_1)\rangle_{\mathrm{PT}} &= \frac{\mathrm{Tr}_{\mathrm{tot}}\left[e^{i\Delta t H_{\mathrm{tot}}}(\sigma_{\mathrm{A}}^{-} \otimes \sigma^{z}) e^{-i\Delta t H_{\mathrm{tot}}} P_{\mathrm{tot}}^{\mathrm{PT}}(\sigma_{\mathrm{A}}^{-} \otimes \sigma^{z}) P_{\mathrm{tot}}^{\mathrm{PT}} e^{-\beta H_{\mathrm{tot}}}\right]}{\mathrm{Tr}_{\mathrm{tot}}\left[e^{i\Delta t H_{\mathrm{tot}}}(\sigma_{\mathrm{A}}^{-} \otimes I) e^{-i\Delta t H_{\mathrm{tot}}} P_{\mathrm{tot}}^{\mathrm{PT}}(\sigma_{\mathrm{A}}^{-} \otimes I) P_{\mathrm{tot}}^{\mathrm{PT}} e^{-\beta H_{\mathrm{tot}}}\right]} \\ &= \frac{\mathrm{Tr}_{\mathrm{tot}}\left\{e^{i\Delta t H_{\mathrm{TFI}}}\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes \sigma^{z}\right] e^{-i\Delta t H_{\mathrm{TFI}}} P'_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes \sigma^{z}\right] P'_{\mathrm{tot}} e^{-\beta H_{\mathrm{TFI}}}\right\}}{\mathrm{Tr}_{\mathrm{tot}}\left\{e^{i\Delta t H_{\mathrm{TFI}}}\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes I\right] e^{-i\Delta t H_{\mathrm{TFI}}} P'_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes I\right] P'_{\mathrm{tot}} e^{-\beta H_{\mathrm{TFI}}}\right\}} \\ &= \frac{\mathrm{Tr}_{\mathrm{tot}}\left\{\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes \sigma^{z}\right]_{\mathrm{TFI}}(\Delta t) P'_{\mathrm{tot}}[(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}) \otimes \sigma^{z}] P'_{\mathrm{tot}} e^{-\beta H_{\mathrm{TFI}}}\right\}}{\mathrm{Tr}_{\mathrm{tot}}\left\{\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes I\right]_{\mathrm{TFI}}(\Delta t) P'_{\mathrm{tot}}\left[\left(\sigma_{\mathrm{A}}^{x} - i\sigma_{\mathrm{A}}^{z}\right) \otimes I\right] P'_{\mathrm{tot}} e^{-\beta H_{\mathrm{TFI}}}\right\}},\end{aligned} \quad (43)$$

where $[O]_{\mathrm{TFI}}(t) = e^{itH_{\mathrm{TFI}}} O e^{-itH_{\mathrm{TFI}}}$. Both the numerator and the denominator are obtained as a linear combination of quantities such as

$$\mathrm{Tr}_{\mathrm{tot}}\{[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t) P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}] P'_{\mathrm{tot}} e^{-\beta H_{\mathrm{TFI}}}\}, \quad (44)$$

where $O_{\mathrm{A}}, O'_{\mathrm{A}} \in \{\sigma_{\mathrm{A}}^{x}, \sigma_{\mathrm{A}}^{z}\}$, and $O_{\mathrm{S}} = \sigma^{z}(I)$ for the numerator (denominator) in Eq. (43). The quantity in Eq. (44) can be evaluated using the polarization identity [107]:

$$\begin{aligned}A^{\dagger}MB = &\tfrac{1}{4}[(A+B)^{\dagger}M(A+B) - (A-B)^{\dagger}M(A-B) \\ &- i(A+iB)^{\dagger}M(A+iB) + i(A-iB)^{\dagger}M(A-iB)].\end{aligned} \quad (45)$$

Indeed, applying this identity to $[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t) P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}] P'_{\mathrm{tot}}$ with $A = I$, $M = [O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t)$, and $B = P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}] P'_{\mathrm{tot}}$, we obtain

$$\begin{aligned}[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t) P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}] P'_{\mathrm{tot}} = \ &\tfrac{1}{4}(I + P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}})^{\dagger}[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t)(I + P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}}) \\ &- \tfrac{1}{4}(I - P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}})^{\dagger}[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t)(I - P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}}) \\ &- \tfrac{i}{4}(I + iP'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}})^{\dagger}[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t)(I + iP'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}}) \\ &+ \tfrac{i}{4}(I - iP'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}})^{\dagger}[O'_{\mathrm{A}} \otimes O_{\mathrm{S}}]_{\mathrm{TFI}}(\Delta t)(I - iP'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}}).\end{aligned} \quad (46)$$

The first term on the right-hand side is obtained by first applying $A + B = I + P'_{\mathrm{tot}}[O_{\mathrm{A}} \otimes O_{\mathrm{S}}]P'_{\mathrm{tot}}$ to the thermal equilibrium state ρ_{TFI} and then measuring $O'_{\mathrm{A}} \otimes O_{\mathrm{S}}$ after an evolution time of Δt. The other terms can also be evaluated similarly.

V. SUMMARY AND FUTURE PERSPECTIVE

We have identified a quantum system which exhibits the Yang-Lee edge singularity on the basis of the quantum-classical correspondence and discussed its realization in an open quantum system. Specifically, we have embedded a non-Hermitian classical system in an extended quantum Hermitian system by introducing an ancilla, and found that the physical origin of the singularity lies in the facts that the physical quantity to be evaluated is the expectation value conditioned on the measurement outcome of the ancilla and that the probability of the successful postselection of events almost vanishes in the vicinity of the critical point. Moreover, we have found unconventional scaling laws for finite-temperature dynamics, which are unique to quantum critical phenomena [60]. We have shown that an expectation value over the canonical ensemble with respect to a non-Hermitian Hamiltonian corresponds to that for an extended Hermitian system with the projection onto specific matrix elements of the reduced density matrix of the ancilla [see Eq. (15)]. It is an interesting future problem to investigate the generality of this correspondence.

The Yang-Lee edge singularity is a prototypical example of nonunitary critical phenomena involving anomalous scaling laws unseen in unitary critical systems. We hope that this work stimulates further investigation on nonunitary critical phenomena in open quantum systems for higher-dimensional systems and other universality classes.

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APPENDIX A: YANG-LEE EDGE SINGULARITY IN THE CLASSICAL ONE-DIMENSIONAL ISING MODEL

We briefly review the Yang-Lee edge singularity [3–9] in the classical one-dimensional ferromagnetic Ising model [6]:

$$H = -J\sum_j \sigma_j\sigma_{j+1} - h\sum_j \sigma_j, \tag{A1}$$

where J is positive and h is complex in general. The transfer matrix of this model is given by

$$\begin{aligned} T &= \begin{pmatrix} e^{\beta J+\beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta h}\end{pmatrix} \\ &= e^{-\beta J}\sigma^x + e^{\beta J}[\cosh(\beta h)I + \sinh(\beta h)\sigma^z], \end{aligned} \tag{A2}$$

and its eigenvalues are given by

$$\lambda_\pm = e^{\beta J}\cosh\beta h \pm \sqrt{e^{2\beta J}\sinh^2\beta h + e^{-2\beta J}}. \tag{A3}$$

Under the periodic boundary condition, the partition function is represented as

$$Z = \mathrm{Tr}[T^N], \tag{A4}$$

where N is the number of sites. In the thermodynamic limit $N\to\infty$, the free-energy density is given by

$$\begin{aligned} f &= -\frac{1}{\beta N}\ln\lambda_+^N \\ &= -\frac{1}{\beta}\ln\left(e^{\beta J}\cosh\beta h + \sqrt{e^{2\beta J}\sinh^2\beta h + e^{-2\beta J}}\right), \end{aligned} \tag{A5}$$

and the correlation length is given by

$$\xi = \frac{1}{\ln\lambda_+ - \ln\lambda_-}. \tag{A6}$$

From Eq. (A6), we find that the Yang-Lee edge singularity manifests itself as the diverging correlation length when the magnetic field satisfies the following condition:

$$e^{2\beta J}\sinh^2\beta h + e^{-2\beta J} = 0, \tag{A7}$$

and hence the critical magnetic field is pure imaginary:

$$h_c = \pm i\beta^{-1}\sin^{-1}(e^{-2\beta J}), \tag{A8}$$

which is in accordance with the Lee-Yang circle theorem [2,61–63].

The Yang-Lee edge singularity involves anomalous scaling laws with no counterparts in unitary critical phenomena. The magnetization density is obtained as $m = -\partial_h f$, which scales in the vicinity of the critical point as

$$m = \frac{\sinh\beta h}{\sqrt{\sinh^2\beta h + e^{-4\beta J}}} = \frac{\sinh\beta h}{\sqrt{C\Delta h + o(\Delta h)}} \propto \Delta h^\sigma, \tag{A9}$$

where $\sigma = -\frac{1}{2}$, $\Delta h := h - h_c$, and C is a nonuniversal constant. By differentiating the magnetization density with respect to the magnetic field, we obtain the scaling law for the magnetic susceptibility:

$$\chi = \frac{dm}{dh} = \frac{\beta\, e^{-4\beta J}\cosh(\beta h)}{(\sinh^2\beta h + e^{-4\beta J})^{3/2}} \propto \frac{1}{\Delta h^\gamma} \tag{A10}$$

with $\gamma = 1-\sigma = \frac{3}{2}$. Correlation functions also exhibit anomalous scaling laws. It follows from Eq. (A6) that the correlation length scales near the critical point as

$$\begin{aligned} \xi^{-1} &\simeq 2e^{-\beta J}(\cosh\beta h)^{-1}\sqrt{e^{2\beta J}\sinh^2\beta h + e^{-2\beta J}} \\ &= 2(\cosh\beta h)^{-1}\sqrt{C\Delta h + o(\Delta h)} \propto \Delta h^\nu, \end{aligned} \tag{A11}$$

where $\nu = 1/2$. Finally, the correlation function $G_{\rm cl}(x)$ at spatial distance x scales as

$$G_{\rm cl}(x) = \frac{(\lambda_-/\lambda_+)^x}{1 + e^{4\beta J}\sinh^2(\beta h)} \propto \frac{e^{-x/\xi}}{\Delta h} \propto \frac{e^{-x/\xi}}{(x/\xi)^2}x^{-(d-2+\eta)}, \tag{A12}$$

where $d = 1$ and $\eta = -1$.

APPENDIX B: QUANTUM-CLASSICAL CORRESPONDENCE

In this section, we discuss the quantum-classical correspondence [32,33] between the classical one-dimensional ferromagnetic Ising model $H_{\rm cl} = -J\sum_j\sigma_j\sigma_{j+1} - ih_{\rm cl}\sum_j\sigma_j$ ($J>0, h_{\rm cl}\in\mathbb{R}$), and a parity-time ($\mathcal{PT}$) symmetric non-Hermitian Hamiltonian [34–36] $H_{\rm Q} = -h_x\sigma^x - ih_z\sigma^z$ with real parameters h_x and h_z. We list the correspondence in Table I. This correspondence is based on the equivalence of the partition functions for $H_{\rm cl}$ and $H_{\rm Q}$. The partition function for $H_{\rm Q}$ is given by

$$Z_{\rm Q} = \mathrm{Tr}[e^{-\beta H_{\rm Q}}] = \sum_{\sigma_0=\pm1}\langle\sigma_0|e^{-(\beta_0 n_{\rm temp})H_{\rm Q}}|\sigma_0\rangle, \tag{B1}$$

where $|\sigma_0\rangle$ is the eigenstate of σ^z with the eigenvalue $\sigma_0\in\{+1,-1\}$, and the inverse temperature β is given by $\beta = \beta_0 n_{\rm temp}$ with an integer $n_{\rm temp}$ and a fixed value β_0. We employ a path-integral representation [37,38] of Eq. (B1) by dividing each segment β_0 into $n_{\rm div}$ subsegments of width $\beta_0/n_{\rm div}$, which results in $N = n_{\rm temp}n_{\rm div}$ subsegments in total. By inserting a complete basis set between subsegments, we obtain

TABLE I. Quantum-classical correspondence between a quantum system with a parity-time symmetric non-Hermitian Hamiltonian H_{Q} and a classical system with a one-dimensional ferromagnetic Ising Hamiltonian H_{cl}. The partition functions for H_{Q} and H_{cl} are denoted as Z_{Q} and Z_{cl}, respectively. Parameters of the two systems are related to each other by $\beta_{\mathrm{cl}}J = -\frac{1}{2}\ln[\tanh(\beta_0 h_x/n_{\mathrm{div}})]$ and $\beta_{\mathrm{cl}}h_{\mathrm{cl}} = \beta_0 h_z/n_{\mathrm{div}}$. The differences between the quantum and classical systems in the partition function, the free-energy density, the magnetization density, the magnetic susceptibility, and the correlation function are evaluated in Eqs. (B7), (B8), (B17), (B18), and (B19), respectively.

	Quantum system	Classical system	Upper bound or leading-order term of the difference between the two systems
$\beta = \beta_0 n_{\mathrm{temp}}$	Inverse temperature	System size	
β_0/n_{div}	Segment width of inverse temperature	Lattice constant	
$N = n_{\mathrm{temp}} n_{\mathrm{div}}$	Number of segments in inverse temperature	Number of sites	
$n_{\mathrm{temp}} \to \infty$	Zero-temperature limit	Thermodynamic limit	
$n_{\mathrm{div}} \to \infty$	Continuum limit for imaginary time	Continuum limit for real space	
Partition function	$Z_{\mathrm{Q}} = \mathrm{Tr}[e^{\beta(h_x\sigma^x + ih_z\sigma^z)}]$	$Z_{\mathrm{cl}} = A^N \sum_{\{\sigma_k\}} e^{\sum_k(\beta_{\mathrm{cl}}J\sigma_{k+1}\sigma_k + i\beta_{\mathrm{cl}}h_{\mathrm{cl}}\sigma_k)}$	$\lvert Z_{\mathrm{Q}} - Z_{\mathrm{cl}}\rvert \leqslant \frac{2n_{\mathrm{temp}}\beta_0^3(\lvert h_x\rvert+\lvert h_z\rvert)^3}{3n_{\mathrm{div}}^2} e^{n_{\mathrm{temp}}\beta_0(\lvert h_x\rvert+\lvert h_z\rvert)}$
Free-energy density	$f_{\mathrm{Q}} = \frac{-1}{\beta_0 n_{\mathrm{temp}}}\ln Z_{\mathrm{Q}}$	$f_{\mathrm{cl}} = -\frac{1}{\beta_{\mathrm{cl}} n_{\mathrm{div}} n_{\mathrm{temp}}}\ln Z_{\mathrm{cl}}$	$\lvert f_{\mathrm{Q}} - f_{\mathrm{cl}}\rvert \leqslant \frac{2\beta_0^2(\lvert h_x\rvert+\lvert h_z\rvert)^3}{3n_{\mathrm{div}}^2 Z_{\mathrm{cl}}} e^{n_{\mathrm{temp}}\beta_0(\lvert h_x\rvert+\lvert h_z\rvert)}$
Magnetization density in $n_{\mathrm{temp}} \to \infty$	$-i\frac{\sin\phi}{\sqrt{\cos 2\phi}}$	$i\frac{\sin(\beta_{\mathrm{cl}}h_{\mathrm{cl}})}{\sqrt{\exp(-4\beta_{\mathrm{cl}}J)-\sin^2(\beta_{\mathrm{cl}}h_{\mathrm{cl}})}}$	$\frac{\cos^2\phi}{(\cos 2\phi)^{3/2}}\frac{\lvert\sin\phi - 3\sin 3\phi\rvert}{24}\left(\frac{\beta_0 R}{n_{\mathrm{div}}}\right)^2 + \mathcal{O}(n_{\mathrm{div}}^{-4})$
Magnetic susceptibility in $n_{\mathrm{temp}} \to \infty$	$-i\frac{\cos^3\phi}{(\cos 2\phi)^{3/2}}$	$i\beta_{\mathrm{cl}}\frac{\exp(-4\beta_{\mathrm{cl}}J)\cos(\beta_{\mathrm{cl}}h_{\mathrm{cl}})}{[\exp(-4\beta_{\mathrm{cl}}J)-\sin^2(\beta_{\mathrm{cl}}h_{\mathrm{cl}})]^{3/2}}$	$\frac{\cos^3\phi}{(\cos 2\phi)^{5/2}}\frac{\lvert 3+12\cos 2\phi+\cos 4\phi\rvert}{16}\left(\frac{\beta_0 R}{n_{\mathrm{div}}}\right)^2 + \mathcal{O}(n_{\mathrm{div}}^{-4})$
Correlation function in $n_{\mathrm{temp}} \to \infty$	$\frac{\cos^2\phi}{\cos 2\phi}e^{-2R\Delta\tau\sqrt{\cos 2\phi}}$	$\frac{1}{1-e^{4\beta_{\mathrm{cl}}J}\sin^2(\beta_{\mathrm{cl}}h_{\mathrm{cl}})} \times \left[\frac{\cos(\beta_{\mathrm{cl}}h_{\mathrm{cl}})-\sqrt{e^{-4\beta_{\mathrm{cl}}J}-\sin^2(\beta_{\mathrm{cl}}h_{\mathrm{cl}})}}{\cos(\beta_{\mathrm{cl}}h_{\mathrm{cl}})+\sqrt{e^{-4\beta_{\mathrm{cl}}J}-\sin^2(\beta_{\mathrm{cl}}h_{\mathrm{cl}})}}\right]^x$	$\frac{\tan^2 2\phi}{24}\lvert 1 + 2R\Delta\tau(\cos^2\phi)\sqrt{\cos 2\phi} + 3\cos 2\phi\rvert \times e^{-2R\Delta\tau\sqrt{\cos 2\phi}}\left(\frac{\beta_0 R}{n_{\mathrm{div}}}\right)^2 + \mathcal{O}(n_{\mathrm{div}}^{-4})$

$$
\begin{aligned}
Z_{\mathrm{Q}} &= \sum_{\sigma_0=\pm 1}\langle\sigma_0|\left\{\left[\exp\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\sigma^x\right)\exp\left(i\frac{\beta_0 h_z}{n_{\mathrm{div}}}\sigma^z\right)\right]^{n_{\mathrm{temp}}n_{\mathrm{div}}} + E_{n_{\mathrm{div}},n_{\mathrm{temp}}}\right\}|\sigma_0\rangle \\
&= \sum_{\sigma_0}\cdots\sum_{\sigma_{N-1}}\prod_{k=0}^{N-1}\langle\sigma_{k+1}|\exp\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\sigma^x\right)\exp\left(i\frac{\beta_0 h_z}{n_{\mathrm{div}}}\sigma^z\right)|\sigma_k\rangle + \mathrm{Tr}[E_{n_{\mathrm{div}},n_{\mathrm{temp}}}] \\
&= \sum_{\sigma_0}\cdots\sum_{\sigma_{N-1}}A^N\exp\left[\sum_{k=0}^{N-1}(\beta_{\mathrm{cl}}J\sigma_{k+1}\sigma_k + i\beta_{\mathrm{cl}}h_{\mathrm{cl}}\sigma_k)\right] + \mathrm{Tr}[E_{n_{\mathrm{div}},n_{\mathrm{temp}}}],
\end{aligned}
\tag{B2}
$$

where $\sigma_N = \sigma_0$, and the coefficients are given by

$$\beta_{\mathrm{cl}}J := -\frac{1}{2}\ln\left[\tanh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right)\right], \tag{B3}$$

$$\beta_{\mathrm{cl}}h_{\mathrm{cl}} := \frac{\beta_0 h_z}{n_{\mathrm{div}}}, \tag{B4}$$

$$A := \sqrt{\cosh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right)\sinh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right)}. \tag{B5}$$

In Eq. (B2), we have evaluated the matrix element as follows:

$$
\begin{aligned}
&\langle\sigma_{k+1}|\exp\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\sigma^x\right)\exp\left(i\frac{\beta_0 h_z}{n_{\mathrm{div}}}\sigma^z\right)|\sigma_k\rangle \\
&= \langle\sigma_{k+1}|\left[\cosh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right) + \sinh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right)\sigma^x\right]|\sigma_k\rangle e^{i\frac{\beta_0 h_z}{n_{\mathrm{div}}}\sigma_k} \\
&= \left[\cosh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right)\delta_{\sigma_{k+1},\sigma_k} + \sinh\left(\frac{\beta_0 h_x}{n_{\mathrm{div}}}\right)(1-\delta_{\sigma_{k+1},\sigma_k})\right]e^{i\frac{\beta_0 h_z}{n_{\mathrm{div}}}\sigma_k} \\
&= A\exp(\beta_{\mathrm{cl}}J\sigma_{k+1}\sigma_k)\exp\left(i\frac{\beta_0 h_z}{n_{\mathrm{div}}}\sigma_k\right).
\end{aligned}
\tag{B6}
$$

The first term on the right-hand side of Eq. (B2) gives the partition function Z_{cl} for the classical one-dimensional ferromagnetic Ising model, which shows the desired quantum-classical correspondence for the partition functions. Here, the difference between the quantum and classical partition functions is bounded from above as [39] (see Table I)

$$
\begin{aligned}
\lvert Z_{\mathrm{Q}} - Z_{\mathrm{cl}}\rvert &= \lvert\mathrm{Tr}[E_{n_{\mathrm{div}},n_{\mathrm{temp}}}]\rvert \\
&\leqslant \frac{2n_{\mathrm{temp}}\beta_0^3(\lvert h_x\rvert+\lvert h_z\rvert)^3}{3n_{\mathrm{div}}^2}e^{n_{\mathrm{temp}}\beta_0(\lvert h_x\rvert+\lvert h_z\rvert)},
\end{aligned}
\tag{B7}
$$

which is vanishingly small in the continuum limit $n_{\rm div} \to \infty$. As a corollary of this bound, we also evaluate the difference in the free-energy density as (see Table I)

$$\begin{aligned} |f_{\rm Q} - f_{\rm cl}| &= \left|\left(-\frac{\ln Z_{\rm Q}}{\beta_0 n_{\rm temp}}\right) - \left(-\frac{\ln Z_{\rm cl}}{\beta_{\rm cl} n_{\rm div} n_{\rm temp}}\right)\right| \\ &= \frac{1}{\beta_0 n_{\rm temp}} \ln\left(1 + \frac{|{\rm Tr}[E_{n_{\rm div},n_{\rm temp}}]|}{Z_{\rm cl}}\right) \\ &\leqslant \frac{|{\rm Tr}[E_{n_{\rm div},n_{\rm temp}}]|}{\beta_0 n_{\rm temp} Z_{\rm cl}} \\ &\leqslant \frac{2\beta_0^2(|h_x| + |h_z|)^3}{3n_{\rm div}^2 Z_{\rm cl}} e^{n_{\rm temp}\beta_0(|h_x|+|h_z|)}, \end{aligned} \tag{B8}$$

where we have set the inverse temperature of the classical system as $\beta_{\rm cl} = \beta_0/n_{\rm div}$.

1. Difference in observables between the quantum and classical systems

We evaluate the difference of some observable quantities between the quantum and classical systems. In the classical system, the magnetization density, the magnetic susceptibility, and the spatial correlation function are given in the thermodynamic limit $n_{\rm temp} \to \infty$ by (see Eqs. (A9), (A10), and (A12))

$$m_{\rm cl} \to i\frac{\sin(\beta_{\rm cl}h_{\rm cl})}{\sqrt{\exp(-4\beta_{\rm cl}J) - \sin^2(\beta_{\rm cl}h_{\rm cl})}}, \tag{B9}$$

$$\chi_{\rm cl} \to i\beta_{\rm cl}\frac{\exp(-4\beta_{\rm cl}J)\cos(\beta_{\rm cl}h_{\rm cl})}{[\exp(-4\beta_{\rm cl}J) - \sin^2(\beta_{\rm cl}h_{\rm cl})]^{3/2}}, \tag{B10}$$

$$G_{\rm cl}(x) \to \frac{1}{1 - e^{4\beta_{\rm cl}J}\sin^2(\beta_{\rm cl}h_{\rm cl})} \times \left(\frac{\cos(\beta_{\rm cl}h_{\rm cl}) - \sqrt{e^{-4\beta_{\rm cl}J} - \sin^2(\beta_{\rm cl}h_{\rm cl})}}{\cos(\beta_{\rm cl}h_{\rm cl}) + \sqrt{e^{-4\beta_{\rm cl}J} - \sin^2(\beta_{\rm cl}h_{\rm cl})}}\right)^x. \tag{B11}$$

On the other hand, in the quantum system, where the same limit $n_{\rm temp} \to \infty$ represents the zero-temperature limit, the corresponding observables are given by (see Eqs. (23)–(25))

$$m_{\rm Q} \to -i\frac{\sin\phi}{\sqrt{\cos 2\phi}}, \tag{B12}$$

$$\chi_{\rm Q} \to -i\frac{\cos^3\phi}{(\cos 2\phi)^{3/2}}, \tag{B13}$$

$$G_{\rm Q}(\Delta\tau) \to \frac{\cos^2\phi}{\cos 2\phi}\exp\left(-2R\Delta\tau\sqrt{\cos 2\phi}\right), \tag{B14}$$

where $\Delta\tau := (\beta_0/n_{\rm div})x$ and the parametrization in Eqs. (2) and (23)–(25) is reproduced by

$$h_x = -R\cos\phi, \tag{B15}$$

$$h_z = -R\sin\phi. \tag{B16}$$

We use the relations in Eqs. (B3) and (B4) to evaluate the difference in each observable between the quantum and classical systems in the same limit $n_{\rm temp} \to \infty$ (see Table I):

$$m_{\rm Q} - m_{\rm cl} \to -\frac{i}{24}\frac{\cos^2\phi}{(\cos 2\phi)^{3/2}}[\sin\phi - 3\sin(3\phi)]\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 + \mathcal{O}(n_{\rm div}^{-4}), \tag{B17}$$

$$\chi_{\rm Q} - \left(-\frac{\beta_{\rm cl}n_{\rm div}}{\beta_0 R\cos\phi}\right)^{-1}\chi_{\rm cl} \to \frac{i}{16}\frac{\cos^3\phi}{(\cos 2\phi)^{5/2}}[3 + 12\cos(2\phi) + \cos(4\phi)]\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 + \mathcal{O}(n_{\rm div}^{-4}), \tag{B18}$$

$$G_{\rm Q}(\Delta\tau) - G_{\rm cl}(x) \to -\frac{\tan^2(2\phi)}{24}\left[1 + 2R\Delta\tau(\cos^2\phi)\sqrt{\cos(2\phi)} + 3\cos(2\phi)\right]e^{-2R\Delta\tau\sqrt{\cos 2\phi}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 + \mathcal{O}(n_{\rm div}^{-4}), \tag{B19}$$

where we have rescaled the magnetic susceptibility for the classical system by taking account of the difference in the magnetic field between the two systems. Here, the scaling $\mathcal{O}(n_{\rm div}^{-2})$ of the leading term in the differences is consistent with the general argument given in Refs. [39,40].

2. Required fineness of division for a given precision

We consider how finely we should divide the inverse temperature in the quantum system to obtain the quantum-classical correspondence with a difference smaller than a given precision ϵ. We argue that the integer $n_{\rm div}$ should be chosen large enough so as to satisfy

$$\frac{n_{\rm div}}{\beta_0 R} > \frac{1}{\sqrt{\epsilon}\,(\cos 2\phi)^{5/4}}. \tag{B20}$$

Under this condition, we can show that (i) higher-order terms in $n_{\rm div}^{-1}$ in the difference of observables can be relatively neglected and that (ii) the leading term is bounded from above by ϵ. To demonstrate (i), we consider the expansion of the difference in the magnetization density in the limit of $n_{\rm temp} \to \infty$:

$$m_{\rm Q} - m_{\rm cl} \to \frac{-i}{24\sqrt{2}}\frac{1}{(\cos 2\phi)^{3/2}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 + \frac{i}{384\sqrt{2}}\frac{1}{(\cos 2\phi)^{5/2}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^4 + \frac{5i}{27648\sqrt{2}}\frac{1}{(\cos 2\phi)^{7/2}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^6 + \cdots. \tag{B21}$$

Here, if $n_{\rm div}$ is large enough to satisfy the condition in Eq. (B20), higher-order terms can be neglected because

$$\frac{1}{\cos 2\phi}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 < \epsilon\,(\cos 2\phi)^{3/2} \ll 1. \tag{B22}$$

The above argument holds also for the magnetic susceptibility and the correlation function. Furthermore, (ii) can be demonstrated near the critical point ($n_{\rm temp} \to \infty$ followed by $\phi \to \pi/4 - 0$) as follows:

$$\frac{|\Delta m|}{\epsilon} = \frac{1}{\epsilon}\frac{1}{24}\frac{\cos^2\phi}{(\cos 2\phi)^{3/2}}|\sin\phi - 3\sin(3\phi)|\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 \to \frac{1}{\epsilon}\frac{1}{24\sqrt{2}}\frac{1}{(\cos 2\phi)^{3/2}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 < \frac{\cos 2\phi}{24\sqrt{2}} < 1, \tag{B23}$$

$$\frac{|\Delta\chi|}{\epsilon} = \frac{1}{\epsilon}\frac{1}{16}\frac{\cos^3\phi}{(\cos 2\phi)^{5/2}}|3 + 12\cos(2\phi) + \cos(4\phi)|\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 \to \frac{1}{\epsilon}\frac{1}{16\sqrt{2}}\frac{1}{(\cos 2\phi)^{5/2}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 < \frac{1}{16\sqrt{2}} < 1, \tag{B24}$$

$$\frac{|\Delta G|}{\epsilon} = \frac{1}{\epsilon}|1 + 2R\Delta\tau(\cos^2\phi)\sqrt{\cos(2\phi)} + 3\cos(2\phi)|\frac{\tan^2(2\phi)}{24}e^{-2R\Delta\tau\sqrt{\cos 2\phi}}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 \to \frac{1}{\epsilon}\frac{1}{24}\frac{1}{\cos^2(2\phi)}\left(\frac{\beta_0 R}{n_{\rm div}}\right)^2 < \frac{\sqrt{\cos 2\phi}}{24} < 1, \tag{B25}$$

where Δm, $\Delta\chi$, and ΔG denote the leading terms of the differences between the quantum and classical systems in the zero-temperature limit $n_{\rm temp} \to \infty$ for the magnetization density, the magnetic susceptibility, and the correlation function, respectively [see Eqs. (B17)–(B19)].

APPENDIX C: DERIVATION OF THE RESULTS IN EQUATIONS (9)–(16) FOR THE EXTENDED HERMITIAN SYSTEM

1. Embedding the non-Hermitian system in an extended Hermitian system

In this section, we discuss how to obtain physical quantities for the canonical ensemble of $H_{\rm PT}$ from the extended Hermitian system. We first consider the dynamics of the total system generated by $H_{\rm tot} = r\sin\theta\, I_{\rm A}\otimes\sigma^x + r\cos\theta\,\sigma^y_{\rm A}\otimes\sigma^z$ in the following two-dimensional subspace of $\mathcal{H}_{\rm tot}$:

$$\mathcal{H}^{\rm PT}_{\rm tot} = \left\{|\psi\rangle^{\rm PT}_{\rm tot} = |{\uparrow}\rangle_{\rm A}\otimes|\psi\rangle + |{\downarrow}\rangle_{\rm A}\otimes(\eta|\psi\rangle)\;\middle|\;|\psi\rangle\in\mathcal{H}_{\rm S}\right\}, \tag{C1}$$

which is the eigenspace of the conserved quantity $\tilde{H}$ with eigenvalue $+1$. The action of $H_{\rm tot}$ is described by

$$\begin{aligned} H_{\rm tot}|\psi\rangle^{\rm PT}_{\rm tot} =\; & |{\uparrow}\rangle_{\rm A}\otimes(r\sin\theta\,\sigma^x - ir\cos\theta\,\sigma^z\eta)|\psi\rangle \\ & + |{\downarrow}\rangle_{\rm A}\otimes(r\sin\theta\,\sigma^x + ir\cos\theta\,\sigma^z\eta^{-1})(\eta|\psi\rangle) \\ =\; & |{\uparrow}\rangle_{\rm A}\otimes\frac{r}{\sin\theta}(\sigma^x - i\cos\theta\,\sigma^z)|\psi\rangle \\ & + |{\downarrow}\rangle_{\rm A}\otimes\frac{r}{\sin\theta}(\sigma^x + i\cos\theta\,\sigma^z)(\eta|\psi\rangle). \end{aligned} \tag{C2}$$

This can be rewritten as

$$H_{\rm tot}|\psi\rangle^{\rm PT}_{\rm tot} = |{\uparrow}\rangle_{\rm A}\otimes H_{\rm PT}|\psi\rangle + |{\downarrow}\rangle_{\rm A}\otimes H^\dagger_{\rm PT}(\eta|\psi\rangle), \tag{C3}$$

where the non-Hermitian Hamiltonian $H_{\rm PT}$ is given by $H_{\rm PT} = R(\cos\phi)\sigma^x + iR(\sin\phi)\sigma^z$ with the parameters $R = r\sqrt{1+\cos^2\theta}/\sin\theta$ and $\phi = -\arctan(\cos\theta)$. The time evolution according to $H_{\rm tot}$ is given by

$$e^{-itH_{\rm tot}}|\psi\rangle^{\rm PT}_{\rm tot} = |{\uparrow}\rangle_{\rm A}\otimes e^{-itH_{\rm PT}}|\psi\rangle + |{\downarrow}\rangle_{\rm A}\otimes\eta e^{-itH_{\rm PT}}|\psi\rangle. \tag{C4}$$

2. Partition function

The partition function for the system qubit with $H_{\rm PT}$ is obtained from the partition function for the total system with $H_{\rm tot}$ under the restriction of the Hilbert space to $\mathcal{H}^{\rm PT}_{\rm tot}$:

$${\rm Tr}_{\rm tot}[P^{\rm PT}_{\rm tot}e^{-\beta H_{\rm tot}}] = {\rm Tr}_{\rm S}[e^{-\beta H_{\rm PT}}] = Z. \tag{C5}$$

To show this, we note that the projection operator can be written as

$$P^{\rm PT}_{\rm tot} = \frac{\sin\theta}{2}\sum_{\{|\sigma\rangle\}}|\sigma\rangle^{\rm PT}_{\rm tot}\,{}^{\rm PT}_{\rm tot}\langle\eta^{-1}\sigma|, \tag{C6}$$

where $\{|\sigma\rangle\}$ is an orthonormal basis of $\mathcal{H}_{\rm S}$ and satisfies $\langle\sigma'|\sigma\rangle = \delta_{\sigma'\sigma}$. Here, the state vector $|\eta^{-1}\sigma\rangle^{\rm PT}_{\rm tot} = |{\uparrow}\rangle_{\rm A}\otimes\eta^{-1}|\sigma\rangle + |{\downarrow}\rangle_{\rm A}\otimes|\sigma\rangle$ $(\in\mathcal{H}^{\rm PT}_{\rm tot})$ satisfies the following relation:

$$\begin{aligned} {}^{\rm PT}_{\rm tot}\langle\eta^{-1}\sigma'|\sigma\rangle^{\rm PT}_{\rm tot} &= \langle\sigma'|(\eta+\eta^{-1})|\sigma\rangle = \frac{2\cos\phi}{\sqrt{\cos 2\phi}}\delta_{\sigma'\sigma} \\ &= \frac{2}{\sin\theta}\delta_{\sigma'\sigma}. \end{aligned} \tag{C7}$$

Using these expressions, the partition function is evaluated as

$$\begin{aligned} &{\rm Tr}_{\rm tot}[P^{\rm PT}_{\rm tot}e^{-\beta H_{\rm tot}}] \\ &= \frac{\sin\theta}{2}\sum_{\{|\sigma\rangle\}}{}^{\rm PT}_{\rm tot}\langle\eta^{-1}\sigma|e^{-\beta H_{\rm tot}}|\sigma\rangle^{\rm PT}_{\rm tot} \\ &= \frac{\sin\theta}{2}\sum_{\{|\sigma\rangle\}}({}_{\rm A}\langle{\uparrow}|\otimes\langle\sigma|\eta^{-1} + {}_{\rm A}\langle{\downarrow}|\otimes\langle\sigma|) \\ &\quad\times(|{\uparrow}\rangle_{\rm A}\otimes e^{-\beta H_{\rm PT}}|\sigma\rangle + |{\downarrow}\rangle_{\rm A}\otimes\eta e^{-\beta H_{\rm PT}}|\sigma\rangle) \\ &= \frac{\sin\theta}{2}{\rm Tr}_{\rm S}[(\eta+\eta^{-1})e^{-\beta H_{\rm PT}}] = {\rm Tr}_{\rm S}[e^{-\beta H_{\rm PT}}] = Z. \end{aligned} \tag{C8}$$

In obtaining the penultimate equality, we have used the following relation (see Eq. (4)):

$$\eta + \eta^{-1} = \frac{2\cos\phi}{\sqrt{\cos 2\phi}} = \frac{2}{\sin\theta}. \tag{C9}$$

3. Expectation values of observables

We consider the four formal expectation values $\langle O\rangle^{mn}_{\rm tot}$ $(m, n \in \{\uparrow, \downarrow\})$ appearing in Eqs. (11)–(14) for the canonical ensemble with respect to $H_{\rm PT}$. We first focus on

$$\langle O\rangle^{\downarrow\uparrow}_{\rm tot} := \frac{{\rm Tr_{tot}}[P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes O)e^{-\beta H_{\rm tot}}]}{{\rm Tr_{tot}}[P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes I)e^{-\beta H_{\rm tot}}]}, \quad \text{(C10)}$$

which exhibits the Yang-Lee edge singularity. Here, $\sigma^-_{\rm A}$ is defined as $\sigma^-_{\rm A} = \frac{1}{2}(\sigma^x_{\rm A} - i\sigma^y_{\rm A}) = |\downarrow\rangle_{\rm AA}\langle\uparrow|$. The numerator of this expression is evaluated with a parameter $0 < x < 1$ as

$$\begin{aligned}{\rm Tr_{tot}}[P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes O)e^{-\beta H_{\rm tot}}] &= \frac{\sin\theta}{2}\sum_{\{|\sigma\rangle\}} {}^{\rm PT}_{\rm tot}\langle\eta^{-1}\sigma|e^{-(1-x)\beta H_{\rm tot}}(\sigma^-_{\rm A}\otimes O)e^{-x\beta H_{\rm tot}}|\sigma\rangle^{\rm PT}_{\rm tot}\\ &= \frac{\sin\theta}{2}\sum_{\{|\sigma\rangle\}}\big({}_{\rm A}\langle\uparrow|\otimes\langle\sigma|\eta^{-1}e^{-(1-x)\beta H^\dagger_{\rm PT}} + {}_{\rm A}\langle\downarrow|\otimes\langle\sigma|e^{-(1-x)\beta H_{\rm PT}}\big)(\sigma^-_{\rm A}\otimes O)\\ &\quad\times\big(|\uparrow\rangle_{\rm A}\otimes e^{-x\beta H_{\rm PT}}|\sigma\rangle + |\downarrow\rangle_{\rm A}\otimes e^{-x\beta H^\dagger_{\rm PT}}\eta|\sigma\rangle\big) = \frac{\sin\theta}{2}{\rm Tr_S}[Oe^{-\beta H_{PT}}],\end{aligned} \quad \text{(C11)}$$

from which we obtain the desired expression:

$$\begin{aligned}\langle O\rangle^{\downarrow\uparrow}_{\rm tot} &= \frac{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes O)e^{-\beta H_{\rm tot}}\big]}{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes I)e^{-\beta H_{\rm tot}}\big]}\\ &= \frac{\frac{\sin\theta}{2}{\rm Tr_S}[Oe^{-\beta H_{PT}}]}{\frac{\sin\theta}{2}Z} = \frac{1}{Z}\sum_p e^{-\beta E_p}\frac{\langle E^L_p|O|E^R_p\rangle}{\langle E^L_p|E^R_p\rangle}.\end{aligned} \quad \text{(C12)}$$

Similar calculations yield the following results:

$$\begin{aligned}\langle O\rangle^{\uparrow\uparrow}_{\rm tot} &:= \frac{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(|\uparrow\rangle_{\rm AA}\langle\uparrow|\otimes O)e^{-\beta H_{\rm tot}}\big]}{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(|\uparrow\rangle_{\rm AA}\langle\uparrow|\otimes I)e^{-\beta H_{\rm tot}}\big]}\\ &= \frac{\frac{\sin\theta}{2}{\rm Tr_S}[\eta^{-1}Oe^{-\beta H_{\rm PT}}]}{\frac{1}{2}Z} = \frac{1}{Z}\sum_p e^{-\beta E_p}\langle E^R_p|O|E^R_p\rangle,\end{aligned} \quad \text{(C13)}$$

$$\begin{aligned}\langle O\rangle^{\downarrow\downarrow}_{\rm tot} &:= \frac{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(|\downarrow\rangle_{\rm AA}\langle\downarrow|\otimes O)e^{-\beta H_{\rm tot}}\big]}{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(|\downarrow\rangle_{\rm AA}\langle\downarrow|\otimes I)e^{-\beta H_{\rm tot}}\big]}\\ &= \frac{\frac{\sin\theta}{2}{\rm Tr_S}[O\eta e^{-\beta H_{\rm PT}}]}{\frac{1}{2}Z} = \frac{1}{Z}\sum_p e^{-\beta E_p}\langle E^L_p|O|E^L_p\rangle,\end{aligned} \quad \text{(C14)}$$

$$\begin{aligned}\langle O\rangle^{\uparrow\downarrow}_{\rm tot} &:= \frac{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(|\uparrow\rangle_{\rm AA}\langle\downarrow|\otimes O)e^{-\beta H_{\rm tot}}\big]}{{\rm Tr_{tot}}\big[P^{\rm PT}_{\rm tot}(|\uparrow\rangle_{\rm AA}\langle\downarrow|\otimes I)e^{-\beta H_{\rm tot}}\big]}\\ &= \frac{\frac{\sin\theta}{2}{\rm Tr_S}\big[Oe^{-\beta H^\dagger_{PT}}\big]}{\frac{\sin\theta}{2}Z} = \frac{1}{Z}\sum_p e^{-\beta E_p}\frac{\langle E^R_p|O|E^L_p\rangle}{\langle E^R_p|E^L_p\rangle}.\end{aligned} \quad \text{(C15)}$$

The two-time correlation function $G(O(t_2), O(t_1)) = \langle O(t_2)O(t_1)\rangle_{\rm PT} - \langle O(t_2)\rangle_{\rm PT}\langle O(t_1)\rangle_{\rm PT}$ can be obtained in a similar manner. In particular, $\langle O(t_2)O(t_1)\rangle_{\rm PT}$ is obtained as

$$\frac{{\rm Tr_{tot}}\big[e^{i\Delta tH_{\rm tot}}(\sigma^-_{\rm A}\otimes O)e^{-i\Delta tH_{\rm tot}}P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes O)P^{\rm PT}_{\rm tot}e^{-\beta H_{\rm tot}}\big]}{{\rm Tr_{tot}}\big[e^{i\Delta tH_{\rm tot}}(\sigma^-_{\rm A}\otimes I)e^{-i\Delta tH_{\rm tot}}P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes I)P^{\rm PT}_{\rm tot}e^{-\beta H_{\rm tot}}\big]}, \quad \text{(C16)}$$

where $\Delta t := t_2 - t_1$. In fact, this quantity is evaluated as

$$\begin{aligned}&\frac{{\rm Tr_{tot}}\big[e^{i\Delta tH_{\rm tot}}(\sigma^-_{\rm A}\otimes O)e^{-i\Delta tH_{\rm tot}}P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes O)P^{\rm PT}_{\rm tot}e^{-\beta H_{\rm tot}}\big]}{{\rm Tr_{tot}}\big[e^{i\Delta tH_{\rm tot}}(\sigma^-_{\rm A}\otimes I)e^{-i\Delta tH_{\rm tot}}P^{\rm PT}_{\rm tot}(\sigma^-_{\rm A}\otimes I)P^{\rm PT}_{\rm tot}e^{-\beta H_{\rm tot}}\big]}\\ &= \frac{{\rm Tr_S}\big[e^{it_2H_{\rm PT}}\frac{\sin\theta}{2}Oe^{-i(t_2-t_1)H_{\rm PT}}\frac{\sin\theta}{2}Oe^{-it_1H_{\rm PT}}e^{-\beta H_{\rm PT}}\big]}{{\rm Tr_S}\big[e^{it_2H_{\rm PT}}\frac{\sin\theta}{2}Ie^{-i(t_2-t_1)H_{\rm PT}}\frac{\sin\theta}{2}Ie^{-it_1H_{\rm PT}}e^{-\beta H_{\rm PT}}\big]}\\ &= \frac{{\rm Tr_S}[O(t_2)O(t_1)e^{-\beta H_{\rm PT}}]}{Z} = \langle O(t_2)O(t_1)\rangle_{\rm PT}.\end{aligned} \quad \text{(C17)}$$

APPENDIX D: EXPECTATION VALUES OF OBSERVABLES FOR FINITE-TEMPERATURE QUANTUM SYSTEMS

1. Expectation values of observables in a $\mathcal{PT}$-symmetric non-Hermitian system

In this section, we calculate the expectation values of observables for a finite-temperature quantum system. First, we derive the expectation values of the magnetization, the magnetic susceptibility, and the two-time correlation function for the $\mathcal{PT}$-symmetric non-Hermitian system. The magnetization is evaluated as

$$\begin{aligned}m &:= \langle\sigma^z\rangle_{\rm PT}\\ &= \frac{\cos\phi}{\sqrt{\cos 2\phi}}\frac{e^{-\beta E_-}\langle E^L_-|\sigma^z|E^R_-\rangle + e^{-\beta E_+}\langle E^L_+|\sigma^z|E^R_+\rangle}{e^{-\beta E_-} + e^{-\beta E_+}}\\ &= -i\frac{\sin\phi}{\sqrt{\cos 2\phi}}\tanh(\beta R\sqrt{\cos 2\phi}).\end{aligned} \quad \text{(D1)}$$

By differentiating this with respect to a $(= \tan\phi)$, we obtain the magnetic susceptibility:

$$\begin{aligned}\chi &:= \frac{dm}{da} = (\cos^2\phi)\frac{\partial m}{\partial\phi} = \frac{-i\cos^3\phi}{(\cos 2\phi)^{\frac{3}{2}}}\\ &\quad\times\left[\tanh(\beta R\sqrt{\cos 2\phi}) - \frac{2\beta R(\sin^2\phi)\sqrt{\cos 2\phi}}{\cosh^2(\beta R\sqrt{\cos 2\phi})}\right].\end{aligned} \quad \text{(D2)}$$

To derive the two-time correlation function

$$G(\sigma^z(t_2),\sigma^z(t_1)) := \langle\sigma^z(t_2)\sigma^z(t_1)\rangle_{\rm PT} - \langle\sigma^z(t_2)\rangle_{\rm PT}\langle\sigma^z(t_1)\rangle_{\rm PT}, \tag{D3}$$

we calculate the first term on the right-hand side as follows:

$$\begin{aligned}\langle\sigma^z(t_2)\sigma^z(t_1)\rangle_{\rm PT} =& \frac{1}{Z}\left(\frac{\cos\phi}{\sqrt{\cos 2\phi}}\right)^2 \sum_{p\in\{+,-\}} \langle E_p^L|e^{-(\beta-it_2)E_p}\sigma^z \\ &\times \sum_{q\in\{+,-\}} |E_q^R\rangle e^{-i(t_2-t_1)E_q}\langle E_q^L|\sigma^z e^{-it_1E_p}|E_p^R\rangle \\ =& \frac{1}{Z}\frac{\cos^2\phi}{\cos 2\phi}\sum_{p,q\in\{+,-\}} e^{-\beta E_p - i(t_2-t_1)(E_q-E_p)} \\ &\times \langle E_p^L|\sigma^z|E_q^R\rangle\langle E_q^L|\sigma^z|E_p^R\rangle. \end{aligned}\tag{D4}$$

Now we split the sum into two parts, one with $p=q$ and the other with $p\neq q$, as follows:

$$\begin{aligned}&\langle\sigma^z(t_2)\sigma^z(t_1)\rangle_{\rm PT}\\ &= \frac{1}{Z}\frac{\cos^2\phi}{\cos 2\phi}\sum_{p\in\{+,-\}} e^{-\beta E_p}\left(\langle E_p^L|\sigma^z|E_p^R\rangle\right)^2 \\ &\quad + \frac{1}{Z}\frac{\cos^2\phi}{\cos 2\phi}\sum_{p\neq q\in\{+,-\}} e^{-\beta E_p - i(t_2-t_1)(E_q-E_p)} \\ &= -\frac{\sin^2\phi}{\cos 2\phi} + \frac{1}{Z}\frac{\cos^2\phi}{\cos 2\phi}\,2\cosh\left[(\beta-2i\Delta t)R\sqrt{\cos 2\phi}\right] \\ &= \frac{\cos^2\phi}{\cos 2\phi}\left(-\tan^2\phi + \frac{\cosh\left[(\beta-2i\Delta t)R\sqrt{\cos 2\phi}\right]}{\cosh\left[\beta R\sqrt{\cos 2\phi}\right]}\right),\end{aligned}\tag{D5}$$

where $\Delta t := t_2 - t_1$. Combining this expression with Eq. (D1), we obtain the two-time correlation function:

$$\begin{aligned}G(t_2,t_1) &= \langle\sigma^z(t_2)\sigma^z(t_1)\rangle_{\rm PT} - \langle\sigma^z(t_2)\rangle_{\rm PT}\langle\sigma^z(t_1)\rangle_{\rm PT} \\ &= \frac{\cos^2\phi}{\cos 2\phi}\Bigg\{(\tan^2\phi)\left(\tanh^2[\beta R\sqrt{\cos 2\phi}]-1\right) \\ &\quad + \frac{\cosh\left[(\beta-2i\Delta t)R\sqrt{\cos 2\phi}\right]}{\cosh[\beta R\sqrt{\cos 2\phi}]}\Bigg\}.\end{aligned}\tag{D6}$$

2. Expectation values of observables in an extended Hermitian system

The results in Appendix D 1 are also obtained from the extended Hermitian system discussed in Appendix C in an equivalent form. In fact, the magnetization m is obtained from Eq. (C12) as

$$\begin{aligned}m = \langle\sigma^z\rangle_{\rm PT} &= \frac{{\rm Tr_{tot}}\left[P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^-\otimes\sigma^z)e^{-\beta H_{\rm tot}}\right]}{{\rm Tr_{tot}}\left[P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^-\otimes I)e^{-\beta H_{\rm tot}}\right]} \\ &= \frac{{\rm Tr_{tot}}\left\{\left(\frac{1}{2}+\frac{\sin\theta}{2}\sigma_{\rm A}^x\otimes I + \frac{\cos\theta}{2}\sigma_{\rm A}^z\otimes\sigma^y\right)(\sigma_{\rm A}^-\otimes\sigma^z)\left[\cosh(\beta r) - \sinh(\beta r)(\sin\theta\, I_{\rm A}\otimes\sigma^x + \cos\theta\,\sigma_{\rm A}^y\otimes\sigma^z)\right]\right\}}{{\rm Tr_{tot}}\left\{\left(\frac{1}{2}+\frac{\sin\theta}{2}\sigma_{\rm A}^x\otimes I + \frac{\cos\theta}{2}\sigma_{\rm A}^z\otimes\sigma^y\right)(\sigma_{\rm A}^-\otimes I)\left[\cosh(\beta r) - \sinh(\beta r)(\sin\theta\, I_{\rm A}\otimes\sigma^x + \cos\theta\,\sigma_{\rm A}^y\otimes\sigma^z)\right]\right\}} \\ &= \frac{{\rm Tr_{tot}}\left\{\frac{1}{2}\left(-\frac{i}{2}\sigma_{\rm A}^y\otimes\sigma^z\right)\left[-\sinh(\beta r)\cos\theta\,\sigma_{\rm A}^y\otimes\sigma^z\right]\right\}}{{\rm Tr_{tot}}\left[\left(\frac{\sin\theta}{2}\sigma_{\rm A}^x\otimes I\right)\left(\frac{1}{2}\sigma_{\rm A}^x\otimes I\right)\cosh(\beta r)\right]} = \frac{i}{\tan\theta}\tanh(\beta r).\end{aligned}\tag{D7}$$

We can also express the magnetic susceptibility with r and θ as

$$\chi = -\frac{i}{\sin^3\theta}\left[\tanh(\beta r) - \frac{2\beta r}{\cosh^2(\beta r)\,[1+(\cos\theta)^{-2}]}\right].\tag{D8}$$

Finally, the two-time correlation function $G(\sigma^z(t_2),\sigma^z(t_1)) = \langle\sigma^z(t_2)\sigma^z(t_1)\rangle_{\rm PT} - \langle\sigma^z(t_2)\rangle_{\rm PT}\langle\sigma^z(t_1)\rangle_{\rm PT}$ is obtained from Eqs. (C17) and (D7) as

$$\begin{aligned}G(\sigma^z(t_2),\sigma^z(t_1)) &= \frac{{\rm Tr_{tot}}\left[e^{i\Delta tH_{\rm tot}}(\sigma_{\rm A}^-\otimes\sigma^z)e^{-i\Delta tH_{\rm tot}}P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^-\otimes\sigma^z)P_{\rm tot}^{\rm PT}e^{-\beta H_{\rm tot}}\right]}{{\rm Tr_{tot}}\left[e^{i\Delta tH_{\rm tot}}(\sigma_{\rm A}^-\otimes I)e^{-i\Delta tH_{\rm tot}}P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^-\otimes I)P_{\rm tot}^{\rm PT}e^{-\beta H_{\rm tot}}\right]} - \left(\frac{{\rm Tr_{tot}}\left[P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^-\otimes\sigma^z)e^{-\beta H_{\rm tot}}\right]}{{\rm Tr_{tot}}\left[P_{\rm tot}^{\rm PT}(\sigma_{\rm A}^-\otimes I)e^{-\beta H_{\rm tot}}\right]}\right)^2 \\ &= \frac{-(\cos^2\theta)\cosh(\beta r) + \cosh\left[(\beta-2i\Delta t)r\right]}{(\sin^2\theta)\cosh(\beta r)} - \left[\frac{i}{\tan\theta}\tanh(\beta r)\right]^2 \\ &= -\frac{1}{\tan^2\theta}\,\frac{1}{\cosh^2(\beta r)} + \frac{\cosh\left[(\beta-2i\Delta t)r\right]}{(\sin^2\theta)\cosh(\beta r)}.\end{aligned}\tag{D9}$$

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