Fractons, non-Riemannian geometry, and double field theory

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We initiate a systematic study of fracton physics within the geometric framework of double field theory. We ascribe the immobility and large degeneracy of the former to the non-Riemannian backgrounds of the latter, in terms of generalized geodesics and infinite-dimensional isometries. A doubled pure Yang-Mills or Maxwell theory reduces to an ordinary one coupled to a strain tensor of elasticity theory and thus rather remarkably provides a unifying description of photons and phonons. Upon a general double field theory background, which consists of Riemannian and non-Riemannian subspaces, the dual photon-phonon pair becomes fractonic over the non-Riemannian subspace. When the elasticity displacement vector condenses, minimally coupled charged particles acquire an effective mass even in the purely Riemannian case, yielding predictions for polaron physics and time crystals. Furthermore, the immobility of neutral particles along the non-Riemannian directions is lifted to a saturation velocity for charged particles. Utilizing the differential geometry of double field theory, we also present curved spacetime extensions which exhibit general covariance.

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I. INTRODUCTION

Fractons are novel quasiparticles with properties that challenge the conventional understanding of topological phases of matter [1–4]. In modern condensed matter physics, it is generically expected that any lattice model with local interactions admits a well-defined continuum field theory limit in the far-infrared regime. However, fractons defy this doctrine and appear to require ingenious or exotic field theories with some manner of UV-IR mixing [5–12]. Fractons have further characteristic properties such as immobility, infinite ground-state degeneracy in the continuum limit, and higher-moment conservation laws. We refer readers to Refs. [13,14] for reviews and Refs. [15–48] for further significant developments.

Recent advances have shown further that the immobility can be explained in terms of certain subsystem symmetries or conserved higher multipole moments. As symmetry has been a successful guiding principle in modern physics, the characteristics of fracton physics can be grasped through the underlying (though rather exotic) symmetry laws. For example, a charged particle with both monopole and dipole conservation in specific directions explains the immobility of the monopole in the corresponding subspace.

Parallel to the endeavors to find continuum field theory limits of all known fracton lattice models, it may be worthwhile to have a formalism which allows us to construct systematically and geometrically new types of quantum field theories featuring fractons.

In this paper, we launch a systematic top-down approach to fracton physics by employing the geometric framework of double field theory (DFT), assuming the $\mathbf{O}(D,D)$ symmetry therein as the first principle. Historically, $\mathbf{O}(d,d;\mathbb{Z})$ was an "emerging" discrete symmetry for string theory compactified on a torus background T^d [49]. However, from the modern DFT point of view, string theory itself "knows" the $\mathbf{O}(D,D)=\mathbf{O}(D,D;\mathbb{R})$ symmetry regardless of the chosen background, with D now denoting the full spacetime dimension. The theory is *ab initio* "covariant" (rather than invariant) under $\mathbf{O}(D,D)$ symmetry rotations. Only a specific individual background breaks it spontaneously, either fully or partially, such as $\mathbf{O}(D,D;\mathbb{R})\rightarrow\mathbf{O}(d,d;\mathbb{Z})$ upon the aforementioned toroidal compactification.

We shall demonstrate in this paper that fracton physics may arise from such fully $\mathbf{O}(D,D)$-symmetric theories when the background is non-Riemannian, meaning that an invertible metric $g_{\mu\nu}$ is not defined even locally. Analogous to general relativity (GR), which describes physics on Riemannian geometries, the (stringy) gravitational theory for more

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general geometries, including both Riemannian and non-Riemannian ones, is DFT. By embedding fracton physics into DFT, it becomes readily possible to further address fermionic extensions, supersymmetrizations, and curved spacetime generalizations, while likely maintaining consistency with string theory or quantum gravity.

DFT was originally conceived [50–55] to make manifest the hidden symmetry of D-dimensional supergravity underlying the so-called "Buscher rule" [56,57]. In order to do so, the theory demands that the coordinates be formally doubled, $x^A = (\tilde{x}_\mu, x^\nu)$, $\partial_A = (\tilde{\partial}^\mu, \partial_\nu)$, and redefines the notion of general covariance: Under infinitesimal doubled diffeomorphisms $\delta x^A = \xi^A(x)$, a covariant tensor density of weight w transforms through a generalized Lie derivative,

$$\hat{\mathcal{L}}_\xi T_{A_1\cdots A_n} = \xi^B\partial_B T_{A_1\cdots A_n} + w\partial_B\xi^B T_{A_1\cdots A_n} + \sum_{j=1}^{n}\left(\partial_{A_j}\xi^B - \partial^B\xi_{A_j}\right)T_{A_1\cdots B\cdots A_n}. \tag{1}$$

Here, $A, B = 1, 2, \ldots, D + D$ are $\mathbf{O}(D, D)$ indices which are raised and lowered by an $\mathbf{O}(D, D)$ invariant metric,

$$\mathcal{J}_{AB} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{2}$$

Closure of (1) requires imposing the so-called "section condition,"

$$\partial_A\partial^A = 0, \tag{3}$$

which enforces that the contraction between any pair of derivatives should be trivial. Decomposing this as $\partial_A\partial^A = \partial_\mu\tilde{\partial}^\mu + \tilde{\partial}^\mu\partial_\mu$, the condition is conveniently solved by switching off any tilde-coordinate dependence, $\tilde{\partial}^\mu = 0$. In this way, the theory is not truly doubled: Rather, it packages various component fields into a unifying $\mathbf{O}(D, D)$ multiplet.

In DFT the "dilaton" d and the "generalized metric" $\mathcal{H}_{AB}$ are the two fundamental variables that constitute the gravitational sector, in analogy with the Riemannian metric $g_{\mu\nu}$ in GR. While the former exponentiates to a unit-weight scalar density e^{-2d}, the latter satisfies its own defining properties,

$$\mathcal{H}_{AB} = \mathcal{H}_{BA}, \quad \mathcal{H}_A{}^C\mathcal{H}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}. \tag{4}$$

This implies that $\det\mathcal{H}_{AB} = \pm 1$; hence the generalized metric alone cannot produce any integral measure like $\sqrt{g}$ in GR. Instead, combined with the $\mathbf{O}(D, D)$ invariant metric, it generates a pair of mutually orthogonal projectors, $P_{AB} = \frac{1}{2}(J + \mathcal{H})_{AB}$ and $\bar{P}_{AB} = \frac{1}{2}(J - \mathcal{H})_{AB}$, satisfying

$$P_A{}^B P_B{}^C = P_A{}^C, \quad \bar{P}_A{}^B\bar{P}_B{}^C = \bar{P}_A{}^C, \quad P_A{}^B\bar{P}_B{}^C = 0.$$

Parallel to general relativity (GR), DFT has its own Christoffel symbols Γ_{ABC}, scalar, Ricci, and Einstein curvatures, etc., all arising from $\{d, \mathcal{H}_{AB}\}$ [58,59]. Moreover, when coupled to extra "matter" $\mathbf{O}(D, D)$ symmetrically, DFT satisfies "Einstein equations" [60],

$$G_{AB} = T_{AB}, \tag{5}$$

which unifies the equations of motion of d and $\mathcal{H}_{AB}$. The left-hand side and right-hand side satisfy a Bianchi identity and on-shell conservation, respectively: $\nabla_A G^{AB} = 0 = \nabla_A T^{AB}$, with $\nabla_A = \partial_A + \Gamma_A$. The extra matter can be quite generic [61–69], such as point particles [70,71], strings [72,73], and the standard model [74].

In the early days of the development of DFT, the generalized metric was simply assumed to be of the form

$$\mathcal{H}_{AB} = \begin{pmatrix} \mathcal{H}^{\mu\nu} & \mathcal{H}^\mu{}_\lambda \\ \mathcal{H}_\kappa{}^\nu & \mathcal{H}_{\kappa\lambda} \end{pmatrix} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\rho}B_{\rho\lambda} \\ B_{\kappa\rho}g^{\rho\nu} & g_{\kappa\lambda} - B_{\kappa\rho}g^{\rho\sigma}B_{\sigma\lambda} \end{pmatrix}. \tag{6}$$

In this case, the D-dimensional diagonal blocks roughly correspond to the inverse metric $g^{\mu\nu}$ and metric $g_{\mu\nu}$, with additional components generated by a skew-symmetric tensor "B field," $B_{\mu\nu}$. Together with the decomposition of the DFT volume element as $e^{-2d} = e^{-2\phi}\sqrt{-g}$, the resulting fields $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ constitute the gravitational multiplet in the supergravity theory which arises as the low-energy effective description of the massless modes of the closed string propagating in Minkowskian spacetime.

However, this is not the most general parametrization of the generalized metric that satisfies the two defining conditions (4). Surprisingly, it turns out that DFT describes not only the Riemannian geometries given in (6) but also non-Riemannian ones where an invertible Riemannian metric cannot be defined even locally [72]. Namely, with respect to the section choice $\tilde{\partial}^\mu = 0$, the upper-indexed $D \times D$ block matrix $\mathcal{H}^{\mu\nu}$ can be degenerate. From the most general solutions to the conditions (4), all possible DFT geometries have been classified by two non-negative integers $(n, \bar{n})$, with $\mathbf{dim}(\mathbf{ker}\,\mathcal{H}^{\mu\nu}) = n + \bar{n}$ [75]. Only those of type (0,0) are Riemannian, while others are intrinsically non-Riemannian. In particular, the maximally non-Riemannian cases of $(D, 0)$ or $(0, D)$ correspond to $\mathcal{H}_{AB} = \pm\mathcal{J}_{AB}$, and thus they are the two perfectly symmetric vacua of DFT, preserving the entire $\mathbf{O}(D, D)$ symmetry with no moduli [76]. Intriguingly then, the Riemannian spacetime (6) may arise after the spontaneous breaking of $\mathbf{O}(D, D)$ symmetry, which identifies $g_{\mu\nu}$ and $B_{\mu\nu}$ as the massless Nambu-Goldstone bosons [77]. Some intermediate types of non-Riemannian geometries, such as (1,1), $(D-1, 0)$, etc. [75,77–79], have also been identified as nonrelativistic or ultrarelativistic gravities and/or strings [80–86].

Splitting the coordinates into three parts,

$$x^\mu = (x^a,\ y^i,\ \bar{y}^{\bar{\imath}}), \quad \partial_\mu = (\partial_a,\ \partial_i,\ \bar{\partial}_{\bar{\imath}}), \tag{7}$$

where $1 \leqslant a \leqslant D-n-\bar{n}$, $1 \leqslant i \leqslant n$, $1 \leqslant \bar{\imath} \leqslant \bar{n}$, *a flat* $(n, \bar{n})$ *background* is given by constant d and, with a subdimensional (Minkowskian) metric η_{ab} [87],

$$\mathcal{H}^{\mu\nu} = \eta^{ab}\delta_a^\mu\delta_b^\nu, \quad \mathcal{H}_{\mu\nu} = \delta_\mu^a\delta_\nu^b\eta_{ab}, \quad \mathcal{H}_\mu{}^\nu = \delta_\mu^i\delta_i^\nu - \delta_\mu^{\bar{\imath}}\delta_{\bar{\imath}}^\nu, \tag{8}$$

while $\mathcal{H}^\mu{}_\nu = \mathcal{H}_\nu{}^\mu$ and $\Gamma_{ABC} = 0$; hence $\nabla_A = \partial_A$. Here, "flat" means simply being constant: Unlike GR, it appears that there is no four-indexed "Riemann curvature" in DFT [58,88]. Nevertheless, any constant background of $\{d, \mathcal{H}_{AB}\}$ solves the vacuum Einstein equations $G_{AB} = 0$; hence in contrast to GR the constant flat geometries are not unique in DFT. The fact that $\mathcal{H}^{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$ are degenerate $D \times D$ matrices with $n + \bar{n} \neq 0$ characterizes the non-Riemannianity.

In this paper, we will further establish connections between double field theory and fracton physics, for generic $(n, \bar{n}) \neq (0, 0)$. We will identify two main points of contact with known fracton models. The first is the key idea that mobility restrictions arise naturally from non-Riemannian geometry *à la* (generalized) geodesics, with infinite-dimensional isometries playing the role of higher-multipole conservation laws, as will be explained in detail in the next section. In addition, we reveal that the DFT generalization of Yang-Mills theory, to be discussed in Sec. IV, secretly contains an elasticity theory. Theories of elasticity are known to be related to fracton models via a duality transformation [26]. Since the elasticity theory is present even for purely Riemannian geometries, this represents a second, independent link to fractons.

The remainder of this paper is organized as follows. In Sec. II, we present three key motivators for our proposal of studying fractons via DFT. In Sec. III, as elementary warm-up exercises, we consider a particle action and scalar field theory on the non-Riemannian $(n,\bar{n})$ constant background and verify their fractonic behaviors. We then turn to our main example, doubled pure Yang-Mills theory, in Sec. IV. We show that it reduces to ordinary Yang-Mills theory coupled to an elasticity theory of a non-Abelian strain tensor, and we spell out its infinite-dimensional Noether symmetries originating from the non-Riemannian isometries. We subsequently couple its Abelian version, i.e., doubled Maxwell theory, to charged particles in Sec. V and study the resulting dynamics. In particular, we observe that the elasticity displacement vector, once condensed, changes the effective mass of the particle. In Sec. VI we extend our results to curved $(n,\bar{n})$ backgrounds through the DFT formalism. We conclude with comments including a connection to polarons in Sec. VII, and we display some technical formulas in the Appendix.

While our primary goal was to explore the fractonic nature of field theories on non-Riemannian DFT backgrounds, during the investigation of the doubled pure Yang-Mills theory, as well as Maxwell theory and its coupling to point particles, we uncovered some remarkable properties genuinely valid even for Riemannian backgrounds, or on Riemannian subspaces. One is that the doubled pure Yang-Mills theory (30) reduces to an ordinary (undoubled) Yang-Mills theory coupled to a (non-Abelian) strain tensor theory (41), such that its Abelian version provides a unifying description of photons and phonons. Furthermore, when minimally coupled to a point particle (53), the particle will acquire an effective mass through the condensation of the displacement vector of phonons (60), suggesting a potential application to polaron physics.

II. THREE KEY MOTIVATORS

The three key motivators for our proposal of studying fractons via DFT are (a) geodesic immobility, (b) infinite-dimensional isometries, and (c) induced Noether currents. All of these assume *non-Riemannian constant backgrounds* (8).

A. Geodesic immobility

The geometric meaning of the section condition (3) advocated in Ref. [89] is that half of the doubled coordinates, e.g., $\tilde{x}_\mu$, are actually gauged as $\mathbf{D}x^A = (d\tilde{x}_\mu - \mathbf{a}_\mu, dx^\nu)$. This enables us to define an $\mathbf{O}(D,D)$-symmetric, *doubled-diffeomorphism-invariant*, proper length [90] and consequently a particle action [70,71],

$$S_{\text{particle}} = \int d\tau\, \frac{1}{2}e^{-1}\mathbf{D}_\tau x^A \mathbf{D}_\tau x^B \mathcal{H}_{AB} - \frac{1}{2}em^2, \tag{9}$$

where e (einbein) and $\mathbf{a}_\mu$ in $\mathbf{D}_\tau x^A$ are auxiliary variables. After Gaussian integration of the $\mathbf{a}_a$'s along the Riemannian directions, the above doubled particle action reduces upon the constant $(n,\bar{n})$ background (8) to an undoubled one [75] (cf. Ref. [22]),

$$S^{(n,\bar{n})}_{\text{particle}} = \int d\tau \frac{1}{2}e^{-1}\dot{x}^a\dot{x}_a - \frac{1}{2}em^2 + \Lambda_i\dot{y}^i + \bar{\Lambda}_{\bar{\imath}}\dot{\bar{y}}^{\bar{\imath}}. \tag{10}$$

Here, Λ_i and $\bar{\Lambda}_{\bar{\imath}}$ originate from the field redefinitions of the gauge components $\mathbf{a}_i$ and $\mathbf{a}_{\bar{\imath}}$, respectively, and crucially play the role of Lagrange multipliers, enforcing immobility along the non-Riemannian directions,

$$\dot{y}^i = 0, \quad \dot{\bar{y}}^{\bar{\imath}} = 0. \tag{11}$$

Similarly, on a string worldsheet [72,73], y^i and $\bar{y}^{\bar{\imath}}$ become chiral and antichiral, respectively [75].

B. Infinite-dimensional isometries

Our second observation is that the isometry of the $(n,\bar{n}) \neq (0,0)$ non-Riemannian constant background (8) is infinite dimensional [87]: The most general solution to the twofold Killing equations,

$$\hat{\mathcal{L}}_\xi \mathcal{H}_{AB} = 0, \quad \hat{\mathcal{L}}_\xi e^{-2d} = 0, \tag{12}$$

is, with $\xi^A = (\lambda_\mu, \xi^\nu)$,

$$\begin{aligned} &\xi^a = w^a{}_b x^b + \zeta^a(y) + \bar{\zeta}^a(\bar{y}), \quad \lambda_a = \zeta_a(y) - \bar{\zeta}_a(\bar{y}),\\ &\xi^i = \omega\bar{n}y^i + \zeta^i(y), \quad \lambda_i = \rho_i(y),\\ &\bar{\xi}^{\bar{\imath}} = -\omega n\bar{y}^{\bar{\imath}} + \bar{\zeta}^{\bar{\imath}}(\bar{y}), \quad \bar{\lambda}_{\bar{\imath}} = \bar{\rho}_{\bar{\imath}}(\bar{y}). \end{aligned} \tag{13}$$

Here, w_{ab} (Lorentz symmetry, $w_{ab} = -w_{ba}$) and ω are constants. All other parameters are arbitrary functions of the non-Riemannian coordinates y^i or $\bar{y}^{\bar{\imath}}$, as displayed in (13). Furthermore, $\zeta^i(y)$ and $\bar{\zeta}^{\bar{\imath}}(\bar{y})$ should be divergenceless,

$$\partial_i\zeta^i(y) = 0, \quad \bar{\partial}_{\bar{\imath}}\bar{\zeta}^{\bar{\imath}}(\bar{y}) = 0, \tag{14}$$

which ensures that $\partial_\mu\xi^\mu = 0$, a requirement following from the Killing equation of the dilaton d (12).

C. Induced Noether currents

The third point of interest relates to the energy-momentum tensor in DFT [60],

$$T^{AB} = -e^{2d}\left[8P^{[A}{}_C\bar{P}^{B]}{}_D\frac{\delta S_{\text{matter}}}{\delta\mathcal{H}_{CD}} + \frac{1}{2}\mathcal{J}^{AB}\frac{\delta S_{\text{matter}}}{\delta d}\right]. \tag{15}$$

By construction, for arbitrary ξ^A, it satisfies the off-shell relation

$$\begin{aligned} \partial_A(e^{-2d}T^A{}_B\xi^B) = {} & e^{-2d}\xi^B\nabla_A T^A{}_B + \frac{1}{2D}T^A{}_A\hat{\mathcal{L}}_\xi e^{-2d}\\ & - \frac{1}{2}e^{-2d}(PT\bar{P})^{AB}\hat{\mathcal{L}}_\xi\mathcal{H}_{AB}. \end{aligned} \tag{16}$$

Thus, for the constant background (8) with the Killing vector (13), we acquire an on-shell conserved current,

$$\mathbb{J}^{\mu}=T^{\mu}{}_{A}\xi^{A}=T^{\mu}{}_{\nu}\xi^{\nu}+T^{\mu\nu}\lambda_{\nu},\quad \partial_{\mu}\mathbb{J}^{\mu}=0, \tag{17}$$

where we have decomposed $T^{\mu}{}_{A}=(T^{\mu\nu},T^{\mu}{}_{\nu})$. Note that there is no special relation between the independent energy-momentum tensor components $T^{\mu\nu}$ and $T^{\mu}{}_{\rho}$: In particular, $T^{\mu\nu}\neq T^{\mu}{}_{\rho}g^{\rho\nu}$, not to mention the absence of an invertible metric $g_{\mu\nu}$ in non-Riemannian geometry. Explicitly, as a collection of independent currents,

$$\begin{aligned}\mathbb{J}^{\mu}=&(T^{\mu}{}_{a}+\eta_{ab}T^{\mu b})\zeta^{a}(y)+(T^{\mu}{}_{a}-\eta_{ab}T^{\mu b})\bar{\zeta}^{a}(\bar{y})\\&+\omega(\bar{n}T^{\mu}{}_{i}y^{i}-nT^{\mu}{}_{\bar{\imath}}\bar{y}^{\bar{\imath}})+T^{\mu}{}_{i}\zeta^{i}(y)+T^{\mu}{}_{\bar{\imath}}\bar{\zeta}^{\bar{\imath}}(\bar{y})\\&+T^{\mu i}\rho_{i}(y)+T^{\mu\bar{\imath}}\bar{\rho}_{\bar{\imath}}(\bar{y})+T^{\mu}{}_{a}w^{a}{}_{b}x^{b}.\end{aligned} \tag{18}$$

Evidently, power-series expansions of the local parameters in the coordinates y^{i} and $\bar{y}^{\bar{\imath}}$ generate infinitely many higher-multipole conservation laws. This includes dipole conservation laws generated by the parameter ω and other linear terms from $\{\zeta^{a}(y),\bar{\zeta}^{a}(\bar{y}),\zeta^{i}(y),\bar{\zeta}^{\bar{\imath}}(\bar{y})\}$, modulo $\mathbf{so}(n)\oplus\mathbf{so}(\bar{n})$ rotations. Among them, the $(D-n-\bar{n})(n+\bar{n})$ linear terms of $\zeta^{a}(y)$ and $\bar{\zeta}^{a}(\bar{y})$ correspond to conventional dipole conservations in the non-Riemannian directions, arising from isometries along the Riemannian subspace. Meanwhile, the linear terms in $\zeta^{i}(y)$ and $\bar{\zeta}^{\bar{\imath}}(\bar{y})$ generate further non-Riemannian dipole symmetries. In all, mobility is restricted in the $(n+\bar{n})$ non-Riemannian directions. In the special cases where $n=1$ or $\bar{n}=1$, the divergenceless condition (14) actually enforces ζ^{i} or $\bar{\zeta}^{\bar{\imath}}$ simply to be constant, which implies the absence of all higher-multipole conservation laws along the non-Riemannian directions. In particular, (1,1) allows only dipole conservation, corresponding to the finite scale transformation

$$y\to e^{\omega}y,\quad \bar{y}\to e^{-\omega}\bar{y}, \tag{19}$$

where the two non-Riemannian directions are inversely related. Note that the symmetry is still "supertranslational" in the Riemannian directions for any $(n,\bar{n})\neq(0,0)$, as $\zeta^{a}(y)$ and $\bar{\zeta}^{a}(\bar{y})$ appearing in (13) are arbitrary functions.

Meanwhile, for the global translational symmetries generated by the constant terms in ξ^{μ} and λ_{ν}, the conservation of the current (17) reduces to that of the energy-momentum tensors,

$$\partial_{\mu}T^{\mu}{}_{\nu}=0, \tag{20}$$

for the untilde x^{μ} directions, and further, inequivalently,

$$\partial_{\mu}T^{\mu\nu}=0, \tag{21}$$

for the tilde $\tilde{x}_{\mu}$ directions. The latter can be nontrivial even after switching off the tilde coordinates, i.e., setting $\tilde{\partial}^{\mu}=0$ as our choice of section: As we shall see later, a scalar field theory has trivial $T^{\mu\nu}$ (27), whereas it is nontrivial for Yang-Mills theory (50).

The three points, motivators (a), (b), and (c), imply that any (double field theorizable) field theory should feature the fractonic properties of higher-multipole conservation [18,91] and a huge degeneracy of quantum states, as there are infinitely many conserved quantities. Intriguingly, the (1,1) non-Riemannian background, corresponding to the nonrelativistic string [80], allows only dipole conservation along the pair of non-Riemannian directions $y,\bar{y}$ (19), which alludes to UV-IR mixing of these two directions. This property is comparable to known fracton field theory models [6,7,15,16]. We stress that all of these are direct consequences of the underlying constant non-Riemannian background. In the following we verify these properties explicitly for several examples, such as particles, scalar fields, doubled Yang-Mills theory, and a strain-Maxwell theory minimally coupled to charged particles. The advantage of embedding fracton physics into DFT is that generalizations to curved geometries, supersymmetry *à la* Ref. [62], and consistent string backgrounds are readily available, by setting $D=10$ or 26 and $n=\bar{n}$ [92].

III. PARTICLE AND SCALAR FIELD

The doubled energy-momentum tensor of the point particle (10) was obtained in Ref. [60] from the variation of the covariant particle action (9) following the prescription (15),

$$T^{\mu\nu}=0,\quad T^{\mu}{}_{\nu}(x)=\int d\tau\,\dot{x}^{\mu}(\tau)p_{\nu}\delta^{D}(x-x(\tau)), \tag{22}$$

where the delta function is defined for the untilde coordinates $x^{\mu}-x^{\mu}(\tau)$ and $p_{\mu}=(e^{-1}\dot{x}_{a},\Lambda_{i},\bar{\Lambda}_{\bar{\imath}})$ is the conjugate momentum of x^{μ}, of which all components are constant on shell. Thus conservation indeed holds,

$$\partial_{\mu}T^{\mu}{}_{\nu}=-\int d\tau\ p_{\nu}\frac{d}{d\tau}\delta^{D}(x-x(\tau))=0. \tag{23}$$

Furthermore, from the on-shell relations

$$T^{a}{}_{c}\eta^{cb}=T^{b}{}_{c}\eta^{ca},\quad T^{i}{}_{\nu}=0=T^{\bar{\imath}}{}_{\nu}, \tag{24}$$

the conservation of the current (18) readily follows. The corresponding Noether symmetries of the reduced particle action (10) inherited from the doubled particle action (9) read, with (13),

$$\begin{aligned}&\delta x^{a}=\xi^{a},\quad \delta y^{i}=\xi^{i},\quad \delta\bar{y}^{\bar{\imath}}=\bar{\xi}^{\bar{\imath}},\quad \delta e=0,\\&\delta\Lambda_{i}=-e\dot{x}^{a}\partial_{i}\zeta_{a}(y)-\omega\bar{n}\Lambda_{i}-\Lambda_{j}\partial_{i}\zeta^{j}(y),\\&\delta\bar{\Lambda}_{\bar{\imath}}=-e\dot{x}^{a}\bar{\partial}_{\bar{\imath}}\bar{\zeta}_{a}(\bar{y})+\omega n\bar{\Lambda}_{\bar{\imath}}-\bar{\Lambda}_{\bar{\jmath}}\bar{\partial}_{\bar{\imath}}\bar{\zeta}^{\bar{\jmath}}(\bar{y}).\end{aligned} \tag{25}$$

As a target-spacetime counterpart to the particle action, we turn to a scalar field theory with Lagrangian (density) $e^{-2d}L_{\Phi}$ (cf. Refs. [6,7]),

$$L_{\Phi}=-\tfrac{1}{2}\mathcal{H}^{AB}\partial_{A}\Phi\partial_{B}\Phi-V(\Phi)=-\tfrac{1}{2}\eta^{ab}\partial_{a}\Phi\partial_{b}\Phi-V(\Phi). \tag{26}$$

The doubled energy-momentum tensor is, from Ref. [60],

$$T^{\mu\nu}=0,\quad T^{\mu}{}_{\nu}=\delta^{\mu}_{a}\partial^{a}\Phi\partial_{\nu}\Phi+\delta^{\mu}_{\nu}L_{\Phi}, \tag{27}$$

which is conserved on shell as

$$\partial_{\mu}T^{\mu}{}_{\nu}=[\partial_{a}\partial^{a}\Phi-V'(\Phi)]\partial_{\nu}\Phi=0. \tag{28}$$

The infinite-dimensional Noether symmetries for the Killing vector (13) are given simply by $\delta\Phi=\xi^{\mu}\partial_{\mu}\Phi$. In particular, when the scalar theory is free with a Lagrangian $L_{\Phi}=\frac{1}{2}\Phi(\eta^{ab}\partial_{a}\partial_{b}\Phi-m^{2}\Phi)$ which vanishes on shell, its energy-momentum tensor also satisfies (24). Thus the usual agreement between a spinless particle and a scalar field generalizes to generic $(n,\bar{n})$ constant non-Riemannian backgrounds. It is also worthwhile to note that massless scalar fields propagate through subdimensional Riemannian spacetime only: $\partial_{a}\partial^{a}\Phi=0$.

IV. DOUBLED YANG-MILLS THEORY

Our next example is a doubled generalization of Yang-Mills theory. This turns out to be a rich theory in its own right (even for purely Riemannian geometries): In the Abelian case, it reduces to a theory of photons and phonons and thus may itself be applicable to systems of lattice vibrations interacting with light. Moreover, this suggests a second pathway linking DFT to fracton physics: Established fracton models such as symmetric tensor gauge theories are known to be dual to phonon systems via fracton-elasticity duality [26].

For a doubled vector potential $\mathcal{V}_A$, the fully covariant field strength $(P\mathcal{F}\bar{P})_{AB} = P_A{}^C\bar{P}_B{}^D\mathcal{F}_{CD}$ is projected from the "semi-covariant" one [58,93],

$$\mathcal{F}_{AB} = \nabla_A\mathcal{V}_B - \nabla_B\mathcal{V}_A - i[\mathcal{V}_A, \mathcal{V}_B]. \quad (29)$$

The doubled pure Yang-Mills Lagrangian $e^{-2d}L_{\rm YM}$ then takes the form [64]

$$L_{\rm YM} = {\rm Tr}[P^{AC}\bar{P}^{BD}\mathcal{F}_{AB}\mathcal{F}_{CD}]. \quad (30)$$

With $\mathcal{D}_A = \nabla_A - i[\mathcal{V}_A, \cdot\,]$, the equations of motion are [59]

$$\mathcal{D}_A(P\mathcal{F}\bar{P})^{[AB]} = \tfrac{1}{2}\mathcal{D}_A[(P\mathcal{F}\bar{P})^{AB} + (\bar{P}\mathcal{F}P)^{AB}] = 0, \quad (31)$$

while the energy-momentum tensor is [60]

$$T_{AB} = -4P_{[A}{}^C\bar{P}_{B]}{}^D{\rm Tr}[(\mathcal{F}\mathcal{H}\mathcal{F})_{CD} + \mathcal{D}_E(\mathcal{F}_{CD}\mathcal{V}^E)] + \mathcal{J}_{AB}L_{\rm YM}. \quad (32)$$

We now compute the Lagrangian explicitly on the constant $(n, \bar{n})$ non-Riemannian background (8), which we denote using $\{H^{\mu\nu}, K_{\mu\nu}, Z_\mu{}^\nu\}$ as

$$H^{\mu\nu} = \eta^{ab}\delta_a^\mu\delta_b^\nu = \mathcal{H}^{\mu\nu},$$
$$K_{\mu\nu} = \delta_\mu^a\delta_\nu^b\eta_{ab} = \mathcal{H}_{\mu\nu},$$
$$Z_\mu{}^\nu = \delta_\mu^j\delta_j^\nu - \delta_\mu^{\bar{j}}\delta_{\bar{j}}^\nu = \mathcal{H}_\mu{}^\nu. \quad (33)$$

Parametrizing the doubled vector as

$$\mathcal{V}_A = (\varphi^\mu, A_\nu), \quad (34)$$

the projectors and the semicovariant Yang-Mills field strength (29) read

$$P_A{}^B = \frac{1}{2}(\delta_A{}^B + \mathcal{H}_A{}^B) = \frac{1}{2}\begin{pmatrix}\delta^\mu{}_\nu + Z_\nu{}^\mu & H^{\mu\sigma}\\ K_{\rho\nu} & \delta_\rho{}^\sigma + Z_\rho{}^\sigma\end{pmatrix},$$
$$\bar{P}_A{}^B = \frac{1}{2}(\delta_A{}^B - \mathcal{H}_A{}^B) = \frac{1}{2}\begin{pmatrix}\delta^\mu{}_\nu - Z_\nu{}^\mu & -H^{\mu\sigma}\\ -K_{\rho\nu} & \delta_\rho{}^\sigma - Z_\rho{}^\sigma\end{pmatrix},$$
$$\mathcal{F}_{AB} = 2\partial_{[A}\mathcal{V}_{B]} - i[\mathcal{V}_A, \mathcal{V}_B] = \begin{pmatrix}-i[\varphi^\mu, \varphi^\nu] & -D_\sigma\varphi^\mu\\ D_\rho\varphi^\nu & f_{\rho\sigma}\end{pmatrix}, \quad (35)$$

respectively, where $D_\mu = \partial_\mu - i[A_\mu, \cdot\,]$ and

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad (36)$$

is the field strength of ordinary undoubled Yang-Mills theory. From these ingredients we obtain the fully covariant field strength,

$$(P\mathcal{F}\bar{P})_{AB} = \begin{pmatrix}(P\mathcal{F}\bar{P})^{\mu\nu} & (P\mathcal{F}\bar{P})^\mu{}_\sigma\\ (P\mathcal{F}\bar{P})_\rho{}^\nu & (P\mathcal{F}\bar{P})_{\rho\sigma}\end{pmatrix},$$
$$(P\mathcal{F}\bar{P})^{\mu\nu} = -\tfrac{1}{4}\widetilde{f}^{\kappa\lambda}(\delta_\kappa{}^\mu + Z_\kappa{}^\mu)(\delta_\lambda{}^\nu - Z_\lambda{}^\nu),$$
$$(P\mathcal{F}\bar{P})^\mu{}_\sigma = \tfrac{1}{4}[H^{\mu\kappa}(\delta_\sigma{}^\lambda - Z_\sigma{}^\lambda)\widehat{f}_{\kappa\lambda} - \Upsilon_\sigma{}^\mu],$$
$$(P\mathcal{F}\bar{P})_\rho{}^\nu = -\tfrac{1}{4}[(\delta_\rho{}^\kappa + Z_\rho{}^\kappa)H^{\nu\lambda}\widehat{f}_{\kappa\lambda} - \bar{\Upsilon}_\rho{}^\nu],$$
$$(P\mathcal{F}\bar{P})_{\rho\sigma} = \tfrac{1}{4}(\delta_\rho{}^\kappa + Z_\rho{}^\kappa)(\delta_\sigma{}^\lambda - Z_\sigma{}^\lambda)\widehat{f}_{\kappa\lambda}. \quad (37)$$

Here, we have introduced the shorthand notation

$$\widetilde{f}^{\mu\nu} = f^{ab}\delta_a{}^\mu\delta_b{}^\nu + i[\varphi^\mu, \varphi^\nu] - (\delta_a{}^\mu D^a\varphi^\nu + \delta_a{}^\nu D^a\varphi^\mu),$$
$$\widehat{f}_{\mu\nu} = f_{\mu\nu} + i\delta_\mu{}^a\delta_\nu{}^b[\varphi_a, \varphi_b] - (\delta_\mu{}^a D_\nu\varphi_a + \delta_\nu{}^a D_\mu\varphi_a),$$
$$\Upsilon_\mu{}^\nu = 2\delta_\mu{}^a D_a^-\varphi^i\delta_i{}^\nu + 4\delta_\mu{}^{\bar{\imath}}D_{\bar{\imath}}\varphi^i\delta_i{}^\nu,$$
$$\bar{\Upsilon}_\mu{}^\nu = 2\delta_\mu{}^a D_a^+\varphi^{\bar{\imath}}\delta_{\bar{\imath}}{}^\nu + 4\delta_\mu{}^i D_i\varphi^{\bar{\imath}}\delta_{\bar{\imath}}{}^\nu, \quad (38)$$

and defining $A_a^\pm = A_a \pm \varphi_a$, we further set

$$D_a^\pm = \partial_a - i[A_a^\pm, \cdot\,],$$
$$f_{ai}^\pm = \partial_a A_i - \partial_i A_a^\pm - i[A_a^\pm, A_i] = f_{ai} \mp D_i\varphi_a,$$
$$f_{a\bar{\imath}}^\pm = \partial_a A_{\bar{\imath}} - \bar{\partial}_{\bar{\imath}} A_a^\pm - i[A_a^\pm, A_{\bar{\imath}}] = f_{a\bar{\imath}} \mp D_{\bar{\imath}}\varphi_a. \quad (39)$$

Substituting these expressions into the Lagrangian (30), after expanding as

$$L_{\rm YM} = 2{\rm Tr}\big[(P\mathcal{F}\bar{P})^{\mu\nu}(P\mathcal{F}\bar{P})_{\mu\nu} + (P\mathcal{F}\bar{P})^\mu{}_\nu(P\mathcal{F}\bar{P})_\mu{}^\nu\big], \quad (40)$$

one arrives at the undoubled Lagrangian,

$$L_{\rm YM}^{(n,\bar{n})} = {\rm Tr}\begin{bmatrix}-\frac{1}{4}(f_{ab}+i[\varphi_a, \varphi_b])(f^{ab}+i[\varphi^a, \varphi^b])\\ -\frac{1}{4}u_{ab}u^{ab} - f_{ai}^- D^{-a}\varphi^i + f_{a\bar{\imath}}^+ D^{+a}\varphi^{\bar{\imath}}\\ -2D_i\varphi^{\bar{\imath}}D_{\bar{\imath}}\varphi^i - 2if_{i\bar{\imath}}[\varphi^i, \varphi^{\bar{\imath}}]\end{bmatrix}, \quad (41)$$

where we have defined a symmetric tensor

$$u_{ab} = D_a\varphi_b + D_b\varphi_a. \quad (42)$$

Identifying φ^μ as the displacement vector in elasticity theory, u_{ab} corresponds to a strain tensor which now interacts with the undoubled Yang-Mills theory. The symmetric strain tensor originates essentially from the projection

$$4(P\mathcal{F}\bar{P})_{ab} = f_{ab} + i[\varphi_a, \varphi_b] - u_{ab} = \partial_a A_b^- - \partial_b A_a^+ - i[A_a^+, A_b^-]. \quad (43)$$

Since A_μ and φ^μ are dual to each other *à la* Buscher [56,57], so are their (Abelian) elementary quanta, photon and phonon (cf. Ref. [94]).

By construction from (30), the Lagrangian (41) enjoys "supertranslational" Noether symmetries given *a priori* by $\delta\mathcal{V}_A = \hat{\mathcal{L}}_\xi\mathcal{V}_A$ (1) with the Killing vector (13), which reduce in terms of the ordinary Lie derivative to

$$\delta A_\mu = \mathcal{L}_\xi A_\mu + 2\partial_{[\mu}\lambda_{\rho]}\varphi^\rho, \quad \delta\varphi^\mu = \mathcal{L}_\xi\varphi^\mu. \quad (44)$$

In particular, under the transformations (44) with the Killing vector (13), $(P\mathcal{F}\bar{P})_{AB}$ transforms covariantly as

$$\delta(P\mathcal{F}\bar{P})^{\mu\nu} = \mathcal{L}_\xi(P\mathcal{F}\bar{P})^{\mu\nu},$$
$$\delta(P\mathcal{F}\bar{P})^\mu{}_\nu = \mathcal{L}_\xi(P\mathcal{F}\bar{P})^\mu{}_\nu + 2\partial_{[\nu}\lambda_{\rho]}(P\mathcal{F}\bar{P})^{\mu\rho},$$
$$\delta(P\mathcal{F}\bar{P})_\mu{}^\nu = \mathcal{L}_\xi(P\mathcal{F}\bar{P})_\mu{}^\nu + 2\partial_{[\mu}\lambda_{\rho]}(P\mathcal{F}\bar{P})^{\rho\nu},$$
$$\delta(P\mathcal{F}\bar{P})_{\mu\nu} = \mathcal{L}_\xi(P\mathcal{F}\bar{P})_{\mu\nu} + 2\partial_{[\mu}\lambda_{\rho]}(P\mathcal{F}\bar{P})^\rho{}_\nu + 2\partial_{[\nu}\lambda_{\rho]}(P\mathcal{F}\bar{P})_\mu{}^\rho, \quad (45)$$

from which the invariance of the action, or (40), follows straightforwardly,

$$\delta L_{\rm YM} = \xi^\mu\partial_\mu L_{\rm YM} = \partial_\mu(\xi^\mu L_{\rm YM}). \quad (46)$$

Note that although (45) can be verified directly by brute force, it can be understood simply as a natural consequence of the $\mathbf{O}(D,D)$-symmetric general covariance of the doubled Yang-Mills field strength (37), encoded via (1).

The equations of motion $\mathcal{D}_A(P\mathcal{F}\bar{P})^{[AB]}=0$ (31) are, exhaustively,

$$\begin{aligned}
0 &= D_b\big(F^b{}_a + i[\varphi^b,\varphi_a]\big) - D^-_a D_i\varphi^i + D^+_a D_{\bar\imath}\varphi^{\bar\imath} \\
&\quad + i[\varphi_b, u^b{}_a] + 2i[\varphi^i, f^-_{ai}] - 2i[\varphi^{\bar\imath}, f^+_{a\bar\imath}],\\
0 &= D^-_a D^{-a}\varphi^i + 4i[\varphi^{\bar\imath}, D_{\bar\imath}\varphi^i] - 2i[\varphi^i, D_{\bar\imath}\varphi^{\bar\imath}],\\
0 &= D^+_a D^{+a}\varphi^{\bar\imath} - 4i[\varphi^i, D_i\varphi^{\bar\imath}] + 2i[\varphi^{\bar\imath}, D_i\varphi^i],\\
0 &= D_b u^b{}_a + D^-_a D_i\varphi^i + D^+_a D_{\bar\imath}\varphi^{\bar\imath} - 2i\big([\varphi^i, f^-_{ai}] + [\varphi^{\bar\imath}, f^+_{a\bar\imath}]\big),\\
0 &= D^{-a} f^-_{ai} + 2D_i D_{\bar\imath}\varphi^{\bar\imath} + 4i[f_{i\bar\imath}, \varphi^{\bar\imath}],\\
0 &= D^{+a} f^+_{a\bar\imath} - 2D_{\bar\imath} D_i\varphi^i + 4i[f_{i\bar\imath}, \varphi^i].
\end{aligned} \tag{47}$$

We directly verified the conservation $\partial_\mu \mathbb{J}^\mu = 0$ (17) by checking, first of all, the conservation of the doubled energy-momentum tensor itself [up to the Bianchi identity, the on-shell equations (31), and the section condition],

$$\begin{aligned}
\partial_B T^B{}_A &= \mathrm{Tr}[6(P\mathcal{F}\bar{P})^{BC} D_{[A}\mathcal{F}_{BC]} - 4\mathcal{F}_{AC} D_B(P\mathcal{F}\bar{P})^{[BC]}]\\
&\quad + 4\partial_C \mathrm{Tr}[\partial^C\mathcal{V}^B(P\mathcal{F}\bar{P})_{[BA]} - \mathcal{V}^C D^B(P\mathcal{F}\bar{P})_{[BA]}] = 0,
\end{aligned} \tag{48}$$

and, secondly, the vanishing of each of

$$\big\{\, T^{ij},\ T^{\bar\imath\bar\jmath},\ T^{ia} + \eta^{ab}T^i{}_b,\ T^{\bar\imath a} - \eta^{ab}T^{\bar\imath}{}_b,\ T^{i\bar\imath} + T^{\bar\imath i} \,\big\}$$

as consequences of the orthogonal projections performed in (32), with $T^i{}_j = \delta^i{}_j L_{\mathrm{YM}}$ and $T^{\bar\imath}{}_{\bar\jmath} = \delta^{\bar\imath}{}_{\bar\jmath} L_{\mathrm{YM}}$; cf. (24). Unlike the previous particle and scalar field theory cases, both $T^{\mu\nu}$ and $T^\mu{}_\nu$ turn out to be nontrivial in Yang-Mills theory: See the Appendix for the full expression, or the upcoming equation (50) for the simple (0,0) Abelian case.

As already mentioned, it is a rather unexpected and remarkable result of the DFT formalism that the doubled Yang-Mills theory produces a unifying description (41) of ordinary Yang-Mills theory and a non-Abelian elasticity theory. We emphasize that this holds true even upon genuine Riemannian backgrounds, i.e., $(n,\bar n) = (0,0)$. The Abelian reduction of (41) on a flat Minkowskian spacetime is simply the sum of Maxwell and strain tensor theories describing free photons and phonons, respectively,

$$L^{(0,0)}_{\text{Maxwell-strain}} = -\tfrac14 f_{ab}f^{ab} - \tfrac14 u_{ab}u^{ab}. \tag{49}$$

The corresponding energy-momentum tensors are

$$\begin{aligned}
T^{ab} &= 2\partial_c\varphi^{[a}f^{b]c} + \partial_c(f^{ab}\varphi^c),\\
T^a{}_b &= f^{ac}f_{bc} + \partial^a\varphi^c\partial_b\varphi_c - \partial_c\varphi^a\partial^c\varphi_b + \partial_c(\varphi^c u^a{}_b)\\
&\quad - \tfrac14\delta^a{}_b\big(f_{cd}f^{cd} + u_{cd}u^{cd}\big).
\end{aligned} \tag{50}$$

While clearly $T^{ab} \neq T^a{}_c\eta^{cb}$, these satisfy the off-shell relations

$$\begin{aligned}
\partial_a T^{ab} &= 2\partial_c\big(\partial_a f^{a[b}\varphi^{c]}\big),\\
\partial_a T^a{}_b &= f_{bd}\partial_c f^{cd} - \tfrac32 f^{cd}\partial_{[b}f_{cd]} + \partial_b\varphi^d\partial_c u^c{}_d + \partial_d(\varphi^d\partial_c u^c{}_b)
\end{aligned} \tag{51}$$

and thus become conserved on shell due to the equations of motion, $\partial_a f^{ab} = 0 = \partial_a u^{ab}$.

Intriguingly, with the decomposition of the displacement vector into temporal and spatial components, $\varphi^a = (\varphi^t, \varphi^{\mathsf a})$, where $\mathsf{a} = 1, 2, \ldots, D-1$, if we truncate the temporal component by setting $\varphi^t = 0$, the strain tensor part in (49) becomes

$$-\tfrac14 u_{ab}u^{ab} = \tfrac12\dot\varphi_{\mathsf a}\dot\varphi^{\mathsf a} - \tfrac14 u_{\mathsf{ab}}u^{\mathsf{ab}}, \tag{52}$$

where the dots denote time derivatives $\frac{d}{dt}$. This is precisely the elasticity theory considered in Ref. [26], shown therein to be dual to a $\mathbf{U}(1)$-symmetric tensor gauge theory. The topological defects of the former map onto charges of the latter, with disclinations and dislocations corresponding to fractons and dipoles, respectively. Our result is in some sense (orthogonally) complementary to Ref. [26], as we are principally focusing on the fracton physics that arises geometrically from non-Riemannian backgrounds. Nevertheless, we remark that both non-Riemannian spaces and topological defects are, after all, singular configurations from a conventional perspective.

V. DOUBLED MAXWELL THEORY COUPLED TO CHARGED PARTICLES

We now consider the doubled Maxwell theory (30) minimally coupled to particles (9) with charge q,

$$\sum_\alpha \int d\tau\, \tfrac12 e^{-1}\mathbf{D}_\tau x^A_\alpha \mathbf{D}_\tau x^B_\alpha \mathcal{H}_{AB} - \tfrac12 e m^2 - q\mathbf{D}_\tau x^A_\alpha \mathcal{V}_A, \tag{53}$$

where α denotes a (negligible) particle index. On the constant $(n,\bar n)$ background (8), the single-particle Lagrangian becomes

$$\begin{aligned}
\mathcal{L}_q &= \tfrac12 e^{-1}\mathbf{D}_\tau x^A\mathbf{D}_\tau x^B\mathcal{H}_{AB} - \tfrac12 e m^2 - q\mathbf{D}_\tau x^A\mathcal{V}_A\\
&= \tfrac12 e^{-1}\dot x^a\dot x_a - \tfrac12 e\big(m^2 + q^2\varphi^a\varphi_a\big) - q\dot x^\mu A_\mu\\
&\quad + \tfrac12 e^{-1}\big(\dot{\tilde x}_a - \mathbf{a}_a - eq\varphi_a\big)\big(\dot{\tilde x}_b - \mathbf{a}_b - eq\varphi_b\big)\eta^{ab}\\
&\quad + (e^{-1}\dot y^i - q\varphi^i)(\dot{\tilde y}_i - \mathbf{a}_i) - (e^{-1}\dot y^{\bar\imath} + q\varphi^{\bar\imath})(\dot{\tilde y}_{\bar\imath} - \mathbf{a}_{\bar\imath}).
\end{aligned} \tag{54}$$

The corresponding action generalizes (10) to

$$S_q = \int d\tau\left[\begin{matrix}\tfrac12 e^{-1}\dot x^a\dot x_a - \tfrac12 e(m^2 + q^2\varphi^a\varphi_a) - q\dot x^\mu A_\mu\\ +\Lambda_i(\dot y^i - eq\varphi^i) + \bar\Lambda_{\bar\imath}(\dot y^{\bar\imath} + eq\varphi^{\bar\imath})\end{matrix}\right], \tag{55}$$

where we have integrated out the auxiliary variables $\mathbf{a}_a$ thereby setting the on-shell value $\mathbf{D}_\tau\tilde x_a = eq\varphi_a$, and we have identified $\Lambda_i = e^{-1}\mathbf{D}_\tau\tilde y_i$ and $\bar\Lambda_{\bar\imath} = -e^{-1}\mathbf{D}_\tau\tilde y_{\bar\imath}$. This action then couples the particle to a strain-Maxwell theory, i.e., the Abelian reduction of (41),

$$L_0 = -\tfrac14 f_{ab}f^{ab} - \tfrac14 u_{ab}u^{ab} - f^-_{ai}\partial^a\varphi^i + f^+_{a\bar\imath}\partial^a\varphi^{\bar\imath} - 2\partial_i\varphi^{\bar\imath}\partial_{\bar\imath}\varphi^i. \tag{56}$$

The Hamiltonian action for (55) is

$$\begin{aligned}
S_H &= \int d\tau\, p_\mu\dot x^\mu - eH,\\
H &= \frac12(p_a + qA_a)(p^a + qA^a) + \tfrac12(m^2 + q^2\varphi_a\varphi^a)\\
&\quad + q(p_i + qA_i)\varphi^i - q(\bar p_{\bar\imath} + qA_{\bar\imath})\varphi^{\bar\imath}.
\end{aligned} \tag{57}$$

Integrating out the p_a's from (57), one recovers (55) with the identification $p_i = \Lambda_i - qA_i$, $\bar{p}_{\bar{\imath}} = \bar{\Lambda}_{\bar{\imath}} - qA_{\bar{\imath}}$.

Clearly, from (55) or (57), the immobility is lifted by the displacement vector to a "saturation velocity,"

$$\dot{y}^i = eq\varphi^i, \quad \dot{\bar{y}}^{\bar{\imath}} = -eq\varphi^{\bar{\imath}}. \tag{58}$$

Furthermore, the Hamiltonian constraint $H = 0$ (57) leads to a modified dispersion relation,

$$k_a k^a + m^2 + q^2\varphi_a\varphi^a + 2qk_i\varphi^i - 2q\bar{k}_{\bar{\imath}}\varphi^{\bar{\imath}} = 0, \tag{59}$$

in which one may identify an effective mass (cf. Refs. [35,36] for an interpretation in terms of a Higgs mechanism),

$$m_{\text{eff}}^2 = m^2 + q^2\varphi^a\varphi_a. \tag{60}$$

We stress that while (58) is a consequence of the non-Riemannian geometry, (60) holds on the Riemannian subspace.

The equations of motion of the photon A_μ are the following generalized Maxwell equations:

$$\begin{aligned} \partial_b f^{ba} - \partial^a\partial_i\varphi^i + \partial^a\bar{\partial}_{\bar{\imath}}\varphi^{\bar{\imath}} = J^a, \\ \partial_b\partial^b\varphi^i = J^i, \quad -\partial_b\partial^b\varphi^{\bar{\imath}} = J^{\bar{\imath}}. \end{aligned} \tag{61}$$

Here, J^μ is the usual electric current,

$$J^\mu(x) = \sum_\alpha \int d\tau\, q\dot{x}^\mu_\alpha(\tau)\delta^D(x - x_\alpha(\tau)), \tag{62}$$

which is identically conserved in the manner of (23), ensuring the consistency of (61) *à la* Maxwell in 1865.

Meanwhile, the phonon φ^μ has equations of motion

$$\begin{aligned} \partial_b u^b{}_a + \partial_a\partial_i\varphi^i + \partial_a\bar{\partial}_{\bar{\imath}}\varphi^{\bar{\imath}} = \tilde{J}_a, \\ \partial_i\partial_b\varphi^b + 2\partial_i\bar{\partial}_{\bar{\imath}}\varphi^{\bar{\imath}} + \partial_b f^b{}_i = \tilde{J}_i, \\ \bar{\partial}_{\bar{\imath}}\partial_b\varphi^b + 2\bar{\partial}_{\bar{\imath}}\partial_i\varphi^i - \partial_b f^b{}_{\bar{\imath}} = \tilde{J}_{\bar{\imath}}, \end{aligned} \tag{63}$$

for which we introduce a (by no means conserved) dual "pseudocurrent,"

$$\tilde{J}_\mu(x) = \sum_\alpha \int d\tau\, q\mathbf{D}_\tau\tilde{x}_{\alpha\mu}(\tau)\delta^D(x - x_\alpha(\tau)). \tag{64}$$

On shell, in terms of φ_a and the conjugate momenta p_i, $\bar{p}_{\bar{\imath}}$ of y^i, $\bar{y}^{\bar{\imath}}$, we can express the dual pseudocurrent without explicitly invoking the tilde coordinates,

$$\begin{aligned} \tilde{J}_a(x) &= \sum_\alpha \int d\tau eq^2\varphi_a\delta^D(x - x_\alpha(\tau)), \\ \tilde{J}_i(x) &= \sum_\alpha \int d\tau eq(p_{\alpha i} + qA_i)\,\delta^D(x - x_\alpha(\tau)), \\ \tilde{J}_{\bar{\imath}}(x) &= \sum_\alpha \int d\tau(-eq)(\bar{p}_{\alpha\bar{\imath}} + qA_{\bar{\imath}})\,\delta^D(x - x_\alpha(\tau)). \end{aligned} \tag{65}$$

Evidently, (61) and (63) generalize the usual Maxwell equations. In particular, in the absence of sources, they describe electromagnetic-strain *waves* that propagate exclusively through the Riemannian subspace, and thus are fractonic,

$$\partial_c\partial^c f_{\mu\nu} = \partial_c\partial^c\varphi^i = \partial_c\partial^c\varphi^{\bar{\imath}} = \partial_c\partial^c\partial_a\varphi^a = \partial_c\partial^c\partial_{[a}\varphi_{b]} = 0. \tag{66}$$

The first term vanishes on shell via the Bianchi identity as $\partial_c\partial^c f_{\mu\nu} = -2\partial_{[\mu}\partial^c f_{\nu]c} = 0$, while the final equality, which pertains to the "rotation tensor" $\partial_{[a}\varphi_{b]}$, follows from the more general relation $\partial_c\partial^c\varphi_a + \partial_a\partial_\lambda\varphi^\lambda = 0$.

The consistent matching between the particle action, (9) or (10), and the scalar field theory (26) generalizes to the interacting theory of a charged particle (54) and a charged complex scalar field. The Lagrangian of the complex scalar field in the fundamental representation is, with $\mathfrak{D}_A\Phi = (\partial_A + iq\mathcal{V}_A)\Phi$ and $\mathfrak{D}_\mu\Phi = (\partial_\mu + iqA_\mu)\Phi$,

$$\begin{aligned} L_{\mathcal{V},\Phi} &= -\tfrac{1}{2}\mathcal{H}^{AB}\mathfrak{D}_A\Phi\mathfrak{D}_B\Phi^\dagger - \tfrac{1}{2}m^2\Phi\Phi^\dagger \\ &= -\tfrac{1}{2}\mathfrak{D}_a\Phi\mathfrak{D}^a\Phi^\dagger - \tfrac{1}{2}(m^2 + q^2\varphi_a\varphi^a)\Phi\Phi^\dagger \\ &\quad - i\tfrac{1}{2}q\varphi^i(\Phi\mathfrak{D}_i\Phi^\dagger - \Phi^\dagger\mathfrak{D}_i\Phi) \\ &\quad + i\tfrac{1}{2}q\varphi^{\bar{\imath}}(\Phi\mathfrak{D}_{\bar{\imath}}\Phi^\dagger - \Phi^\dagger\mathfrak{D}_{\bar{\imath}}\Phi). \end{aligned} \tag{67}$$

Remarkably, the resulting equation of motion agrees with the Hamiltonian constraint of the charged particle (57), including the mass enhancement,

$$[-\mathfrak{D}_a\mathfrak{D}^a + m^2 + q^2\varphi_a\varphi^a - iq\{\mathfrak{D}_i, \varphi^i\} + iq\{\mathfrak{D}_{\bar{\imath}}, \varphi^{\bar{\imath}}\}]\Phi = 0, \tag{68}$$

while it further produces a symmetric "prescription" for the Hamiltonian constraint at the quantum level,

$$\{p_i, \varphi^i\} = p_i\varphi^i + \varphi^i p_i, \quad \{\bar{p}_{\bar{\imath}}, \varphi^{\bar{\imath}}\} = \bar{p}_{\bar{\imath}}\varphi^{\bar{\imath}} + \varphi^{\bar{\imath}}\bar{p}_{\bar{\imath}},$$

after identifying $-i\mathfrak{D}_\mu = -i\partial_\mu + qA_\mu$, with $p_\mu + qA_\mu$.

VI. ON CURVED NON-RIEMANNIAN BACKGROUNDS

Owing to the geometric $\mathbf{O}(D, D)$ formalism applicable to non-Riemannian geometries that has been developed in the literature [75,95], all results in the previous sections can be readily generalized to curved backgrounds. In the context of DFT this also includes the possibility of a nontrivial B field and dilaton. Here, we present such curved results in full generality: In doing so, it is necessary to define many covariant quantities while carefully distinguishing upper and lower D-dimensional curved indices, μ, ν. Though this may at first glance seem overelaborate, we remind and warn the reader that raising and lowering indices is generically not possible in the absence of an invertible metric.

Reference [17] and two recent works [96,97] already considered fracton physics on curved backgrounds using sub-Riemannian or Carrollian (or Aristotelian) geometries, which in the present framework would correspond to the (1,0) or $(D-1, 0)$ non-Riemannian geometries. Such scenarios, not being organized under the generalized metric, break $\mathbf{O}(D, D)$ symmetry. The consequences of this remain to be seen. Here, we merely present our $\mathbf{O}(D, D)$-symmetric, curved spacetime extension of the previous particle, scalar field, and Yang-Mills theories.

On general curved backgrounds, it is convenient to factorize out the B-field contribution via an $\mathbf{O}(D, D)$ transformation,

$$\mathcal{B}_A{}^B = \begin{pmatrix} \delta^\mu{}_\sigma & 0 \\ B_{\rho\sigma} & \delta_\rho{}^\tau \end{pmatrix}, \quad \mathcal{B}_A{}^C\mathcal{B}_B{}^D\mathcal{J}_{CD} = \mathcal{J}_{AB}. \tag{69}$$

The $\mathbf{O}(D,D)$-covariant generalized metric on a general curved background then takes the form [75]

$$\mathcal{H}_{AB}=\mathcal{B}_A{}^C\mathcal{B}_B{}^D\mathring{\mathcal{H}}_{CD},\quad \mathring{\mathcal{H}}_{AB}=\begin{pmatrix}H^{\mu\nu} & Z_\lambda{}^\mu\\ Z_\kappa{}^\nu & K_{\kappa\lambda}\end{pmatrix},\tag{70}$$

where, with $1\leqslant i\leqslant n$, $1\leqslant\bar{\imath}\leqslant\bar{n}$ as before,

$$Z_\mu{}^\nu=X^i_\mu Y^\nu_i-\bar{X}^{\bar{\imath}}_\mu\bar{Y}^\nu_{\bar{\imath}}.\tag{71}$$

The vectors $X^i_\mu,\bar{X}^{\bar{\imath}}_\mu$ and $Y^\mu_i,\bar{Y}^\mu_{\bar{\imath}}$ belong to the kernels of $H^{\mu\nu}$ and $K_{\mu\nu}$, respectively,

$$H^{\mu\nu}X^i_\nu=0=H^{\mu\nu}\bar{X}^{\bar{\imath}}_\nu,\quad K_{\mu\nu}Y^\nu_i=0=K_{\mu\nu}\bar{Y}^\nu_{\bar{\imath}},\tag{72}$$

and correspond to the $n+\bar{n}$ non-Riemannian directions. These objects satisfy a completeness relation,

$$H^{\mu\rho}K_{\rho\nu}+Y^\mu_iX^i_\nu+\bar{Y}^\mu_{\bar{\imath}}\bar{X}^{\bar{\imath}}_\nu=\delta^\mu{}_\nu.\tag{73}$$

From (72), (73), and the linear independence of the null eigenvectors, it follows that

$$X^i_\mu Y^\mu_j=\delta^i_j,\quad \bar{X}^{\bar{\imath}}_\mu\bar{Y}^\mu_{\bar{\jmath}}=\delta^{\bar{\imath}}_{\bar{\jmath}},\quad X^i_\mu\bar{Y}^\mu_{\bar{\imath}}=0=\bar{X}^{\bar{\jmath}}_\mu Y^\mu_j,$$
$$(HKH)^{\mu\nu}=H^{\mu\nu},\quad (KHK)_{\mu\nu}=K_{\mu\nu}.\tag{74}$$

Further from (69) and (70), a similar factorization of the projectors holds,

$$P_{AB}=\mathcal{B}_A{}^C\mathcal{B}_B{}^D\mathring{P}_{CD},\quad \bar{P}_{AB}=\mathcal{B}_A{}^C\mathcal{B}_B{}^D\mathring{\bar{P}}_{CD},\tag{75}$$

where $\mathring{P}_{AB}=\frac{1}{2}(\mathcal{J}+\mathring{\mathcal{H}})_{AB}$ and $\mathring{\bar{P}}_{AB}=\frac{1}{2}(\mathcal{J}-\mathring{\mathcal{H}})_{AB}$.

Remarkably, the doubled metric $\mathring{\mathcal{H}}_{AB}$ is invariant under $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ local rotations, which act on the unbarred $i,j,\dots$ and barred $\bar{\imath},\bar{\jmath},\dots$ indices, and further under the "Milne-shift" symmetry—generalizing the "Galilean boost" in the Newtonian gravity literature [98,99]—which acts with arbitrary local parameters, $V_{\mu i}$ and $\bar{V}_{\mu\bar{\imath}}$, as [75]

$$\begin{aligned}
&\delta_{\rm M}H^{\mu\nu}=0,\quad \delta_{\rm M}X^i_\mu=0,\quad \delta_{\rm M}\bar{X}^{\bar{\imath}}_\mu=0,\\
&\delta_{\rm M}Y^\mu_i=H^{\mu\nu}V_{\nu i},\quad \delta_{\rm M}\bar{Y}^\mu_{\bar{\imath}}=H^{\mu\nu}\bar{V}_{\nu\bar{\imath}},\\
&\delta_{\rm M}K_{\mu\nu}=-2X^i_{(\mu}K_{\nu)\rho}H^{\rho\sigma}V_{\sigma i}-2\bar{X}^{\bar{\imath}}_{(\mu}K_{\nu)\rho}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}},\\
&\delta_{\rm M}B_{\mu\nu}=-2X^i_{[\mu}V_{\nu]i}+2\bar{X}^{\bar{\imath}}_{[\mu}\bar{V}_{\nu]\bar{\imath}}\\
&\qquad+2X^i_{[\mu}\bar{X}^{\bar{\imath}}_{\nu]}(Y^\rho_i\bar{V}_{\rho\bar{\imath}}+\bar{Y}^\rho_{\bar{\imath}}V_{\rho i}).
\end{aligned}\tag{76}$$

In fact, both local symmetries are part of the local Lorentz symmetries in DFT and should not be broken.

First let us briefly comment on the scalar field case. Upon the generic $(n,\bar{n})$ curved background above, with the choice of section $\tilde{\partial}^\mu=0$, the scalar field kinetic term reduces to

$$-\tfrac{1}{2}e^{-2d}\mathcal{H}^{AB}\partial_A\Phi\partial_B\Phi=-\tfrac{1}{2}e^{-2d}H^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi,\tag{77}$$

which obviously generalizes (26) into a covariant form.

Now we turn to the doubled Yang-Mills theory on curved backgrounds. We first factorize the doubled gauge potential, similarly to (70), as

$$\mathcal{V}_A=\mathcal{B}_A{}^B\mathring{\mathcal{V}}_B=\begin{pmatrix}\varphi^\mu\\ A_\nu+B_{\nu\rho}\varphi^\rho\end{pmatrix},\quad \mathring{\mathcal{V}}_A=\begin{pmatrix}\varphi^\mu\\ A_\nu\end{pmatrix}.\tag{78}$$

Like the doubled metric, the doubled vector potential $\mathcal{V}_A$ should be invariant under the DFT local Lorentz symmetries, and thus the Milne-shift transformations of the component fields are

$$\delta_{\rm M}\varphi^\mu=0,\quad \delta_{\rm M}A_\mu=-(\delta_{\rm M}B_{\mu\nu})\varphi^\nu.\tag{79}$$

The semicovariant Yang-Mills field strength is

$$\begin{aligned}\mathcal{F}_{AB}&=\nabla_A\mathcal{V}_B-\nabla_B\mathcal{V}_A-i[\mathcal{V}_A,\mathcal{V}_B]\\ &=2\partial_{[A}\mathcal{V}_{B]}+2\Gamma_{[AB]}{}^C\mathcal{V}_C-i[\mathcal{V}_A,\mathcal{V}_B],\end{aligned}\tag{80}$$

where Γ_{CAB} are the the DFT Christoffel symbols [58]. From the torsionless property $\Gamma_{[ABC]}=0$, it follows that $\Gamma_{[AB]C}=-\frac{1}{2}\Gamma_{CAB}$. Since the fully covariant field strength is $(P\mathcal{F}\bar{P})_{AB}$, we only need the projection

$$P_M{}^A\bar{P}_N{}^B\Gamma_{CAB}=(P\partial_CP\bar{P})_{MN}=\tfrac{1}{2}(P\partial_C\mathcal{H}\bar{P})_{MN}.\tag{81}$$

Using (69), (70), (75), and (81) in (80), the fully covariant field strength is given by

$$\begin{aligned}&(\mathcal{B}^{-1}P)_A{}^C(\mathcal{B}^{-1}\bar{P})_B{}^D\mathcal{F}_{CD}\\ &\quad=\mathring{P}_A{}^C\mathring{\bar{P}}_B{}^D\big(2\partial_{[C}\mathring{\mathcal{V}}_{D]}-i[\mathring{\mathcal{V}}_C,\mathring{\mathcal{V}}_D]-\tfrac{1}{2}\mathring{\mathcal{V}}^E\partial_E\mathring{\mathcal{H}}_{CD}\\ &\qquad+3\mathring{\mathcal{V}}^E\partial_{[E}\mathcal{B}_{CD]}\big).\end{aligned}\tag{82}$$

This is the curved generalization of (37). After a lengthy calculation we may obtain the explicit components,

$$\begin{aligned}
&(\mathcal{B}^{-1}P\mathcal{F}(\mathcal{B}^{-1}\bar{P})^t)^{\mu\nu}=-\tfrac{1}{4}\widetilde{f}^{\kappa\lambda}(\delta_\kappa{}^\mu+Z_\kappa{}^\mu)(\delta_\lambda{}^\nu-Z_\lambda{}^\nu),\\
&(\mathcal{B}^{-1}P\mathcal{F}(\mathcal{B}^{-1}\bar{P})^t)^\mu{}_\sigma=\tfrac{1}{4}\big[H^{\mu\kappa}(\delta_\sigma{}^\lambda-Z_\sigma{}^\lambda)\widehat{f}_{\kappa\lambda}-\Upsilon_\sigma{}^\mu\big],\\
&(\mathcal{B}^{-1}P\mathcal{F}(\mathcal{B}^{-1}\bar{P})^t)_\rho{}^\nu=-\tfrac{1}{4}\big[(\delta_\rho{}^\kappa+Z_\rho{}^\kappa)H^{\nu\lambda}\widehat{f}_{\kappa\lambda}-\bar{\Upsilon}_\rho{}^\nu\big],\\
&(\mathcal{B}^{-1}P\mathcal{F}(\mathcal{B}^{-1}\bar{P})^t)_{\rho\sigma}=\tfrac{1}{4}(\delta_\rho{}^\kappa+Z_\rho{}^\kappa)(\delta_\sigma{}^\lambda-Z_\sigma{}^\lambda)\widehat{f}_{\kappa\lambda},
\end{aligned}\tag{83}$$

where the constituent tensors are now defined as

$$\begin{aligned}
\widetilde{f}^{\mu\nu}&=H^{\mu\rho}H^{\nu\sigma}\big(f_{\rho\sigma}+H_{\rho\sigma\tau}\varphi^\tau\big)+i[\varphi^\mu,\varphi^\nu]-\widetilde{u}^{\mu\nu},\\
\widehat{f}_{\mu\nu}&=f_{\mu\nu}+iK_{\mu\rho}K_{\nu\sigma}[\varphi^\rho,\varphi^\sigma]+\varphi^\rho H_{\rho\mu\nu}-u_{\mu\nu},\\
\Upsilon_\mu{}^\nu&=2K_{\mu\rho}\big(\mathcal{D}^\rho\varphi^i+i[\varphi^\rho,\varphi^\sigma]X^i_\sigma\big)Y^\nu_i+4\bar{X}^{\bar{\imath}}_\mu\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\rho X^i_\rho Y^\nu_i,\\
\bar{\Upsilon}_\mu{}^\nu&=2K_{\mu\rho}\big(\bar{\mathcal{D}}^\rho\varphi^{\bar{\imath}}-i[\varphi^\rho,\varphi^\sigma]\bar{X}^{\bar{\imath}}_\sigma\big)\bar{Y}^\nu_{\bar{\imath}}+4X^i_\mu\mathcal{D}_i\varphi^\rho\bar{X}^{\bar{\imath}}_\rho\bar{Y}^\nu_{\bar{\imath}}.
\end{aligned}\tag{84}$$

Further explanation is in order. Every term in (84) is covariant under (ordinary undoubled) diffeomorphisms and Yang-Mills gauge symmetry, and also invariant under $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ local rotations, but not under the Milne shift: Only the whole set of components of $(P\mathcal{F}\bar{P})_{AB}$ is so. Specifically, $f_{\mu\nu}$ is the usual Yang-Mills field strength (36), while $\widetilde{u}^{\mu\nu}=\widetilde{u}^{\nu\mu}$, $u_{\mu\nu}=u_{\nu\mu}$ are the *strain tensors* for the vector field φ^λ, carrying upper or lower symmetric indices, which can be expressed as symmetrizations of appropriate covariant derivatives,

$$\widetilde{u}^{\mu\nu}=\mathfrak{D}^\mu\varphi^\nu+\mathfrak{D}^\nu\varphi^\mu,\quad u_{\mu\nu}=\mathcal{D}_\mu\varphi_\nu+\mathcal{D}_\nu\varphi_\mu,\tag{85}$$

where, with $\Omega^{\mu\nu}{}_\rho$ to be explained later (91),

$$\begin{aligned}
\mathfrak{D}^\mu\varphi^\nu&=H^{\mu\rho}(\partial_\rho\varphi^\nu-i[A_\rho,\varphi^\nu])+\Omega^{\mu\nu}{}_\rho\varphi^\rho,\\
\mathcal{D}_\mu\varphi_\nu&=(\partial_\mu\varphi^\rho-i[A_\mu,\varphi^\rho])K_{\rho\nu}\\
&\quad+\tfrac{1}{2}(\partial_\rho K_{\mu\nu}+\partial_\mu K_{\nu\rho}-\partial_\nu K_{\mu\rho})\varphi^\rho.
\end{aligned}\tag{86}$$

Furthermore, we have defined various covariant derivatives which are nontrivial only in genuine non-Riemannian cases, i.e., $(n,\bar{n})\neq(0,0)$:

$$\begin{aligned}
\mathcal{D}^\mu\varphi^i &= H^{\mu\rho}\big((\partial_\rho\varphi^\sigma - i[A_\rho,\varphi^\sigma])X_\sigma^i + \varphi^\sigma\partial_\sigma X_\rho^i\big),\\
\bar{\mathcal{D}}^\mu\varphi^{\bar{\imath}} &= H^{\mu\rho}\big((\partial_\rho\varphi^\sigma - i[A_\rho,\varphi^\sigma])\bar{X}_\sigma^{\bar{\imath}} + \varphi^\sigma\partial_\sigma \bar{X}_\rho^{\bar{\imath}}\big),\\
\mathcal{D}_i\varphi^\mu\bar{X}_\mu^{\bar{\imath}} &= \big(Y_i^\rho(\partial_\rho\varphi^\mu - i[A_\rho,\varphi^\mu]) - \varphi^\rho\partial_\rho Y_i^\mu\big)\bar{X}_\mu^{\bar{\imath}},\\
\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\mu X_\mu^i &= \big(\bar{Y}_{\bar{\imath}}^\rho(\partial_\rho\varphi^\mu - i[A_\rho,\varphi^\mu]) - \varphi^\rho\partial_\rho \bar{Y}_{\bar{\imath}}^\mu\big)X_\mu^i. \qquad (87)
\end{aligned}$$

As alternatives to the latter two expressions in (87), we can also write

$$\begin{aligned}
\mathcal{D}_i\varphi^\lambda K_{\lambda\mu} &= \big(Y_i^\rho\big(\partial_\rho\varphi^\lambda - i[A_\rho,\varphi^\lambda]\big) - \varphi^\rho\partial_\rho Y_i^\lambda\big)K_{\lambda\mu},\\
\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\lambda K_{\lambda\mu} &= \big(\bar{Y}_{\bar{\imath}}^\rho\big(\partial_\rho\varphi^\lambda - i[A_\rho,\varphi^\lambda]\big) - \varphi^\sigma\partial_\sigma \bar{Y}_{\bar{\imath}}^\lambda\big)K_{\lambda\mu}. \qquad (88)
\end{aligned}$$

All expressions for covariant derivatives given in (86)–(88) are symmetric under diffeomorphisms, local $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ rotations, and Yang-Mills gauge transformations. However, without the contractions as in (87) and (88), the bare derivatives

$$\begin{aligned}
\mathcal{D}_i\varphi^\mu &= Y_i^\rho(\partial_\rho\varphi^\mu - i[A_\rho,\varphi^\mu]) - \varphi^\rho\partial_\rho Y_i^\mu,\\
\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\mu &= \bar{Y}_{\bar{\imath}}^\rho(\partial_\rho\varphi^\mu - i[A_\rho,\varphi^\mu]) - \varphi^\rho\partial_\rho \bar{Y}_{\bar{\imath}}^\mu \qquad (89)
\end{aligned}$$

are anomalous under local $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ rotations due to the final terms containing derivatives of Y_i^μ and $\bar{Y}_{\bar{\imath}}^\mu$. Among these definitions, note that the upper indexed covariant derivative,

$$\mathfrak{D}^\mu = H^{\mu\rho}\partial_\rho - iH^{\mu\rho}[A_\rho,\ \ ] + \Omega^\mu, \qquad (90)$$

was proposed in Ref. [95]. It can act on an arbitrary (undoubled) tensor, as it is equipped with a generalized Christoffel connection,

$$\begin{aligned}
\Omega^{\mu\nu}{}_\lambda = &-\tfrac{1}{2}\partial_\lambda H^{\mu\nu} - H^{\rho[\mu}\partial_\rho Y_i^{\nu]}X_\lambda^i - H^{\rho[\mu}\partial_\rho \bar{Y}_{\bar{\imath}}^{\nu]}\bar{X}_\lambda^{\bar{\imath}}\\
&-H^{\rho[\mu}\partial_\rho H^{\nu]\sigma}K_{\sigma\lambda} + \big(2H^{\rho[\mu}Y_i^{\nu]}\partial_{[\tau}X_{\rho]}^i\\
&-2H^{\rho[\mu}\bar{Y}_{\bar{\imath}}^{\nu]}\partial_{[\tau}\bar{X}_{\rho]}^{\bar{\imath}}\big)\big(Y_j^\tau X_\lambda^j - \bar{Y}_{\bar{\jmath}}^\tau\bar{X}_\lambda^{\bar{\jmath}}\big). \qquad (91)
\end{aligned}$$

Note that according to (85), only the symmetric part, $\Omega^{(\mu\nu)}{}_\lambda = -\frac{1}{2}\partial_\lambda H^{\mu\nu}$, contributes to the strain tensor $\widetilde{u}^{\mu\nu}$.

The Lagrangian for the doubled Yang-Mills theory on general curved and non-Riemannian backgrounds,

$$L_{\rm YM} = 2\,{\rm Tr}\big[(P\mathcal{F}\bar{P})^{\mu\nu}(P\mathcal{F}\bar{P})_{\mu\nu} + (P\mathcal{F}\bar{P})^\mu{}_\nu(P\mathcal{F}\bar{P})_\mu{}^\nu\big], \qquad (92)$$

can be obtained analogously to the flat case from the components in (83). The full result is

$$L_{\rm YM} = {\rm Tr}\begin{bmatrix} -\frac{1}{4}H^{\mu\rho}H^{\nu\sigma}\big(f_{\mu\nu}+\varphi^\kappa H_{\kappa\mu\nu}\big)\big(f_{\rho\sigma}+\varphi^\lambda H_{\lambda\rho\sigma}\big) - \frac{1}{4}H^{\mu\rho}H^{\nu\sigma}u_{\mu\nu}u_{\rho\sigma} + \frac{1}{4}K_{\mu\rho}K_{\nu\sigma}[\varphi^\mu,\varphi^\nu][\varphi^\rho,\varphi^\sigma]\\ -\big\{\mathcal{D}^\mu\varphi^iY_i^\nu - \bar{\mathcal{D}}^\mu\varphi^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu + i\frac{1}{2}[\varphi^\rho,\varphi^\sigma](1+Z)_\rho{}^\mu(1-Z)_\sigma{}^\nu\big\}\big(f_{\mu\nu}+\varphi^\kappa H_{\kappa\mu\nu}\big)\\ -K_{\mu\nu}\mathcal{D}_i\varphi^\mu\big(\mathcal{D}^\nu\varphi^i + i[\varphi^\nu,\varphi^\rho]X_\rho^i\big) - K_{\mu\nu}\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\mu\big(\bar{\mathcal{D}}^\nu\varphi^{\bar{\imath}} - i[\varphi^\nu,\varphi^\rho]\bar{X}_\rho^{\bar{\imath}}\big) - 2\mathcal{D}_i\varphi^\mu\bar{X}_\mu^{\bar{\imath}}\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\nu X_\nu^i \end{bmatrix}. \qquad (93)$$

By construction, this action is invariant under diffeomorphisms, B-field and Yang-Mills gauge symmetries, $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ local rotations, and Milne shifts (76) and (79). This follows naturally in the doubled formalism but appears nontrivial from the undoubled perspective.

In deriving the action, it is worthwhile to note the following relations among the derivatives $\mathcal{D}^\mu\varphi^i$, $\bar{\mathcal{D}}^\mu\varphi^{\bar{\imath}}$, and $\mathfrak{D}^\mu\varphi^\rho$,

$$\begin{aligned}
&(\mathfrak{D}^\mu\varphi^\rho)X_\rho^iY_i^\nu - (\mathcal{D}^\mu\varphi^i)Y_i^\nu\\
&\quad= H^{\mu\rho}\partial_{[\rho}X_{\sigma]}^i\big(H^{\sigma\kappa}K_{\kappa\lambda} + 2\bar{Y}_{\bar{\imath}}^\sigma\bar{X}_\lambda^{\bar{\imath}}\big)\varphi^\lambda Y_i^\nu,\\
&(\mathfrak{D}^\mu\varphi^\rho)\bar{X}_\rho^{\bar{\imath}}\bar{Y}_{\bar{\imath}}^\nu - (\bar{\mathcal{D}}^\mu\varphi^{\bar{\imath}})\bar{Y}_{\bar{\imath}}^\nu\\
&\quad= H^{\mu\rho}\partial_{[\rho}\bar{X}_{\sigma]}^{\bar{\imath}}\big(H^{\sigma\kappa}K_{\kappa\lambda} + 2Y_i^\sigma X_\lambda^i\big)\varphi^\lambda \bar{Y}_{\bar{\imath}}^\nu, \qquad (94)
\end{aligned}$$

as well as some projection properties,

$$\begin{aligned}
H^{\mu\rho}H^{\nu\sigma}\widehat{f}_{\rho\sigma} &= (HK)^\mu{}_\rho(HK)^\nu{}_\sigma\widetilde{f}^{\rho\sigma},\\
H^{\mu\rho}H^{\nu\sigma}u_{\rho\sigma} &= (HK)^\mu{}_\rho(HK)^\nu{}_\sigma\widetilde{u}^{\rho\sigma},\\
H^{\mu\rho}u_{\rho\sigma}Y_i^\sigma &= (HK)^\mu{}_\rho\mathcal{D}_i\varphi^\rho,\\
H^{\mu\rho}u_{\rho\sigma}\bar{Y}_{\bar{\imath}}^\sigma &= (HK)^\mu{}_\rho\bar{\mathcal{D}}_{\bar{\imath}}\varphi^\rho \qquad (95)
\end{aligned}$$

and

$$\begin{aligned}
&K_{\mu\rho}\widetilde{u}^{\rho\sigma}X_\sigma^i = K_{\mu\rho}\mathcal{D}^\rho\varphi^i, \quad K_{\mu\rho}\widetilde{u}^{\rho\sigma}\bar{X}_\sigma^{\bar{\imath}} = K_{\mu\rho}\bar{\mathcal{D}}^\rho\varphi^{\bar{\imath}},\\
&\Upsilon_\mu{}^\nu K_{\nu\rho} = 0 = \bar{\Upsilon}_\mu{}^\nu K_{\nu\rho}, \quad Y_i^\mu u_{\mu\nu}\bar{Y}_{\bar{\imath}}^\nu = 0 = X_\mu^i\widetilde{u}^{\mu\nu}\bar{X}_\nu^{\bar{\imath}}. \qquad (96)
\end{aligned}$$

Upon taking the flat-spacetime limit (33) with vanishing B field, the connection Ω vanishes and the covariant derivatives $\mathfrak{D}$, $\mathcal{D}$ all reduce to the Yang-Mills covariant derivative, $D = \partial - i[A,\ \ ]$. Thus the field strength (83) and (84) and the strain tensor (85) simplify to (37), (38), and (42), respectively, while from the Lagrangian (93) we recover (41).

Finally, as a curved spacetime generalization of (54), we present the full particle action minimally coupled to the doubled Maxwell vector potential on a generic $(n,\bar{n})$ curved background,

$$\begin{aligned}
\mathcal{L}_q &= \frac{1}{2}e^{-1}{\rm D}_\tau x^A{\rm D}_\tau x^B\mathcal{H}_{AB} - \tfrac{1}{2}em^2 - q{\rm D}_\tau x^A\mathcal{V}_A\\
&= \frac{1}{2}e^{-1}\dot{x}^\mu\dot{x}^\nu K_{\mu\nu} - \tfrac{1}{2}e\big(m^2 + q^2\varphi^\mu\varphi^\nu K_{\mu\nu}\big) - q\dot{x}^\mu A_\mu\\
&\quad+\frac{1}{2}e^{-1}(\dot{\tilde{x}}_\mu - \mathbf{a}_\mu - B_{\mu\kappa}\dot{x}^\kappa - eqK_{\mu\rho}\varphi^\rho)H^{\mu\nu}\\
&\quad\times\big(\dot{\tilde{x}}_\nu - \mathbf{a}_\nu - B_{\nu\lambda}\dot{x}^\lambda - eqK_{\nu\sigma}\varphi^\sigma\big)\\
&\quad+X_\mu^i\big(e^{-1}\dot{x}^\mu - q\varphi^\mu\big)(\dot{\tilde{x}}_\nu - \mathbf{a}_\nu - B_{\nu\rho}\dot{x}^\rho)Y_i^\nu\\
&\quad-\bar{X}_\mu^{\bar{\imath}}\big(e^{-1}\dot{x}^\mu + q\varphi^\mu\big)(\dot{\tilde{x}}_\nu - \mathbf{a}_\nu - B_{\nu\rho}\dot{x}^\rho)\bar{Y}_{\bar{\imath}}^\nu. \qquad (97)
\end{aligned}$$

The first line on the right-hand side of the second equality is essentially the usual (undoubled) action for a charged "relativistic" particle in the Riemannian subspace, with effective mass generated by the displacement vector field,

$$m_{\rm eff}^2 = m^2 + q^2\varphi^\mu\varphi^\nu K_{\mu\nu}. \qquad (98)$$

The second and third lines are quadratic in the auxiliary variable $\mathbf{a}_\mu$ on the Riemannian subspace and hence are to be negligibly integrated out, resulting in

$$H^{\mu\nu}\big(\dot{\tilde{x}}_\nu - \mathbf{a}_\nu - B_{\nu\rho}\dot{x}^\rho - eqK_{\nu\rho}\varphi^\rho\big) = 0. \tag{99}$$

The last two lines are linear in $\mathbf{a}_\mu$ and hence impose constraints,

$$X^i_\mu(e^{-1}\dot{x}^\mu - q\varphi^\mu) = 0, \quad \bar{X}^{\bar{\imath}}_\mu(e^{-1}\dot{x}^\mu + q\varphi^\mu) = 0, \tag{100}$$

which can be unified into a single expression,

$$e^{-1}\dot{x}^\rho Z_\rho{}^\mu = q\varphi^\rho(1 - KH)_\rho{}^\mu. \tag{101}$$

It is worthwhile to note the conjugate momenta of x^μ,

$$\begin{aligned} p_\mu = {}& e^{-1}\big(K_{\mu\nu}+B_{\mu\rho}Z_\nu{}^\rho\big)\dot{x}^\nu - qA_\mu - qB_{\mu\nu}\big(Y_i^\nu X^i_\rho + \bar{Y}_{\bar{\imath}}^\nu \bar{X}^{\bar{\imath}}_\rho\big)\varphi^\rho \\ &+ e^{-1}\big(B_{\mu\rho}H^{\rho\nu} + Z_\mu{}^\nu\big)\big(\dot{\tilde{x}}_\nu - \mathbf{a}_\nu - B_{\nu\sigma}\dot{x}^\sigma - eqK_{\nu\sigma}\varphi^\sigma\big), \end{aligned} \tag{102}$$

and some on-shell values for the tilde coordinates,

$$e^{-1}(\mathrm{D}_\tau\tilde{x}_\mu - B_{\mu\nu}\dot{x}^\nu) = Z_\mu{}^\nu(p_\nu + qA_\nu) + qK_{\mu\nu}\varphi^\nu, \tag{103}$$

where actually only the momenta along the non-Riemannian directions, p_i, $\bar{p}_{\bar{\imath}}$, are relevant.

Clearly, (100) generalizes the saturation velocity (58) as well as the immobility constraint (11) of the constant background (8) to the case of a curved background. Specifically, for a neutral particle of $q = 0$, from (100) we obtain the vanishing of the velocity along the (curved non-Riemannian) X^i_μ and $\bar{X}^{\bar{\imath}}_\nu$ directions,

$$X^i_\mu\dot{x}^\mu = 0, \quad \bar{X}^{\bar{\imath}}_\mu\dot{x}^\mu = 0. \tag{104}$$

Taking the τ derivative of these gives expressions that may be viewed as "non-Riemannian geodesic equations,"

$$X^i_\mu\ddot{x}^\mu + \partial_{(\mu}X^i_{\nu)}\dot{x}^\mu\dot{x}^\nu = 0, \quad \bar{X}^{\bar{\imath}}_\mu\ddot{x}^\mu + \partial_{(\mu}\bar{X}^{\bar{\imath}}_{\nu)}\dot{x}^\mu\dot{x}^\nu = 0, \tag{105}$$

For genuine curved backgrounds where $\partial_{(\mu}X^i_{\nu)}$ or $\partial_{(\mu}\bar{X}^{\bar{\imath}}_{\nu)}$ are nontrivial, this indicates that the immobility of a fracton is rather nontrivial. Related to this, it is worthwhile to note that the first curved non-Riemannian DFT background reported in Ref. [72] was shown in Ref. [87] to admit only a finite number of isometries, implying the absence of higher-moment conservations. Further investigation with more examples is desired.

VII. DISCUSSION

Existing field theoretical intuition on fractons is largely based on dipole conservation for charged particles. Contradistinctly, in our scheme the immobility is universal regardless of charge, since it originates from the "geodesic" particle action (9). Accordingly, our current (17) contains the energy-momentum tensor rather than a charge density of any sort.

Analysis on a spinor field is also possible, following Refs. [60,63]. We merely comment that, since DFT vielbeins square to projectors like $V_A{}^p V_{Bp} = P_{AB}$, on the flat background (8) the doubled Dirac equation

$$\gamma\psi = V^A{}_p\gamma^p\partial_A\psi = 0 \tag{106}$$

gives

$$0 = (\gamma)^2\psi = P^{AB}\partial_A\partial_B\psi = \tfrac{1}{2}\mathcal{H}^{AB}\partial_A\partial_B\psi = \tfrac{1}{2}\partial_a\partial^a\psi.$$

Thus the massless spinor is also fractonic, like (66). Dualization of the full strain-Maxwell model including the non-Riemannian sector (56) and also the connection to D-branes, following Refs. [26–34,91,100] and Ref. [23], respectively, deserve further study.

The charged particle action (55) and (97) is of interest even upon a genuine Riemannian or Minkowskian (0,0) flat background,

$$S^{(0,0)}_q = \int d\tau\,\frac{1}{2}e^{-1}\dot{x}^a\dot{x}_a - \frac{1}{2}e(m^2 + q^2\varphi^a\varphi_a) - q\dot{x}^aA_a. \tag{107}$$

Minimally coupled to the doubled vector potential $\mathcal{V}_A$, this particle action naturally interacts with the Maxwell vector potential of photons A_a, and further with the elasticity displacement vector of phonons φ^b, satisfying, from (61), (63), (62), and (65),

$$\begin{aligned} \partial_c f^{ca} = J^a, \quad & J^a(x) = \sum_\alpha\int d\tau\, q\dot{x}^a_\alpha(\tau)\delta^D(x - x_\alpha(\tau)), \\ \partial_c u^c{}_a = \tilde{J}_a, \quad & \tilde{J}_a(x) = \sum_\alpha\int d\tau\, eq^2\varphi_a\delta^D(x - x_\alpha(\tau)). \end{aligned} \tag{108}$$

This set of equations may provide an effective description of polarons [101–104]. Deformations of a periodic potential of a crystal lattice are described by phonons, or the displacements of atoms from their equilibrium positions. Electrons moving inside the crystal interact with the displacements, which is known as electron-phonon coupling. Such electrons with the accompanying deformation are called polarons. They move freely across the crystal, but with increased effective mass. This polaron picture essentially agrees with (107) and (108) above. The charged particles can correspond to both atomic nuclei and electrons. From (108), the lattice structure of the atomic nuclei naturally sets the dual pseudocurrent $\tilde{J}_a$ and also the strain tensor u^{ab} to be discretely crystallized on the lattice, while the electrons acquire an effective mass (107) from the condensation of the displacement vector φ^a. We recall the effective mass formula (60) and expand the square root,

$$\begin{aligned} m_{\text{eff}} &= \sqrt{m^2 + q^2\varphi^a\varphi_a} \\ &= m\Big[1 + \frac{1}{2}\Big(\frac{q}{m}\Big)^2\varphi^a\varphi_a - \frac{1}{8}\Big(\frac{q}{m}\Big)^4\big(\varphi^a\varphi_a\big)^2 + \cdots\Big]. \end{aligned} \tag{109}$$

We compare this with a well-known formula for the effective mass of a polaron obtained from estimating its self-energy [105,106],

$$m_{\text{known}} \simeq m\big[1 + \tfrac{1}{6}\alpha_{\text{e-ph}} + 0.0236(\alpha_{\text{e-ph}})^2\big], \tag{110}$$

where $\alpha_{\text{e-ph}}$ is a dimensionless electron-phonon coupling constant. From the leading-order terms in the two formulas, we identify $(\frac{q}{m})^2\varphi^a\varphi_a = \frac{1}{3}\alpha_{\text{e-ph}}$, which in turn gives, from (109), $m_{\text{eff}}/m \simeq 1 + \frac{1}{6}\alpha_{\text{e-ph}} - \frac{1}{72}(\alpha_{\text{e-ph}})^2$ and hence differs from (110) at subleading order. We call for experimental verification.

For the Riemannian subspace we have mostly envisaged a Minkowskian signature (8), such that time can flow without suffering from immobility and that the effective mass (60) is not necessarily bigger than the true mass. Intriguingly then, in the case of time crystals [107], where $\varphi^a\varphi_a$ would be timelike or negative, our formula seems to predict that the effective mass of polarons in time crystals should become smaller. Note that a time crystal is a quantum system of particles for which the ground state is characterized by repetitive periodic motion of the particles.

On the other hand, if we were to choose the Euclidean signature $\eta_{ab} = \delta_{ab}$ for the Riemannian subspace and let $(n, \bar{n}) = (1, 1)$, thereby including two non-Riemannian directions, one temporal and the other spatial [73,75,77–79,87,95,108,109], the corresponding fracton physics would match that of the nonrelativistic string [80,81,110] and Newton-Cartan gravities [86,111–117]. Equation (58) then further implies that time therein can start to flow if the timelike displacement vector condenses, setting

$$\dot{t} = eq\varphi^{t}. \tag{111}$$

It would be of utmost interest if any of the non-Riemannian geometries underlying the modified Maxwell equations (61) and (63) are realized in nature. Some well-known singularities in GR [118–120] have recently been identified as regular non-Riemannian geometries [121]. Approaching them, geodesics indeed become immobile. Extra dimensions, if any, might be non-Riemannian [75] and therefore fractonic.

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APPENDIX: DOUBLED YANG-MILLS ENERGY-MOMENTUM TENSOR ON A CONSTANT FLAT BACKGROUND

In this Appendix, we write down the explicit components of the energy-momentum tensor (32) for the doubled Yang-Mills theory (30). In terms of undoubled spacetime indices, the relevant pieces appearing in the on-shell conserved current (17) are

$$\begin{aligned}
T^{\mu\nu} &= \mathrm{Tr}\big[(\mathcal{FHF})^{\mu}{}_{\rho}H^{\rho\nu} - H^{\mu\rho}(\mathcal{FHF})_{\rho}{}^{\nu} + (\mathcal{FHF})^{\mu\rho}Z_{\rho}{}^{\nu} - Z_{\rho}{}^{\mu}(\mathcal{FHF})^{\rho\nu}\big] - 2\partial_{\lambda}\mathrm{Tr}\big[\varphi^{\lambda}\big(P\mathcal{F}\bar{P}+\bar{P}\mathcal{F}P\big)^{\mu\nu}\big],\\
T^{\mu}{}_{\nu} &= \mathrm{Tr}\big[(\mathcal{FHF})^{\mu\rho}K_{\rho\nu} - H^{\mu\rho}(\mathcal{FHF})_{\rho\nu} + (\mathcal{FHF})^{\mu}{}_{\rho}Z_{\nu}{}^{\rho} - Z_{\rho}{}^{\mu}(\mathcal{FHF})^{\rho}{}_{\nu}\big] - 2\partial_{\lambda}\mathrm{Tr}\big[\varphi^{\lambda}\big(P\mathcal{F}\bar{P}+\bar{P}\mathcal{F}P\big)^{\mu}{}_{\nu}\big] + \delta^{\mu}{}_{\nu}L_{\mathrm{YM}}.
\end{aligned} \tag{A1}$$

To evaluate these, we need the explicit expressions

$$\begin{aligned}
(\mathcal{FHF})^{\mu\nu} &= -D_{\rho}\varphi^{\mu}H^{\rho\sigma}D_{\sigma}\varphi^{\nu} + iZ_{\rho}{}^{\sigma}\big(D_{\sigma}\varphi^{\mu}[\varphi^{\rho},\varphi^{\nu}] - [\varphi^{\mu},\varphi^{\rho}]D_{\sigma}\varphi^{\nu}\big) - [\varphi^{\mu},\varphi^{\rho}]K_{\rho\sigma}[\varphi^{\sigma},\varphi^{\nu}],\\
(\mathcal{FHF})^{\mu}{}_{\nu} &= -D_{\rho}\varphi^{\mu}H^{\rho\sigma}f_{\sigma\nu} + i[\varphi^{\mu},\varphi^{\rho}]\big(K_{\rho\sigma}D_{\nu}\varphi^{\sigma} - Z_{\rho}{}^{\sigma}f_{\sigma\nu}\big) + Z_{\rho}{}^{\sigma}D_{\sigma}\varphi^{\mu}D_{\nu}\varphi^{\rho},\\
(\mathcal{FHF})_{\mu}{}^{\nu} &= f_{\mu\rho}H^{\rho\sigma}D_{\sigma}\varphi^{\nu} - i\big(D_{\mu}\varphi^{\rho}K_{\rho\sigma} + f_{\mu\rho}Z_{\sigma}{}^{\rho}\big)[\varphi^{\sigma},\varphi^{\nu}] + D_{\mu}\varphi^{\rho}Z_{\rho}{}^{\sigma}D_{\sigma}\varphi^{\nu},\\
(\mathcal{FHF})_{\mu\nu} &= f_{\mu\rho}H^{\rho\sigma}f_{\sigma\nu} - D_{\mu}\varphi^{\rho}D_{\nu}\varphi^{\sigma}K_{\rho\sigma} + Z_{\rho}{}^{\sigma}\big(D_{\mu}\varphi^{\rho}f_{\sigma\nu} - f_{\mu\sigma}D_{\nu}\varphi^{\rho}\big),
\end{aligned} \tag{A2}$$

as well as

$$\begin{aligned}
-2\big(P\mathcal{F}\bar{P}+\bar{P}\mathcal{F}P\big)^{\mu\nu} &= H^{\mu\rho}H^{\nu\sigma}f_{\rho\sigma} + 2H^{\rho[\mu}Z_{\sigma}{}^{\nu]}D_{\rho}\varphi^{\sigma} + i[\varphi^{\mu},\varphi^{\nu}] - i[\varphi^{\rho},\varphi^{\sigma}]Z_{\rho}{}^{\mu}Z_{\sigma}{}^{\nu},\\
-2\big(P\mathcal{F}\bar{P}+\bar{P}\mathcal{F}P\big)^{\mu}{}_{\nu} &= D_{\nu}\varphi^{\mu} + H^{\mu\rho}D_{\rho}\varphi^{\sigma}K_{\sigma\nu} - i[\varphi^{\rho},\varphi^{\sigma}]Z_{\rho}{}^{\mu}K_{\sigma\nu} + \big(H^{\mu\rho}f_{\rho\sigma} - Z_{\rho}{}^{\mu}D_{\sigma}\varphi^{\rho}\big)Z_{\nu}{}^{\sigma}.
\end{aligned} \tag{A3}$$

Substituting these into (A1), we acquire all the components of the doubled Yang-Mills energy-momentum tensor,

$$\begin{aligned}
T^{a}{}_{b} &= \mathrm{Tr}\big[f^{ac}f_{bc} + D^{a}\varphi^{c}D_{b}\varphi_{c} - D_{c}\varphi^{a}D^{c}\varphi_{b} + [\varphi^{a},\varphi^{c}][\varphi_{b},\varphi_{c}] + \partial_{\lambda}\big(\varphi^{\lambda}u^{a}{}_{b}\big)\\
&\quad - 2Z_{\rho}{}^{\sigma}\big(f^{(a}{}_{\sigma}D^{c)}\varphi^{\rho} + iD_{\sigma}\varphi^{(a}[\varphi^{c)},\varphi^{\rho}]\big)\eta_{cb}\big] + \delta^{a}{}_{b}L_{\mathrm{YM}},\\
T^{a}{}_{i} &= \mathrm{Tr}\big[\big(f^{ac}+D^{c}\varphi^{a}\big)f_{ic} + D^{-a}\varphi^{c}D_{i}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}f^{-a}{}_{i}\big) + Z_{\rho}{}^{\sigma}\big(f^{-a}{}_{\sigma}D_{i}\varphi^{\rho} + D^{-a}\varphi^{\rho}f_{i\sigma}\big)\big],\\
T^{a}{}_{\bar{\imath}} &= \mathrm{Tr}\big[\big(f^{ac}-D^{c}\varphi^{a}\big)f_{\bar{\imath}c} + D^{+a}\varphi^{c}D_{\bar{\imath}}\varphi_{c} - \partial_{\lambda}\big(\varphi^{\lambda}f^{+a}{}_{\bar{\imath}}\big) + Z_{\rho}{}^{\sigma}\big(f^{+a}{}_{\sigma}D_{\bar{\imath}}\varphi^{\rho} + D^{+a}\varphi^{\rho}f_{\bar{\imath}\sigma}\big)\big],\\
T^{i}{}_{a} &= \mathrm{Tr}\big[-D^{c}\varphi^{i}\big(D_{c}\varphi_{a}+f_{ac}\big) - i[\varphi^{i},\varphi^{c}]D^{-}_{a}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}D^{-}_{a}\varphi^{i}\big) - Z_{\rho}{}^{\sigma}\big(D_{\sigma}\varphi^{i}D^{-}_{a}\varphi^{\rho} + i[\varphi^{i},\varphi^{\rho}]f^{-}_{a\sigma}\big)\big],\\
T^{\bar{\imath}}{}_{a} &= \mathrm{Tr}\big[-D^{c}\varphi^{\bar{\imath}}\big(D_{c}\varphi_{a}+f_{ca}\big) + i[\varphi^{\bar{\imath}},\varphi^{c}]D^{+}_{a}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}D^{+}_{a}\varphi^{\bar{\imath}}\big) + Z_{\rho}{}^{\sigma}\big(D_{\sigma}\varphi^{\bar{\imath}}D^{+}_{a}\varphi^{\rho} + i[\varphi^{\bar{\imath}},\varphi^{\rho}]f^{+}_{a\sigma}\big)\big],\\
T^{i}{}_{\bar{\imath}} &= 2\mathrm{Tr}\big[D^{c}\varphi^{i}f_{c\bar{\imath}} - i[\varphi^{i},\varphi^{c}]D_{\bar{\imath}}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}D_{\bar{\imath}}\varphi^{i}\big) - Z_{\rho}{}^{\sigma}\big(D_{\sigma}\varphi^{i}D_{\bar{\imath}}\varphi^{\rho} + i[\varphi^{i},\varphi^{\rho}]f_{\sigma\bar{\imath}}\big)\big],\\
T^{\bar{\imath}}{}_{i} &= 2\mathrm{Tr}\big[-D^{c}\varphi^{\bar{\imath}}f_{ci} + i[\varphi^{\bar{\imath}},\varphi^{c}]D_{i}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}D_{i}\varphi^{\bar{\imath}}\big) + Z_{\rho}{}^{\sigma}\big(D_{\sigma}\varphi^{\bar{\imath}}D_{i}\varphi^{\rho} - i[\varphi^{\bar{\imath}},\varphi^{\rho}]f_{\sigma i}\big)\big],
\end{aligned} \tag{A4}$$

$$\begin{aligned}
T^{ab} &= \mathrm{Tr}\big[2D_{c}\varphi^{[a}f^{b]c} - 2i[\varphi^{c},\varphi^{[a}]D^{b]}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}\big(f^{ab} + i[\varphi^{a},\varphi^{b}]\big)\big) + 2Z_{\rho}{}^{\sigma}\big(D_{\sigma}\varphi^{[a}D^{b]}\varphi^{\rho} - i[\varphi^{\rho},\varphi^{[a}]f^{b]}{}_{\sigma}\big)\big],\\
T^{ai} &= \mathrm{Tr}\big[-\big(f^{ac}+D^{c}\varphi^{a}\big)D_{c}\varphi^{i} - i[\varphi^{i},\varphi^{c}]D^{-a}\varphi_{c} + \partial_{\lambda}\big(\varphi^{\lambda}D^{-a}\varphi^{i}\big) - Z_{\rho}{}^{\sigma}\big(D^{-a}\varphi^{\rho}D_{\sigma}\varphi^{i} - i[\varphi^{\rho},\varphi^{i}]f^{-a}{}_{\sigma}\big)\big],\\
T^{a\bar{\imath}} &= \mathrm{Tr}\big[-\big(f^{ac}-D^{c}\varphi^{a}\big)D_{c}\varphi^{\bar{\imath}} - i[\varphi^{\bar{\imath}},\varphi^{c}]D^{+a}\varphi_{c} - \partial_{\lambda}\big(\varphi^{\lambda}D^{+a}\varphi^{\bar{\imath}}\big) - Z_{\rho}{}^{\sigma}\big(D^{+a}\varphi^{\rho}D_{\sigma}\varphi^{\bar{\imath}} - i[\varphi^{\rho},\varphi^{\bar{\imath}}]f^{+a}{}_{\sigma}\big)\big],\\
T^{i\bar{\imath}} &= 2\mathrm{Tr}\big[D^{c}\varphi^{i}D_{c}\varphi^{\bar{\imath}} - [\varphi^{i},\varphi^{c}][\varphi^{\bar{\imath}},\varphi_{c}] + i\partial_{\lambda}\big(\varphi^{\lambda}[\varphi^{i},\varphi^{\bar{\imath}}]\big) + 2iZ_{\rho}{}^{\sigma}D_{\sigma}\varphi^{(i}[\varphi^{\bar{\imath})},\varphi^{\rho}]\big],
\end{aligned} \tag{A5}$$

and, from the projection properties,

$$\begin{aligned}&T^{ij}=0, \quad T^{\bar{i}\bar{j}}=0, \quad T^{i}{}_{j}=\delta^{i}{}_{j}L_{\mathrm{YM}}, \quad T^{\bar{i}}{}_{\bar{j}}=\delta^{\bar{i}}{}_{\bar{j}}L_{\mathrm{YM}},\\ &T^{ia}=-\eta^{ab}T^{i}{}_{b}, \quad T^{\bar{i}a}=\eta^{ab}T^{\bar{i}}{}_{b}, \quad T^{\bar{i}i}=-T^{i\bar{i}}, \quad T^{a}{}_{c}\eta^{cb}=T^{b}{}_{c}\eta^{ca}.\end{aligned} \tag{A6}$$

[1] C. Chamon, Quantum Glassiness, Phys. Rev. Lett. **94**, 040402 (2005).

[2] J. Haah, Local stabilizer codes in three dimensions without string logical operators, Phys. Rev. A **83**, 042330 (2011).

[3] S. Vijay, J. Haah, and L. Fu, A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations, Phys. Rev. B **92**, 235136 (2015).

[4] S. Vijay, J. Haah, and L. Fu, Fracton topological order, generalized lattice gauge theory and duality, Phys. Rev. B **94**, 235157 (2016).

[5] N. Seiberg, Field theories with a vector global symmetry, SciPost Phys. **8**, 050 (2020).

[6] N. Seiberg and S. H. Shao, Exotic symmetries, duality, and fractons in 2+1-dimensional quantum field theory, SciPost Phys. **10**, 027 (2021).

[7] N. Seiberg and S. H. Shao, Exotic $U(1)$ symmetries, duality, and fractons in 3+1-dimensional quantum field theory, SciPost Phys. **9**, 046 (2020).

[8] N. Seiberg and S. H. Shao, Exotic $\mathbb{Z}_N$ symmetries, duality, and fractons in 3+1-dimensional quantum field theory, SciPost Phys. **10**, 003 (2021).

[9] P. Gorantla, H. T. Lam, N. Seiberg, and S. H. Shao, More exotic field theories in 3+1 dimensions, SciPost Phys. **9**, 073 (2020).

[10] T. Rudelius, N. Seiberg, and S. H. Shao, Fractons with twisted boundary conditions and their symmetries, Phys. Rev. B **103**, 195113 (2021).

[11] P. Gorantla, H. T. Lam, N. Seiberg, and S. H. Shao, A modified Villain formulation of fractons and other exotic theories, J. Math. Phys. **62**, 102301 (2021).

[12] P. Gorantla, H. T. Lam, N. Seiberg, and S. H. Shao, The low-energy limit of some exotic lattice theories and UV/IR mixing, Phys. Rev. B **104**, 235116 (2021).

[13] R. M. Nandkishore and M. Hermele, Fractons, Annu. Rev. Condens. Matter Phys. **10**, 295 (2019).

[14] M. Pretko, X. Chen, and Y. You, Fracton phases of matter, Int. J. Mod. Phys. A **35**, 2030003 (2020).

[15] M. Pretko, Subdimensional particle structure of higher rank $U(1)$ spin liquids, Phys. Rev. B **95**, 115139 (2017).

[16] M. Pretko, Generalized electromagnetism of subdimensional particles: A spin liquid story, Phys. Rev. B **96**, 035119 (2017).

[17] K. Slagle, A. Prem, and M. Pretko, Symmetric tensor gauge theories on curved spaces, Ann. Phys. (Amsterdam) **410**, 167910 (2019).

[18] A. Gromov, Towards Classification of Fracton Phases: The Multipole Algebra, Phys. Rev. X **9**, 031035 (2019).

[19] M. Pretko, Emergent gravity of fractons: Mach's principle revisited, Phys. Rev. D **96**, 024051 (2017).

[20] H. Yan, Hyperbolic fracton model, subsystem symmetry, and holography, Phys. Rev. B **99**, 155126 (2019).

[21] A. Gromov, A. Lucas, and R. M. Nandkishore, Fracton hydrodynamics, Phys. Rev. Research **2**, 033124 (2020).

[22] R. Casalbuoni, J. Gomis, and D. Hidalgo, Worldline description of fractons, Phys. Rev. D **104**, 125013 (2021).

[23] H. Geng, S. Kachru, A. Karch, R. Nally, and B. C. Rayhaun, Fractons and exotic symmetries from branes, Fortschr. Phys. **69**, 2100133 (2021).

[24] M. Qi, L. Radzihovsky, and M. Hermele, Fracton phases via exotic higher-form symmetry-breaking, Ann. Phys. (Amsterdam) **424**, 168360 (2021).

[25] J. Distler, A. Karch, and A. Raz, Spontaneously broken subsystem symmetries, J. High Energy Phys. 03 (2022) 016.

[26] M. Pretko and L. Radzihovsky, Fracton-Elasticity Duality, Phys. Rev. Lett. **120**, 195301 (2018).

[27] A. Gromov, Chiral Topological Elasticity and Fracton Order, Phys. Rev. Lett. **122**, 076403 (2019).

[28] M. Pretko and L. Radzihovsky, Symmetry Enriched Fracton Phases from Supersolid Duality, Phys. Rev. Lett. **121**, 235301 (2018).

[29] L. Radzihovsky and M. Hermele, Fractons from Vector Gauge Theory, Phys. Rev. Lett. **124**, 050402 (2020).

[30] M. Pretko, Z. Zhai, and L. Radzihovsky, Crystal-to-fracton tensor gauge theory dualities, Phys. Rev. B **100**, 134113 (2019).

[31] M. Fruchart and V. Vitelli, Symmetries and Dualities in the Theory of Elasticity, Phys. Rev. Lett. **124**, 248001 (2020).

[32] D. X. Nguyen, A. Gromov, and S. Moroz, Fracton-elasticity duality of two-dimensional superfluid vortex crystals: defect interactions and quantum melting, SciPost Phys. **9**, 076 (2020).

[33] Y. Choi, C. Córdova, P.-S. Hsin, H. T. Lam, and S.-H. Shao, Non-invertible duality defects in 3+1 dimensions, Phys. Rev. D **105**, 125016 (2022).

[34] Y. Hirono and Y. H. Qi, Effective field theories for gapless phases with fractons via a coset construction, Phys. Rev. B **105**, 205109 (2022).

[35] D. Bulmash and M. Barkeshli, The Higgs mechanism in higher-rank symmetric $U(1)$ gauge theories, Phys. Rev. B **97**, 235112 (2018).

[36] H. Ma, M. Hermele, and X. Chen, Fracton topological order from the Higgs and partial-confinement mechanisms of rank-two gauge theory, Phys. Rev. B **98**, 035111 (2018).

[37] K. Slagle, Foliated Quantum Field Theory of Fracton Order, Phys. Rev. Lett. **126**, 101603 (2021).

[38] N. Manoj, R. Moessner, and V. B. Shenoy, Fractonic View of Folding and Tearing Paper: Elasticity of Plates Is Dual to a Gauge Theory with Vector Charges, Phys. Rev. Lett. **127**, 067601 (2021).

[39] L. Radzihovsky, Quantum Smectic Gauge Theory, Phys. Rev. Lett. **125**, 267601 (2020).

[40] K. Sun and X. Mao, Fractional Excitations in Non-Euclidean Elastic Plates, Phys. Rev. Lett. **127**, 098001 (2021).
[41] J. K. Yuan, S. A. Chen, and P. Ye, Fractonic superfluids, Phys. Rev. Research **2**, 023267 (2020).
[42] S. A. Chen, J. K. Yuan, and P. Ye, Fractonic superfluids. II. Condensing subdimensional particles, Phys. Rev. Research **3**, 013226 (2021).
[43] H. Li and P. Ye, Renormalization group analysis on emergence of higher rank symmetry and higher moment conservation, Phys. Rev. Research **3**, 043176 (2021).
[44] J. Distler, M. Jafry, A. Karch, and A. Raz, Interacting fractons in 2+1-dimensional quantum field theory, J. High Energy Phys. 03 (2022) 070.
[45] J. K. Yuan, S. A. Chen, and P. Ye, Higher rank symmetry defects: Defect bound states and hierarchical proliferation, arXiv:2201.08597 [cond-mat.str-el].
[46] P. Gorantla, H. T. Lam, N. Seiberg, and S.-H. Shao, Global dipole symmetry, compact Lifshitz theory, tensor gauge theory, and fractons, Phys. Rev. B **106**, 045112 (2022).
[47] J. K. Yuan, S. A. Chen, and P. Ye, Quantum hydrodynamics of fractonic superfluids with lineon condensate: From Navier-Stokes-like equations to Landau-like criterion, Chin. Phys. Lett. **39**, 057101 (2022).
[48] A. Pérez and S. Prohazka, Asymptotic symmetries and soft charges of fractons, Phys. Rev. D **106**, 044017 (2022).
[49] A. Giveon, M. Porrati, and E. Rabinovici, Target space duality in string theory, Phys. Rep. **244**, 77 (1994).
[50] W. Siegel, Two vierbein formalism for string inspired axionic gravity, Phys. Rev. D **47**, 5453 (1993).
[51] W. Siegel, Superspace duality in low-energy superstrings, Phys. Rev. D **48**, 2826 (1993).
[52] C. Hull and B. Zwiebach, Double field theory, J. High Energy Phys. 09 (2009) 099.
[53] C. Hull and B. Zwiebach, The gauge algebra of double field theory and Courant brackets, J. High Energy Phys. 09 (2009) 090.
[54] O. Hohm, C. Hull, and B. Zwiebach, Background independent action for double field theory, J. High Energy Phys. 07 (2010) 016.
[55] O. Hohm, C. Hull, and B. Zwiebach, Generalized metric formulation of double field theory, J. High Energy Phys. 08 (2010) 008.
[56] T. H. Buscher, A symmetry of the string background field equations, Phys. Lett. B **194**, 59 (1987).
[57] T. H. Buscher, Path integral derivation of quantum duality in nonlinear sigma models, Phys. Lett. B **201**, 466 (1988).
[58] I. Jeon, K. Lee, and J. H. Park, Stringy differential geometry, beyond Riemann, Phys. Rev. D **84**, 044022 (2011).
[59] J. H. Park, S. J. Rey, W. Rim, and Y. Sakatani, $\mathbf{O}(D,D)$ covariant Noether currents and global charges in double field theory, J. High Energy Phys. 11 (2015) 131.
[60] S. Angus, K. Cho, and J. H. Park, Einstein double field equations, Eur. Phys. J. C **78**, 500 (2018).
[61] I. Jeon, K. Lee, and J. H. Park, Supersymmetric double field theory: Stringy reformulation of supergravity, Phys. Rev. D **85**, 081501(R) (2012).
[62] I. Jeon, K. Lee, J. H. Park, and Y. Suh, Stringy unification of type IIA and IIB supergravities under $\mathcal{N}=2$ $D=10$ supersymmetric double field theory, Phys. Lett. B **723**, 245 (2013).
[63] I. Jeon, K. Lee, and J. H. Park, Incorporation of fermions into double field theory, J. High Energy Phys. 11 (2011) 025.
[64] I. Jeon, K. Lee, and J. H. Park, Double field formulation of Yang-Mills theory, Phys. Lett. B **701**, 260 (2011).
[65] I. Jeon, K. Lee, and J. H. Park, Ramond-Ramond cohomology and $\mathbf{O}(D,D)$ T-duality, J. High Energy Phys. 09 (2012) 079.
[66] O. Hohm, S. K. Kwak, and B. Zwiebach, Unification of Type II Strings and T-duality, Phys. Rev. Lett. **107**, 171603 (2011).
[67] O. Hohm, S. K. Kwak, and B. Zwiebach, Double field theory of type II strings, J. High Energy Phys. 09 (2011) 013.
[68] S. Angus, K. Cho, G. Franzmann, S. Mukohyama, and J. H. Park, $\mathbf{O}(D,D)$ completion of the Friedmann equations, Eur. Phys. J. C **80**, 830 (2020).
[69] E. Lescano and N. Mirón-Granese, Double field theory with matter and its cosmological application, arXiv:2111.03682 [hep-th].
[70] S. M. Ko, J. H. Park, and M. Suh, The rotation curve of a point particle in stringy gravity, J. Cosmol. Astropart. Phys. **2017**, 002 (2017).
[71] T. Basile, E. Joung, and J. H. Park, A note on Faddeev–Popov action for doubled-yet-gauged particle and graded Poisson geometry, J. High Energy Phys. 02 (2020) 022.
[72] K. Lee and J. H. Park, Covariant action for a string in doubled yet gauged spacetime, Nucl. Phys. B **880**, 134 (2014).
[73] J. H. Park, Green-Schwarz superstring on doubled-yet-gauged spacetime, J. High Energy Phys. 11 (2016) 005.
[74] K. S. Choi and J. H. Park, Standard Model as a Double Field Theory, Phys. Rev. Lett. **115**, 171603 (2015).
[75] K. Morand and J. H. Park, Classification of non-Riemannian doubled-yet-gauged spacetime, Eur. Phys. J. C **77**, 685 (2017).
[76] K. Cho, K. Morand, and J. H. Park, Kaluza–Klein reduction on a maximally non-Riemannian space is moduli-free, Phys. Lett. B **793**, 65 (2019).
[77] D. S. Berman, C. D. A. Blair, and R. Otsuki, Non-Riemannian geometry of M-theory, J. High Energy Phys. 07 (2019) 175.
[78] S. M. Ko, C. Melby-Thompson, R. Meyer, and J. H. Park, Dynamics of perturbations in double field theory & non-relativistic string theory, J. High Energy Phys. 12 (2015) 144.
[79] C. D. A. Blair, A worldsheet supersymmetric Newton-Cartan string, J. High Energy Phys. 10 (2019) 266.
[80] J. Gomis and H. Ooguri, Nonrelativistic closed string theory, J. Math. Phys. **42**, 3127 (2001).
[81] U. H. Danielsson, A. Guijosa, and M. Kruczenski, IIA/B, wound and wrapped, J. High Energy Phys. 10 (2000) 020.
[82] R. Andringa, E. Bergshoeff, J. Gomis, and M. de Roo, 'Stringy' Newton–Cartan gravity, Classical Quantum Gravity **29**, 235020 (2012).
[83] E. Bergshoeff, J. Gomis, and G. Longhi, Dynamics of Carroll particles, Classical Quantum Gravity **31**, 205009 (2014).
[84] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time, Classical Quantum Gravity **31**, 085016 (2014).
[85] X. Bekaert and K. Morand, Connections and dynamical trajectories in generalised Newton-Cartan gravity II. An ambient perspective, J. Math. Phys. **59**, 072503 (2018).
[86] T. Harmark, J. Hartong, and N. A. Obers, Nonrelativistic strings and limits of the AdS/CFT correspondence, Phys. Rev. D **96**, 086019 (2017).

[87] C. D. A. Blair, G. Oling, and J. H. Park, Non-Riemannian isometries from double field theory, J. High Energy Phys. 04 (2021) 072.

[88] O. Hohm and B. Zwiebach, On the Riemann tensor in double field theory, J. High Energy Phys. 05 (2012) 126.

[89] J. H. Park, Comments on double field theory and diffeomorphisms, J. High Energy Phys. 06 (2013) 098.

[90] J. H. Park, Stringy gravity: Solving the dark problems at 'short' distance, EPJ Web Conf. **168**, 01010 (2018).

[91] V. Cvetkovic, Z. Nussinov, and J. Zaanen, Topological kinematic constraints: quantum dislocations and the glide principle, Philos. Mag. **86**, 2995 (2006).

[92] J. H. Park and S. Sugimoto, String Theory and Non-Riemannian Geometry, Phys. Rev. Lett. **125**, 211601 (2020).

[93] I. Jeon, K. Lee, and J. H. Park, Differential geometry with a projection: Application to double field theory, J. High Energy Phys. 04 (2011) 014.

[94] S. Guddala, F. Komissarenko, S. Kiriushechkina, A. Vakulenko, M. Li, V. M. Menon, A. Alù, and A. B. Khanikaev, Topological phonon-polariton funneling in midinfrared metasurfaces, Science **374**, 225 (2021).

[95] K. Cho and J. H. Park, Remarks on the non-Riemannian sector in double field theory, Eur. Phys. J. C **80**, 101 (2020).

[96] L. Bidussi, J. Hartong, E. Have, J. Musaeus, and S. Prohazka, Fractons, dipole symmetries and curved spacetime, SciPost Phys. **12**, 205 (2022).

[97] A. Jain and K. Jensen, Fractons in curved space, SciPost Phys. **12**, 142 (2022).

[98] E. A. Milne, A Newtonian expanding universe, Q. J. Math. **os-5**, 64 (1934).

[99] C. Duval, On Galileian isometries, Classical Quantum Gravity **10**, 2217 (1993).

[100] A. J. Beekman, J. Nissinen, K. Wu, R.-J Slager, Z. Nussinov, V. Cvetkovic, and J. Zaanen, Dual gauge field theory of quantum liquid crystals in two dimensions, Phys. Rep. **683**, 1 (2017).

[101] L. D. Landau, Über die Bewegung der Elektronen in Kristallgitter, Phys. Z. Sowjetunion **3**, 644 (1933).

[102] L. D. Landau and S. I. Pekar, Effective mass of a polaron, Zh. Eksp. Teor. Fiz. **18**, 419 (1948).

[103] H. Fröhlich, Electrons in lattice fields, Adv. Phys. **3**, 325 (1954).

[104] R. P. Feynman, Slow Electrons in a Polar Crystal, Phys. Rev. **97**, 660 (1955).

[105] J. Roseler, A new variational ansatz in the polaron theory, Phys. Status Solidi B **25**, 311 (1968).

[106] D. M. Larsen, Intermediate-Coupling Polaron Effective Mass, Phys. Rev. **174**, 1046 (1968).

[107] A. D. Shapere and F. Wilczek, Regularizations of time-crystal dynamics, Proc. Natl. Acad. Sci. USA **116**, 18772 (2019).

[108] A. D. Gallegos, U. Gürsoy, S. Verma, and N. Zinnato, Non-Riemannian gravity actions from double field theory, J. High Energy Phys. 06 (2021) 173.

[109] C. D. A. Blair, D. Gallegos, and N. Zinnato, A non-relativistic limit of M-theory and 11-dimensional membrane Newton-Cartan, J. High Energy Phys. 10 (2021) 015.

[110] J. Gomis, J. Gomis, and K. Kamimura, Non-relativistic superstrings: A new soluble sector of $AdS_5 \times S^5$, J. High Energy Phys. 12 (2005) 024.

[111] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, Torsional Newton-Cartan geometry and Lifshitz holography, Phys. Rev. D **89**, 061901 (2014).

[112] J. Hartong and N. A. Obers, Hořava-Lifshitz gravity from dynamical Newton-Cartan geometry, J. High Energy Phys. 07 (2015) 155.

[113] T. Harmark, J. Hartong, L. Menculini, N. A. Obers, and Z. Yan, Strings with non-relativistic conformal symmetry and limits of the AdS/CFT correspondence, J. High Energy Phys. 11 (2018) 190.

[114] E. Bergshoeff, J. Gomis, and Z. Yan, Nonrelativistic string theory and T-duality, J. High Energy Phys. 11 (2018) 133.

[115] E. A. Bergshoeff, J. Gomis, J. Rosseel, C. Şimşek, and Z. Yan, String theory and string Newton-Cartan geometry, J. Phys. A: Math. Theor. **53**, 014001 (2020).

[116] T. Harmark, J. Hartong, L. Menculini, N. A. Obers, and G. Oling, Relating non-relativistic string theories, J. High Energy Phys. 11 (2019) 071.

[117] E. A. Bergshoeff, J. Lahnsteiner, L. Romano, J. Rosseel, and C. Şimşek, A non-relativistic limit of NS-NS gravity, J. High Energy Phys. 06 (2021) 021.

[118] E. Witten, On string theory and black holes, Phys. Rev. D **44**, 314 (1991).

[119] C. P. Burgess, R. C. Myers, and F. Quevedo, On spherically symmetric string solutions in four-dimensions, Nucl. Phys. B **442**, 75 (1995).

[120] G. T. Horowitz and A. Strominger, Black strings and P-branes, Nucl. Phys. B **360**, 197 (1991).

[121] K. Morand, J. H. Park, and M. Park, Identifying Riemannian Singularities with Regular Non-Riemannian Geometry, Phys. Rev. Lett. **128**, 041602 (2022).