

Dynamical generation of chiral W and Greenberger-Horne-Zeilinger states in laser-controlled Rydberg-atom trimers

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Motivated by the significantly improved scalability of optically trapped neutral-atom systems, extensive efforts have been devoted in recent years to quantum-state engineering in Rydberg-atom ensembles. Here we investigate the problem of engineering generalized (“twisted”) W states, as well as Greenberger-Horne-Zeilinger (GHZ) states, in the strongly interacting regime of a neutral-atom system. We assume that each atom in the envisioned system initially resides in its ground state and is subject to several external laser pulses that are close to being resonant with the same internal atomic transition. In particular, in the special case of a three-atom system (Rydberg-atom trimer) we determine configurations of field alignments and atomic positions that enable the realization of *chiral* W states—a special type of twisted three-qubit W states of interest for implementing noiseless-subsystem qubit encoding. Using chiral W states as an example we also address the problem of deterministically converting twisted W states into their GHZ counterparts in the same three-atom system, thus significantly generalizing recent works that involve only ordinary W states. We show that starting from twisted—rather than ordinary— W states is equivalent to renormalizing downward the relevant Rabi frequencies. While this leads to somewhat longer state-conversion times, we also demonstrate that those times are at least two orders of magnitude shorter than typical lifetimes of relevant Rydberg states.

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I. INTRODUCTION

Maximally entangled multiqubit states are of special interest for quantum-information processing (QIP) [1]. Two particularly prominent classes of such states are W [2] and Greenberger-Horne-Zeilinger (GHZ) [3] states, for which it is known that they cannot be transformed into each other through local operations and classical communication (LOCC inequivalence [1]). Owing to their proven usefulness in various QIP protocols [4–7], several different schemes for the preparation of W [8–18] and GHZ states [19–23] in various physical platforms have been proposed in recent years.

One of the currently most promising platforms for QIP is based on ensembles of neutral atoms in Rydberg states [24,25]. The scalability of these systems, confined in arrays of individual optical-dipole microtraps (tweezers), has improved significantly in recent years [26–31]. This development has been interwoven with other important feats, such as high-fidelity state preparation/readout and accurate realization of quantum logic gates [32,33]. As a result, neutral-atom-based platforms currently allow for controlled quantum dynamics of more than 100 qubits, with the prospect of reliable QIP with even much larger qubit systems [34–36]

deemed to be realistic in the not too distant future [37]. This has, in turn, reinvigorated research interest in quantum-state engineering in this class of atomic systems [16,38–45].

An ordinary (prototype) N -qubit W state [46] is an equal superposition of all N -qubit basis states with exactly one qubit in their “up” state, and all the remaining ones in their “down” state. In systems with a periodic spatial arrangement (i.e., a lattice) [47] of qubits, it makes sense to consider generalized W states, which represent linear combinations of the same N -qubit basis states but with a constant phase difference between contributions corresponding to adjacent lattice sites. This phase difference corresponds to a quasimomentum from the Brillouin zone of the underlying lattice. The physical meaning of such states—which in the following will be referred to as *twisted* W states—becomes fully transparent upon switching from spin-1/2 to spinless-fermion degrees of freedom using the Jordan-Wigner transformation [48]. Namely, these states are equivalent to Bloch states of spinless-fermion excitations.

Aside from realizing W [13] and GHZ states [41], interconversion between those states is another relevant problem of quantum-state engineering. This problem was first addressed in the context of a photonic system [49], where such interconversion can be carried out only in a nondeterministic fashion. In recent years, the same problem was investigated in a system of three equidistant Rydberg atoms with van der Waals type interaction, which are at the same time acted upon by several external laser fields [44,45]. This last system was first studied [44] using the method of shortcuts to adiabaticity [50], more precisely Lewis-Riesenfeld invariants [51], followed by an alternative treatment [45] that made use of a dynamical-symmetry-based approach [52].

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In this paper, we consider the system of three neutral atoms in the Rydberg-blockade (RB) regime [53–56], interacting with external laser fields, with our research objective being twofold. We first present a deterministic preparation scheme for *chiral* W states, a special class of twisted three-qubit W states that are of relevance for implementing noiseless-subsystem qubit encoding [14,57]. We then address the problem of deterministically converting twisted W states into their GHZ counterparts via different intermediate states. Both of these dynamical generation schemes rely heavily on relative alignments of the laser fields involved and precise positioning of the atoms, these ingredients being within reach of current technology [58–61]. Furthermore, we show that even without such an experimental precision a conversion of a twisted W state is still possible. A strong laser driving field, which introduces light shifts, can determine the specific twisted states participating in a conversion scheme. The scheme also makes use of additional weaker fields, which address the lifted degeneracies of internal energy levels of the Rydberg trimer.

Our principal result in the context of the state-conversion problem—relative to previous studies of this problem [44,45] that concentrated only on ordinary W states—is that starting from twisted W states is equivalent to renormalizing downward the relevant Rabi frequencies of external laser pulses. This renormalization, which depends only on the relative alignment between the laser fields used, leads to somewhat longer state-conversion times (for the same laser-pulse energy used) than in the case with ordinary W states. However, we also demonstrate that the latter times are at least two orders of magnitude shorter than the typical lifetimes of relevant Rydberg states.

The remainder of this paper is organized as follows. In Sec. II, we introduce the neutral-atom system under consideration and briefly describe its interaction with external laser fields. In Sec. III, we introduce several classes of entangled multiqubit states of interest for the present work (W , GHZ, and Dicke states) and the notation to be used throughout the paper. Section IV is devoted to the derivation of effective Hamiltonians of the system that serve as the point of departure for the state-engineering schemes discussed in the present work. In Sec. V, we provide a discussion of specific alignments of laser fields and relative atom positions that are required for the generation of twisted W states from the atomic-ensemble ground state via π pulses of a single laser field resonant with the Rydberg transition. In particular, we describe in detail a preparation scheme for chiral W states in a Rydberg trimer. In Sec. VI, we present two different schemes for the conversion of twisted W states into GHZ states, which are respectively based on degenerate Dicke manifolds of states and lifted degeneracies. We conclude, with a summary of the obtained results and a short survey of possible directions for future investigation, in Sec. VII. For the sake of completeness, some relevant mathematical details are presented in detail in Appendixes A and B.

II. SYSTEM AND ATOM-FIELD INTERACTION

We consider a system that consists of N identical neutral atoms (e.g., of ^{87}Rb) located at positions determined by the

vectors \mathbf{x}_n ($n = 1, 2, \dots, N$). Anticipating the use of external laser pulses that are all close to being resonant with the same internal atomic transition—namely, the one between the ground state $|g\rangle_n$ (with energy E_g) and a highly excited Rydberg state $|r\rangle_n$ (energy E_r)—the atoms can be treated as effective two-level systems with the atomic frequency $\omega_A = (E_r - E_g)/\hbar$ as resonance frequency. In the following, we will treat E_g as the origin of the energy scale, i.e., set $E_g = 0$. In the QIP context, each atom in this system represents a *gr*-type qubit [37], where the atomic states $|g\rangle_n$ and $|r\rangle_n$ play the role of the logical “down” ($|0\rangle_n$) and “up” ($|1\rangle_n$) states of the n th qubit, respectively. Recalling that the typical energy splitting of *gr*-type qubits is in the range between 900 and 1500 THz [37], manipulations of such qubits require either an ultraviolet laser or a combination of visible and infrared lasers in a ladder configuration.

We also assume that the atoms are pairwise coupled through off-resonant dipole-dipole (van der Waals) interaction. In the special case of equidistant atoms—the physical situation of primary relevance in the remainder of the present work—the magnitude $V_{pq} = C_6/d_{pq}^6$ of this interaction (where d_{pq} is the distance between atoms p and q and C_6 the van der Waals interaction constant) is the same for all pairs of atoms, and we denote $V_{pq} = V$. For $N = 3$, the case of equidistant atoms corresponds to their arrangement in the form of an equilateral triangle [for a pictorial illustration, see Fig. 1(a)], while for $N = 4$ they are located at the vertices of a regular tetrahedron.

Importantly, our envisioned system is also assumed to be in the RB regime [53–56], which is equivalent to demanding that the interaction-induced energy shift V far exceeds the Fourier-limited width of all the utilized laser pulses (i.e., $|V|T_{\text{int}}/\hbar \gg 1$, where T_{int} is the pulse duration). Thus, the state-preparation and conversion schemes to be presented in what follows are applicable in the regime of primary interest for QIP, as the phenomenon of RB provides the conditional logic that enables neutral-atom quantum computing [62]. The suitability of our envisioned system for quantum-state engineering is further underscored by its reliance on *gr*-type qubits, which—owing to their straightforward initialization, manipulation, and measurements—represent the preferred neutral-atom qubit type for fast, high-fidelity entangling operations [37].

The total Hamiltonian of the system at hand is given by $H = H_A + H_F + H_{\text{int}}$, where H_A describes the atomic ensemble, H_F the free external fields, and H_{int} the atom-field interaction. The form of these three contributions to the total system Hamiltonian will be discussed in detail in the following.

The Hamiltonian of the atomic ensemble is given by

$$H_A = \sum_{n=1}^N \hbar\omega_A |r\rangle_n \langle r| + \sum_{p < q}^N V |r\rangle_p \langle r|_q \langle r|_p \langle r|. \quad (1)$$

The energy eigenvalues of the atomic ensemble are given by $E_a = a \hbar\omega_A + V \binom{a}{2}$, where $a \leq N$ is the number of atoms in the excited state. The energy level E_a has a degeneracy of $\binom{N}{a}$ and the energy gap between adjacent excitation subspaces is given by $\Delta E_a \equiv E_a - E_{a-1} = \hbar\omega_A + V(a-1)$. In particular,

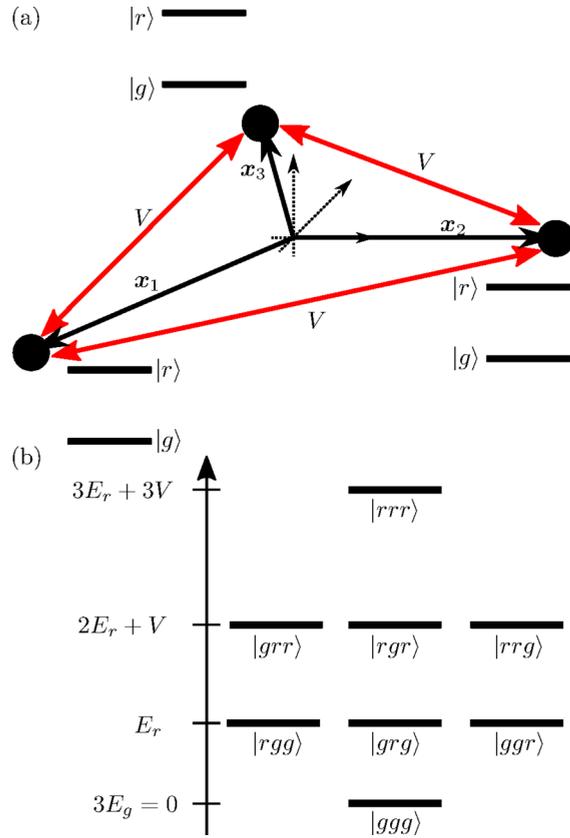


FIG. 1. (a) Schematic illustration of a Rydberg-atom trimer ($N = 3$). The three Rydberg atoms, located at the positions specified by the vectors \mathbf{x}_n ($n = 1, 2, 3$), form an equilateral triangle. The ground and Rydberg states of each atom are denoted by $|g\rangle$ and $|r\rangle$, respectively, while V stands for the magnitude of their pairwise van der Waals interaction. (b) Energy-level scheme of a Rydberg-atom trimer. The origin of the energy scale is chosen such that $E_g = 0$. We are considering long interaction times T , i.e., $|E_r - E_g|T/\hbar \gg 1$, $|V|T/\hbar \gg 1$, and large electronic excitation energies, i.e., $|E_r - E_g| \gg |V|$.

the energy-level scheme in the Rydberg-trimer case ($N = 3$) is pictorially illustrated in Fig. 1(b).

The atomic ensemble is subject to multiple plane-wave laser fields with respective wave vectors \mathbf{k}_j and frequencies ω_j ($j = 0, 1, \dots, J$). The fields are quantized using creation and annihilation operators a_j^\dagger and a_j , such that the free-field Hamiltonian is given by

$$H_F = \sum_{j=0}^J \hbar\omega_j a_j^\dagger a_j, \quad (2)$$

where for the sake of convenience the ground-state energies $\hbar\omega_j/2$ of all modes are omitted. We treat the interaction between laser pulses and the atomic ensemble in the dipole and rotating-wave approximations (RWA), with the corresponding coupling strengths being denoted by d_j . All laser fields are assumed to resemble classical fields. Hence, they can be described as coherent field states [63] of high mean photon numbers M_j , such that the coupling constants define (resonant) Rabi frequencies $\Omega_j = \sqrt{M_j} d_j/\hbar$. Here $d_j = -i\sqrt{\hbar\omega_j/(2\epsilon_0\tilde{V})} \langle r|\mathbf{d} \cdot \boldsymbol{\epsilon}_j|g\rangle$, where ϵ_0 is the vacuum dielectric

constant, \tilde{V} is the quantization volume of the field modes, $\boldsymbol{\epsilon}_j$ is the polarization unit vector of mode j (orthogonal to its propagation direction), and \mathbf{d} is the atomic dipole operator.

The atom-field interaction in this system is described by the Hamiltonian

$$H_{\text{int}} = \sum_{n=1}^N \sum_{j=1}^J (|g\rangle_{nm} \langle r| d_j^* a_j^\dagger e^{-i\mathbf{k}_j \cdot \mathbf{x}_n} + \text{H.c.}), \quad (3)$$

where the site-dependent phases $\mathbf{k}_j \cdot \mathbf{x}_n$ result from evaluating the mode function for plane waves at the distinct atom positions. This Hamiltonian can be recast in the form

$$H_{\text{int}} = \sum_{n=1}^N \sum_{j=1}^J [U(\mathbf{k}_j) |g\rangle_{nm} \langle r| d_j^* a_j^\dagger U^\dagger(\mathbf{k}_j) + \text{H.c.}] \quad (4)$$

by introducing the transformation

$$U(\mathbf{k}) = \bigotimes_{n=1}^N (e^{i\mathbf{k} \cdot \mathbf{x}_n} |r\rangle_{nm} \langle r| + |g\rangle_{nm} \langle g|). \quad (5)$$

In the following, the transformation $U(\mathbf{k})$ will be used repeatedly in order to simplify the description of the system under consideration. For the sake of brevity, we will just use \mathbf{k} to parametrize this transformation, because the atoms are assumed to be located at fixed positions.

III. RELEVANT MULTIQUBIT STATES

In what follows, we introduce several classes of entangled multiqubit states of interest in the remainder of this work. In Sec. III A, we discuss generalized W states, together with their GHZ counterparts. We also briefly introduce Dicke states and their twisted counterparts that play an auxiliary role in our further discussion. In Sec. III B, we specialize to the three-qubit systems, introducing first the chiral W states, and then explaining their connection to specific twisted W states in the system under consideration (Rydberg trimer). The notation used will be the one appropriate for gr -type Rydberg-atom qubits [64], with $\{|g\rangle, |r\rangle\}$ being the relevant computational basis of a single qubit.

A. Generalized N -qubit W , GHZ, and Dicke states

The most general W -type states, not necessarily maximally entangled, represent linear combinations of states in which exactly one qubit is in the state $|r\rangle$, with all the remaining ones being in the state $|g\rangle$ (i.e., all states corresponding to Hamming-weight-1 bit strings). They are given by

$$|W_N(A_1, \dots, A_N)\rangle = \frac{1}{\sqrt{A}} \sum_{n=1}^N A_n |g \dots r_n \dots g\rangle, \quad (6)$$

where $A = \sum_{n=1}^N |A_n|^2$ and A_n are N arbitrary complex numbers, with at least two of them being unequal to zero. In the special case with $|A_n| = 1/\sqrt{N}$ ($n = 1, \dots, N$), one can substitute $A_n = e^{i\varphi_n}/\sqrt{N}$ and recast the last states in the form

$$|W_N(\varphi_1, \dots, \varphi_N)\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{i\varphi_n} |g \dots r_n \dots g\rangle. \quad (7)$$

Here the phases $\varphi_1, \dots, \varphi_N$ are defined modulo 2π , and as usual the state $|W_N\rangle$ is defined up to a global phase. In particular, the special case $\varphi_1 = \dots = \varphi_N = 0$ of the latter maximally-entangled states are the most often used, “ordinary” W states

$$|W_N\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N |g \dots r_n \dots g\rangle. \quad (8)$$

The most notable property of W states is that they are the most robust ones to particle loss among all N -qubit states [46]. The entanglement inherent to W states is fundamentally different than that of GHZ states

$$|\text{GHZ}_N\rangle = \frac{1}{\sqrt{2}} (|rr \dots rr\rangle + e^{i\varphi} |gg \dots gg\rangle), \quad (9)$$

no matter whether one considers pairwise or distributed entanglement. For instance, in the $N = 3$ case, the W state are characterized by a strong pairwise entanglement (as quantified by the corresponding concurrences) while the essential three-way entanglement (as quantified by the 3-tangle) vanishes [65]. On the other hand, its GHZ counterpart has maximal essential three-way entanglement, while pairwise entanglements vanish [66].

In the case of a periodic spatial arrangement (lattice) of qubits, it is pertinent to introduce the “twisted” W states as a special case of the states in Eq. (7) where $\varphi_n \equiv \mathbf{k} \cdot \mathbf{x}_n$, with \mathbf{k} being a quasimomentum from the Brillouin zone corresponding to the underlying lattice of qubits with positions \mathbf{x}_n . Those states are given by

$$|W_N(\mathbf{k})\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{i\mathbf{k} \cdot \mathbf{x}_n} |g \dots r_n \dots g\rangle. \quad (10)$$

For instance, if qubits form a regular one-dimensional lattice, then the quasimomentum, expressed in units of the inverse lattice period, belongs to $(-\pi, \pi]$. The special significance of the state in Eq. (10) rests on the notion that using Jordan-Wigner transformation from pseudospin-1/2 to spinless fermion (or hardcore boson) degrees of freedom [48], this state is mapped onto a bare-excitation Bloch state with quasimomentum \mathbf{k} . In particular, the ordinary W state with $\varphi_1 = \varphi_2 = \dots = \varphi_N = 0$ —the special case of Eq. (10) with $\mathbf{k} \cdot \mathbf{x}_n = 0$ —corresponds to the $\mathbf{k} = 0$ Bloch state.

To describe all possible N -qubit states, we consider different numbers a of excited qubits. A generic state in the subspace of states with a excitations can be parameterized as $|\{n_1, \dots, n_a\}\rangle$, where the atoms enumerated with n_1, \dots, n_a are in the excited state while the remaining ones are in the ground state. The Dicke state

$$|D_a^N\rangle = \binom{N}{a}^{-1/2} \sum_{n_1 < \dots < n_a} |\{n_1, \dots, n_a\}\rangle \quad (11)$$

represents the equal superposition of all states $|\{n_1, \dots, n_a\}\rangle$ spanning that subspace, where the sum in the last equation runs over all $\binom{N}{a}$ combinations of a atoms out of N .

In a completely analogous way as in the case of W states, one can introduce twisting, i.e., \mathbf{k} -dependent relative phases between different N -qubit basis states with equal excitation

number a . These phases are captured by the unitary transformation of Eq. (5), which maps the Dicke states into their twisted counterparts

$$|D_a^N(\mathbf{k})\rangle = \sqrt{\frac{1}{\binom{N}{a}}} \sum_{n_1 < \dots < n_a} e^{i\mathbf{k} \cdot \sum_{l=1}^a \mathbf{x}_{n_l}} |\{n_1, \dots, n_a\}\rangle. \quad (12)$$

Obviously, twisted W states are a special case ($a = 1$) of twisted Dicke states, i.e., $|W_N(\mathbf{k})\rangle \equiv |D_1^N(\mathbf{k})\rangle$.

B. Chiral three-qubit W states

Given that all of our numerical calculations in the following will pertain to the $N = 3$ case, it is prudent to devote special attention to three-qubit systems and introduce a special notation that allows one to conveniently denote the three-qubit states. For this purpose, we introduce the self-adjoint *chirality* operator [67]

$$\chi = \frac{1}{2\sqrt{3}} \sum_{\alpha, \beta, \gamma} \epsilon_{\alpha\beta\gamma} \sigma_{1\alpha} \sigma_{2\beta} \sigma_{3\gamma}. \quad (13)$$

Here the indices α, β , and γ run over x, y , and z , with $\epsilon_{\alpha\beta\gamma}$ being the totally antisymmetric Levi-Civita symbol defined in terms of these indices. $\sigma_{n\alpha}$ is a Pauli operator acting on qubit n ($n = 1, 2, 3$). Orthonormal eigenstates $|\zeta_{as}\rangle$ of this chirality operator are explicitly given by

$$\begin{aligned} |\zeta_{00}\rangle &= |ggg\rangle, & |\zeta_{10}\rangle &= (|rgg\rangle + |grg\rangle + |ggr\rangle)/\sqrt{3}, \\ |\zeta_{1+}\rangle &= (w^*|rgg\rangle + |grg\rangle + w|ggr\rangle)/\sqrt{3}, \\ |\zeta_{1-}\rangle &= (w|rgg\rangle + |grg\rangle + w^*|ggr\rangle)/\sqrt{3}, \\ |\zeta_{20}\rangle &= (|grr\rangle + |rgr\rangle + |rrg\rangle)/\sqrt{3}, \\ |\zeta_{2+}\rangle &= (w^*|grr\rangle + |rgr\rangle + w|rrg\rangle)/\sqrt{3}, \\ |\zeta_{2-}\rangle &= (w|grr\rangle + |rgr\rangle + w^*|rrg\rangle)/\sqrt{3}, & |\zeta_{30}\rangle &= |rrr\rangle, \end{aligned} \quad (14)$$

with $w \equiv \exp(2\pi i/3)$ [14]. They constitute a basis of the state space of three qubits. The quantum number $a = 0, \dots, 3$ denotes the number of qubits in the $|r\rangle$ state (i.e., the Hamming weight of the corresponding bit string), and the additional quantum number $s = 0, \pm$ identifies the eigenstates uniquely. Among these eight basis states there are three W states, i.e., states corresponding to Hamming-weight-1 bit strings ($a = 1$)—the ordinary W state $|\zeta_{10}\rangle \equiv |W_3(\mathbf{k} = 0)\rangle$ and two *chiral* W states $|\zeta_{1+}\rangle$ and $|\zeta_{1-}\rangle$. The mutual orthogonality of these three W states is thus a consequence of the fact that they belong to three different eigensubspaces of the chirality operator [cf. Eq. (13)].

It is important at this point to establish a connection between the two chiral W states and the general twisted three-qubit states [as defined by Eq. (10)], which is of interest for our treatment of the Rydberg-atom system under consideration. Our assumed spatial arrangement of three neutral-atom qubits in the form of an equilateral triangle [cf. Fig. 1(a)], which implies that these three qubits are symmetrically positioned on a circle, is equivalent to an array (i.e., a one-dimensional lattice) of three qubits with periodic boundary conditions imposed. In particular, it is

straightforward to verify that the state $|\zeta_{1+}\rangle$ is—up to an irrelevant global phase—equivalent to the twisted state $|W_3(k = 2\pi/3)\rangle$ of a three-qubit array that corresponds to the quasimomentum $k = 2\pi/3$ (expressed in units of the inverse lattice spacing). Similarly, $|\zeta_{1-}\rangle$ is equivalent to the twisted state $|W_3(k = -2\pi/3)\rangle$. Having established the correspondence between the two chiral states and the twisted W states of a one-dimensional array of three qubits, we will in the following use $|\zeta_{1+}\rangle$ and $|\zeta_{1-}\rangle$ as our primary examples for the latter class of generalized three-qubit W states.

IV. EFFECTIVE HAMILTONIANS

In what follows, we describe the derivation of effective system Hamiltonians that constitute the basis for designing various state-engineering schemes in the present work. These effective Hamiltonians, which are derived with reference to twisted (rather than ordinary) W states, constitute a generalization of the effective four-level Hamiltonian that was first presented in Ref. [44]. In particular, we first treat the case of resonant laser fields (Sec. IV A), followed by a discussion of the off-resonant ones (Sec. IV B).

To realize different state-preparation and conversion schemes in the neutral-atom system under consideration, we derive different effective Hamiltonians using the resolvent formalism (see, e.g., Ref. [68]). The effective Hamiltonian can most generally be written in the form

$$H_{\text{eff}} = \sum_{E \in \sigma(H_0)} P_E \left(H + H Q_E \frac{1}{E - Q_E H Q_E} Q_E H \right) P_E, \quad (15)$$

where the sum runs over the whole energy spectrum $\sigma(H_0)$ of the noninteracting part $H_0 = H_A + H_F$ of the total Hamiltonian of the system. This equation is a direct consequence of the general relation for the resolvent $G(E)$ of the Hamiltonian H , i.e., $G(E) = (E - H)^{-1}$ and of the definition of the effective Hamiltonian H_{eff} in terms of the orthogonal projection operators P_E and $Q_E = \mathbb{1} - P_E$, i.e., $P_E G(E) P_E = (E - H_{\text{eff}})^{-1} P_E$. Thus, the effective Hamiltonian H_{eff} describes the dynamics inside the subspace of the Hilbert space only, which is characterized by the projection operator P_E . By choosing the orthogonal projection operators P_E and Q_E appropriately, effective Hamiltonians can be determined systematically within a perturbative framework so that secular terms are avoided in the time evolution. If a single mode is considered, the (unperturbed) energy eigenvalues are $E_a^m = E_a + m\hbar\omega$, where a is the number of excited atoms and m is the photon number of this mode, and the projection operators of Eq. (15) project onto the corresponding (degenerate) energy subspaces. For a perturbative approach to first order, the denominator of the resolvent of Eq. (15) can be approximated by the unperturbed Hamiltonian.

The concrete form of the projectors P_E depends on the considered fields. We discuss two distinct cases—off-resonant and resonant laser fields. The crucial difference between them is that off-resonant laser fields do not introduce additional energy degeneracies, i.e., $E_a^m = E_{a'}^{m'}$ if and only if the two atomic-excitation numbers are the same ($a = a'$) and the two photon numbers as well ($m = m'$). As long as degeneracies due to different fields are well separated, we can split up the

sum over all fields in the system Hamiltonian H and treat each field separately. By tracing out the field's degrees of freedom, we will derive effective Hamiltonians describing the dynamics of the atomic ensemble via corrections to the atomic ensemble Hamiltonian H_A . In the following, we will discuss the two cases of a single laser field separately. The resulting effective Hamiltonians and their combinations will then be used in Secs. V and VI for state preparation and conversion schemes, respectively.

A. Off-resonant laser field

We first consider a single off-resonant laser field (enumerated with $j = 0$) with wave vector \mathbf{k}_0 , assuming that its detuning $\Delta_0 = \omega_0 - \omega_A \equiv \omega_0 - (E_r - E_g)/\hbar$ is much larger in absolute value than the corresponding Rabi frequency Ω_0 , i.e., $|\Delta_0| \gg |\Omega_0|$.

Because the field is assumed to be off-resonant, the degenerate energy subspaces are completely determined by the number of excitations in the atomic ensemble a and the number of laser-field excitations, i.e., the number of photons m_0 . The projectors onto a subspace of energy $E_a^{m_0}$ is given by

$$P_a^{m_0} = \sum_{n_1 < \dots < n_a}^N |\{n_1, \dots, n_a\}\rangle \langle\{n_1, \dots, n_a\}| \otimes |m_0\rangle \langle m_0|. \quad (16)$$

A detailed derivation of the effective Hamiltonian [cf. Eq. (15)] is relegated to Appendix A 1. Here we only state the resulting corrections to the Hamiltonian H_A of the atomic ensemble, which are obtained by assuming a coherent field state of high mean photon number M_0 and tracing out the field degrees of freedom. To succinctly write down these corrections, we make use of the operator

$$\text{Hd}_2(N) = \sum_{n_1, n_2}^N (1 - \delta_{n_1 n_2}) |r\rangle_{n_1 n_1} \langle g| \otimes |g\rangle_{n_2 n_2} \langle r|, \quad (17)$$

which transforms every state into an equal (not necessarily normalized) superposition of all other states connected to it via precisely one excitation and one de-excitation at different atoms. In other words, speaking in terms of bit strings with $g \equiv 0$ and $r \equiv 1$, this operator connects all bit strings of equal Hamming weight but with a Hamming distance (Hd) of two.

The lowest order corrections to the atomic ensemble Hamiltonian are captured by

$$H_N^{\text{off}}(\mathbf{k}_0) = \sum_{a=0}^N \hbar^2 |\Omega_0|^2 P_a \left[\frac{U(\mathbf{k}_0) \text{Hd}_2(N) U^\dagger(\mathbf{k}_0) + N - a}{\hbar \Delta_0 - aV} - \frac{U(\mathbf{k}_0) \text{Hd}_2(N) U^\dagger(\mathbf{k}_0) + a}{\hbar \Delta_0 - (a-1)V} \right] P_a, \quad (18)$$

such that the effective Hamiltonian of the atomic ensemble is given by $H_{\text{eff}} = H_A + H_N^{\text{off}}(\mathbf{k}_0)$. In the last equation

$$P_a = \sum_{n_1 < \dots < n_a}^N |\{n_1, \dots, n_a\}\rangle \langle\{n_1, \dots, n_a\}| \quad (19)$$

stands for the projector onto the subspace of a excitations, while the effect of the site-dependent phase shifts is captured by the unitary-transformation operators $U(\mathbf{k}_0)$ [cf. Eq. (5)].

It is important to note that for $a = 0, 1, N - 1, N$ the corresponding terms in the Hamiltonian of Eq. (18) can further be simplified using the following identities:

$$\begin{aligned} P_0 \text{Hd}_2(N) P_0 &= P_N \text{Hd}_2(N) P_N = 0, \\ P_1 \text{Hd}_2(N) P_1 &= N |D_1^N\rangle \langle D_1^N| - P_1, \\ P_{N-1} \text{Hd}_2(N) P_{N-1} &= N |D_{N-1}^N\rangle \langle D_{N-1}^N| - P_{N-1}. \end{aligned} \quad (20)$$

For Rydberg trimers ($N = 3$), all terms of Hd_2 reduce to one of the above special cases and we obtain

$$\begin{aligned} H_3^{\text{off}}(\mathbf{k}_0) &= 3s_0 |ggg\rangle \langle ggg| - 3s_2 |rrr\rangle \langle rrr| \\ &+ (-3s_0 + 3s_1) |D_1^3(\mathbf{k}_0)\rangle \langle D_1^3(\mathbf{k}_0)| \\ &+ (-3s_1 + 3s_2) |D_2^3(\mathbf{k}_0)\rangle \langle D_2^3(\mathbf{k}_0)| + s_1(P_1 - P_2), \end{aligned} \quad (21)$$

where $s_a = \hbar^2 |\Omega_0|^2 / (\hbar \Delta_0 - aV)$ is a shorthand for the energy shifts. Hence, the off-resonant laser field just shifts the energy levels of the atomic ensemble, but up to first order does not contribute any off-diagonal elements. This result does not include jump operators between different levels. They would appear in higher order terms of the resolvent expansion, but are not considered here. Therefore, we neglect small oscillatory behavior in the level populations of the atomic ensemble of the order of $\max\{[s_a / (\hbar |\Omega_0|)]^2; 0 \leq a \leq N - 1\}$. Due to the induced energy shifts, the effective Hamiltonian lifts the energy degeneracies of the subspaces with $a = 1, 2$ excitations, such that $|D_1^3(\mathbf{k})\rangle$ and $|D_2^3(\mathbf{k})\rangle$ differ in energy from the corresponding orthogonal states of the same total excitation number a . A suitable eigenbasis of this effective Hamiltonian are the $|\zeta_{as}\rangle$ states in Eqs. (14). The energy shifts can be set to drive specific transitions by choosing appropriate detunings of additional fields as will be discussed in Sec. VIB below.

B. Resonant laser fields

If a field (enumerated $j = a$) is in resonance with a specific transition $a \leftrightarrow a - 1$ of the atomic ensemble, the subspaces $P_a^{m_a}$ and $P_{a-1}^{m_a+1}$ become energetically degenerate. By equating the energies $E_a^{m_a}$ and $E_{a-1}^{m_a+1}$, one obtains the condition

$$\hbar \Delta_a = \hbar(\omega_a - \omega_A) = (a - 1)V, \quad (22)$$

where ω_a is the frequency of the laser field. Given this degeneracy, the last two subspaces have to be jointly considered within the framework of the resolvent formalism. Assuming the field to be classical, i.e., in a coherent state of high mean photon number M_a , and tracing over the degrees of freedom of the field, we obtain an effective Hamiltonian for the atomic system alone. If we further assume that $V \gg \hbar |\Omega_a|$, we can neglect all terms scaling with $|\hbar \Omega_a|^2 / V$, thus leaving—apart from H_A —just parts containing the subspace ladder operator

$$\sigma_a^- = \sum_{n_1 < \dots < n_a} \sum_{n=1}^N |g\rangle_{mn} \langle r | \{n_1, \dots, n_a\} \langle \{n_1, \dots, n_a\} | \quad (23)$$

and its Hermitian conjugate. This results in the effective Hamiltonian $H_A + H_N^{a \leftrightarrow a-1}$, where

$$H_N^{a \leftrightarrow a-1}(\mathbf{k}_a) = \hbar \Omega_a^* U(\mathbf{k}_a) \sigma_a^- U^\dagger(\mathbf{k}_a) + \text{H.c.} \quad (24)$$

In the last equation, whose detailed derivation is presented in Appendix A2, the effect of the site-dependent phase shifts is once again encoded into the unitary transformation $U(\mathbf{k}_a)$. The special case $a = 1$ corresponds to the well-known effect of enhanced Rabi oscillations [37] and leads to a simple preparation scheme for twisted W states, as discussed in Sec. V below.

Combining N laser fields with detunings $\Delta_a = (a - 1)V$ with $a \in \{1, \dots, N\}$, such that every laser field is in resonance with one specific transition between eigenstates of the atomic Hamiltonian H_A , we can construct an effective Hamiltonian connecting stepwise all $N + 1$ degenerate energy levels of the atomic ensemble. The corrections added to H_A in this case are

$$H_N^L(\{\mathbf{k}_a\}) = \sum_{a=1}^N [\hbar \Omega_a^* U(\mathbf{k}_j) \sigma_a^- U^\dagger(\mathbf{k}_a) + \text{H.c.}], \quad (25)$$

where $\{\mathbf{k}_a\} = \{\mathbf{k}_1, \dots, \mathbf{k}_N\}$ and each laser field $j = 1, \dots, N$ connects the subspace of $a' = j - 1$ and $a = j$ excitations like a step on a ladder (L). These fields in general have different corresponding wave vectors \mathbf{k}_a , thus introducing different site-dependent phase shifts. Because in general $U(\mathbf{k}_a) \neq U(\mathbf{k}_{a \pm 1})$, these fields do not necessarily form a ladder Hamiltonian of $N + 1$ states. To what extent the steps match is described by the overlaps

$$\langle D_a^N(\mathbf{k}_a) | D_a^N(\mathbf{k}_{a+1}) \rangle = \binom{N}{a}^{-1} \sum_{n_1 < \dots < n_a} e^{i(\mathbf{k}_{a+1} - \mathbf{k}_a) \cdot \sum_{j=1}^a \mathbf{x}_{n_j}}. \quad (26)$$

Overlaps smaller than unity result in offsets. One way to deal with the latter is to control laser alignments and set precise atom positions, such that special atomic ensemble states with their specific relative phase are selected in the effective Hamiltonian. The easiest case is to avoid phase differences between different atom positions in the first place. If all laser fields are properly aligned such that $\mathbf{k}_a \cdot \mathbf{x}_n = 0 \pmod{2\pi}$ for all combinations of $a, n = 1, \dots, N$, the effective Hamiltonian is characterized by perfect overlaps and connects all $N + 1$ different $|D_a^N\rangle$ states succeedingly. The possibility to select different states for the state conversion is discussed in Sec. VIA. Alternatively, such a strong off-resonant field can be used to lift some of the degeneracies in H_A . The energy shifts introduced by this field ($j = 0$) can be used to select certain parts of the Hamiltonian dynamics by a fine detuning of the ladder fields $j = 1, \dots, J$. Such compensation of unwanted terms will be carried out for the Rydberg-trimer case in Sec. VIB.

V. PREPARATION OF CHIRAL W STATES

In this section, we present a state-preparation scheme for twisted W states for the Rydberg system under consideration by making use of the effective Hamiltonian for a single resonant laser field derived in Sec. IV B. It includes the preparation of the chiral states $|\zeta_{1+}\rangle$ and $|\zeta_{1-}\rangle$ as special cases.

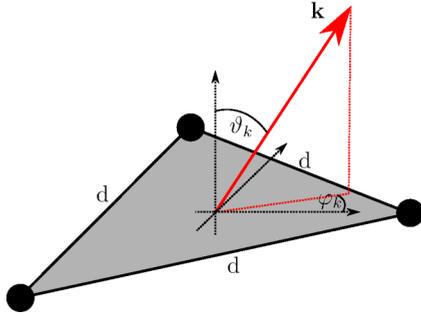


FIG. 2. Schematic drawing of the orientation of the laser field and the atomic plane for the $N = 3$ case. The atoms form an equilateral triangle with interatomic distance d . The orientation of a plane-wave laser field with wave vector \mathbf{k} is defined through the angles ϑ_k and φ_k .

With a single laser field resonant to the Rydberg transition, i.e., $\Delta_1 = 0$, we immediately recognize enhanced Rabi oscillations in Eq. (24), because

$$H_N^{0 \leftrightarrow 1}(\mathbf{k}_1)/\hbar = \Omega_1^* \sqrt{N} |g \dots g\rangle \langle D_1^N(\mathbf{k}_1) | + \text{H.c.} \quad (27)$$

describes the well-known effect of collective Rabi enhancement, with $\Omega_N = \sqrt{N}\Omega_1$ [37]. This effect was first experimentally observed in Ref. [56] and more recently discussed, for example, in Ref. [69]. Due to the site-dependent phases, the oscillations appear between the ground state and the twisted W state $|D_1^N(\mathbf{k}_1)\rangle$.

Regarding the total number of excitations, the different twisted W states are indistinguishable. With the atomic ground state $|g \dots g\rangle$ as initial state, and assuming control of the alignment of the resonant laser field, it is straightforward to prepare a specific class of twisted W states by applying a laser pulse such that half a Rabi oscillation with Rabi frequency Ω_N is carried out. We explicitly determine the phases in the Rydberg-trimer case ($N = 3$), because it is straightforward to implement them on purely geometrical grounds.

If the three atoms are positioned such that they form an equilateral triangle of interatomic distance d , their positions relative to the center of mass can be described via the radial coordinate $r = d/\sqrt{3}$ and three azimuthal angles with relations $\varphi_1 - \varphi_2 = 4\pi/3$ and $\varphi_3 - \varphi_2 = 2\pi/3$. The site-dependent phases of the laser field are given by

$$\mathbf{k}_1 \cdot \mathbf{x}_n = \frac{\omega_A}{c} \frac{d}{\sqrt{3}} \sin \vartheta_k \cos(\varphi_k - \varphi_n), \quad (28)$$

where ϑ_k is the angle between the propagation direction of the laser field and the atomic plane, while φ_k is the azimuthal angle describing the projection onto this plane [cf. Fig. 2]. Because we treat the interaction in the RWA, the resonance frequency is always much higher than the absolute values of the detunings, i.e., $|\Delta_j|/\omega_A \ll 1$, and thus the phases are solely determined by the propagation direction, i.e., ϑ_k and φ_k and the interatomic distance d (for a schematic illustration of the laser orientation with respect to the atomic plane, see Fig. 2).

Since $\sum_n \cos(\varphi_k - \varphi_n) = 0$, we can only describe symmetric twisting in this setup where $\sum_n \mathbf{k} \cdot \mathbf{x}_n = 0$. For example, we can choose the interatomic distance to be twice

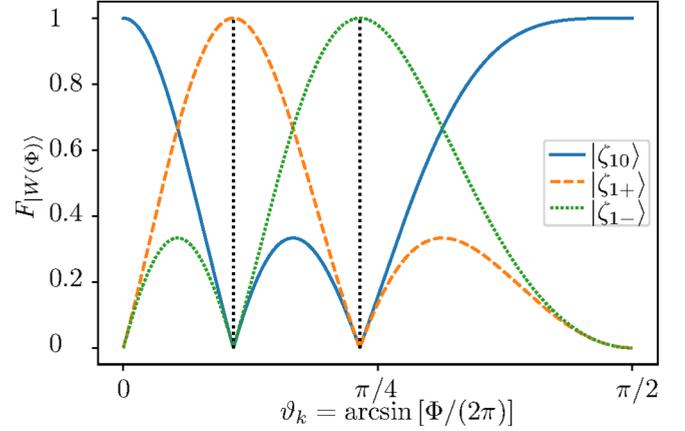


FIG. 3. Fidelity $F_{|W(\Phi)\rangle} = |\langle W(\Phi) | \zeta_{1s} \rangle|$ ($s = 0, \pm$) of the twisted W state $|W(\Phi)\rangle$ corresponding to the chiral basis states $|\zeta_{10}\rangle = |W_3(\mathbf{k} = 0)\rangle$ (solid line), $|\zeta_{1+}\rangle$ (dashed line), and $|\zeta_{1-}\rangle$ (dotted line) as the target states, for different polar angles $\vartheta_k = \arcsin(\Phi/(2\pi))$ and azimuthal angle $\varphi_k - \varphi_2 = \pi/2$. The vertical dotted lines indicate the configurations required for the preparation of the two chiral states.

the resonance wavelength (i.e., $d = 4\pi c/\omega_A$) and the relative polar angle $\varphi_k - \varphi_2 = \pi/2$, such that $\cos(\varphi_k - \varphi_n) = -\sqrt{3}/2, 0, \sqrt{3}/2$. With this setup, the whole range of relative phases $\mathbf{k} \cdot (\mathbf{x}_{1,3} - \mathbf{x}_2) = \mp\Phi$ with $0 \leq \Phi \leq 2\pi$ is achievable by tilting the laser field accordingly with respect to the atomic plane such that $0 \leq \vartheta_k \leq \pi/2$ and $\vartheta_k = \arcsin[\Phi/(2\pi)]$. In the envisioned scheme, half a Rabi oscillation drives the ground state $|ggg\rangle$ into the symmetrically twisted W state

$$|W(\Phi)\rangle = \frac{1}{\sqrt{3}} (e^{-i\Phi} |rgg\rangle + |grg\rangle + e^{i\Phi} |ggr\rangle). \quad (29)$$

In particular, the two chiral states $|\zeta_{1+}\rangle$ and $|\zeta_{1-}\rangle$ can be realized by tilting the propagation direction of the laser field such that $\vartheta_k = \arcsin(1/3)$ and $\vartheta_k = \arcsin(2/3)$, respectively (note that the corresponding values of Φ are $2\pi/3$ and $-2\pi/3$). This is illustrated in Fig. 3, which shows the fidelities $|\langle W(\Phi) | \zeta_{1s} \rangle|$ of the state $|W(\Phi)\rangle$ corresponding to the ordinary W state ($s = 0$) and the two chiral states ($s = \pm$) dependent on the polar angle $\vartheta_k \equiv \arcsin[\Phi/(2\pi)]$ of the laser field. Figure 4 shows an example of a time evolution for the preparation of a twisted W state from the ground state $|ggg\rangle$ via a π pulse.

The fact that the external field only allows the generation of a W state with one specific twisted relative phase [represented by Φ in Eq. (29)] can be seen as a selection rule. Namely, this twisted phase has to match the one characterizing the field itself. In other words, the field only connects the ground state—for which the analog of this twisted phase is zero—to one particular (field-specific) W state.

Given that the states $|\zeta_{10}\rangle \equiv |W_3\rangle$, $|\zeta_{1+}\rangle$, and $|\zeta_{1-}\rangle$ form an orthonormal basis of the $a = 1$ subspace of the total three-atom (qubit) Hilbert space, a combination of three laser fields with these specific alignments could drive the ground state toward an arbitrary linear superposition of these states. In other words, with three laser fields, the preparation of an arbitrary twisted W state of three qubits is possible, since all three orthogonal transitions are addressed.

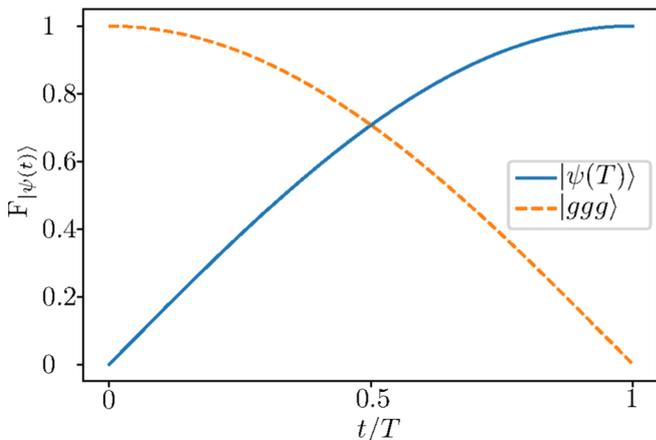


FIG. 4. Typical time evolution of fidelities $F_{|\psi(t)\rangle} = |\langle \psi(t) | \Psi \rangle|$ ($|\Psi\rangle = |ggg\rangle, |\psi(T)\rangle$) corresponding to a π pulse with $\Omega_1 = \pi/(2\sqrt{3}T)$, where T is the conversion time and $|\psi(T)\rangle = |W_3(\mathbf{k}_1)\rangle$ the resulting twisted W state. The time evolution corresponds to the $N = 3$ Hamiltonian [cf. Eq. (37) in Sec. VI A] and randomly chosen phases $\mathbf{k}_1 \cdot \mathbf{x}_n$, with $V/\hbar = 3000/T$ and $\Omega_2 = \Omega_3 = 0$.

One specific application of chiral W states in QIP pertains to implementing noiseless-subsystem (NSS) qubit encoding [57]. NSS encoding is one of the well-known encoding schemes for logical qubits that are inherently robust to noise and constitute an alternative to active error correction. This type of encoding represents a three-qubit generalization of a two-qubit decoherence-free subspace (DFS) encoding [70]. While the latter is robust against global σ_z dephasing, NSS encoding is insensitive to any global Pauli operator [57]. In particular, a dissipative preparation of chiral W states in a trapped-ion system, along with the implementation of noiseless-subsystem encoding, has quite recently been reported [14]. On the other hand, the preparation of such states and the implementation of NSS encoding with neutral atoms in Rydberg states has never been reported before, and thus the scheme proposed here may serve as the basis for an experimental realization.

VI. CONVERSIONS FROM TWISTED W TO GHZ STATES

Having considered the generation of special types of twisted W states in the system at hand, we now turn our attention to the conversion of W states into their GHZ counterparts. While the creation of W -type states, characterized by a single excitation that is shared by all the atoms in an ensemble, represents the hallmark of the RB regime [37], any realization of a GHZ state with strongly interacting Rydberg atoms can be viewed as an antiblockade-type phenomenon [71]. While Rydberg antiblockade for two or more atoms can result from different scenarios, in the strongly interacting regime ($|V| \geq 10 \hbar|\Omega|$, where Ω is the relevant Rabi frequency of the external laser) it entails a dispersive interaction with the specific value of the detuning Δ of the external laser from the relevant internal transition ($\hbar\Delta = V/2$ in the two-atom case; $\hbar\Delta = (N-1)V/N$ in the general case [71]). Our scheme for realizing GHZ states, which involves multiple lasers with differing detunings, is far more complicated than this

conventional scenario. Yet, because it results in a finite probability to have a state with more than one atom simultaneously excited to the Rydberg state in the strongly interacting regime it can be considered as a generalized form of the Rydberg antiblockade.

The structure of the ladder Hamiltonian in Eq. (25) is such that it only connects adjacent energy levels of the atomic ensemble. However, due to the existence of nontrivial offsets, the dynamics inherent to this Hamiltonian are not necessarily enclosed within a subspace of $N+1$ states. In the following, we discuss two approaches whereby one can ensure subspace-enclosed dynamics by selecting one state for each excitation number a . In these cases, the system can be described by an effective Hamiltonian connecting adjacent levels, thus inheriting the already existing solutions for systems described by Hamiltonians of that type. In all the following cases, this is accomplished through a readjustment of Rabi frequencies. In what follows, we will either make use of a simple π pulse to drive half a Rabi oscillation, or an adaption of a more complicated pulse scheme that was utilized for conversions between W and GHZ states in Ref. [45].

In Sec. VI A below, we discuss a scheme where the alignments of the laser fields and the positioning of the atoms are adjusted to ensure a subspace-enclosed dynamics of $N+1$ states via selection rules as in the state-preparation scheme of Sec. V. By contrast, in Sec. VI B we consider an alternative scheme in which an additional laser field lifts some of the degeneracies and the states that participate in the dynamics are singled out via fine detunings of the remaining laser fields and the attendant hierarchies of timescales. This last scheme is then discussed in a broader context in Sec. VI C, where we also demonstrate its soundness by showing that typical state-conversion times are much shorter than the relevant Rydberg-state lifetimes.

A. Conversion schemes involving degenerate Dicke manifolds of states

In contrast to the preparation scheme in Sec. V, the presence of several laser fields complicates the situation as they have, in principle, different propagation directions (i.e., different wave vectors \mathbf{k}_j). Equation (25) can in this case be written as

$$U^\dagger(\mathbf{k}_1) H_N^L(\{\mathbf{k}_a\}) U(\mathbf{k}_1) = \sum_{a=1}^N \hbar\Omega_a^* U(\mathbf{k}_a - \mathbf{k}_1) \sigma_a^- U^\dagger(\mathbf{k}_a - \mathbf{k}_1) + \text{H.c.} \quad (30)$$

The alignment of the laser field resonant to the Rydberg transition ($\Delta_1 = 0$) sets a reference frame insofar that it is the only one connecting the remaining levels to the ground state. Any state conversion scheme naturally starts with all atoms in the ground state or is preceded by a preparation scheme of the kind proposed in Sec. V. Because of that, and without loss of generality, we set the twisting induced by the first laser as the reference one; i.e., we set $\mathbf{k}_1 \cdot \mathbf{x}_n = 0 \pmod{2\pi}$. This transforms all wave vectors \mathbf{k}_j to $\mathbf{k}_j - \mathbf{k}_1$. If all laser fields are properly aligned such that $(\mathbf{k}_j - \mathbf{k}_1) \cdot \mathbf{x}_n = 0 \pmod{2\pi}$, we can split up the a th rising operator into a parallel and orthogonal part with respect to $|D_a^N\rangle$ such that the dynamics

of $|D_a^N\rangle$ states decouples from their orthogonal counterparts:

$$H_N^L/\hbar = \sum_{a=1}^N \Omega_a^* [\sqrt{a(N-a+1)} |D_{a-1}^N\rangle\langle D_a^N| + \sigma_a^- (P_a - |D_a^N\rangle\langle D_a^N|)] + \text{H.c.} \quad (31)$$

Now with $P_D = \sum_{a=0}^N |D_a^N\rangle\langle D_a^N|$ we project onto the subspace just containing the $N+1$ different $|D_a^N\rangle$ states. Hence, we calculate $P_D H_N^L P_D$ and get

$$H_N^{\text{DL}}/\hbar = \sum_{a=1}^N \Omega_a^* \sqrt{a(N-a+1)} |D_{a-1}^N\rangle\langle D_a^N| + \text{H.c.} \quad (32)$$

The resulting effective Hamiltonian is a matching ladder of Dicke states (DL); hence, it connects stepwise all $N+1$ energy levels, such that any state conversion involving adjacent energy levels that are connected via Rabi frequencies can be carried out. For example, pulses not overlapping in time, which induce Rabi half-oscillations corresponding to adjacent transitions would drive the system from the ground- to the highest excited state. Alternative schemes with temporally overlapping pulses are also possible.

We now discuss some special cases of Eq. (32). For $N=3$ we obtain

$$H_3^{\text{DL}}/\hbar = \sqrt{3}\Omega_1^* |ggg\rangle\langle D_1^3| + 2\Omega_2^* |D_1^3\rangle\langle D_2^3| + \sqrt{3}\Omega_3^* |D_2^3\rangle\langle rrr| + \text{H.c.} \quad (33)$$

Similarly, for $N=4$ we have

$$H_4^{\text{DL}}/\hbar = 2\Omega_1^* |gggg\rangle\langle D_1^4| + \sqrt{6}\Omega_2^* |D_1^4\rangle\langle D_2^4| + \sqrt{6}\Omega_3^* |D_2^4\rangle\langle D_3^4| + 2\Omega_4^* |D_3^4\rangle\langle rrrr| + \text{H.c.} \quad (34)$$

These are the same effective Hamiltonians as used in Refs. [44,45] for W -to-GHZ state conversion. However, it is important to point out that a strong off-resonant laser field, as it was utilized in this previous studies, is not a prerequisite for obtaining these effective Hamiltonians, as long as the state of the atomic ensemble fulfills $P_D|\psi(t)\rangle = |\psi(t)\rangle$ for all times t during the conversion process. This can be achieved by properly aligning all laser fields as discussed above. With other alignments, effective Hamiltonians which include orthogonal chiral states can be designed exploiting selection rules and the additional twisting induced by the laser fields. We will carry this out explicitly for the Rydberg-trimer case ($N=3$).

In the basis of $|\zeta_{as}\rangle$ states [cf. Eqs. (14)], we can recast Eq. (30) in the form

$$H_3^L(\{\mathbf{k}_a\})/\hbar = \Omega_1^* \sqrt{3} |ggg\rangle\langle \zeta_{10}| + \Omega_3^* \sqrt{3} U(\mathbf{k}_3) |\zeta_{20}\rangle\langle rrr| U^\dagger(\mathbf{k}_3) + \Omega_2^* U(\mathbf{k}_2) [2|\zeta_{10}\rangle\langle \zeta_{20}| - |\zeta_{1+}\rangle\langle \zeta_{2+}| - |\zeta_{1-}\rangle\langle \zeta_{2-}|] U^\dagger(\mathbf{k}_2) + \text{H.c.} \quad (35)$$

Atom positions and laser-field alignment chosen such that

$$U(\mathbf{k}_2)|\zeta_{1\pm}\rangle = |\zeta_{10}\rangle, \quad U(\mathbf{k}_2)|\zeta_{2\pm}\rangle = U(\mathbf{k}_3)|\zeta_{20}\rangle \quad (36)$$

would single out a $-|\zeta_{10}\rangle\langle \zeta_{2-}|$ ($-|\zeta_{10}\rangle\langle \zeta_{2+}|$) transition operator in the Hamiltonian in the upper-sign (lower-sign) case. If the initial state lies in the subspace spanned by the four

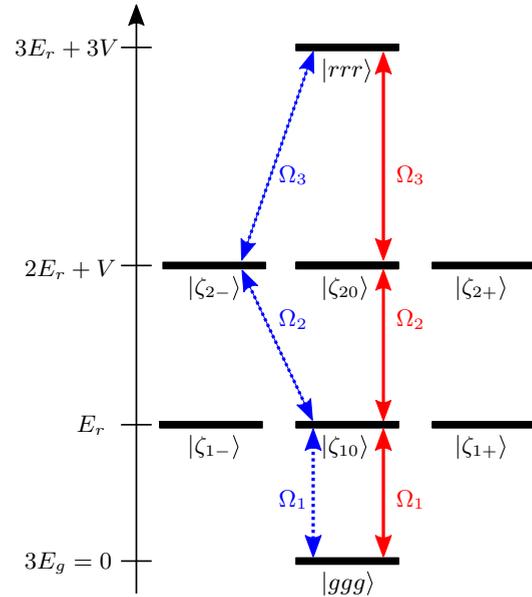


FIG. 5. Energies of the atomic ensemble with $N=3$ corresponding to the states $|\zeta_{as}\rangle$ [cf. Eqs. (14)]. The solid arrows indicate transitions driven in the case when all laser fields are aligned such that $(\mathbf{k}_j - \mathbf{k}_1) \cdot \mathbf{x}_n = 0$, while the dotted ones indicate those corresponding to the choice of laser-field alignments described by the upper-sign case of Eq. (36). Here $E_g = 0$ is chosen as origin for the energy scale.

states $\{|ggg\rangle, |W\rangle, |\zeta_{2-}\rangle, |\zeta_{2+}\rangle, |rrr\rangle\}$ the unitary time evolution of the atomic ensemble is enclosed in this subspace. State conversion schemes where the underlying Hamiltonian connects adjacent levels can easily be adapted by adjusting the Rabi frequencies. Effective Hamiltonians including different states of the $a=2$ subspace are indicated in the level scheme of Fig. 5.

To illustrate the differences in the effective Hamiltonian with respect to the alignments, the evolution of the initial state $|\psi(t=0)\rangle = |W\rangle$ is calculated numerically [72–75] based on the interaction Hamiltonian

$$H_I(t) = \sum_{j=1}^3 \sum_{n=1}^3 (\hbar\Omega_j e^{i(\mathbf{k}_j \cdot \mathbf{x}_n - \Delta_j t)} |r\rangle_{nm} \langle g| + \text{H.c.}) + \sum_{p<q} V_{pq} |rr\rangle_{pq} \langle rr| \quad (37)$$

with constant Rabi frequencies Ω_i realizing W to GHZ conversion via $|\zeta_{20}\rangle$.

In the case when all laser fields are aligned, the values of the Rabi frequencies are $T_0\Omega_1 = 1.22/\sqrt{3}$, $T_0\Omega_2 = 1.42/2$, and $T_0\Omega_3 = 2.35/\sqrt{3}$, where T_0 is the conversion time. These specific values of the constant Rabi frequencies are determined in Ref. [45] and are based on the observation that under the assumption of real-valued Rabi frequencies state conversion in a four-level system is characterized by the dynamical symmetry $\text{su}(2) \oplus \text{su}(2) \cong \text{so}(4)$. Therefore, it can effectively be described in the form of two pseudospin-1/2 degrees of freedom. The fact that only terms connecting adjacent excitation subspaces ($a-1 \leftrightarrow a \leftrightarrow a+1$) appear in the effective

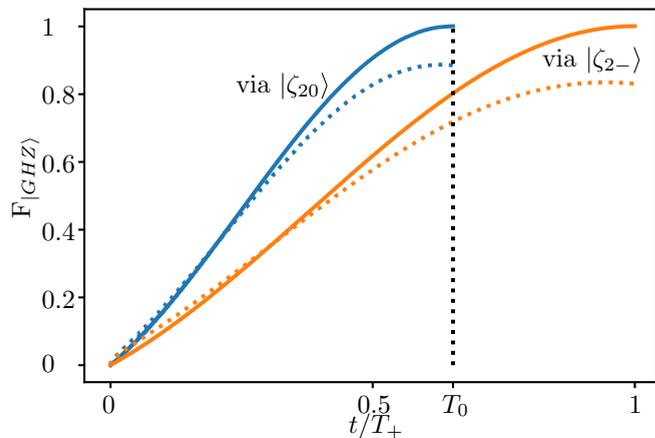


FIG. 6. Time dependence of target-state fidelities $F_{|GHZ\rangle} = \sqrt{\langle GHZ | \rho(t) | GHZ \rangle}$ [cf. Eq. (39)] for the W -to- GHZ state conversions via two different intermediate states $|\zeta_{20}\rangle$ and $|\zeta_{2-}\rangle$, both shown for $VT_+/\hbar = 3000$. The solid lines correspond to the unitary dynamics where $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$. The dotted lines correspond to the open-system dynamics with the dephasing and spontaneous-decay rates $\Gamma = \gamma = 0.1/T_+$. The two conversion pathways are adjusted such that their respective total laser-pulse energy consumptions are mutually equal.

Hamiltonian introduces constraints to the full dynamics of the two pseudospins.

For the conversion between W and GHZ states via $|\zeta_{2-}\rangle$ [cf. the upper-sign case of Eq. (36)] to be carried out in the same time T_0 , the second Rabi frequency would have to be doubled. In order to be able to compare the two conversion paths, we adjust all Rabi frequencies such that the total laser-pulse energy over the corresponding conversion time is the same in both cases. The total laser-pulse energy is given by

$$A(t) = \int_0^t \sum_{j=1}^3 |\Omega_j(t')|^2 dt'. \quad (38)$$

(The time dependence of the Rabi frequencies is introduced here only for later convenience.) Both schemes allow one to carry out the desired state conversion, but the conversion via the achiral state is faster under the assumption of equal laser-pulse energy consumption. In both schemes, only one of the $a = 2$ states acts as intermediate state in the conversion process while the other such states are never occupied. The target state in both cases is

$$|GHZ\rangle = \frac{1}{2}(|ggg\rangle + e^{-i3Vt}|rrr\rangle), \quad (39)$$

where the time-dependent relative phases account for the energy shift arising due to the constant energy difference between levels in Eq. (37).

The results obtained in numerical calculations, which correspond to $V_{pq} = V$ and $V/\hbar = 3000/T_+$ in both cases (where T_+ is the conversion time in the upper-sign case), are shown in Fig. 6. What can be inferred from these results is that—while both conversion schemes realize the target state—the scheme that makes use of $|\zeta_{20}\rangle$ as intermediate state requires a significantly shorter time than the one where

$|\zeta_{2-}\rangle$ plays the analogous role. For the sake of completeness, it should be stressed that yet another state-conversion pathway—equivalent to the second one—that makes use of $|\zeta_{2+}\rangle$ as its intermediate state, is also possible [lower-sign case of Eq. (36)].

In realistic experimental setups spontaneous decay of the Rydberg state and dephasing, caused, e.g., by atomic motion [76], limit the lifetime and the accuracy of the proposed state-conversion schemes. To take such effects into account, we characterize the corresponding open-system dynamics within the framework of the Lindblad master equation [77]. In this framework, the dynamics of the density operator $\rho(t)$ is governed by the equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_I(t), \rho(t)] + \sum_{l=1}^2 \frac{1}{2}([\rho(t)L_l, L_l^\dagger] + [L_l\rho(t), L_l^\dagger]), \quad (40)$$

where the two relevant Lindblad operators are given by

$$L_1 = \sqrt{\Gamma} \sum_{n=1}^3 |g\rangle_{nm}\langle r|, \quad L_2 = \sqrt{\gamma} \sum_{n=1}^3 (|g\rangle_{nm}\langle g| - |r\rangle_{nm}\langle r|). \quad (41)$$

Here L_1 describes spontaneous decay from the Rydberg to the ground state of an atom with decay rate Γ , while L_2 describes the dephasing of these states with the rate γ . We solved the last Lindblad master equation numerically [72–75], choosing rather high rates $\Gamma = \gamma = 0.1/T_+$. Needless to say, the target-state fidelity [cf. Fig. 6] in the open-system scenario is smaller than those found in the closed-system treatment. The obtained results for the fidelity speak in favor of using the faster conversion path, as the debilitating effects of spontaneous decay and dephasing are weaker for that path.

The preparation of chiral states discussed in Sec. V and the state-conversion scheme presented here, rely heavily on setting the site-dependent phases $\mathbf{k}_j \cdot \mathbf{x}_n$. Therefore, it is necessary to control the orientation of each laser field relative to the atomic plane and the position of the atoms to a precision of the order of the laser wavelength. Owing to the recent advances in manipulation and control of cold neutral atoms in optical tweezers [59–61], this last requirement is within experimental reach.

To describe the influence of fluctuations in the atomic positions on the conversion scheme, we consider random variations of atomic positions. These variations affect not only the phases $\mathbf{k}_j \cdot \mathbf{x}_n$, but also the interatomic potentials V_{pq} , because they cause the arrangement of atoms to deviate from the original equilateral triangle. To differentiate these two effects, it is prudent to concentrate on the faster conversion scheme via $|\zeta_{20}\rangle$. Because in this case all laser fields are aligned, we have $\mathbf{k}_j \cdot \mathbf{x}_n = \mathbf{k}_j \cdot \mathbf{x}_n$. As already discussed above, $\mathbf{k}_1 \cdot \mathbf{x}_n$ sets the reference phase. Hence, if we neglect a misalignment of laser fields, the conversion scheme via $|\zeta_{20}\rangle$ is not affected by the random phases. No further matching conditions as in Eq. (36) have to be fulfilled. Furthermore, due to the scaling of $V_{pq} = C_6/d_{pq}^6$, the influence of varying interaction potentials V_{pq} can be expected to dominate over small variations in the phase-matching conditions.

In order to quantify the effect of the deviation from its original value $V = C_6/d^6$ at interatomic distance d , we computed

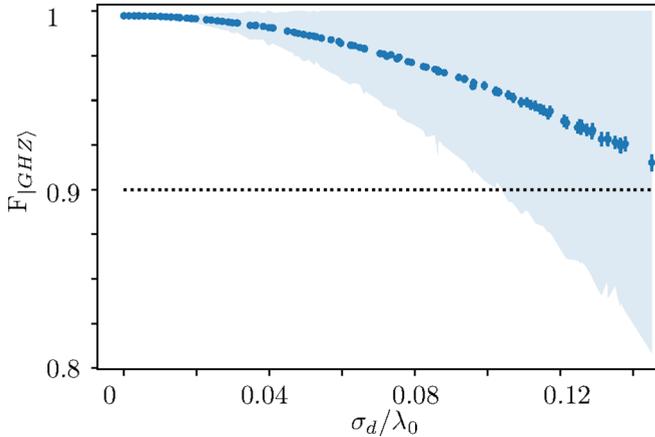


FIG. 7. Mean values of the GHZ-state fidelity $F_{|GHZ\rangle}$ and their corresponding standard deviations σ_F (shaded area) corresponding to the W -to-GHZ state conversion via $|\zeta_{20}\rangle$, computed from a sample of $S = 500$ results, for varying standard deviation σ_d of the interatomic distance. The error bars show the standard error σ_F/\sqrt{S} of the mean fidelity. The parameter values used are $V/\hbar = 30.86/T_0$ and $d = 40\lambda_0$, where T_0 is the state-conversion time and λ_0 is the resonance wavelength.

the different interatomic potentials $V_{pq} = Vd^6/d_{pq}^6$ according to randomly sampled atomic positions. We introduce random errors for each of the spatial coordinates of the three atoms. Accordingly, $\mathbf{x}_n \rightarrow \mathbf{x}_n + \boldsymbol{\epsilon}_n$ differs for all atoms n ($n = 1, 2, 3$). In each realization, the nine components of the three error vectors $\boldsymbol{\epsilon}_n$ were independently drawn from a standard normal distribution of standard deviation σ resulting in varying distances d_{pq} and, accordingly, three different V_{pq} per realization. We then numerically computed the time evolution according to Eq. (37), where random positioning error vectors $\boldsymbol{\epsilon}_n$ were drawn componentwise from a standard normal distribution with $\sigma \in [0, 0.1\lambda_0]$, where λ_0 is the resonance wavelength, and sample size $S = 500$. This numerical evaluation was repeated for different choices of the standard deviation σ resulting in different standard deviations $\sigma_d = \sqrt{(d_{pq} - \bar{d}_{pq})^2}$ of all $3S$ different values of d_{pq} per sample (where \bar{d}_{pq} is the mean of all $3S$ d_{pq} per sample). The parameter values used in these calculations were $V/\hbar = 30.86/T_0$, where T_0 is the conversion time, and $d = 40\lambda_0$.

Figure 7 shows the obtained mean values of the GHZ-state fidelity $F_{|GHZ\rangle} = |\langle GHZ | \psi(T_0) \rangle|$ and its corresponding standard deviation $\sigma_F = \sqrt{(F_{|GHZ\rangle} - \bar{F}_{|GHZ\rangle})^2}$ for different values of σ_d at $t = T_0$ for the W -to-GHZ state conversion via $|\zeta_{20}\rangle$. It can be inferred from the obtained results that the mean values of the fidelity are above 0.9 for the whole range of considered values of σ . The chosen simulation parameters were assumed to have values characteristic of alkali atoms most often used in optical-tweezer experiments, with the principal quantum number $n = 50$ and the interatomic distance $d = 4 \mu\text{m}$ [25]. This speaks in favor of the experimental feasibility of the proposed state-conversion scheme.

In addition to the already presented conversion scheme, we discuss an alternative approach in Sec. VI B. That approach makes use of an additional laser field to set energy

shifts in the atomic Hamiltonian. This singles out a specific twisted W state determined by the site-dependent phases of this strong driving field. In turn, this allows one to address specific atomic states via fine detunings δ_j of the other laser fields ($j = 1, 2, 3$).

B. Conversion schemes involving lifted degeneracies

The effective Hamiltonians derived for state-conversion tasks in Sec. VI A depend on proper relative alignment of the resonant laser fields involved and precise positioning of the Rydberg atoms. Misalignment or errors in the positioning of the atoms result in unwanted phase shifts. We can use a combination of nearly resonant fields (enumerated by $j = 1, 2, 3$) and an additional stronger field ($j = 0$), where the latter sets energy shifts such that it lifts some of the degeneracies of H_A . Fine detunings δ_i added onto the detunings Δ_i can then address specific transitions. This procedure is inspired by the derivation of the effective Hamiltonian in Ref. [44] but realizes a generalized version including relative twisting.

Again, we explicitly calculate it for the Rydberg trimer case $N = 3$. The corrections to H_A in this case are given by

$$H_3^{\text{shift}} = H_3^{\text{off}}(\mathbf{k}_0) + H_3^L(\{\mathbf{k}_a\}). \quad (42)$$

It is the combination of the off-resonant case from Eq. (21) and the ladder Hamiltonian Eq. (25) for $N = 3$ being

$$\begin{aligned} H_3^L(\{\mathbf{k}_a\})/\hbar = & \left[\Omega_1^* \sqrt{3} |ggg\rangle \langle D_1^3(\mathbf{k}_1)| \right. \\ & + \Omega_2^* \left(3 |D_1^3(\mathbf{k}_2)\rangle \langle D_2^3(\mathbf{k}_2)| \right. \\ & \left. \left. - \Phi^*(\mathbf{k}_2) \sum_{n=1}^3 e^{2ik_2 \cdot \mathbf{x}_n} |gg\rangle \langle r|_{nm} \langle g| \langle rr| \right) \right. \\ & \left. + \sqrt{3} \Omega_3^* |D_2^3(\mathbf{k}_3)\rangle \langle rrr| \Phi^*(\mathbf{k}_3) \right] + \text{H.c.} \quad (43) \end{aligned}$$

with $\Phi(\mathbf{k}) = e^{ik \cdot \sum_{n=1}^N \mathbf{x}_n}$. The overlaps of twisted states corresponding to different fields are

$$\begin{aligned} \langle D_1^3(\mathbf{k}_1) | D_1^3(\mathbf{k}_2) \rangle &= \frac{1}{3} \Sigma_{\mathbf{k}_2 - \mathbf{k}_1}, \\ \langle D_2^3(\mathbf{k}_2) | D_2^3(\mathbf{k}_3) \rangle &= \frac{e^{i(\mathbf{k}_3 - \mathbf{k}_2) \cdot \sum_{n=1}^3 \mathbf{x}_n}}{3} \Sigma_{\mathbf{k}_3 - \mathbf{k}_2}, \quad (44) \end{aligned}$$

with $0 \leq \Sigma_{\mathbf{k}} = \sum_{n=1}^3 e^{ik \cdot \mathbf{x}_n} \leq 3$, which describes the amount of relative twisting between two different twisted states of the same excitation number. If there are no relative phase differences, $\Sigma_{\mathbf{k}_j - \mathbf{k}_i} \rightarrow 3$. If $\Sigma_{\mathbf{k}_j - \mathbf{k}_i} = 0$, both laser fields address orthogonal states and the effective Hamiltonian would split in several unconnected dynamics. However, as long as the overlaps do not vanish, we can compensate for it by driving transitions with higher Rabi frequencies.

To show that, we transform the effective Hamiltonian to an interaction picture with respect to the stronger off-resonant laser field with Rabi frequency $\Omega_0 \gg \Omega_j$ ($j = 1, 2, 3$). Since

$[H_A, H_3^{\text{off}}(\mathbf{k}_0)] = 0$, H_A is not affected by this transformation, and we evaluate the remaining parts as

$$\begin{aligned} & e^{iH_3^{\text{off}}(\mathbf{k}_0)t/\hbar} H_3^L(\{\mathbf{k}_a\}) e^{-iH_3^{\text{off}}(\mathbf{k}_0)t/\hbar} \\ &= U(\mathbf{k}_0) e^{iH_3^{\text{off}}t/\hbar} \sum_{a=1}^3 [\Omega_a^* U(\mathbf{k}_a - \mathbf{k}_0) \sigma_a^- U^\dagger(\mathbf{k}_a - \mathbf{k}_0) \\ &+ \text{H.c.}] e^{-iH_3^{\text{off}}t/\hbar} U^\dagger(\mathbf{k}_0), \end{aligned} \quad (45)$$

where H^{off} without wave-vector argument stands for the operator without any twisting. Now we can introduce small fine detunings δ_j to the resonant fields such that the total detunings are $\Delta_j^{\text{total}} = \Delta_j + \delta_j$. If $|\delta_j| \ll |\Delta_0|, V/\hbar$, the fine detunings do not change the calculation of the effective Hamiltonians as discussed in Sec. IV B and Appendix A 2, since δ_j never contributes significantly.

However, in an interaction picture with respect to $H_0 = H_A + H_F$ the Rabi frequencies are shifted $\Omega_j \rightarrow \Omega_j e^{-i\delta_j t}$ (associated with the atomic rising operator) due to the fine detunings being part of the time dependencies of the field operators. Now these fine detunings can be used to compensate the oscillatory behavior of one term per laser field appearing in Eq. (45). Unwanted terms will still oscillate with different residual frequencies ω_R . However, if $\min\{|\omega_R|\} \gg T^{-1}$, where $\{|\omega_R|\}$ is the set of all the relevant residual frequencies and T is the conversion time in question, we can ignore all terms with nonvanishing exponents in Eq. (45). By choosing

$$\begin{aligned} \delta_1 &= (-6s_0 + 4s_1)/\hbar, & \delta_2 &= (3s_0 - 8s_1 + 3s_2)/\hbar, \\ \delta_3 &= (4s_1 - 6s_2)/\hbar, \end{aligned} \quad (46)$$

with $s_a = \hbar^2 |\Omega_0|^2 / (\hbar \Delta_0 - aV)$ ($a = 0, 1, 2$), we obtain the following twisted-ladder (TL) Hamiltonian:

$$\begin{aligned} \tilde{H}_3^{\text{TL}}/\hbar &= \sqrt{3}\Omega_1 \frac{|\Sigma_{\mathbf{k}_0 - \mathbf{k}_1}|}{3} |\widetilde{ggg}\rangle \langle D_1^3(\mathbf{k}_0)| \\ &+ \sqrt{3}\Omega_3 \frac{|\Sigma_{\mathbf{k}_0 - \mathbf{k}_3}|}{3} |\widetilde{D}_2^3(\mathbf{k}_0)\rangle \langle \widetilde{rrr}| \\ &+ 2\Omega_2 \frac{|\Sigma_{\mathbf{k}_0 - \mathbf{k}_2}|}{3} |D_1^3(\mathbf{k}_0)\rangle \langle \widetilde{D}_2^3(\mathbf{k}_0)| + \text{H.c.} \end{aligned} \quad (47)$$

Here, in order to ensure that Rabi frequencies are real valued, we included additional phases into the redefined atomic states

$$\begin{aligned} |\widetilde{ggg}\rangle &= e^{i\varphi(\Sigma_{\mathbf{k}_0 - \mathbf{k}_1})} |ggg\rangle, \\ |\widetilde{D}_2^3(\mathbf{k}_0)\rangle &= e^{-i\varphi(\Sigma_{\mathbf{k}_0 - \mathbf{k}_2})} |D_2^3(\mathbf{k}_0)\rangle, \\ |\widetilde{rrr}\rangle &= e^{i[\varphi(\Sigma_{\mathbf{k}_0 - \mathbf{k}_2}) - \varphi(\Sigma_{\mathbf{k}_0 - \mathbf{k}_3}) + \mathbf{k}_0 \cdot \sum_{n=1}^3 \mathbf{x}_n]} |rrr\rangle, \end{aligned} \quad (48)$$

where $\varphi(z)$ is the argument of the complex number z . The driven transitions are indicated in Fig. 8. A more detailed derivation of the twisted-ladder Hamiltonian can be found in Appendix B.

The result is a Hamiltonian connecting twisted states with adjacent numbers of excitations. The twisting is solely determined by the site-dependent phases of the strong laser field ($j = 0$). Choosing other fine detunings δ_j would result in different residual frequencies and other effective Hamiltonians, e.g., including chiral states [relative to $U(\mathbf{k}_0)|\zeta_{10}\rangle$]. Yet, because the chiral states are still energetically degenerate with

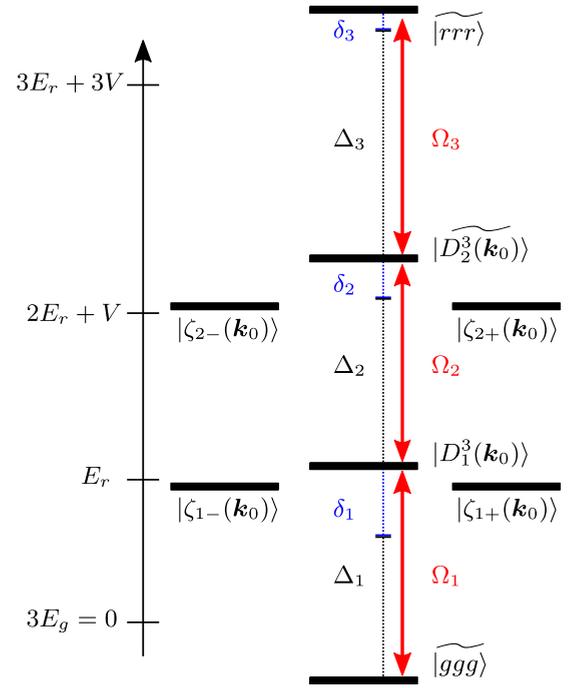


FIG. 8. Energy-level scheme corresponding to the Hamiltonian $H_A + H_3^{\text{off}}(\mathbf{k}_0)$ for Rydberg trimers. E_g is chosen as origin of the energy scale. The arrows indicate the transitions driven by laser fields with detunings $\Delta_i + \delta_i$ as determined by Eq. (46) [cf. Eq. (47)].

respect to $H_3^{\text{off}}(\mathbf{k}_0)$, the effective Hamiltonian would be of higher dimension. If all $\mathbf{k}_a \cdot \mathbf{x}_n \rightarrow 0$, i.e., without any twisting, this final result reproduces the effective Hamiltonian as discussed in Ref. [44]. However, here we derived a generalized version which includes relative twisting due to the different laser fields. The smaller the specific $|\Sigma_{\mathbf{k}_0 - \mathbf{k}_a}|$ becomes, the higher the corresponding Rabi frequency Ω_a has to be for a specific conversion to be possible in a given time frame. Those adjustments are only possible as long as all $|\Sigma_{\mathbf{k}_0 - \mathbf{k}_a}|$ are not too small, because with increasing Rabi frequencies the perturbative treatment eventually breaks down. Alternatively, the conversion time has to be increased accordingly which allows the Rabi frequencies to remain sufficiently small.

With the effective Hamiltonian (47) we can consider the preparation of twisted W states as in Sec. V, but now the amount of twisting is solely determined by the stronger off-resonant field ($j = 0$). With $\Omega_2 = \Omega_3 = 0$ we have an effective Hamiltonian of Eq. (47) describing Rabi oscillations between $|ggg\rangle$ and $|D_1^3(\mathbf{k}_0)\rangle$ with effective Rabi frequency $\Omega' = \sqrt{3}\Omega_1 |\Sigma_{(\mathbf{k}_0 - \mathbf{k}_1)}|/3$. Therefore, under the assumptions of equal laser-pulse energy consumption [cf. Eq. (38)] the conversion time increases as $T = 9T_0 / |\Sigma_{\mathbf{k}_0 - \mathbf{k}_1}|^2$, where T_0 is the reference conversion time without any relative twisting. Furthermore, the constant laser fields from Ref. [45] as already used in the last subsection implement a state conversion from twisted $|W(\mathbf{k}_0)\rangle = |D_1^3(\mathbf{k}_0)\rangle$ to the GHZ state

$$|\text{GHZ}(\mathbf{k}_0)\rangle = \frac{1}{\sqrt{2}} (|\widetilde{ggg}\rangle + e^{i\varphi} |\widetilde{rrr}\rangle), \quad (49)$$

if Rabi frequencies are adjusted by $3/|\Sigma_{\mathbf{k}_0 - \mathbf{k}_a}|$ for $a = 1, 2, 3$, respectively.

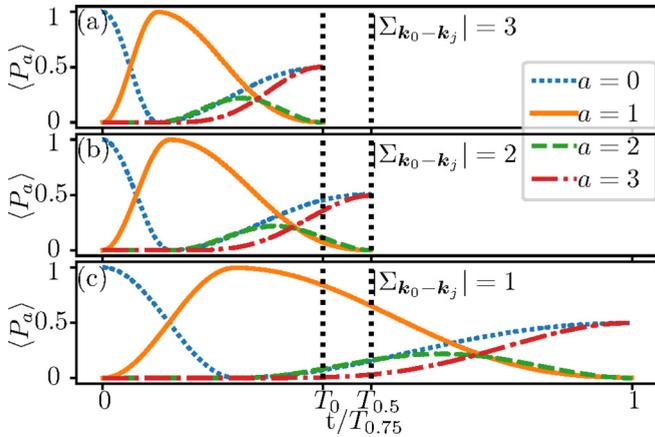


FIG. 9. Expectation values $\langle P_a \rangle = \langle \psi(t) | P_a | \psi(t) \rangle$ for the atomic excitation number a over the conversion time for a conversion scheme $|ggg\rangle \xrightarrow{\pi\text{-pulse}} |D_1^3(\mathbf{k}_0)\rangle \xrightarrow{[45]} \text{GHZ state}$ for different amounts of relative twisting corresponding to $|\Sigma_{k_0-k_j}| = 3, 2, 1$ [(a)–(c)]. $T_{0,0.5,0.75}$ is the respective conversion time under the assumption of equal laser-pulse energy consumption [cf. Eq. (38)].

To illustrate the discussed adjustments, we give an example combining preparation of the twisted state $|D_1^3(\mathbf{k}_0)\rangle$ from the ground state, followed by its conversion into a GHZ state for different amounts of relative twisting. The laser fields $j = 1, 2, 3$ are all aligned with the same polar angles ϑ_k (cf. Fig. 2) and azimuthal angle φ_k . We compare three different alignments labeled via $s = 3 \sin(\vartheta_k) = 0, 0.5, 0.75$, such that $|\Sigma_{k_0-k_j}| = 3, 2, 1$. The first part is executed via a π pulse of the field $j = 1$ and the second one via the constant Rabi frequencies as mentioned before. The π pulse is set to take a quarter of the total respective conversion time T_s . All values of Rabi frequencies ($j = 1, 2, 3$) are adjusted such that the total laser-pulse energy consumption is the same in all cases [cf. Eq. (38)]. We numerically evaluate [72–75] the dynamics governed by the interaction Hamiltonian

$$H_I(t) = H_3^{\text{off}}(\mathbf{k}_0) + \sum_{j=1}^3 \sum_{n=1}^3 [\hbar \Omega_j(t) e^{i(k_j \cdot x_n - \delta_j t)} |r\rangle_{nm} \langle 0| + \text{H.c.}] \quad (50)$$

with initial state $|\psi(t=0)\rangle = |ggg\rangle$ and the time dependencies of $\Omega_j(t)$ chosen in the form of step functions, such that they equal the respective constant values at all times. Here we disregard the fast dynamics due to H_A and the strong laser field with Rabi frequency Ω_0 and detuning Δ_0 , because we are only interested in the slower dynamics introduced by the three fields $j = 1, 2, 3$.

For definiteness, we set $\Omega_0 = -0.03\Delta_0$ and $\hbar\Delta_0/V = -0.7$. The negative detuning Δ_0 with respect to the Rydberg transition ensures that the field is even more detuned with respect to transitions involving the RB, and hence $|s_0\rangle > |s_1\rangle > |s_2\rangle$ with $T_0 s_0/\hbar = -1247$ where T_0 is the considered conversion time in the case without relative twisting. The relation between detuning and the interaction energy shift V ensures that all residual frequencies satisfy the condition $\min\{|\omega_R|\}T_0 > 600$. The obtained numerical results are presented in Fig. 9. Similar to the atomic Hamiltonian H_A which

is not considered here, the first-order correction $H_3^{\text{off}}(\mathbf{k}_0)$ and the relative twisting give rise to phases [cf. Eqs. (48)]. Since we are not interested in such relative phases, we just show expectation values $\langle \psi(t) | P_a | \psi(t) \rangle$. As was to be expected, the population is transferred to the one-excitation subspace ($a = 1$) via the π pulse and the ensuing conversion scheme leads to a GHZ state, such that $\langle \psi(T_k) | P_0 | \psi(T_k) \rangle = \langle \psi(T_k) | P_3 | \psi(T_k) \rangle = 0.5$. The required conversion time becomes significantly longer for a larger relative twisting, i.e., for smaller $|\Sigma_{k_0-k_j}|$.

With this approach by an additional laser field, the reference frame of the twisted states participating in the effective dynamics can be set by the alignment of the strong field ($j = 0$). Furthermore, even if—for experimental reasons—perfect alignment of the other fields or perfect positioning of the atoms is not possible, this can be partially compensated for by an adjustment of the Rabi frequencies. This speaks in favor of the flexibility of the proposed scheme.

To summarize, within the proposed state-conversion scheme, suitable combinations of interatomic distances and laser orientations allow control of site-dependent phases. This opens up the possibility to address different states during the conversion process, with only a slight adjustment of the Rabi frequencies of external lasers. It should be emphasized that, while a strong field is not necessary for our scheme, such a field can still be used for selecting—in combination with fine detunings addressing lifted degeneracies—twisted states that participate in the laser-controlled dynamics.

C. Timescale hierarchies and conversion times

Various schemes for generating entanglement in quantum systems can be divided into those based on controlled dissipation [78,79] and those that are governed by timescale hierarchies [80]. Our state-conversion scheme in Sec. VI B belongs to the latter group of schemes, which generically entail the application of a strong “dressing drive” at rate G simultaneously with other interactions that are characterized by rates g_i . The linchpin of such schemes is that the dressing drive creates resonances that are resolved by the other drives in the limit $g_i \ll G$, and the corresponding hierarchy of timescales $g_i^{-1} \gg G^{-1}$ is what protects the entangled target state.

It is important to stress that for all schemes based on timescale hierarchies the steady-state entanglement fidelity only asymptotically approaches unity upon increasing the relative strength $G/\max\{g_i\}$ of the dressing drive. At the same time, timescale hierarchies limit the entanglement-generation speed, because the other interactions g_i populating the entangled target state must be driven slowly compared to experimentally achievable rates for G .

To demonstrate the soundness of our proposed state-conversion scheme, it is important to show that despite the limitations imposed by the aforementioned timescale hierarchies our characteristic state-conversion times are significantly shorter than the relevant Rydberg-state lifetimes. The latter scale as $\tau_n \propto n^3$, where n is the principal quantum number, so that for $n \approx 50$ one has $\tau_n \approx 100 \mu\text{s}$ [24]. In particular, the hierarchy of timescales in the system at hand dictates the following inequalities for the pulse duration T_{int} ,

the Rabi frequencies g_i , the spontaneous decay rate κ/n^3 of the n th Rydberg state with κ denoting a typical spontaneous decay rate of an energetically low-lying bound state, and the Stark-induced level shift G (all expressed in frequency units):

$$\kappa/n^3 \ll T_{\text{int}}^{-1} \lesssim |g_i| \ll |G| \ll E_1/(\hbar n^3). \quad (51)$$

Here $E_1/(\hbar n^3)$ is the frequency corresponding to the level spacing between Rydberg states n and $n+1$, with $E_1/\hbar = 10^{16} \text{ s}^{-1}$ being its counterpart corresponding to the ionization energy $E_1 \approx 13, 6 \text{ eV}$ of the hydrogen atom, and $\kappa = 10^9 \text{ s}^{-1}$ (note that κ is seven orders of magnitude smaller than E_1/\hbar due to $\hbar\kappa/E_1$ being proportional to the third power of the fine-structure constant $\alpha \approx 1/137$).

The last conditions can be fulfilled, for example, for a Rydberg state with $n = 50$ by choosing the relevant parameters such that $|G| = 10^{10}/125 \text{ s}^{-1}$ and $|g_i| = 10^8/125 \text{ s}^{-1}$. This further yields

$$125 \times 10^{-10} \ll 125 \times 10^{-8} \text{ s} \lesssim T_{\text{int}} \ll 125 \times 10^{-6} \text{ s}. \quad (52)$$

Because the difference between spontaneous-decay rates and typical optical transition frequencies always involves a factor of $\alpha^3 \approx 10^{-7}$, these last conditions imply that the characteristic state-conversion times in the system at hand are much smaller than the typical Rydberg-state lifetimes even if the rates g_i and G differ by a factor of 100. Therefore, typical conversion times in a neutral Rydberg-atom system are of the order of microseconds.

VII. SUMMARY AND CONCLUSIONS

To summarize, in this paper we addressed the problem of dynamical state generation (i.e., state preparation and conversion) in the Rydberg-blockade regime of a neutral-atom system in which the atomic ensemble is subject to multiple external laser fields. We presented a preparation scheme for twisted W states, which assumes precise control over the alignment of resonant laser fields and the positioning of

atoms. We illustrated this scheme in the special case of three-qubit chiral W states, a special type of W states of importance for implementing noiseless-subsystem qubit encoding [57]. In addition, we showed that conversions from twisted W to GHZ states are possible by adopting already known pulse schemes for ordinary W states. We further showed, that even if such a precision in positioning of the atoms is not possible, a state conversion starting from twisted W states is still possible. It involves a downward renormalization of the relevant Rabi frequencies. Thus, somewhat higher laser-pulse energies are required to carry out the desired state conversion within the same time frame. We demonstrated the soundness of our state-conversion scheme by showing that the typical state-conversion times are much smaller than relevant Rydberg-state lifetimes.

Several possible directions of future work can be envisioned. First, while all the examples of state engineering in the present work pertained to a three-atom (qubit) system, the preparation of general twisted W states in lattice-periodic systems is of utmost importance in the area of analog quantum simulation [81,82]. Namely, owing to their known connection with single-excitation Bloch states, such states represent the desired states of analog simulators [83–86] prior to performing interaction quenches of various types [87]. Second, while our proposed state-conversion scheme is characterized by timescale hierarchies, it would be instructive to devise its counterparts based on controlled dissipation [78,79]. Last but not least, the state preparation scheme proposed here can be extended to other classes of generalized W states, different from the twisted ones. For instance, an interesting W -type state was proposed in the past for applications in quantum teleportation and superdense coding [5]. However, this state has never been realized with Rydberg-atom-based qubits.

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APPENDIX A: DERIVATION OF THE EFFECTIVE HAMILTONIANS

1. Off-resonant field

In what follows, we provide a detailed derivation of the effective Hamiltonian for a field (enumerated by $j = 0$), which is far from resonance to any transition in H_A .

With the projectors P_a^m discussed at the beginning of Sec. IV A, we have $P_a^m H_{\text{int}} P_a^m = 0$ and $\sum_{m=0}^{\infty} \sum_{a=0}^N P_a^m H P_a^m = H_0$, such that the first-order effective Hamiltonian is

$$H_{\text{eff}} = H_0 + \sum_{m=0}^{\infty} \sum_{a=0}^N P_a^m H_{\text{int}} Q_a^m \frac{1}{E_a - Q_a^m H_0 Q_a^m} Q_a^m H_{\text{int}} P_a^m, \quad (A1)$$

where $Q_a^m = 1 - P_a^m$. Furthermore, since the interaction Hamiltonian H_{int} connects only atomic subspaces which differ by one excitation, we obtain

$$P_a^m H_{\text{int}} Q_a^m = P_a^m H_{\text{int}} P_{a-1}^{m+1} + P_a^m H_{\text{int}} P_{a+1}^{m-1} \quad (A2)$$

and we can write down the corresponding terms of the effective Hamiltonian in Eq. (A1) as

$$P_a^m H_{\text{int}} Q_a^m \frac{1}{E_a - Q_a^m H_0 Q_a^m} Q_a^m H_{\text{int}} P_a^m = P_a^m H_{\text{int}} \left(\frac{P_{a-1}^{m+1}}{E_a^m - E_{a-1}^{m+1}} + \frac{P_{a+1}^{m-1}}{E_a^m - E_{a+1}^{m-1}} \right) H_{\text{int}} P_a^m. \quad (A3)$$

The energy differences in the denominators are given by

$$E_a^m - E_{a\mp 1}^{m\pm 1} = \hbar(\mp\omega \pm \omega_A) + V \left[\binom{a}{2} - \binom{a\mp 1}{2} \right] = \begin{cases} -\hbar\Delta_0 + \binom{a-1}{1}V = -\hbar\Delta_0 + (a-1)V \\ +\hbar\Delta_0 - \binom{a}{1}V = \hbar\Delta_0 - aV \end{cases} \quad (\text{A4})$$

with $\Delta_0 = \omega - \omega_A$. Since $[P_a^m, U(k) \otimes \mathbb{1}_F] = 0$, we can compensate the site-dependent phases via the unitary transformation $U(\mathbf{k}_0)$ and have to introduce them back into the equation at the end. By setting $P_{a<0}^m = P_{a>N}^m = P_a^{m<0} = 0$ we can write down the general term

$$\begin{aligned} C_a^m(\mathbf{k}_0) &= U^\dagger(\mathbf{k}_0) P_a^m H_{\text{int}} P_{a\mp 1}^{m\pm 1} H_{\text{int}} P_a^m U(\mathbf{k}_0) \\ &= U^\dagger(\mathbf{k}_0) P_a^m H_{\text{int}} \sum_{n_1 < \dots < n_{a\mp 1}}^N |\{n_1, \dots, n_{a\mp 1}\}\rangle \langle \{n_1, \dots, n_{a\mp 1}\}| \otimes |m \pm 1\rangle \langle m \pm 1| \left(\sum_{n'=1}^N d_0^* |g\rangle_{n'} \langle r| a_0^\dagger + \text{H.c.} \right) P_a^m \\ &= U^\dagger(\mathbf{k}_0) P_a^m H_{\text{int}} \sum_{n'=1}^N \sum_{n_1 < \dots < n_{a\mp 1}}^N \left[(1 - \chi_{\{n_1, \dots, n_{a\mp 1}\}}^{\{n'\}}) d_0^* \sqrt{m \pm 1} |\{n_1, \dots, n_{a\mp 1}\}\rangle \langle \{n_1, \dots, n_{a\mp 1}\} \cup \{n'\}| \otimes |m \pm 1\rangle \langle m \pm 1 - 1| \right. \\ &\quad \left. + \chi_{\{n_1, \dots, n_{a\mp 1}\}}^{\{n'\}} d_0 \sqrt{m \pm 1 + 1} |\{n_1, \dots, n_{a\mp 1}\}\rangle \langle \{n_1, \dots, n_{a\mp 1}\} \setminus \{n'\}| \otimes |m \pm 1\rangle \langle m \pm 1 + 1| \right] P_a^m, \end{aligned} \quad (\text{A5})$$

where we used the characteristic function ($\chi_B^A = 1$ if $A \subseteq B$ and 0 otherwise) to encode the annihilation effect of the atomic rising and lowering operators. Calculating the action of H_{int} from the left results in four terms for each combination of $(\{n_1, \dots, n_a\}, n, n')$.

Since P_a^m projects onto the subspace containing m photons and a atomic excitations only one term per case, i.e., per sign \pm , survives, resulting in

$$C_a^m(\mathbf{k}_0) = \sum_{n_1 < \dots < n_{a\mp 1}}^N \left\{ \begin{array}{l} |d_0|^2 (m+1) \sum_{n=1}^N (1 - \chi_{\{n_1, \dots, n_{a-1}\}}^{\{n\}}) |\{n_1, \dots, n_{a-1}\} \cup \{n\}\rangle \\ \sum_{n'=1}^N (1 - \chi_{\{n_1, \dots, n_{a-1}\}}^{\{n'\}}) \langle \{n_1, \dots, n_{a-1}\} \cup \{n'\}| \\ |d_0|^2 m \sum_{n=1}^N \chi_{\{n_1, \dots, n_{a+1}\}}^{\{n\}} |\{n_1, \dots, n_{a+1}\} \setminus \{n\}\rangle \\ \sum_{n'=1}^N \chi_{\{n_1, \dots, n_{a+1}\}}^{\{n'\}} \langle \{n_1, \dots, n_{a+1}\} \setminus \{n'\}| \end{array} \right\} \otimes |m\rangle \langle m|. \quad (\text{A6})$$

Reintroducing the site-dependent phases and dividing the resulting expression into diagonal elements that commute with $U(\mathbf{k}_0) \otimes \mathbb{1}_F$ and appear a times, and off-diagonal ones which do not commute with $U(\mathbf{k}_0) \otimes \mathbb{1}_F$ and appear just once, we obtain

$$P_a^m H_{\text{int}} P_{a\mp 1}^{m\pm 1} H_{\text{int}} P_a^m = \begin{cases} |d_0|^2 (m+1) P_a [U(\mathbf{k}_0) \text{Hd}_2(N) U^\dagger(\mathbf{k}_0) + a] P_a \otimes |m\rangle \langle m|, \\ |d_0|^2 m P_a [U(\mathbf{k}_0) \text{Hd}_2(N) U^\dagger(\mathbf{k}_0) + (N-a)] P_a \otimes |m\rangle \langle m|. \end{cases} \quad (\text{A7})$$

Here we used the operator Hd_2 , as defined in Sec. IV A, to write down all off-diagonal elements. Thus, the effective Hamiltonian in this case reads

$$H_{\text{eff},N}^{\text{off}} = H_0 + \sum_{a=0}^N \sum_{m=0}^{\infty} |d_0|^2 P_a^m \left[m \frac{U(\mathbf{k}_0) \text{Hd}_2(N) U^\dagger(\mathbf{k}_0) + N - a}{\hbar\Delta_0 - aV} - (m+1) \frac{U(\mathbf{k}_0) \text{Hd}_2(N) U^\dagger(\mathbf{k}_0) + a}{\hbar\Delta_0 - (a-1)V} \right] P_a^m. \quad (\text{A8})$$

Assuming a coherent state of high mean photon number M_0 and tracing over the field degrees of freedom, we derive the atomic ensemble Hamiltonian of Eq. (18) with Rabi frequency $\Omega_0^2 = |d_0|^2 M_0 / \hbar^2$.

2. Resonant field

In what follows, we present a detailed derivation of the effective Hamiltonian for a field which is resonant with one of the transitions in H_A .

As discussed in Sec. IV B, we have to join the two subspaces resonantly connected by the laser field. Hence,

$$P_{a,a-1}^{m,m+1} = P_a^m + P_{a-1}^{m+1} = \mathbb{1} - Q_{a,a-1}^{m,m+1}, \quad (\text{A9})$$

which joins the terms containing P_a^m and P_{a-1}^{m+1} in the effective Hamiltonian. For this resonance projector, we have [cf. Eq. (15)]

$$P_{a,a-1}^{m,m+1} H P_{a,a-1}^{m,m+1} = P_{a,a-1}^{m,m+1} H_0 P_{a,a-1}^{m,m+1} + (P_a^m H_{\text{int}} P_{a-1}^{m+1} + \text{H.c.}), \quad (\text{A10})$$

where in comparison to the other nonresonance projectors an additional term appears, which contains H_{int} . Compensating for the site-dependent phases, this term can be written down as

$$\begin{aligned}
C_{a,a-1}^{m,m+1} &= U^\dagger(\mathbf{k}_a) P_a^m H_{\text{int}} P_{a-1}^{m+1} U(\mathbf{k}_a) \\
&= \sum_{n_1 < \dots < n_a}^N |\{n_1, \dots, n_a\}\rangle \sum_{n'_1 < \dots < n'_{a-1}}^N d_a \sqrt{m+1} \sum_{n=1}^N \chi_{\{n'_1, \dots, n'_{a-1}\} \cup \{n\}}^{\{n_1, \dots, n_a\}} (1 - \chi_{\{n'_1, \dots, n'_{a-1}\}}^{\{n\}}) |\{n'_1, \dots, n'_{a-1}\}\rangle \otimes |m\rangle \langle m+1| \\
&= d_a \sqrt{m+1} \sum_{n'_1 < \dots < n'_{a-1}}^N \sum_{n=1}^N |r\rangle_{nm} \langle g | \{n'_1, \dots, n'_{a-1}\}\rangle \langle \{n'_1, \dots, n'_{a-1}\} | \otimes |m\rangle \langle m+1| \\
&= d_a \sqrt{m+1} \sigma_{a-1}^+ \otimes |m\rangle \langle m+1|.
\end{aligned} \tag{A11}$$

Here, we have introduced the rising operator of the atomic subspace with $a-1$ excitations, connecting this subspace to its counterpart with a excitations. We can define a lowering operator in an equivalent fashion. These two operators are given by

$$\begin{aligned}
\sigma_a^- &= \sum_{n_1 < \dots < n_a}^N \sum_{n=1}^N |g\rangle_{nm} \langle r | \{n_1, \dots, n_a\}\rangle \langle \{n_1, \dots, n_a\} | = (\sigma_{a-1}^+)^{\dagger}, \\
\sigma_a^+ &= \sum_{n_1 < \dots < n_a}^N \sum_{n=1}^N |r\rangle_{nm} \langle g | \{n_1, \dots, n_a\}\rangle \langle \{n_1, \dots, n_a\} | = (\sigma_{a+1}^-)^{\dagger},
\end{aligned} \tag{A12}$$

and act on the states $|D_a^N\rangle$ according to

$$\sigma_a^+ |D_a^N\rangle = \sqrt{(N-a)(a+1)} |D_{a+1}^N\rangle, \quad \sigma_a^- |D_a^N\rangle = \sqrt{a(N-a+1)} |D_{a-1}^N\rangle. \tag{A13}$$

In addition, for the second part of the effective Hamiltonian we have to compute

$$\begin{aligned}
P_{a,a-1}^{m,m+1} H Q_{a,a-1}^{m,m+1} \frac{1}{E_a^m - Q_{a,a-1}^{m,m+1} H Q_{a,a-1}^{m,m+1}} Q_{a,a-1}^{m,m+1} H P_{a,a-1}^{m,m+1} &= P_{a,a-1}^{m,m+1} H_{\text{int}} Q_{a,a-1}^{m,m+1} \frac{1}{E_a^m - Q_{a,a-1}^{m,m+1} H_0 Q_{a,a-1}^{m,m+1}} Q_{a,a-1}^{m,m+1} H_{\text{int}} P_{a,a-1}^{m,m+1} \\
&= P_a^m H_{\text{int}} \frac{P_{a+1}^{m-1}}{E_a^m - E_{a+1}^{m-1}} H_{\text{int}} P_a^m + P_{a-1}^{m+1} H_{\text{int}} \frac{P_{a-2}^{m+2}}{E_{a-1}^{m+1} - E_{a-2}^{m+2}} H_{\text{int}} P_{a-1}^{m+1}.
\end{aligned} \tag{A14}$$

With an index shift $a \rightarrow a-1$ and $m \rightarrow m+1$ to match the second term, we can use the results pertaining to the off-resonant field in Appendix A 1. For all other projectors with $a' \neq a, a-1$, we can also use the results corresponding to the off-resonant case, because for other transitions the resonance condition is not fulfilled. Putting everything together yields

$$\begin{aligned}
U^\dagger(\mathbf{k}_a) H_{\text{eff},N}^{a \leftrightarrow a-1} U(\mathbf{k}_a) &= H_0 + \sum_{m=0}^{\infty} \left[(d^* \sqrt{m+1} \sigma_a^- \otimes |m+1\rangle \langle m+1| + \text{H.c.}) - \frac{|d_a|^2 m}{V} (\text{Hd}_2(N) + (N-a)) P_a^m \right. \\
&\quad \left. - \frac{|d_a|^2 (m+2)}{V} (\text{Hd}_2(N) + a-1) P_{a-1}^{m+1} \right. \\
&\quad \left. + \sum_{a,a-1 \neq a'=0}^N |d_a|^2 \left(m \frac{\text{Hd}_2(N) + (N-a')}{\hbar \Delta_a - a'V} - (m+1) \frac{\text{Hd}_2(N) + a'}{\hbar \Delta_a - (a'-1)V} \right) P_{a'}^m \right].
\end{aligned} \tag{A15}$$

By further assuming that $V \gg \hbar |\Omega_a|$ for all fields, we can ignore all terms scaling with $|d_a|^2/V$ and only the first line contributes. Tracing out the field degrees of freedom and assuming a coherent field state results in the effective Hamiltonian as given by Eq. (24).

APPENDIX B: DETAILED DERIVATION OF THE TWISTED-LADDER HAMILTONIAN

In the following, we provide detailed derivation of the twisted-ladder Hamiltonian in Eq. (47) starting from Eq. (45).

We can evaluate the three terms ($a = 1, 2, 3$) of Eq. (45) separately using the well-known operator identity

$$e^A B e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} [A, B]_m, \tag{B1}$$

where $[A, B]_m$ is a shorthand for the repeated commutator of A and B with m appearances of the operator A . This is relatively straightforward for $a = 1, 2, 3$. For the sake of brevity, we omit the argument $\mathbf{k}_a - \mathbf{k}_0$ of $U(\mathbf{k}_a - \mathbf{k}_0)$, because in each a term the argument is the same. We reinstate this argument at the end of the derivation.

We first evaluate

$$[H^{\text{off}}, U\sigma_1^-U^\dagger] = (3s_0 - s_1)U\sigma_1^-U^\dagger + (s_0 - s_1)\Sigma_{k_0-k_1}\sigma_1^-, \tag{B2}$$

where $s_a = \hbar^2|\Omega_0|^2/(\hbar\Delta_0 - aV)$ are the energy shifts introduced by H_3^{off} and $H^{\text{off}} = H_3^{\text{off}}(\mathbf{k} = 0)$. We can see that the commutator partially reproduces the operator and adds an additional term without twisting. Therefore, we can write down the m th commutator using a triangular matrix as

$$e^{iH^{\text{off}}t/\hbar}U\sigma_1^-U^\dagger e^{-iH^{\text{off}}t/\hbar} = \sum_{m=0}^{\infty} \frac{(it/\hbar)^m}{m!} [H^{\text{off}}, U\sigma_1^-U^\dagger]_m = \sum_{m=0}^{\infty} \frac{(it/\hbar)^m}{m!} \mathbf{v}_1 \cdot (A_1^m \mathbf{e}_1) = \mathbf{v}_1 \cdot (e^{itA_1/\hbar} \mathbf{e}_1) \tag{B3}$$

with $\mathbf{e}_1 = (1 \ 0)^T$ being a unit vector of appropriate dimension and

$$\mathbf{v}_1 = \begin{pmatrix} U\sigma_1^-U^\dagger \\ \sigma_1^- \end{pmatrix}; \quad A_1 = \begin{pmatrix} 3s_0 - s_1 & 0 \\ (s_0 - s_1)\Sigma_{k_0-k_1} & 6s_0 - 4s_1 \end{pmatrix} \tag{B4}$$

with eigenvalues $\sigma(A_1) = \{3s_0 - s_1, 6s_0 - 4s_1\}$. Similarly, for $a = 3$ we obtain

$$[H^{\text{off}}, U\sigma_3^-U^\dagger] = (-s_1 + 3s_2)U\sigma_3^-U^\dagger + \sigma_3^-(-s_1 + s_2)\Sigma_{k_0-k_3} \tag{B5}$$

and the analogous expression with a matrix exponential and

$$\mathbf{v}_3 = \begin{pmatrix} U\sigma_3^-U^\dagger \\ \sigma_3^- \end{pmatrix}; \quad A_3 = \begin{pmatrix} -s_1 + 3s_2 & 0 \\ (-s_1 + s_2)\Sigma_{k_0-k_3} & -4s_1 + 6s_2 \end{pmatrix} \tag{B6}$$

with eigenvalues $\sigma(A_3) = \{-s_1 + 3s_2, -4s_1 + 6s_2\}$. The case $a = 2$ is more complicated and we will compute the commutator for a more general case involving the unitary transformations $U' = U(\mathbf{k}')$ and $U^\dagger = U^\dagger(\mathbf{k})$ for different wave vectors. This will be helpful later on for defining the matrix A_2 . We calculate

$$\begin{aligned} [H^{\text{off}}, U'\sigma_2^-U^\dagger] &= \left[H^{\text{off}}, 3|D_1^3(\mathbf{k}')\rangle\langle D_2^3(\mathbf{k})| - \Phi(-\mathbf{k}) \sum_{n=1}^3 e^{i(\mathbf{k}'+\mathbf{k})\cdot\mathbf{x}_n} |gg\rangle|r\rangle_m \langle g| \langle rr| \right] \\ &= 2s_1 U'\sigma_2^-U^\dagger + 3\Sigma_{\mathbf{k}'}(-s_0 + s_1)|D_1^3(0)\rangle\langle D_2^3(\mathbf{k})| + 3\Sigma_{\mathbf{k}}(s_1 - s_2)\Phi(-\mathbf{k})|D_1^3(\mathbf{k}')\rangle\langle D_2^3(0)| \\ &\quad + 3\Phi(\mathbf{k}')(s_0 - s_1)|D_1^3(0)\rangle\langle D_2^3(\mathbf{k}'+\mathbf{k})| + 3\Phi(-\mathbf{k})(-s_1 + s_2)|D_1^3(\mathbf{k}'+\mathbf{k})\rangle\langle D_2^3(0)|, \end{aligned} \tag{B7}$$

with $\Phi(\mathbf{k}) = e^{i\mathbf{k}\cdot\sum_n^N \mathbf{x}_n}$. Since all but the self-reproducing part (first line on the right-hand side) contains at least one generalized $D_{1,2}^3$ state (either ket or bra) with $\mathbf{k} = 0$, the next commutator will accumulate terms of the form $|D_1^3(0)\rangle\langle D_2^3(0)|$. Omitting the argument $\mathbf{k} = 0$ in the following, we can write down the transformed $a = 2$ part similar as the other ones by using $\mathbf{k}' = \mathbf{k}$, resulting in an analogous matrix exponential equation with

$$\begin{aligned} \mathbf{v}_2 &= \begin{pmatrix} U\sigma_2^-U^\dagger \\ |D_1^3\rangle\langle D_2^3(\mathbf{k})| \\ |D_1^3(\mathbf{k})\rangle\langle D_2^3| \\ |D_1^3\rangle\langle D_2^3(2\mathbf{k})| \\ |D_1^3(2\mathbf{k})\rangle\langle D_2^3| \\ |D_1^3\rangle\langle D_2^3| \end{pmatrix}; \quad \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \\ A_2 &= \begin{pmatrix} 2s_1 & 0 & 0 & 0 & 0 & 0 \\ 3\Sigma_{\mathbf{k}}(-s_0 + s_1) & -3s_0 + 5s_1 & 0 & 0 & 0 & 0 \\ 3\Phi(-\mathbf{k})\Sigma_{\mathbf{k}}(s_1 - s_2) & 0 & 5s_1 - 3s_2 & 0 & 0 & 0 \\ 3\Phi(\mathbf{k})(s_0 - s_1) & 0 & 0 & -3s_0 + 5s_1 & 0 & 0 \\ 3\Phi(-\mathbf{k})(-s_1 + s_2) & 0 & 0 & 0 & 5s_1 - 3s_2 & 0 \\ 0 & \Phi(-\mathbf{k})\Sigma_{\mathbf{k}}(s_1 - s_2) & \Sigma_{\mathbf{k}}(-s_0 + s_1) & \Phi(-2\mathbf{k})\Sigma_{2\mathbf{k}}(s_1 - s_2) & \Sigma_{2\mathbf{k}}(-s_0 + s_1) & -3s_0 + 8s_1 - 3s_2 \end{pmatrix} \end{aligned} \tag{B8}$$

and substitute $\mathbf{k} = \mathbf{k}_2 - \mathbf{k}_0$. The corresponding eigenvalues are $\{2s_1, -3s_0 + 5s_1, 5s_1 - 3s_2, -3s_0 + 8s_1 - 3s_2\}$. Given that in the three equations for $a = 1, 2, 3$ only the transformed unit vectors are of interest, we can write the transformed ladder

Hamiltonian [cf. Eq. (45)] in the form

$$\begin{aligned}
& U(\mathbf{k}_0) e^{iH^{\text{off}}_t/\hbar} U^\dagger(\mathbf{k}_0) H_3^L(\{\mathbf{k}_a\}) U(\mathbf{k}_0) e^{-iH^{\text{off}}_t/\hbar} U^\dagger(\mathbf{k}_0) \\
&= U(\mathbf{k}_0) \left[\sum_{a=1}^3 \Omega_a^* \mathbf{v}_a (e^{itA_a/\hbar} \mathbf{e}_1) + \text{H.c.} \right] U^\dagger(\mathbf{k}_0) \\
&= \left[\begin{aligned}
& \sqrt{3} \Omega_1^* \left(\frac{|ggg\rangle\langle D_1^3(\mathbf{k}_1)|}{|ggg\rangle\langle D_1^3(\mathbf{k}_0)|} \right) \cdot \left(\frac{e^{it(3s_0-s_1)/\hbar}}{(e^{it(6s_0-4s_1)/\hbar} - e^{it(3s_0-s_1)/\hbar}) \Sigma_{\mathbf{k}_0-\mathbf{k}_1}/3} \right) \\
& + \sqrt{3} \Omega_3^* \left(\frac{|D_2^3(\mathbf{k}_3)\rangle\langle rrr|\Phi(-\mathbf{k}_0)}{|D_2^3(\mathbf{k}_0)\rangle\langle rrr|\Phi(-\mathbf{k}_0)} \right) \cdot \left(\frac{e^{it(-s_1+3s_2)/\hbar}}{(e^{it(-4s_1+6s_2)/\hbar} - e^{it(-s_1+3s_2)/\hbar}) \Sigma_{\mathbf{k}_0-\mathbf{k}_3}/3} \right) \\
& + \Omega_2^* \left(\begin{array}{c} 3|D_1^3(\mathbf{k}_2)\rangle\langle D_2^3(\mathbf{k}_2)| \\ |D_1^3(\mathbf{k}_0)\rangle\langle D_2^3(\mathbf{k}_2)| \\ |D_1^3(\mathbf{k}_2)\rangle\langle D_2^3(\mathbf{k}_0)| \\ |D_1^3(\mathbf{k}_0)\rangle\langle D_2^3(2\mathbf{k}_2-\mathbf{k}_0)| \\ |D_1^3(2\mathbf{k}_2-\mathbf{k}_0)\rangle\langle D_2^3(\mathbf{k}_0)| \\ |D_1^3(\mathbf{k}_0)\rangle\langle D_2^3(\mathbf{k}_0)| \end{array} \right) \cdot \left(\begin{array}{c} e^{it2s_1/\hbar} \\ (e^{it(-3s_0+5s_1)/\hbar} - e^{it2s_1/\hbar}) \Sigma_{\mathbf{k}_2-\mathbf{k}_0} \\ (e^{it(5s_1-3s_2)/\hbar} - e^{it2s_1/\hbar}) \Sigma_{\mathbf{k}_2-\mathbf{k}_0} \Phi(\mathbf{k}_0-\mathbf{k}_2) \\ (-e^{it(-3s_0+5s_1)/\hbar} + e^{it2s_1/\hbar}) \Phi(\mathbf{k}_2-\mathbf{k}_0) \\ (e^{it2s_1/\hbar} - e^{it(5s_1-3s_2)/\hbar}) \Phi(\mathbf{k}_0-\mathbf{k}_2) \\ \eta(\mathbf{k}_2-\mathbf{k}_0) \end{array} \right) \\
& - \Omega_2^* e^{it2s_1/\hbar} \Phi(-\mathbf{k}_2) \sum_{n=1}^3 e^{i2k_2 \cdot \mathbf{x}_n} |gg\rangle|r\rangle_{mn} \langle g|\langle rr| \end{aligned} \right] + \text{H.c.} \tag{B9}
\end{aligned}$$

with

$$\eta(\mathbf{k}) = \frac{2\Sigma-\mathbf{k}}{3} [e^{it2s_1/\hbar} - e^{it(-3s_0+5s_1)/\hbar} - e^{it(5s_1-3s_2)/\hbar} + e^{it(-3s_0+8s_1-3s_2)/\hbar}]. \tag{B10}$$

From the last equation, we can identify one relevant term per field ($j = 1, 2, 3$) and compensate the exponential time dependence via fine detunings. Choosing

$$\delta_1 = (-6s_0 + 4s_1)/\hbar, \quad \delta_2 = (3s_0 - 8s_1 + 3s_2)/\hbar, \quad \delta_3 = (4s_1 - 6s_2)/\hbar \tag{B11}$$

results in the effective Hamiltonian in which terms oscillating with nonvanishing residual frequencies are neglected (cf. Sec. VI B). The set of residual frequencies $\{\omega_R\}$ is given by

$$\begin{aligned}
-3s_0 + s_1 - \hbar\delta_1 &= 3s_0 - 3s_1, \\
-2s_1 - \hbar\delta_2 &= -3s_0 + 6s_1 - 3s_2, \quad 3s_0 - 5s_1 - \hbar\delta_2 = 3s_1 - 3s_2, \quad -5s_1 + 3s_2 - \hbar\delta_2 = -3s_0 + 3s_2, \\
s_1 - 3s_2 - \hbar\delta_3 &= -3s_1 + 3s_2, \tag{B12}
\end{aligned}$$

where each equation corresponds to one residual energy $\hbar\omega_R$ and each line corresponds to one value of a ($a = 1, 2, 3$).

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Correction: The previously published Figure 5 was a duplicate of Figure 6 and has been replaced with the correct figure.