Real-space construction of crystalline topological superconductors and insulators in 2D interacting fermionic systems

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The construction and classification of crystalline symmetry protected topological (SPT) phases in interacting bosonic and fermionic systems have been intensively studied in the past few years. Crystalline SPT phases are not only of conceptual importance, but also provide us great opportunities towards experimental realization since space group symmetries naturally exist for any realistic material. In this paper, we systematically classify the crystalline topological superconductors (TSC) and topological insulators (TI) in 2D interacting fermionic systems by using an explicit real-space construction. In particular, we discover an intriguing fermionic crystalline topological superconductor that can only be realized in interacting fermionic systems (i.e., not in free-fermion or interacting bosonic systems). Moreover, we also verify the recently conjectured crystalline equivalence principle for generic 2D interacting fermionic systems.

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I. INTRODUCTION

A. The goal of this paper

In the past decade, a lot of efforts have been made on the theoretical prediction and experimental searching for topological superconductors (TSC) and topological insulators (TI) in noninteracting or weakly-interacting systems [1,2]. However, in realistic materials, strong electronic interactions typically play a very important role and can not be neglected or treated as perturbations, especially in low-dimensional systems. Therefore, a complete construction and classification of TSC/TI in interacting fermionic systems become a very important but challenging problem. It turns out that a large class of TSC/TI require certain symmetry protection and they can be connected to a trivial disorder phase (e.g., an s-wave BCS-superconductor or an atomic insulator) in the absence of global symmetry. Such kind of "integer" TSC/TI are short-range entangled quantum states and they are actually the simplest examples of symmetry-protected topological (SPT) phases [3].

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Thanks to the cutting-edge breakthrough in the classification and construction of SPT phases for interacting bosonic and fermionic systems recently [4–19], a complete understanding of "integer" TSC/TI protected by internal symmetry (e.g., time-reversal symmetry or spin rotational symmetry) for interacting electronic systems has been achieved [11,20–24]. In general, by "gauging" the internal (unitary) symmetry [25–37] and investigating the braiding statistics of the corresponding gauge fluxes/flux lines, different SPT phases can be uniquely identified. Moreover, gapless edge states or anomalous surface topological orders have also been proposed as another very powerful way to characterize different SPT phases in interacting systems [23,38–45].

In recent years, the notion of SPT phases was further extended to systems with crystalline symmetry protection and the so-called crystalline SPT phases have been intensively studied [46–69]. Crystalline SPT phases are not only of conceptual importance, but also provide us great opportunities towards experimental realization since space group symmetries naturally exist for any realistic material. The crystalline TI first proposed in free fermion systems is the simplest example of crystalline SPT phases, and it has already been realized in many different materials [70–73]. For free fermion systems, there are two systematic methods for classifying and characterizing the crystalline TI: one is the so-called *symmetry indicators* [53,67,74–77], which classifies and characterizes the crystalline TI by symmetry representations of band structures at high-symmetry momenta; another is a real-space construction based on the concept of topological crystal [49,60]. Very recently, boundary modes [78–80] of the so-called higher-order TSC/TI [81–86] protected by crystalline symmetry (with additional time-reversal symmetry in certain cases) also attract a lot of interest in both 2D and 3D. In general, an nth-order TSC/TI protects gapless modes at the system boundary of codimension n. For example, a second-order 3D TI has gapless states on its hinges, while its surfaces are gapped, and a third-order 3D TI has gapless states on its corners, while both its surfaces and hinges are gapped. Nevertheless, most of these studies are still focusing on free fermion systems and it is not quite clear whether the corresponding gapless boundary modes are stable or not against interactions. On the other hand, for interacting bosonic systems, it was pointed out that the classification of crystalline SPT phases is closely related to the SPT phases with internal symmetry. In Ref. [51], a *crystalline equivalence principle* was proposed with a rigorous mathematical proof: i.e., crystalline topological phases with space group symmetry G are in one-to-one correspondence with topological phases protected by the same internal symmetry G, but acting in a twisted way, where if an element of G is a mirror reflection (orientation-reversing symmetry), it should be regarded as a time-reversal symmetry (antiunitary symmetry). This principle indicates the profound relationship between crystalline SPT phases and SPT phases protected by internal symmetry. Thus, the classification of crystalline SPT phases for free-fermion and interacting bosonic systems can be computed systematically.

Despite the huge success in understanding crystalline SPT phases for free-fermion and interacting bosonic systems, a systematical understanding of crystalline SPT phases for interacting fermionic systems is still lacking. Although it has been believed that the strategy of classification schemes [51,59–61] should still work and some simple examples have been studied [62,64,87], most studies focus on the systems with point group symmetry only and the generic cases are unclear. Recent study on generalizing *crystalline equivalence principle* into interacting fermionic systems shed new light towards a complete understanding of crystalline SPT phases for interacting fermion. In Ref. [87], by some explicit calculations for both crystalline SPT phases and SPT phases protected by internal symmetry, it has been demonstrated that the crystalline equivalence principle is still valid for 2D crystalline SPT phases protected by point group symmetry, but in a twisted way, where spinless (spin-1/2) fermions should be mapped into spin-1/2 (spinless) fermions.

In this paper, we aim at systematically constructing and classifying crystalline TSC/TI for 2D interacting fermionic systems and establishing a general paradigm of real-space construction for interacting fermionic crystalline SPT phases. We will consider both spinless and spin-1/2 fermionic systems. In particular, we obtain an intriguing fermionic TSC that cannot be realized in either free-fermion or interacting bosonic systems: a $p4m$ (#11 wallpaper group) symmetric 2D system with spinless fermions. These TSC can be realized in systems with coplanar spin order and might have very interesting experimental implementations. Furthermore, we compare all our results with the classifications of 2D fermionic SPT (FSPT) phases protected by corresponding internal symmetries. We confirm the crystalline equivalence principle for generic 2D interacting fermionic systems, where a mirror reflection symmetry action should be mapped onto a time-reversal symmetry action, and that spinless (spin-1/2) fermionic systems should be mapped into spin-1/2 (spinless) fermionic systems.

Our general real-space construction scheme includes following three major steps:

Cell decomposition: For a specific wallpaper group, firstly we can divide it into an assembly of unit cells; then we divide each unit cell into an assembly of lower-dimensional blocks.

Block-state decoration: For a specific wallpaper group with cell decomposition, we can decorate lower-dimensional block state on different blocks. A gapped assembly of block states is called *obstruction free* decoration.

Bubble equivalence: For a specific obstruction-free decoration, we need to further examine whether such a decoration can be trivialized or not. Finally, the obstruction and trivialization free block state decoration corresponds to a 2D fermionic crystalline SPT phase.

In addition, we also need to examine the possible nontrivial stacking relation between block states with different dimensions to determine the actual group structure of 2D fermionic crystalline SPT phases.

B. Space group symmetry for spinless and spin-1/2 systems

Here we would like to clarify the precise meaning of "spinless" and "spin-1/2" fermions for systems with and without $U^f(1)$ charge conservation symmetry.

For a fermionic system with total symmetry group G_f, there is always a subgroup $\mathbb{Z}_2^f = \{1, P_f = (-1)^F\}$, where F is the total number of fermions. $\mathbb{Z}_2^f$ is the center of G_f because all physical symmetries commute with P_f, i.e., cannot change fermion parity of the system. In particular, for systems without $U^f(1)$ charge conservation symmetry, we can define the bosonic (physical) symmetry group by a quotient group $G_b = G_f/\mathbb{Z}_2^f$. In reverse, for a given physical symmetry group G_b, there are many different fermionic symmetry groups G_f, which are the central extension of G_b by $\mathbb{Z}_2^f$. It can be expressed by the following short exact sequence:

$$0 \to \mathbb{Z}_2^f \to G_f \to G_b \to 0 \tag{1}$$

and different extensions G_f are characterized by different factor systems of Eq. (1) that are 2-cocycles $\omega_2 \in \mathcal{H}^2(G_b, \mathbb{Z}_2)$. Consequently, we denote G_f as $\mathbb{Z}_2^f \times_{\omega_2} G_b$.

For systems with additional $U^f(1)$ charge conservation, the group element is $U_\theta = e^{i\theta F}$. Aforementioned fermion parity operator $P_f = U_\pi$ is the order 2 element of $U^f(1)$, hence we denote this charge conservation symmetry by $U^f(1)$ with a superscript f. It is easy to notice that $U^f(1)$ charge conservation is a normal subgroup of the total symmetry group G_f, which can be expressed by the following short exact sequence:

$$0 \to U^f(1) \to G_f \to G \to 0 \tag{2}$$

where $G := G_f/U^f(1)$. In reverse, for a given physical symmetry group G, we can define $G_f = U^f(1) \rtimes_{\omega_2} G$. Here ω_2 is related to the extension of the physical symmetry group G. The multiplication of the total symmetry group G_f is defined as

$$(1, g) \times (1, h) = (e^{2\pi i\omega_2(g,h)F}, gh) \in G_f \tag{3}$$

with $\omega_2 \in \mathbb{R}/\mathbb{Z} = [0, 1)$ as a $U(1)$ phase, associated with $g, h \in G$. Therefore ω_2 is a 2-cocycle in $\mathcal{H}^2(G, \mathbb{R}/\mathbb{Z})$.

The spin of fermions (spinless or spin-1/2) is characterized by different choices of 2-cocycles ω_2, i.e., the spinless corresponds to a trivial ω_2 while spin-1/2 fermion corresponds to specific choice of nontrivial ω_2.

For example, consider even-fold dihedral group D_{2n} symmetry with two generators $\boldsymbol{R}$ and $\boldsymbol{M}$ satisfying $\boldsymbol{R}^{2n} = \boldsymbol{M}^2 = I$ ($n \in \mathbb{Z}$ and I is identity) for systems without $U^f(1)$ symmetry. Different extensions of fermion parity are characterized by different 2-cocycles ω_2,

$$\omega_2 \in \mathcal{H}^2(D_{2n}, \mathbb{Z}_2) = \mathbb{Z}_2^3. \tag{4}$$

In particular, the spinless fermions corresponding to the trivial 2-cocycle ω_2 satisfy

$$\begin{aligned} \boldsymbol{R}^{2n} &= 1, \\ \boldsymbol{M}^2 &= 1, \end{aligned} \tag{5}$$

while the spin-1/2 fermions corresponding to the 2-cocycle ω_2 satisfy

$$\begin{aligned} \boldsymbol{R}^{2n} &= P_f, \\ \boldsymbol{M}^2 &= P_f, \\ \boldsymbol{M}\boldsymbol{R}\boldsymbol{M}^{-1}\boldsymbol{R} &= 1. \end{aligned} \tag{6}$$

To satisfy these conditions, we consider the 2-cocycle ω_2 as following. For $\forall a_g, b_h \in D_{2n}$ defined as

$$D_{2n} = \{(a, g) = a_g | 0 \leqslant a \leqslant (2n-1), 0 \leqslant g \leqslant 1\}, \tag{7}$$

we choose

$$\begin{aligned} \omega_2(a_g, b_h) = &\left\lfloor \frac{[(-1)^{g+h}a]_{2n} + [(-1)^h b]_{2n}}{2n} \right\rfloor \\ &+ (1 - \delta_a)(a+1)h + g \cdot h, \end{aligned} \tag{8}$$

where we define $[x]_n \equiv x(\mathrm{mod}\, n)$, $\lfloor x \rfloor$ as the greatest integer less than or equal to x, and

$$\delta_a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

For systems with $U^f(1)$ symmetry, spinless and spin-1/2 fermions are characterized by different 2-cocycle ω_2,

$$\omega_2 \in \mathcal{H}^2(D_{2n}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2. \tag{10}$$

In particular, the spinless fermions corresponding to the trivial 2-cocycle ω_2 satisfy

$$(MR^n)^2 = 1 \tag{11}$$

while the spin-1/2 fermions corresponding to the 2-cocycle ω_2 satisfy

$$(MR^n)^2 = P_f. \tag{12}$$

C. Summary of main results

Here we first summarize all classification results of 2D crystalline TSC for both spinless and spin-1/2 fermionic systems. We label the classification attributed to p-dimensional block-state decorations by $E^{d\mathrm{D}}$. For the systems with spinless fermions, the classification results are summarized in Table I, and the classification data are listed layer by layer, i.e., classification contributed by 0D/1D block-state decorations, respectively. For the systems with spin-1/2 fermions, the classification results are summarized in Table II layer by layer. Furthermore, we also study the group structure of the classifications by explicitly investigating the possible nontrivial stacking relation between 1D and 0D block states: For certain cases, stacking of several 1D block states can be deformed into a 0D block state, hence the total group could be a nontrivial extension between 1D and 0D block states. In particular, we label the classification indices with fermionic root phase by red, and the classification indices with bosonic root phase by blue.

TABLE I. Interacting classification of 2D crystalline TSC for spinless fermionic systems. The results are listed layer by layer, together with their group structures (represented by $\mathcal{G}_0$). We label the classification indices with fermionic/bosonic root phases with red/blue. The fermionic $\mathbb{Z}_4$ indices are obtained from nontrivial extensions between 1D and 0D block states, thus stacking two root phases will become another fermionic crystalline TSC. In particular, 1D block state of the $p4m$ case is an intriguing fermionic SPT phase that cannot be realized by free-fermion and interacting bosonic systems.

G_b	$E_0^{1\mathrm{D}}$	$E_0^{0\mathrm{D}}$	$\mathcal{G}_0$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$p2$	$\mathbb{Z}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pm	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^5 \times \mathbb{Z}_2$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
cm	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pmm	$\mathbb{Z}_1$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$
pmg	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
pgg	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
cmm	$\mathbb{Z}_1$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
$p4$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$p4m$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^3$
$p4g$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$
$p3$	$\mathbb{Z}_1$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
$p31m$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$
$p6$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$
$p6m$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$

For 2D crystalline TI protected by both wall paper group and $U^f(1)$ charge conservation symmetry, we generalize the procedures of real-space construction highlighted in Sec. II to include the internal $U^f(1)$ symmetry. It turns out that 1D block-state decoration does not contribute any nontrivial crystalline topological phase because of the absence of nontrivial 1D root phase in the presence of $U^f(1)$ symmetry. All classification results are summarized in Table III. Again we label the classification indices with fermionic root phase by red, and the classification indices with bosonic root phase by blue.

The rest of the paper is organized as follows: In Sec. II, we introduce the general paradigm of the real-space construction of crystalline SPT phases protected by wallpaper group in

TABLE II. Interacting classification of 2D crystalline TSC for spin-1/2 fermionic systems. The results are listed layer by layer, together with their group structure (represented by $\mathcal{G}_{1/2}$). We label the classification indices with fermionic/bosonic root phases with red/blue. We note that except for $p1$ and pg cases(spinless fermion and spin-1/2 fermion are the same for these two cases), all $\mathbb{Z}_4$ indices are from twofold rotation(the on-site symmetry group of arbitrary 0D block is $\mathbb{Z}_4^f$, which is the nontrivial $\mathbb{Z}_2^f$ extension of $\mathbb{Z}_2$.) and stacking two root phases will become a bosonic crystalline SPT phase. Similarly, for $p4$ case, two of three $\mathbb{Z}_8$ fermionic indices are from fourfold rotation and stacking two root phases will also become a bosonic SPT phase. The fermionic $\mathbb{Z}_8$ index of the $p4g$ case can be understood in the same way. All other fermionic $\mathbb{Z}_8$ indices are obtained from nontrivial extension between 1D and 0D block states. For these cases, stacking two fermionic root phases will become another fermionic crystalline TSC. In addition, the $\mathbb{Z}_{12}$ index of $p6$ case is obtained from sixfold rotation and stacking two fermionic root phases will also lead to a bosonic phase.

G_b	$E_{1/2}^{1D}$	$E_{1/2}^{0D}$	$\mathcal{G}_{1/2}$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2$
$p2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_4^4$	$\mathbb{Z}_4\times\mathbb{Z}_8^3$
pm	$\mathbb{Z}_2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_4\times\mathbb{Z}_8$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2$
cm	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_4$
pmm	$\mathbb{Z}_1$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$
pmg	$\mathbb{Z}_2^2$	$\mathbb{Z}_4^3$	$\mathbb{Z}_4\times\mathbb{Z}_8^2$
pgg	$\mathbb{Z}_2^2$	$\mathbb{Z}_4^2$	$\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8$
cmm	$\mathbb{Z}_2$	$\mathbb{Z}_4\times\mathbb{Z}_2^4$	$\mathbb{Z}_8\times\mathbb{Z}_2^4$
$p4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_8^2\times\mathbb{Z}_4$	$\mathbb{Z}_2\times\mathbb{Z}_8^3$
$p4m$	$\mathbb{Z}_1$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$
$p4g$	$\mathbb{Z}_2$	$\mathbb{Z}_8\times\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_8\times\mathbb{Z}_2^2$
$p3$	$\mathbb{Z}_1$	$\mathbb{Z}_2\times\mathbb{Z}_3^3$	$\mathbb{Z}_2\times\mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4$
$p31m$	$\mathbb{Z}_1$	$\mathbb{Z}_4\times\mathbb{Z}_3$	$\mathbb{Z}_4\times\mathbb{Z}_3$
$p6$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_4\times\mathbb{Z}_3$	$\mathbb{Z}_{12}\times\mathbb{Z}_8\times\mathbb{Z}_3$
$p6m$	$\mathbb{Z}_1$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$

2D interacting fermionic systems. In Sec. III, we explicitly show how to construct and classify the crystalline TSC in 2D interacting fermionic systems for five different crystallographic systems by using real-space construction, for both spinless and spin-1/2 fermions. All classification results are summarized in Tables I and II. Furthermore, we also classify the crystalline TI in 2D interacting fermionic systems with additional $U^f(1)$ charge conservation by using similar real-space construction scheme in Sec. IV, and the results are summarized in Table III. In Sec. V, by comparing these results with the classification results of 2D FSPT phases protected by the corresponding on-site symmetry groups, we verify the *crystalline equivalence principle* for generic 2D interacting fermionic systems. Finally, conclusions and discussions about further applications of real-space construction and experimental implications are presented in Sec. VI. In the Supplemental Material [88], we first discuss the 2D crystalline TI protected by point group symmetry and compare the results with the classifications of 2D FSPT phases protected by the corresponding internal symmetry, then we discuss the real-space construction of TSC and TI for all remaining cases of wallpaper groups.

TABLE III. The interacting classification of crystalline TI for 2D interacting fermionic systems. The results for both spinless and spin-1/2 fermions are summarized together. We note that the classifications are the same for those wall paper groups with only one reflection axis. We label the classification indices with fermionic/bosonic root phases with red/blue.

G_b	Spinless	Spin-1/2
$p1$	$\mathbb{Z}$	$\mathbb{Z}$
$p2$	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2$
pm	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$
pg	$\mathbb{Z}$	$\mathbb{Z}$
cm	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_2$
pmm	$\mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2^4$	$2\mathbb{Z}\times\mathbb{Z}_2^8$
pmg	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2$
pgg	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2$
cmm	$\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2^2$	$2\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2^4$
$p4$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_4\times\mathbb{Z}_2$
$p4m$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2^3$	$2\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_2^6$
$p4g$	$\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_2^2$	$\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_2^2$
$p3$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_3^3$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_3^2\times\mathbb{Z}_2$
$p31m$	$\mathbb{Z}\times\mathbb{Z}_3\times\mathbb{Z}_6$	$\mathbb{Z}\times\mathbb{Z}_3\times\mathbb{Z}_6$
$p6$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_3$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_6\times\mathbb{Z}_3$
$p6m$	$\mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_2^2$	$2\mathbb{Z}\times\mathbb{Z}_6\times\mathbb{Z}_2^3$

II. GENERAL PARADIGM OF REAL-SPACE CONSTRUCTION

In this section, we highlight the general paradigm of real-space construction of crystalline SPT phases for 2D interacting fermionic systems. There are three major steps: Firstly, we decompose the whole system into an assembly of unit cells, each of which is composed by several lower-dimensional blocks; secondly, we decorate some proper lower-dimensional block states on them and check their validity (for SPT phases, we require a fully gapped bulk ground state without ground-state degeneracy), that is, if the bulk state of a block-state construction cannot be fully gapped, we call such a decoration *obstructed*; finally, we consider the so-called bubble equivalence to investigate all possible *trivializations* (We note that certain block-states decorations actually lead to a trivial crystalline SPT phase). An obstruction-free and trivialization-free decoration corresponds to a nontrivial crystalline SPT phase. Below we demonstrate these procedures in full details by using the #14 wallpaper group $p3m1$ as an example.

A. Cell decomposition

For a 2D system with an arbitrary wallpaper group symmetry, we can divide the whole system into an assembly of unit cells, where different unit cells are identical and related

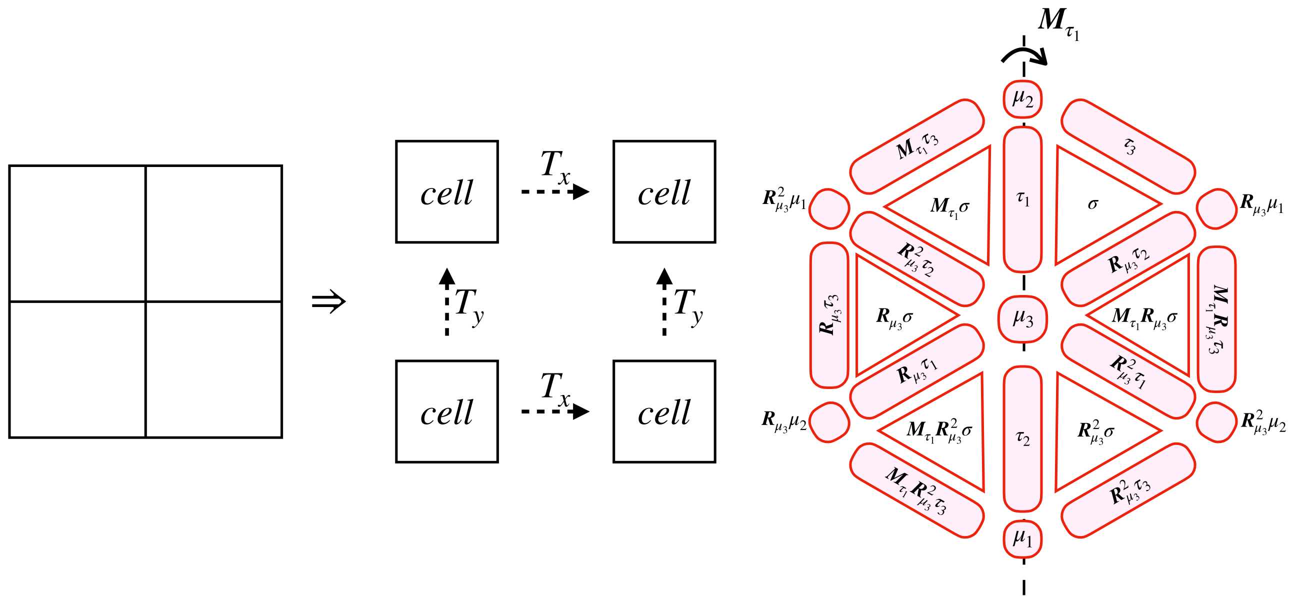


FIG. 1. Cell decomposition of #14 wallpaper group $p3m1$. Left panel illustrates the intercell decomposition that decomposes a lattice to an assembly of unit cells; right panel illustrates the intracell decomposition that decompose a unit cell to an assembly of lower-dimensional blocks.

by translation symmetries, as illustrated in the left panel of Fig. 1. Therefore, we should only specify the physics in each unit cell.

Then we decompose a specific unit cell of the wallpaper group $p3m1$ into an assembly of lower-dimensional blocks (see the right panel of Fig. 1). Here $\boldsymbol{R}_{\mu_3}$ represents threefold rotational symmetry operation centered at the 0D block labeled by μ_3, and $\boldsymbol{M}_{\tau_1}$ represents the reflection symmetry operation with the axis (indicated by the vertical dashed line in right panel of Fig. 1) coincided with the 1D block labeled by τ_1.

The physical background of the "intracell" decomposition is the "extended trivialization" in each cell [49]. Suppose $|\psi\rangle$ is an SPT state that cannot be trivialized by a symmetric finite-depth local unitary transformation. Due to the translational symmetry, $|\psi\rangle$ can be expressed in terms of a direct product of the wavefunctions of all cells,

$$|\psi\rangle = \bigotimes_c |\psi_c\rangle. \tag{13}$$

Because of the translational symmetry, investigation of a specific $|\psi_c\rangle$ in a cell is enough for understanding the SPT state $|\psi\rangle$. As a consequence, $|\psi_c\rangle$ will inherit the property that cannot be trivialized by a symmetric finite-depth local unitary transformation O^{loc}. Nevertheless, we can still define an alternative local unitary to *extensively* trivialize $|\psi_c\rangle$. First we can trivialize the region σ (see the right panel of Fig. 1): restrict O^{loc} to σ as O_σ^{loc} and act it on $|\psi_c\rangle$,

$$O_\sigma^{\mathrm{loc}}|\psi_c\rangle = |T_\sigma\rangle \otimes \left|\psi_c^{\bar{\sigma}}\right\rangle, \tag{14}$$

where the system is in the product state $|T_\sigma\rangle$ in region σ and the remainder of the system $\bar{\sigma}$ is in the state $|\psi_c^{\bar{\sigma}}\rangle$. To trivialize the system symmetrically, we denote that $V_g O_\sigma^{\mathrm{loc}} V_g^{-1}$ trivializes the region $g\sigma$, where $g \in D_3$. Therefore, we act on $|\psi_c\rangle$ with

$$O^{\mathrm{loc}} = \bigotimes_{g\in D_3} V_g O_\sigma^{\mathrm{loc}} V_g^{-1}, \tag{15}$$

which results in an *extensively trivialized* wavefunction

$$\begin{aligned} |\psi_c'\rangle &= O_R^{\mathrm{loc}}|\psi_c\rangle \\ &= \bigotimes_{g\in D_3} |T_{g\sigma}\rangle \otimes \bigotimes_{j=1,h\in D_3}^{3} \left|\psi_{h\tau_j}\right\rangle \otimes \bigotimes_{k=1,p\in D_3}^{3} \left|\psi_{p\mu_k}\right\rangle, \end{aligned} \tag{16}$$

where τ_j, $j=1,2,3$ and μ_k, $k=1,2,3$ label the 1D and 0D blocks as illustrated in the right panel of Fig. 1. Now all nontrivial topological properties of $|\psi_c\rangle$ are encoded in lower-dimensional block states $|\psi_{h\tau_j}\rangle$ and $|\psi_{p\mu_k}\rangle$, hence all nontrivial properties of $|\psi\rangle$ are encoded in lower-dimensional blocks in different unit cells.

B. Block-state decoration

Subsequently, with cell decompositions, we can decorate some proper lower-dimensional block states on the corresponding lower-dimensional blocks. Some symmetry operations act internally on some lower-dimensional blocks, hence the lower-dimensional block states should respect the corresponding on-site symmetry on which they decorate. As an example, we still consider the #14 wallpaper group $p3m1$ with the cell decomposition as illustrated in Fig. 1, the threefold rotational symmetry operations act on $g\mu_j$ ($g \in D_3$ and $j=1,2,3$) internally, and reflection symmetry operations act on $h\tau_k$ ($h \in D_3$ and $k=1,2,3$) internally, hence the root phases decorated on 0D and 1D blocks are 0D FSPT phases protected by $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$ on-site symmetry and 1D FSPT phases protected by $\mathbb{Z}_2$ on-site symmetry, respectively. All dD block states form the group $\{\mathrm{BS}\}^{d\mathrm{D}}$, and all block states form the following group:

$$\{\mathrm{BS}\} = \bigotimes_{d=0}^{1} \{\mathrm{BS}\}^{d\mathrm{D}}. \tag{17}$$

Here "BS" is the abbreviation of "block states".

Furthermore, the decorated states should respect the *no-open-edge condition*. Once we decorate some lower-dimensional block states on the corresponding blocks, they might leave several gapless modes on the edge of the corresponding blocks, and there are several gapless edge modes coinciding near the blocks with lower dimension. Repeatedly consider the wallpaper group $p3m1$ as an example, if we decorate a Majorana chain on the 1D block labeled by τ_1 (because of the rotational symmetry, there are also two Majorana chains decorated at the 1D blocks labeled by $\boldsymbol{R}_{\mu_3}\tau_1$ and $\boldsymbol{R}_{\mu_3}^2\tau_1$, respectively), leaving three dangling Majorana modes near the 0D block labeled by μ_3. In order to contribute an SPT state, the bulk of the system should be fully gapped, hence the aforementioned gapless modes should be gapped out (by some proper interactions, mass terms, entanglement pairs, etc.) in a symmetric way. If the bulk of the system cannot be fully gapped (i.e., several aforementioned 0D modes cannot be gapped in a symmetric way), we call the corresponding decoration *obstructed*. Equivalently, an obstruction-free decoration should satisfy the no-open-edge condition. All obstruction-free dD block states form the group $\{\mathrm{OFBS}\}^{d\mathrm{D}} \subset \{\mathrm{BS}\}^{d\mathrm{D}}$ as a subgroup of $\{\mathrm{BS}\}^{d\mathrm{D}}$, and all obstruction-free block states form the following group:

$$\{\mathrm{OFBS}\} = \bigotimes_{d=0}^{1} \{\mathrm{OFBS}\}^{d\mathrm{D}} \subset \{\mathrm{BS}\}. \tag{18}$$

Here "OFBS" is the abbreviation of "obstruction-free block states", and {OFBS} is a subgroup of {BS}.

C. Bubble equivalence

In order to obtain a nontrivial SPT state from obstruction-free block state decorations, we should further consider possible trivializations. For blocks with dimension larger than 0, we can further decorate some codimension 1 degree of freedom that could be trivialized when they shrink to a point. This construction is called *bubble equivalence*, and we demonstrate it for different dimensions:

a. 2D bubble equivalence. For 2D blocks, we can consider a 1D chain, which can be shrunk to a point inside each 2D block, and there is no on-site symmetry on them for all possible cases. In fermionic systems, the only possible state we can decorate is Majorana chain. There are two distinct boundary conditions: periodic boundary condition (PBC) with odd fermion parity and antiperiodic boundary condition (anti-PBC) with even fermion parity, as seen Fig. 2. According to the definition of bubble equivalence, we only choose the "Majorana bubbles" with anti-PBC because it can be trivialized if we shrink it into a point: If we decorate a Majorana chain with anti-PBC on a 2D block, we can shrink it to a smaller one by a 2D local unitary (LU) transformation without breaking any symmetry. Repeatedly apply this LU transformation on "Majorana" bubble, we can shrink it to a point and eliminate

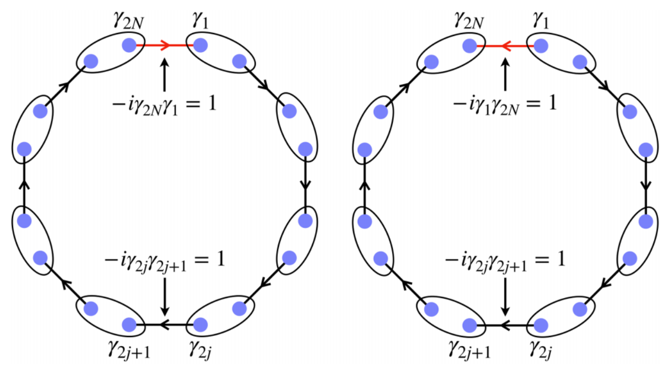


FIG. 2. Majorana chain with periodic boundary condition (PBC, left panel) and antiperiodic boundary condition (anti-PBC, right panel). The boundary conditions are indicated by the red arrows in both panels. Here ellipses represent the physical sites, and the solid oriented line from j to k indicates the paring direction, corresponding to the $i\gamma_j\gamma_k$ term in parent Hamiltonian. For the Majorana chain with PBC, the graph is not Kasteleyn oriented, and the ground state has odd fermion parity; for the Majorana chain with anti-PBC, the graph is Kasteleyn oriented, thus the ground state has even fermion parity.

it (because a Majorana chain with anti-PBC has even fermion parity) by a symmetric finite-depth circuit.

Technically, it is well known that for two Majorana modes γ_j and γ_k, their entanglement pair $i\gamma_j\gamma_k$ can be created by the following projection operator [18,19]:

$$P_{j,k} = \tfrac{1}{2}(1 - i\gamma_j\gamma_k) \tag{19}$$

and the direction is from γ_j to γ_k. Consequently the creation operator of a Majorana chain containing $2N$ Majorana modes with anti-PBC on the 2D block σ can be generated by an assembly of these projection operators

$$A_\sigma = \prod_{i=1}^{N-1} P_{2i,2i+1} \times \frac{1}{2}(1 + i\gamma_{2N}\gamma_1). \tag{20}$$

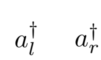
Here the last bracket shows the direction of the Majorana entanglement pair $\langle\gamma_1, \gamma_{2N}\rangle$ is from γ_1 to γ_{2N}, as an explicit indication of the anti-PBC of the Majorana chain we have created. Finally the operator of creating a 2D Majorana bubble in the entire lattice is

$$A = \bigotimes_\sigma A_\sigma. \tag{21}$$

In particular, 2D Majorana bubble cannot change the parity of Majorana chains on 1D blocks: each 1D block is the shared border of two nearby 2D blocks, hence the number of Majorana chains on this 1D block can only be changed by 0 or 2 by 2D Majorana bubble.

b. 1D bubble equivalence. For 1D blocks, we can consider two 1D irreducible representation of the corresponding total on-site symmetry of the 1D blocks that should be trivialized if they shrunk to a point. There are two possibilities:

The first one is fermionic 1D bubble; consider two complex fermions with the following geometry:



. (22)

Where yellow and red dots represent two complex fermions $a_l^\dagger$ and $a_r^\dagger$ who are trivialized when they are fused, i.e., $a_l^\dagger a_r^\dagger|0\rangle$ is a trivial atomic insulating state with even fermion parity. We demonstrate that this 1D bubble can be shrunk to a point and trivialized by a finite-depth circuit: If we decorate a 1D bubble, we can enclose $a_l^\dagger$ and $a_r^\dagger$ by an LU transformation. Repeatedly apply this LU transformation, we can shrink these two modes to a point. Therefore, the creation operator of fermionic 1D bubbles in the entire lattice is

$$B_j^f = \bigotimes_\tau (a_l^\tau)^\dagger (a_r^\tau)^\dagger. \tag{23}$$

Another one is bosonic 1D bubble: Consider two complex fermions with the geometry indicated in Eq. (22), where yellow and red dots represent two bosons $b_l^\dagger$ and $b_r^\dagger$ that carry 1D irreducible representations of the physical symmetry group (total symmetry group quotient by fermion parity $\mathbb{Z}_2^f$) of corresponding 1D blocks. They should be trivialized by shrinking them to a point: $b_l^\dagger b_r^\dagger|0\rangle$ carries trivial 1D irreducible representation of the physical symmetry group. The creation operator of bosonic 1D bubbles in the entire lattice is

$$B_j^b = \bigotimes_\tau (b_l^\tau)^\dagger (b_r^\tau)^\dagger. \tag{24}$$

And the creation operator of general 1D bubbles is

$$B_j = B_j^f \otimes B_j^b. \tag{25}$$

Enlarge these bubbles and approximate to the nearby lower-dimensional blocks, the FSPT phases decorated on the bubble can be fused with the original states on the nearby lower-dimensional blocks, which leads to some possible *trivializations* of lower-dimensional block-state decorations.

Suppose there are m different kinds of 1D bubble constructions, labeled by B_j, $j = 1, ..., m$. With this notation we can label an arbitrary bubble construction by an operator,

$$A^{l_0}\prod_{j=1}^{\beta} B_j^{l_j},\ l_0, l_j \in \mathbb{Z},$$

where l_0/l_j means that we take 2D/1D bubble construction A/B_j by l_0/l_j times. According to the definition of the bubble construction, taking an arbitrary bubble construction on the trivial state will lead to another trivial state, and all these trivial states form another group as following:

$$\{\text{TBS}\} = \left\{A^{l_0}\prod_{j=1}^{\beta} B_j^{l_j}|0\rangle \middle| l_0, l_j \in \mathbb{Z}\right\}. \tag{26}$$

Here "TBS" is the abbreviation of "trivial block states", and $\{\text{TBS}\} \subset \{\text{OFBS}\}$ because all trivial block states are obstruction-free block states. {TBS} includes trivial block states with different dimensions: $\{\text{TBS}\}^{d\text{D}}$ ($d = 0, 1$). Therefore, an obstruction and trivialization free block state can be labeled by a group element of the following quotient group:

$$\mathcal{G} = \{\text{OFBS}\}/\{\text{TBS}\} \tag{27}$$

and all group elements in G are not equivalent because we have already divided all trivial states connected by bubble constructions. Equivalently, group G gives the classification of the corresponding crystalline topological phases.

In particular, we note that the block states are constructed layer-by-layer. Therefore, we should specify the d-dimensional obstruction-free and trivialization-free block states,

$$E^{d\text{D}} = \{\text{OFBS}\}^{d\text{D}}/\{\text{TBS}\}^{d\text{D}}. \tag{28}$$

We should note that $E^{d\text{D}}$ is not a group in the sense of SPT classification, because in order to obtain the ultimate classification of SPT phases, we should further consider the possible stacking between block states with different dimensions. $E^{d\text{D}}$ can only be treated as a group only in the sense of dD block states.

With all obstruction and trivialization free block states with different dimensions, the ultimate classification with accurate group structure of 2D crystalline fSPT phases is extension between $E^{1\text{D}}$ and $E^{0\text{D}}$,

$$\mathcal{G} = E^{1\text{D}} \times_{\omega_2} E^{0\text{D}}, \tag{29}$$

here the symbol $\times_{\omega_2}$ depicts the possible extensions of $E^{1\text{D}}$ and $E^{0\text{D}}$ that is characterized by following short exact sequence:

$$0 \to E^{1\text{D}} \to \mathcal{G} \to E^{0\text{D}} \to 0. \tag{30}$$

In the following, we explicitly apply these procedures to calculate the classification of crystalline TSC and TI by several representative examples for each crystallographic systems.

III. CONSTRUCTION AND CLASSIFICATION OF CRYSTALLINE TOPOLOGICAL SUPERCONDUCTOR

In this section, we describe the details of real-space construction for crystalline TSC in 2D interacting fermionic systems by analyzing several typical examples. It is well known that all 17 wallpaper groups can be divided into five different crystallographic systems:

Square lattice: with rotational symmetry of order 4, including $p4$, $p4m$, $p4g$.

Parallelogrammatic lattice: with only rotational symmetry of order 2, and no other symmetry than translational, including $p1$, $p2$.

Rhombic lattice: with reflection combined with glide reflection, including cm, cmm

Rectangle lattice: with reflection or glide reflection, but not both, including pm, pg, pmm, pmg, pgg.

Hexagonal lattice with rotational symmetry of order 3 or 6, including $p3$, $p3m1$, $p31m$, $p6$, $p6m$.

The key distinction between different crystallographic systems is 0D blocks as centers of different point group.

In particular, we apply the general paradigm of real-space construction highlighted in Sec. II to investigate five representative cases that belong to different crystallographic systems:

(1) square lattice: $p4m$;
(2) parallelogrammatic lattice: $p2$;
(3) rhombic lattice: cmm;
(4) rectangle lattice: pgg;
(5) hexagonal lattice: $p6m$.

All other cases are assigned in the Supplemental Material [88]. The classification results are summarized in Tables I and

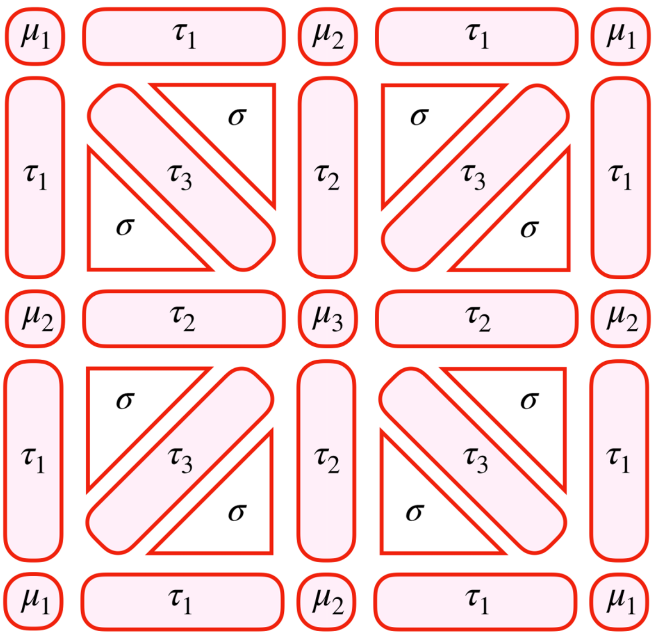


FIG. 3. #11 wallpaper group $p4m$ and its cell decomposition.

II, for spinless and spin-1/2 fermions, respectively. Furthermore, for a 2D spinless system with $p4m$ wallpaper group symmetry, there is an intrinsic interacting fermionic crystalline TSC that cannot be realized in free fermion systems or interacting bosonic systems.

A. Square lattice: $p4m$

For square lattice, we demonstrate the TSC protected by $p4m$ symmetry as an example. In the remainder of this paper, we use the same label of p-dimensional blocks that can be related by symmetry actions for abbreviation. The corresponding point group is dihedral group D_4, and for 2D blocks σ, there is no on-site symmetry group; for 1D blocks τ_1, τ_2, τ_3, the on-site symmetry group is $\mathbb{Z}_2$ via the reflection symmetry acting internally; for 0D blocks μ_1 and μ_3, the on-site symmetry group is $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$ via the D_4 symmetry acting internally; for 0D blocks μ_2, the on-site symmetry group is $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ via the $D_2 \subset D_4$ symmetry acting internally, as seen in Fig. 3.

We discuss systems with spinless and spin-1/2 fermions separately. The "spinless"/"spin-1/2" fermion means that the point subgroup is extended trivially/nontrivially by fermion parity $\mathbb{Z}_2^f$ [87].

1. Spinless fermions

For spinless systems, we first consider the 0D block-state decoration, for 0D blocks μ_j ($j = 1, 2, 3$), the classification data of the corresponding 0D block states can be characterized by different 1D irreducible representations of the full symmetry group ($n = 2, 4$),

$$\mathcal{H}^1\left[\mathbb{Z}_2^f \times (\mathbb{Z}_n \rtimes \mathbb{Z}_2), U(1)\right] = \mathbb{Z}_2^3. \tag{31}$$

For arbitrary 0D block [whose classification data are calculated in Eq. (31)], three $\mathbb{Z}_2$ have different physical meanings: The first $\mathbb{Z}_2$ represents the parity of complex fermion (even or odd), the second $\mathbb{Z}_2$ represents the rotation eigenvalue -1, and the third $\mathbb{Z}_2$ represents the reflection eigenvalue -1. So at each 0D block, the block state can be labeled by $(\pm, \pm, \pm)$, where these three $\pm$ represent the fermion parity and eigenvalues of two independent reflection generators, respectively. We should note that even-fold dihedral group can also be generated by two independent reflection operations: For 0D blocks μ_1/μ_3, D_4 symmetry can be generated by reflection operations $\boldsymbol{M}_{\tau_1}/\boldsymbol{M}_{\tau_2}$ and $\boldsymbol{M}_{\tau_3}$ ($\boldsymbol{M}_{\tau_1}, \boldsymbol{M}_{\tau_2}, \boldsymbol{M}_{\tau_3}$ represent the reflection operation with respect to the axis, which coincide with the 1D block labeled by τ_1, τ_2, τ_3); for 0D block μ_2, D_2 symmetry can be generated by reflection operations $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_2}$. According to this notation, the obstruction-free 0D block states form the following group:

$$\{\text{OFBS}\}_{p4m,0}^{\text{0D}} = \mathbb{Z}_2^9 \tag{32}$$

where the group elements can be labeled by

$$[(\pm, \pm, \pm), (\pm, \pm, \pm), (\pm, \pm, \pm)],$$

here three brackets represent the block states at μ_1, μ_2, and μ_3, respectively.

Subsequently we consider the 1D block-state decoration. For τ_1, τ_2, and τ_3, the total symmetry group is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, so there are two possible 1D block states: Majorana chain and 1D FSPT state, and all 1D block states form a group,

$$\{\text{BS}\}_{p4m,0}^{\text{1D}} = \mathbb{Z}_2^6. \tag{33}$$

Below we discuss the decorations of these two root phases separately.

a. Majorana chain decoration. Consider Majorana chain decoration on 1D blocks labeled by τ_1, which leaves four dangling Majorana modes at each 0D block μ_1/μ_3, and two dangling Majorana modes at each 0D block μ_2. Near μ_1, Majorana modes have the following rotation and reflection symmetry (all subscripts are taken with modulo 4):

$$\boldsymbol{R}_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \;\; \boldsymbol{M}_{\tau_2} : \gamma_j \mapsto \gamma_{4-j.} \tag{34}$$

The local fermion parity operator and its symmetry properties read

$$P_f = -\prod_{j=1}^{4} \gamma_j, \quad \boldsymbol{R}_{\mu_1}, \boldsymbol{M}_{\tau_2} : P_f \mapsto -P_f. \tag{35}$$

Hence these four Majorana modes break the fermion parity. Thus Majorana chain decoration on τ_1 does not contribute to nontrivial crystalline TSC because of the violation of the no-open-edge condition. It is similar for 1D blocks τ_2 and τ_3, so all types of Majorana chain decoration are obstructed.

b. 1D FSPT state decoration. The 1D FSPT state decoration on τ_1, τ_2 and τ_3 will leave eight dangling Majorana modes (ξ_j, ξ_j', $j = 1, 2, 3, 4$) at each 0D block labeled by μ_1/μ_3 and four dangling Majorana modes (η_j, η_j', $j = 1, 2$) at each 0D block labeled by μ_2. At μ_1/μ_3 (we discuss μ_1 as an example), the corresponding eight Majorana modes have the following rotation and reflection symmetry properties (all subscripts are taken under modulo 4, e.g., $\xi_5 \equiv \xi_1$ and $\xi_5' \equiv \xi_1'$),

$$\begin{aligned} &\boldsymbol{R}_{\mu_1} : \xi_j \mapsto \xi_{j+1}, \; \xi_j' \mapsto \xi_{j+1}' \\ &\boldsymbol{M}_{\tau_1} : \xi_j \mapsto \xi_{6-j}, \; \xi_j' \mapsto -\xi_{6-j}' \end{aligned}, \quad j = 1, 2, 3, 4. \tag{36}$$

We can define four complex fermions from these eight dangling Majorana modes,

$$c_j^\dagger = \tfrac{1}{2}(\xi_j + i\xi_j'), \quad j = 1, 2, 3, 4. \tag{37}$$

And from the point group symmetry properties (36), we can obtain the point group symmetry properties of the above complex fermions as

$$\begin{aligned}\boldsymbol{R}_{\mu_1}&: (c_1^\dagger, c_2^\dagger, c_3^\dagger, c_4^\dagger) \mapsto (c_2^\dagger, c_3^\dagger, c_4^\dagger, c_1^\dagger),\\ \boldsymbol{M}_{\tau_1}&: (c_1^\dagger, c_2^\dagger, c_3^\dagger, c_4^\dagger) \mapsto (c_1, c_4, c_3, c_2).\end{aligned} \tag{38}$$

We denote the fermion number operators $n_j = c_j^\dagger c_j$, $j = 1, 2, 3, 4$. Firstly we consider the Hamiltonian with Hubbard interaction ($U > 0$) that can gap out these dangling Majorana modes,

$$H_U = U\sum_{j=1,2}\left(n_j - \frac{1}{2}\right)\left(n_{j+2} - \frac{1}{2}\right). \tag{39}$$

And it can also be expressed in terms of Majorana modes with symmetry properties as shown in Eq. (36),

$$H_U = -\frac{U}{4}(\xi_1\xi_1'\xi_3\xi_3' + \xi_2\xi_2'\xi_4\xi_4'). \tag{40}$$

It is easy to verify that H_U respects all symmetries. There is a fourfold ground-state degeneracy from (n_1, n_3) and (n_2, n_4), which can be viewed as two spin-1/2 degrees of freedom,

$$\tau_{13}^\mu = (c_1^\dagger, c_3^\dagger)\sigma^\mu\begin{pmatrix}c_1\\ c_3\end{pmatrix} \tag{41}$$

and

$$\tau_{24}^\mu = (c_2^\dagger, c_4^\dagger)\sigma^\mu\begin{pmatrix}c_2\\ c_4\end{pmatrix} \tag{42}$$

where σ^μ, $\mu = x, y, z$ are Pauli matrices. In order to lift this ground-state degeneracy (GSD), we should further consider the interactions between these two spins. The symmetry properties of these two spins can be easily obtained from (38),

$$\begin{aligned}\boldsymbol{R}_{\mu_1}&: \begin{array}{l}(\tau_{13}^x, \tau_{13}^y, \tau_{13}^z) \mapsto (\tau_{24}^x, \tau_{24}^y, \tau_{24}^z)\\ (\tau_{24}^x, \tau_{24}^y, \tau_{24}^z) \mapsto (\tau_{13}^x, -\tau_{13}^y, -\tau_{13}^z)\end{array}\\ \boldsymbol{M}_{\tau_1}&: \begin{array}{l}(\tau_{13}^x, \tau_{13}^y, \tau_{13}^z) \mapsto (-\tau_{13}^x, \tau_{13}^y, -\tau_{13}^z)\\ (\tau_{24}^x, \tau_{24}^y, \tau_{24}^z) \mapsto (-\tau_{24}^x, -\tau_{24}^y, \tau_{24}^z).\end{array}\end{aligned} \tag{43}$$

Then we can further add a spin Hamiltonian ($J > 0$),

$$H_J = J(\tau_{13}^x\tau_{24}^x + \tau_{13}^y\tau_{24}^z - \tau_{13}^z\tau_{24}^y). \tag{44}$$

According to the symmetry properties of spin operations (43), we can easily verify that the spin Hamiltonian H_J is symmetric under all symmetries. We can also verify the symmetry properties in Majorana representations by expressing H_J in terms of Majorana modes as

$$\begin{aligned}H_J = &-\frac{J}{4}(\xi_1\xi_3' - \xi_1'\xi_3)(\xi_2\xi_4' - \xi_2'\xi_4)\\ &-\frac{J}{4}(\xi_1\xi_3 + \xi_1'\xi_3')(\xi_2\xi_2' - \xi_4\xi_4')\\ &+\frac{J}{4}(\xi_1\xi_1' - \xi_3\xi_3')(\xi_2\xi_4 + \xi_2'\xi_4')\end{aligned} \tag{45}$$

and it is invariant under the symmetry properties defined in Eq. (36). The GSD is lifted by a symmetric Hamiltonian $H_U + H_J$, and the nondegenerate ground state is

$$|\psi\rangle_{0\mathrm{D}} = -\tfrac{1}{2}(|\uparrow,\uparrow\rangle + i|\uparrow,\downarrow\rangle - i|\downarrow,\uparrow\rangle - |\downarrow,\downarrow\rangle) \tag{46}$$

where $\uparrow$ and $\downarrow$ represent spin-up and spin-down of two spin-1/2 degrees of freedom ($\vec{\tau}_{13}$ and $\vec{\tau}_{24}$), and the ground-state energy is $-3J$. It is easy to verify that this state is invariant under arbitrary symmetry actions because $|\psi\rangle_{0\mathrm{D}}$ is the eigenstate of the operators $\boldsymbol{R}_{\mu_1}$ and $\boldsymbol{M}_{\tau_1}$ as two generators of D_4 group at each μ_1,

$$\begin{aligned}\boldsymbol{R}_{\mu_1}|\psi\rangle_{0\mathrm{D}} &= i|\psi\rangle_{0\mathrm{D}},\\ \boldsymbol{M}_{\tau_1}|\psi\rangle_{0\mathrm{D}} &= -|\psi\rangle_{0\mathrm{D}}.\end{aligned} \tag{47}$$

Thus the corresponding eight Majorana modes are gapped out by interactions in a symmetric way.

Next we consider the dangling Majorana modes from the 1D FSPT decorations on τ_1 at μ_2 with the rotation and reflection symmetry properties,

$$\begin{aligned}\boldsymbol{R}_{\mu_2}&: (\eta_1, \eta_1', \eta_2, \eta_2') \mapsto (\eta_2, \eta_2', \eta_1, \eta_1'),\\ \boldsymbol{M}_{\tau_1}&: (\eta_1, \eta_1', \eta_2, \eta_2') \mapsto (\eta_1, -\eta_1', \eta_2, -\eta_2').\end{aligned} \tag{48}$$

We can define two complex fermions from these four dangling Majorana modes,

$$c^\dagger = \tfrac{1}{2}(\eta_1 + i\eta_2),\ c'^\dagger = \tfrac{1}{2}(\eta_1' + i\eta_2') \tag{49}$$

and from the symmetry properties (48), we can obtain the point group symmetry properties of the above complex fermions,

$$\begin{aligned}\boldsymbol{R}&: (c^\dagger, c'^\dagger) \mapsto (ic, ic'),\\ \boldsymbol{M}&: (c^\dagger, c'^\dagger) \mapsto (c^\dagger, -c'^\dagger).\end{aligned} \tag{50}$$

We denote the fermion number operators $n = c^\dagger c$ and $n' = c'^\dagger c'$. First we consider the Hamiltonian with Hubbard interaction ($U' > 0$) that can gap out these dangling Majorana modes,

$$H_U' = U'\left(n - \tfrac{1}{2}\right)\left(n' - \tfrac{1}{2}\right). \tag{51}$$

And it is easy to verify that H_U' respects all symmetries according to the symmetry properties of defined complex fermions (50). There is a twofold ground-state degeneracy from (n, n') that can be viewed as a spin-1/2 degree of freedom,

$$\tau^\mu = (c^\dagger, c'^\dagger)\sigma^\mu\begin{pmatrix}c\\ c'\end{pmatrix}. \tag{52}$$

In order to investigate whether the degenerate ground states can be gapped out or not, we focus on the projective Hilbert space spanned by two states $c^\dagger|0\rangle$ and $c'^\dagger|0\rangle$. In this projective

Hilbert space, two generators of D_2 symmetry on each μ_2, $\boldsymbol{R}_{\mu_2}$ and $\boldsymbol{M}_{\tau_1}$ can be represented by two 2×2 matrices,

$$\boldsymbol{R}_{\mu_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x,$$
$$\boldsymbol{M}_{\tau_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z. \tag{53}$$

It is obvious that these two generators are anticommute,

$$\boldsymbol{R}_{\mu_2}\boldsymbol{M}_{\tau_1} = -\boldsymbol{M}_{\tau_1}\boldsymbol{R}_{\mu_2},$$

i.e., a sufficient condition shows that the projective Hilbert space is a projective representation of the symmetry group D_2 at each 0D block labeled by μ_2. Hence, the twofold ground-state degeneracy cannot be lifted.

We demonstrate this conclusion in Majorana representation and elucidate that all possible mass terms are not compatible with symmetries. A mass term is formed by two Majorana operators, and all possible mass terms are

$$\eta_1\eta_2,\ \eta_1\eta_1',\ \eta_1\eta_2',\ \eta_2\eta_1',\ \eta_2\eta_2',\ \eta_1'\eta_2'$$

and their linear combinations. Under twofold rotation $\boldsymbol{R}_{\mu_2}$, these mass terms will be transformed to

$$-\eta_1\eta_2,\ \eta_2\eta_2',\ \eta_2\eta_1',\ \eta_1\eta_2',\ \eta_1\eta_1',\ -\eta_1'\eta_2'$$

so there are only two mass terms that are symmetric under $\boldsymbol{R}_{\mu_2}$: $\eta_1\eta_1'+\eta_2\eta_2'$ and $\eta_1\eta_2'+\eta_2\eta_1'$ and their linear combinations. Subsequently, under the reflection $\boldsymbol{M}_{\tau_1}$, these terms are not symmetric,

$$-(\eta_1\eta_1'+\eta_2\eta_2'),\ -(\eta_1\eta_2'+\eta_2\eta_1').$$

Therefore, there is no symmetric mass term to lift the GSD. Accordingly, 1D FSPT state decoration on τ_1 is *obstructed* because of the degenerate ground state, and similar arguments can also be held on 1D blocks labeled by τ_2 (and the obstruction also happens at 0D block μ_2, as the center of D_2 symmetry). 1D FSPT state decoration on τ_3 is *obstruction-free* because this decoration leaves eight dangling Majorana modes at each 0D block labeled by μ_1 and μ_3, and both of them are centers of D_4 symmetry.

There is one exception: If we decorate a 1D FSPT phase on each 0D block labeled by τ_1 and τ_2 simultaneously, it leaves eight dangling Majorana modes at each 0D block μ_2 ($\eta_j, \eta_j', j=1,2,3,4$), with the following rotation and reflection symmetry properties:

$$\boldsymbol{R}_{\mu_2}:\begin{cases}(\eta_1,\eta_1',\eta_2,\eta_2')\mapsto(\eta_2,\eta_2',\eta_1,\eta_1')\\ (\eta_3,\eta_3',\eta_4,\eta_4')\mapsto(\eta_4,\eta_4',\eta_3,\eta_3'),\end{cases}$$
$$\boldsymbol{M}_{\tau_1}:\begin{cases}(\eta_1,\eta_1',\eta_2,\eta_2')\mapsto(\eta_1,-\eta_1',\eta_2,-\eta_2')\\ (\eta_3,\eta_3',\eta_4,\eta_4')\mapsto(\eta_4,-\eta_4',\eta_3,-\eta_3').\end{cases} \tag{54}$$

This situation is quite similar with aforementioned gapping situation at each 0D block labeled by μ_1 or μ_3, with lower point group symmetry ($D_2 \in D_4$). Thus eight dangling Majorana modes at each 0D block μ_2 from decorating a 1D FSPT state on each τ_1 and τ_2 can be gapped by previously discussed interactions $H_U + H_J$ [cf. Eqs. (39) and (44)] in a symmetric way, and the 1D FSPT state decoration on τ_1 and τ_2 simultaneously is *obstruction-free*. We should note that this block state has no free-fermion realization because as aforementioned, we should introduce some interactions to satisfy the no-open-edge condition, as noninteracting mass terms cannot gap them out. Hence the crystalline TSC realized here is an intrinsic interacting FSPT phase. In summary, all obstruction-free 1D block states are

(i) 1D FSPT state decoration on τ_1 and τ_2 simultaneously;
(ii) 1D FSPT state decoration on τ_3.

and they form the following group:

$$\{\mathrm{OFBS}\}_{p4m,0}^{\mathrm{1D}} = \mathbb{Z}_2^2 \tag{55}$$

where the group elements can be labeled by

$$[m_1 = m_2, m_3],$$

here $m_j = 0, 1$ ($j = 1, 2, 3$) represents the number of decorated 1D FSPT states on τ_j, respectively. According to aforementioned discussions, a necessary condition of an obstruction-free block state is $m_1 = m_2$.

With all obstruction-free block states, below we will discuss all possible trivializations. First we consider the 2D bubble equivalences: we decorate a Majorana bubble on each 2D block σ (see Fig. 4), and then demonstrate that they can be deformed into double Majorana chains at each nearby 1D block, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry. Fig. 4(b) shows that these Majorana bubbles can be deformed to double Majorana chains. For $p4m$ case, all 1D blocks are lying on the reflection axis, and reflection operation are acting on them internally: reflection operation (on-site $\mathbb{Z}_2$ symmetry on 1D blocks) exchanges two Majorana chains deformed from "Majorana" bubble constructions, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry. Equivalently, we can say that 1D FSPT state decorations on all 1D blocks can be deformed to a trivial state via 2D bubble equivalence. Next, we further investigate whether 2D bubble equivalence can change the fermion parity of 0D blocks or not. We have already seen that 2D Majorana bubble equivalence leaves double Majorana chains on all 1D blocks. Correspondingly, it leaves 16 Majorana modes at each 0D block μ_1/μ_3 and eight Majorana modes at each 0D block μ_2 as the edge modes of double Majorana chains on 1D blocks. Apparently, these Majorana modes cannot be connected to Majorana chains with PBC surrounding the 0D blocks (with fermion parity odd): It is well known that Majorana chain is not compatible with reflection symmetry; however, the Majorana chain with PBC surrounding 0D block must across at least one reflection axis. As a result, the overall effect of 2D Majorana bubble equivalence is deforming the 1D FSPT phase (protected by on-site $\mathbb{Z}_2$ symmetry) decorations on all 1D blocks to a trivial state.

Subsequently we consider the 1D bubble equivalences. For instance, we decorate a pair of complex fermions [cf. Eq. (22)] on each 1D block τ_1: Near each 0D block μ_1, there are four complex fermions forming the following atomic insulator:

$$|\psi\rangle_{p4m}^{\mu_1} = c_1^\dagger c_2^\dagger c_3^\dagger c_4^\dagger|0\rangle \tag{56}$$

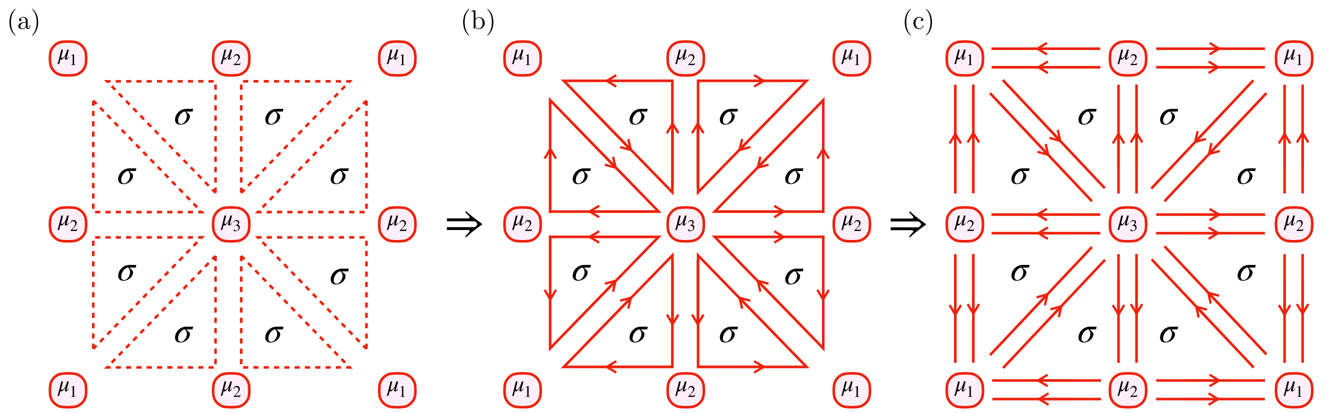


FIG. 4. Deformation of "Majorana" bubble construction. (a) 1D vacuum block state that all 1D blocks are in vacuum state. Here μ_1, μ_2, and μ_3 label different 0D blocks and σ enclosed by a dashed triangle labels 2D blocks in one unit cell. (b) Decorate a "Majorana" bubble in each 2D block in a symmetric way. Each solid oriented triangle expresses a Majorana chain with anti-PBC. (c) Enlarge the "Majorana" bubbles, they can be deformed to the 1D block state with 1D FSPT states protected by on-site $\mathbb{Z}_2$ symmetry on all 1D blocks. Here each double oriented lines expresses a 1D FSPT states protected by on-site $\mathbb{Z}_2$ symmetry that can be constructed by two Majorana chains.

with two independent reflection properties,

$$\begin{aligned} \boldsymbol{M}_{\tau_1}|\psi\rangle_{p4m}^{\mu_1} &= c_1^\dagger c_4^\dagger c_3^\dagger c_2^\dagger|0\rangle = -|\psi\rangle_{p4m}^{\mu_1}, \\ \boldsymbol{M}_{\tau_3}|\psi\rangle_{p4m}^{\mu_1} &= c_3^\dagger c_4^\dagger c_1^\dagger c_2^\dagger|0\rangle = |\psi\rangle_{p4m}^{\mu_1}, \end{aligned} \tag{57}$$

i.e., at 0D blocks μ_1, 1D bubble construction on τ_1 changes the reflection eigenvalue of $\boldsymbol{M}_{\tau_1}$, and leaves the reflection eigenvalue of $\boldsymbol{M}_{\tau_2}$ invariant. Near each 0D block μ_2, there are two complex fermions forming another atomic insulator,

$$|\psi\rangle_{p4m}^{\mu_2} = c_1'^\dagger c_2'^\dagger|0\rangle \tag{58}$$

with two independent reflection properties,

$$\begin{aligned} \boldsymbol{M}_{\tau_1}|\psi\rangle_{p4m}^{\mu_2} &= c_1'^\dagger c_2'^\dagger|0\rangle = |\psi\rangle_{p4m}^{\mu_2}, \\ \boldsymbol{M}_{\tau_2}|\psi\rangle_{p4m}^{\mu_2} &= c_2'^\dagger c_1'^\dagger|0\rangle = -|\psi\rangle_{p4m}^{\mu_2}, \end{aligned} \tag{59}$$

i.e., at 0D blocks μ_2 1D bubble construction on τ_1 changes the reflection eigenvalue of $\boldsymbol{M}_{\tau_2}$, and leaves the reflection eigenvalue of $\boldsymbol{M}_{\tau_1}$ invariant. This 1D bubble equivalence is illustrated in Fig. 5. Similar 1D bubble constructions can be held on 1D blocks τ_2 and τ_3, and we summarize the effects of 1D bubble constructions as following:

(1) 1D bubble construction on τ_1: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_1}$ at μ_1 and $\boldsymbol{M}_{\tau_2}$ at μ_2;

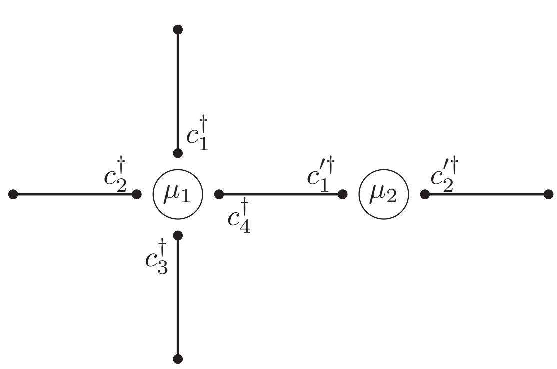


FIG. 5. 1D bubble equivalence on τ_1. Atomic insulators (56) and (58) are constructed by this procedure.

(2) 1D bubble construction on τ_2: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_1}$ at μ_2 and $\boldsymbol{M}_{\tau_2}$ at μ_3;

(3) 1D bubble construction on τ_3: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_3}$ at μ_1 and $\boldsymbol{M}_{\tau_3}$ at μ_3;

With all possible trivializations, we are ready to study the trivial states. Start from the original 0D trivial block state

$$[(+,+,+),(+,+,+),(+,+,+)].$$

If we take 1D bubble constructions on τ_j by l_j times ($j = 1, 2, 3$), the above trivial 0D block state will be transformed to a new 0D block state labeled by

$$[(+,(-1)^{l_1},(-1)^{l_3}),(+,(-1)^{l_2},(-1)^{l_1}),(+,(-1)^{l_2},(-1)^{l_3})]. \tag{60}$$

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only three independent quantities (l_j, $j = 1, 2, 3$) in Eq. (60). Together with the 2D Majorana bubble construction that deforms the vacuum 1D block state to 1D FSPT states decorated on all 1D blocks, all these trivial states form the group,

$$\{\mathrm{TBS}\}_{p4m,0} = \{\mathrm{TBS}\}_{p4m,0}^{1\mathrm{D}} \times \{\mathrm{TBS}\}_{p4m,0}^{0\mathrm{D}} = \mathbb{Z}_2 \times \mathbb{Z}_2^3 = \mathbb{Z}_2^4, \tag{61}$$

where $\{\mathrm{TBS}\}_{p4m,0}^{1\mathrm{D}}$ represents the group of trivial states with nonvacuum 1D blocks (i.e., 1D FSPT phase decorations on all 1D blocks), and $\{\mathrm{TBS}\}_{p4m,0}^{0\mathrm{D}}$ represents the group of trivial states with nonvacuum 0D blocks.

Therefore, all independent nontrivial block states with different dimensions are classified by

$$\begin{aligned} E_{p4m,0}^{1\mathrm{D}} &= \{\mathrm{OFBS}\}_{p4m,0}^{1\mathrm{D}}/\{\mathrm{TBS}\}_{p4m,0}^{1\mathrm{D}} = \mathbb{Z}_2, \\ E_{p4m,0}^{0\mathrm{D}} &= \{\mathrm{OFBS}\}_{p4m,0}^{0\mathrm{D}}/\{\mathrm{TBS}\}_{p4m,0}^{0\mathrm{D}} = \mathbb{Z}_2^6, \end{aligned} \tag{62}$$

where one $\mathbb{Z}_2$ is from the nontrivial 1D block state, and other six $\mathbb{Z}_2$ are from the nontrivial 0D block states.

With all nontrivial block states, we consider the group structure of the ultimate classification. The physical meaning of the group structure is whether stacking of 1D block state extends to 0D block state or not. We argue that there is no stacking between block states with different dimensions for p4m symmetry. In order to investigate the possible stacking, we consider two identical 1D block states: for example, we decorate two copies of 1D FSPT states on each 1D block labeled by τ_3, which leaves 16 dangling Majorana modes at each 0D block labeled by μ_1/μ_3. It is easy to verify that two copies of 1D FSPT states should be a trivial 1D block state because the root phase has a $\mathbb{Z}_2$ structure. First of all, according to previous discussions, these decoration cannot be deformed to a Majorana chain surrounding the 0D block to change the corresponding fermion parity because the Majorana chain is not compatible with the reflection symmetry. Subsequently at each 0D block μ_1/μ_3, we can treat these 16 Majorana modes as eight complex fermions: c_j and c_j' ($j=1,2,3,4$) form two atomic insulators,

$$\begin{aligned}|\phi\rangle &= a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger|0\rangle\\ |\phi'\rangle &= a_1'^\dagger a_2'^\dagger a_3'^\dagger a_4'^\dagger|0\rangle\end{aligned}\tag{63}$$

and the wavefunction of these eight complex fermions is direct product of $|\phi\rangle$ and $|\phi'\rangle$,

$$|\Phi\rangle = |\phi\rangle\otimes|\phi'\rangle.\tag{64}$$

$|\phi\rangle$ and $|\phi'\rangle$ are eigenstates of two generators of D_4 symmetry, $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_3}$,

$$\begin{aligned}\boldsymbol{M}_{\tau_1}|\phi\rangle &= a_2^\dagger a_1^\dagger a_4^\dagger a_3^\dagger|0\rangle = |\phi\rangle,\\ \boldsymbol{M}_{\tau_1}|\phi'\rangle &= a_2'^\dagger a_1'^\dagger a_4'^\dagger a_3'^\dagger|0\rangle = |\phi'\rangle,\\ \boldsymbol{M}_{\tau_3}|\phi\rangle &= a_1^\dagger a_4^\dagger a_3^\dagger a_2^\dagger|0\rangle = -|\phi\rangle,\\ \boldsymbol{M}_{\tau_3}|\phi'\rangle &= a_1'^\dagger a_4'^\dagger a_3'^\dagger a_2'^\dagger|0\rangle = -|\phi'\rangle.\end{aligned}\tag{65}$$

Then the eigenvalues of $|\Phi\rangle$ under $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_3}$ are trivial,

$$\begin{aligned}\boldsymbol{M}_{\tau_1}|\Phi\rangle &= |\Phi\rangle,\\ \boldsymbol{M}_{\tau_3}|\Phi\rangle &= |\Phi\rangle.\end{aligned}\tag{66}$$

Therefore, stacking of 1D block state cannot extend to 0D block state, and the ultimate classification of 2D crystalline SPT phases with $p4m$ symmetry for spinless fermions is

$$\mathcal{G}_{p4m,0} = E^{\mathrm{1D}}_{p4m,0}\times E^{\mathrm{0D}}_{p4m,0} = \mathbb{Z}_2^7.\tag{67}$$

2. Spin-1/2 fermions

Now we turn to discuss systems with spin-1/2 fermions. We first consider the 0D block-state decoration. For each 0D block μ_j ($j=1,2,3$), the classification data can also be characterized by different 1D irreducible representations of the full symmetry group $\mathbb{Z}_2^f\times_{\omega_2}(\mathbb{Z}_n\rtimes\mathbb{Z}_2)$ ($n=2,4$, and the symbol $\times_{\omega_2}$ means that the physical symmetry group is nontrivially extended by fermion parity $\mathbb{Z}_2^f$, which is characterized by a 2-cocycle ω_2, see Sec. I B):

$$\mathcal{H}^1\big[\mathbb{Z}_2^f\times_{\omega_2}(\mathbb{Z}_n\rtimes\mathbb{Z}_2), U(1)\big] = \mathbb{Z}_2^2.\tag{68}$$

To calculate this, we should firstly calculate the following two cohomologies:

$$\begin{aligned}\mathcal{H}^0(\mathbb{Z}_n\rtimes\mathbb{Z}_2,\mathbb{Z}_2) &= \mathbb{Z}_2\\ \mathcal{H}^1[\mathbb{Z}_n\rtimes\mathbb{Z}_2, U(1)] &= \mathbb{Z}_2^2.\end{aligned}\tag{69}$$

But the 0-cocycle $n_0\in\mathcal{H}^0(\mathbb{Z}_n\rtimes\mathbb{Z}_2,\mathbb{Z}_2)$ does not contribute a nontrivial 0D block state: a specific n_0 is obstructed if and only if $(-1)^{\omega_2\smile n_0}\in\mathcal{H}^2[\mathbb{Z}_4\rtimes\mathbb{Z}_2, U(1)]$ is a nontrivial 2-cocycle with $U(1)$ coefficient. From Refs. [19] and [87] we know that nontrivial 0-cocycle $n_0=1$ (fermion parity odd) leads to a nontrivial 2-cocycle $(-1)^{\omega_2\smile n_0}\in\mathcal{H}^2[\mathbb{Z}_4\rtimes\mathbb{Z}_2, U(1)]$, and the 0D block states at μ_j with odd fermion parity are obstructed. Hence different $\mathbb{Z}_2$'s in the classification data represent the rotation and reflection eigenvalues at each D_4 or D_2 center. As a consequence, all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}^{\mathrm{0D}}_{p4m,1/2} = \mathbb{Z}_2^6.\tag{70}$$

Then we demonstrate that there is no trivialization. For spinless fermions, we have demonstrated that the eigenvalue -1 of $\boldsymbol{M}_{\tau_1}/\boldsymbol{M}_{\tau_2}$ is trivialized by atomic insulator $|\psi\rangle^{\mu_1}_{p4m}/|\psi\rangle^{\mu_2}_{p4m}$ [cf. Eqs. (56) and (58), and Fig. 5]. Nevertheless, to fulfill the spin-1/2 condition ($\boldsymbol{M}_{\tau_1}^2=\boldsymbol{M}_{\tau_2}^2=-1$), there is an additional minus sign under reflections,

$$\begin{aligned}\boldsymbol{M}_{\tau_1}|\psi\rangle^{\mu_1}_{p4m} &= c_1^\dagger c_4^\dagger c_3^\dagger(-c_2^\dagger)|0\rangle = |\psi\rangle^{\mu_1}_{p4m},\\ \boldsymbol{M}_{\tau_2}|\psi\rangle^{\mu_2}_{p4m} &= c_1'^\dagger(-c_2'^\dagger) = |\psi\rangle^{\mu_2}_{p4m},\end{aligned}\tag{71}$$

i.e., eigenvalues of $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_2}$ remain invariant. Equivalently, there is no trivialization for 0D block states,

$$\{\mathrm{TBS}\}^{\mathrm{0D}}_{p4m,1/2} = \mathbb{Z}_1.\tag{72}$$

As the consequence, the classification attributed to 0D block states is

$$E^{\mathrm{0D}}_{p4m,1/2} = \mathbb{Z}_2^6.\tag{73}$$

Subsequently we consider the 1D block-state decoration. For arbitrary 1D blocks, the total symmetry group is $\mathbb{Z}_4^f$, hence there is no nontrivial 1D block state due to the trivial classification of the corresponding 1D FSPT phases, and the classification attributed to 1D block-state decorations is trivial,

$$E^{\mathrm{1D}}_{p4m,1/2} = \{\mathrm{OFBS}\}^{\mathrm{1D}}_{p4m,1/2} = \mathbb{Z}_1.\tag{74}$$

Therefore it is obvious that there is no stacking between 1D and 0D block states because of the trivial contribution from 1D block state. The ultimate classification with accurate group structure is

$$\mathcal{G}_{p4m,1/2} = E^{\mathrm{0D}}_{p4m,1/2}\times E^{\mathrm{1D}}_{p4m,1/2} = \mathbb{Z}_2^6.\tag{75}$$

B. Parallelogrammatic lattice: *p*2

For parallelogrammatic lattice, we demonstrate the crystalline TSC protected by $p2$ symmetry as an example. The corresponding point group of $p2$ is rotation group C_2. For 1D and 2D blocks, there is no on-site symmetry group, but the rotational subgroup C_2 acts on each 0D blocks internally, just identical with on-site $\mathbb{Z}_2$ symmetry, as seen Fig. 6. Below we discuss systems with spinless and spin-1/2 fermions separately.

1. Spinless fermions

For spinless fermions, the total on-site symmetry of each 0D block labeled by μ_j, $j=1,2,3,4$, is $\mathbb{Z}_2^f\times\mathbb{Z}_2$, and the

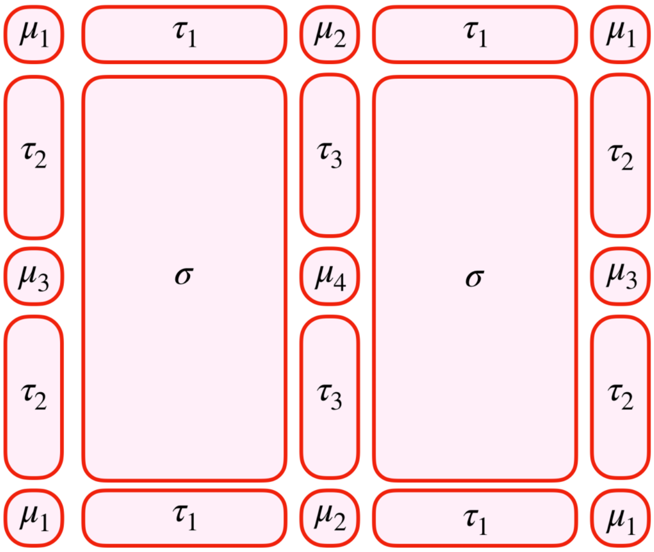


FIG. 6. #2 wallpaper group $p2$ and its cell decomposition.

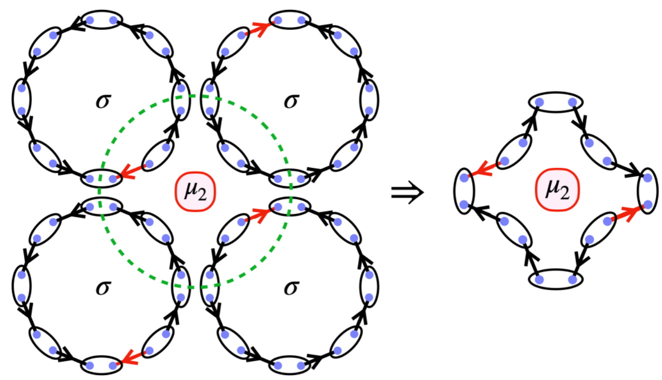


FIG. 7. 2D bubble equivalence for #2 wallpaper group $p2$. Near each 0D block (μ_2 for example), Majorana modes surrounded by green-dashed circle are deformed toward an enclosed Majorana chain surrounding the 0D block μ_2. Left panel shows the bubble construction, and right panel is the deformed Majorana chain, which is not Kasteleyn oriented, and the state has odd fermion parity.

classification data can be characterized by different 1D irreducible representations of the symmetry group $\mathbb{Z}_2^f \times \mathbb{Z}_2$,

$$\mathcal{H}^1\big[\mathbb{Z}_2^f \times \mathbb{Z}_2, U(1)\big] = \mathbb{Z}_2^2. \tag{76}$$

Here one $\mathbb{Z}_2$ is from the fermion parity, and the other is from the rotation eigenvalue -1. Thus at each 0D block, the block state can be labeled by $(\pm,\pm)$ where one $\pm$ represents the fermion parity and the other represents rotation eigenvalue, respectively. According to this notation, the obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{p2,0}^{0\mathrm{D}} = \mathbb{Z}_2^8 \tag{77}$$

and the group elements can be labeled by (four brackets represent the block states at μ_j, $j = 1, 2, 3, 4$)

$$[(\pm,\pm), (\pm,\pm), (\pm,\pm), (\pm,\pm)].$$

Subsequently we consider the 1D block-state decorations. The unique possible 1D block state is Majorana chain due to the absence of on-site symmetry on arbitrary 1D block, and all 1D block states form a group,

$$\{\mathrm{BS}\}_{p2,0}^{1\mathrm{D}} = \mathbb{Z}_2^3. \tag{78}$$

Then we consider the possible obstructions: Majorana chain decoration on τ_1 leaves two dangling Majorana modes at each 0D block labeled by μ_2, which can be glued by an entanglement pair $i\gamma_1\gamma_2$. Nevertheless, this entanglement pair breaks C_2 symmetry:

$$\boldsymbol{R}_{\mu_2} : i\gamma_1\gamma_2 \mapsto i\gamma_2\gamma_1 = -i\gamma_1\gamma_2, \tag{79}$$

hence this decoration is obstructed, and does not contribute nontrivial crystalline TSC because of the violation of the no-open-edge condition. It is similar for all other 1D blocks. As a consequence, 1D block-state decorations do not contribute any nontrivial crystalline TSC because all block states are obstructed,

$$E_{p2,0}^{1\mathrm{D}} = \{\mathrm{OFBS}\}_{p2,0}^{1\mathrm{D}} = \mathbb{Z}_1. \tag{80}$$

With all obstruction-free block states, we consider possible trivializations via bubble construction. First of all, we consider the 2D bubble equivalence: as illustrated in Fig. 7, we decorate a Majorana chain with anti-PBC on each 2D block that can be trivialized if it shrinks to a point. At each nearby 1D block, we can see that these Majorana bubbles can be deformed into double Majorana chains. Consequently, Majorana bubble construction has no effect on 1D blocks. At each nearby 0D block (μ_2 as an example, see Fig. 7), these Majorana bubbles can be deformed into an alternative Majorana chain with *odd* fermion parity surrounding it. Distinct from the $p4m$ case, this Majorana chain respects all symmetries of $p2$, so this Majorana bubble construction can change the fermion parities of all 0D blocks simultaneously.

Furthermore, consider 1D bubble equivalence on τ_1: on each 1D block labeled by τ_1, we decorate a pair of complex fermions [cf. Eq. (22)]: Near each 0D block μ_2, there are two complex fermions forming an atomic insulator,

$$|\psi\rangle_{p2}^{\mu_2} = c_1^\dagger c_2^\dagger|0\rangle \tag{81}$$

with rotation property

$$\boldsymbol{R}_{\mu_2}|\psi\rangle_{p2}^{\mu_2} = c_2^\dagger c_1^\dagger|0\rangle = -|\psi\rangle_{p2}^{\mu_2}. \tag{82}$$

Hence the rotation eigenvalue -1 can be trivialized by atomic insulator $|\psi\rangle_{p2}^{\mu_2}$. Similar for μ_1, and we can conclude that rotation eigenvalues at 0D blocks labeled by μ_1 and μ_2 are not independent. Similar bubble equivalences can be held on arbitrary 1D blocks τ_j, $j = 1, 2, 3, 4$, and rotation eigenvalues at all 0D blocks are not independent.

With all possible bubble constructions, we are ready to study the trivial states. Start from the original trivial state

$$[(+,+), (+,+), (+,+), (+,+)],$$

if we take 2D bubble construction l_0 times, and take 1D bubble constructions on τ_j with l_j times ($j = 1, 2, 3$), above trivial state will be transformed to a new 0D block state labeled by

$$\begin{aligned}&[((-1)^{l_0}, (-1)^{l_1+l_2}), ((-1)^{l_0}, (-1)^{l_1+l_3}),\\ &\times ((-1)^{l_0}, (-1)^{l_2}), ((-1)^{l_0}, (-1)^{l_3})].\end{aligned} \tag{83}$$

According to the definition of bubble equivalence, all these states should be trivial. Alternatively, all 0D block states can be viewed as a vector of an 8-dimensional $\mathbb{Z}_2$-valued vector space, and all trivial 0D block states with the form as Eq. (83) can be viewed as a 4-dimensional vector subspace generated by l_0, l_1, l_2, l_3. Hence all trivial 0D block states form the group,

$$\{\text{TBS}\}_{p2,0}^{0\text{D}} = \mathbb{Z}_2^4. \tag{84}$$

Therefore, all independent nontrivial 0D block states are labeled by different group elements of the following quotient group:

$$E_{p2,0}^{0\text{D}} = \{\text{OFBS}\}_{p2,0}^{0\text{D}} / \{\text{TBS}\}_{p2,0}^{0\text{D}} = \mathbb{Z}_2^4. \tag{85}$$

It is obvious that there is no stacking between 1D and 0D block states, and the ultimate classification with accurate group structure is

$$\mathcal{G}_{p2,0} = E_{p2,0}^{1\text{D}} \times E_{p2,0}^{0\text{D}} = \mathbb{Z}_2^4. \tag{86}$$

2. Spin-1/2 fermions

For spin-$1/2$ fermions, first we consider the 0D block-state decoration, whose candidate states can be characterized by different 1D irreducible representations of the symmetry group $\mathbb{Z}_4^f$ (nontrivial $\mathbb{Z}_2^f$ extension of $\mathbb{Z}_2$ on-site symmetry),

$$\mathcal{H}^1\left[\mathbb{Z}_4^f, U(1)\right] = \mathbb{Z}_4. \tag{87}$$

All root phases are characterized by eigenvalues $\{i, -1, -i, 1\}$ of $\mathbb{Z}_4^f$. So at each 0D block, the block state can be labeled by $\nu \in \{i, -1, -i, 1\}$. According to this notation, the obstruction-free 0D block states form the following group:

$$\{\text{OFBS}\}_{p2,1/2}^{0\text{D}} = \mathbb{Z}_4^4 \tag{88}$$

and different group elements can be labeled by

$$[\nu_1, \nu_2, \nu_3, \nu_4]$$

where ν_j labels the 0D block state at μ_j (j=1,2,3,4). It is easy to see that there is no trivialization on 0D block states (i.e., $\{\text{TBS}\}_{p2,1/2}^{0\text{D}} = \mathbb{Z}_1$), so the classification attributed to 0D block-state decoration is

$$E_{p2,1/2}^{0\text{D}} = \{\text{OFBS}\}_{p2,1/2}^{0\text{D}} / \{\text{TBS}\}_{p2,1/2}^{0\text{D}} = \mathbb{Z}_4^4. \tag{89}$$

Subsequently consider the 1D block-state decorations. The unique possible 1D block state is still the Majorana chain due to the absence of on-site symmetry on each 1D block. The Majorana chain decoration on τ_1 leaves two dangling Majorana modes at each 0D block labeled by μ_2, which can be glued by an entanglement pair $i\gamma_1\gamma_2$, and it respects rotational symmetry,

$$\boldsymbol{R}_{\mu_2}:\ i\gamma_1\gamma_2 \mapsto -i\gamma_2\gamma_1 = i\gamma_1\gamma_2. \tag{90}$$

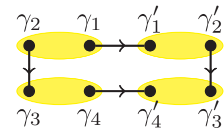
Hence Majorana chain decoration on τ_1 is an obstruction-free block state because of the satisfaction of the no-open-edge condition. It is similar for 1D blocks labeled by τ_2 and τ_3. Hence all obstruction-free 1D block states form the following group:

$$\{\text{OFBS}\}_{p2,1/2}^{1\text{D}} = \mathbb{Z}_2^3. \tag{91}$$

We have demonstrated that the 2D Majorana bubble construction cannot change the parity of Majorana chains on each 1D block in Sec. II, hence there is no trivialization (i.e., $\{\text{TBS}\}_{p2,1/2}^{1\text{D}} = \mathbb{Z}_1$), so the classification attributed to 1D block-state decorations is

$$E_{p2,1/2}^{1\text{D}} = \{\text{OFBS}\}_{p2,1/2}^{1\text{D}} / \{\text{TBS}\}_{p2,1/2}^{1\text{D}} = \mathbb{Z}_2^3. \tag{92}$$

With the classification data Eqs. (89) and (92), we consider the group structure of the corresponding classification. Equivalently, we investigate whether 1D block state extends 0D block state or not. As an example, we decorate two copies of Majorana chains on each 1D block labeled by τ_1, which leaves four dangling Majorana fermions at each 0D block labeled by μ_1/μ_2. Similar with Ref. [62], these Majorana chains can be smoothly deformed to another assembly of Majorana chains surrounding 0D blocks labeled by μ_1 and μ_2 as follows (each yellow ellipse represents a physical site):



(93)

with rotational symmetry properties: $\gamma_j \mapsto \gamma_j'$ and $\gamma_j' \mapsto -\gamma_j$. The gapped Hamiltonian corresponding to the graph in Eq. (93) is

$$H = -i\gamma_1\gamma_1' - i\gamma_4\gamma_4' - i\gamma_2\gamma_3 - i\gamma_2'\gamma_3'. \tag{94}$$

We can further define four complex fermions according to eight Majorana modes in Eq. (93) as follows:

$$\begin{aligned} c_1 &= (\gamma_2 + i\gamma_1)/2 \qquad c_2 = (\gamma_3 + i\gamma_4)/2, \\ c_1' &= (\gamma_2' + i\gamma_1')/2 \qquad c_2' = (\gamma_3' + i\gamma_4')/2. \end{aligned} \tag{95}$$

It is easy to find the ground state of Eq. (94),

$$\begin{aligned} |\phi\rangle_{0\text{D}} = &(c_1^\dagger - c_2^\dagger - ic_1'^\dagger + ic_2'^\dagger - c_1^\dagger c_1'^\dagger c_2'^\dagger + c_2^\dagger c_1'^\dagger c_2'^\dagger \\ &+ ic_1^\dagger c_2^\dagger c_1'^\dagger - ic_2^\dagger c_2^\dagger c_1'^\dagger)|0\rangle \end{aligned} \tag{96}$$

with the twofold rotation property

$$\boldsymbol{R}_{\mu_1}|\phi\rangle_{0\text{D}} = i|\phi\rangle_{0\text{D}}. \tag{97}$$

If a 0D block state with eigenvalue $e^{i\pi q/2}$ under twofold rotation is attached to each 1D block state near each 0D block labeled by μ_1, the rotation eigenvalue r of the obtained 0D block state becomes

$$r = e^{i\pi/2 + i\pi q}, \quad q \in \mathbb{Z}, \tag{98}$$

and there is no solution to the formula $r = 1$. Therefore, near each 0D block labeled by μ_1/μ_2, 1D block states extend 0D block states, hence the 0D block states at μ_1/μ_2 have the group structure $\mathbb{Z}_8$ as the nontrivial extension of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ that should be attributed to 0D and 1D block-state decorations, respectively.

Similar for other 1D and 0D block states, we can obtain that the 0D block states have the group structure $\mathbb{Z}_8$ for an arbitrary 0D block. Nevertheless, stacking between 1D and 0D block states at different 0D blocks are not independent. For instance, if we decorate two copies of Majorana chain on 1D blocks τ_1, these two Majorana chains extend the 0D block states at both μ_1 and μ_2. It is not hard to verify that only three 0D blocks

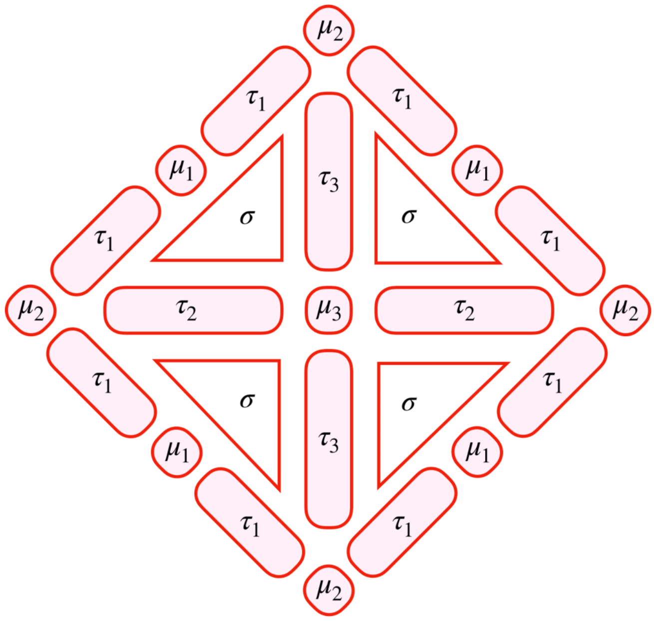


FIG. 8. #9 wallpaper group *cmm* and its cell decomposition.

have independent stacking between 1D and 0D block states, hence the ultimate classification with accurate group structure is

$$\mathcal{G}_{p2,1/2} = E^{\text{1D}}_{p2,1/2} \times_{\omega_2} E^{\text{0D}}_{p2,1/2} = \mathbb{Z}_4 \times \mathbb{Z}_8^3, \tag{99}$$

here the symbol $\times_{\omega_2}$ means that independent nontrivial 1D and 0D block states $E^{\text{1D}}_{p2,1/2}$ and $E^{\text{0D}}_{p2,1/2}$ have nontrivial extension, characterized by the following short exact sequence:

$$0 \to E^{\text{1D}}_{p2,1/2} \to G_{p2,1/2} \to E^{\text{0D}}_{p2,1/2} \to 0. \tag{100}$$

C. Rhombic lattice: *cmm*

For rhombic lattice, we demonstrate the crystalline TSC protected by *cmm* symmetry as an example. The corresponding point group of *cmm* is dihedral group D_2, and for 2D blocks σ and 1D blocks τ_1, there is no on-site symmetry; and for 1D blocks τ_2/τ_3 and 0D blocks μ_1, the on-site symmetry is $\mathbb{Z}_2$ via the reflection symmetry acting internally; for 0D blocks μ_2 and μ_3, the on-site symmetry group is $\mathbb{Z}_2 \rtimes \mathbb{Z}_2$ via the D_2 symmetry acting internally. The cell decomposition of *cmm* is illustrated in Fig. 8.

1. Spinless fermions

For spinless fermions, consider the 0D block-state decoration: For 0D blocks μ_1, the total symmetry group of each is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, and candidate states can be characterized by different 1D irreducible representations of the symmetry group,

$$\mathcal{H}^1\big[\mathbb{Z}_2^f \times \mathbb{Z}_2, U(1)\big] = \mathbb{Z}_2^2. \tag{101}$$

So at each 0D block labeled by μ_1, the block state can be labeled by $(\pm, \pm)$, and these two $\pm$'s represent the fermion and rotation eigenvalue, respectively. For 0D blocks μ_2 and μ_3, the classification data can be characterized by different irreducible representations of the full symmetry group $\mathbb{Z}_2^f \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$,

$$\mathcal{H}^1\big[\mathbb{Z}_2^f \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2), U(1)\big] = \mathbb{Z}_2^3. \tag{102}$$

So at each 0D block, the block state can be labeled by $(\pm, \pm, \pm)$, and these three $\pm$'s represent the fermion parity and eigenvalues of two independent reflection generators $\boldsymbol{M}_{\tau_2}$ and $\boldsymbol{M}_{\tau_3}$, respectively. According to this notation, the obstruction-free 0D block states form the group

$$\{\text{OFBS}\}^{\text{0D}}_{cmm,0} = \mathbb{Z}_2^8 \tag{103}$$

where the group elements can be labeled by

$$[(\pm, \pm), (\pm, \pm, \pm), (\pm, \pm, \pm)], \tag{104}$$

here three brackets represent the block states at μ_1, μ_2, and μ_3, respectively.

Subsequently we consider the 1D block-state decoration. For 1D block τ_1, the total symmetry group is just fermion parity $\mathbb{Z}_2^f$, so the only nontrivial 1D block state is Majorana chain; for 1D blocks τ_2 and τ_3, the total symmetry group is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, so there are two possible 1D block states: Majorana chain and 1D FSPT state (composed by double Majorana chains), so all 1D block states form a group

$$\{\text{BS}\}^{\text{1D}}_{cmm,0} = \mathbb{Z}_2^5. \tag{105}$$

Then we discuss the decorations of these two root phases separately.

a. Majorana chain decoration. First we consider the Majorana chain decoration on 1D blocks τ_1, which leaves two/four dangling Majorana modes at each 0D block μ_1/μ_2. Near μ_1, Majorana modes have following rotational symmetry properties:

$$\boldsymbol{R}_{\mu_1} : \gamma_1 \leftrightarrow \gamma_2 \tag{106}$$

with local fermion parity and its symmetry property

$$P_f = i\gamma_1\gamma_2, \quad \boldsymbol{R}_{\mu_1} : P_f \mapsto -P_f. \tag{107}$$

Hence these two Majorana modes break the fermion parity on 0D block μ_1. Thus Majorana chain decoration on 1D block τ_1 is obstructed because of the violation of the no-open-edge condition.

Then we consider the Majorana chain decoration on 1D blocks τ_2 that leaves two Majorana modes at each 0D block μ_2/μ_3. Near μ_2, Majorana modes have following reflection symmetry properties:

$$\boldsymbol{M}_{\tau_3} : \gamma_1 \leftrightarrow \gamma_2 \tag{108}$$

with local fermion parity and its symmetry property

$$P_f = i\gamma_1\gamma_2, \boldsymbol{M}_{\tau_3} : P_f \mapsto -P_f. \tag{109}$$

Hence these two Majorana modes break the fermion parity on 0D block μ_2. Thus Majorana chain decoration on τ_2 is obstructed because of the violation of the no-open-edge condition. Majorana chain decoration on 1D blocks τ_3 is similar, hence all types of Majorana chain decorations are obstructed.

b. 1D FSPT state decoration. First we consider the 1D FSPT state decoration on 1D blocks τ_2 that leaves four dangling Majorana modes $(\xi_j, \xi_j', j = 1, 2)$ at each 0D block μ_2/μ_3. Near μ_2/μ_3, the corresponding four Majorana modes have the symmetry properties ($j=1,2$),

$$\begin{aligned}\boldsymbol{M}_{\tau_2}&: \xi_j \mapsto \xi_j,\ \xi_j' \mapsto -\xi_j',\\ \boldsymbol{M}_{\tau_3}&: \xi_1 \leftrightarrow \xi_2,\ \xi_1' \leftrightarrow \xi_2'.\end{aligned} \tag{110}$$

Similar with the 1D block-state decorations for $p4m$ case, these four Majorana modes cannot be gapped out because they form a projective representation of D_2 group at each corresponding 0D block, and a nondegenerate ground state is forbidden. Accordingly, the 1D FSPT state decoration on τ_2 or τ_3 is obstructed because of the degenerate ground state.

There is one exception: If we decorate 1D FSPT phases on 1D blocks τ_2 and τ_3 simultaneously, it leaves eight dangling Majorana modes at each μ_2 and μ_3. Similar with the 0D block labeled by μ_2 in $p4m$ case (see Fig. 3), these eight dangling Majorana modes can be gapped symmetrically. As a consequence, the only nontrivial obstruction-free 1D block state is 1D FSPT state decorations on τ_2 and τ_3 simultaneously, and all obstruction-free 1D block states form a group

$$\{\mathrm{OFBS}\}_{cmm,0}^{1\mathrm{D}} = \mathbb{Z}_2 \tag{111}$$

where the group elements can be labeled by $m_2 = m_3$ (m_2/m_3 represents the number of decorated 1D FSPT states on τ_2/τ_3).

With all obstruction-free block states, we discuss all possible trivializations. First we consider the 2D bubble equivalences: we decorate a Majorana chain with anti-PBC on each 2D block and enlarge all Majorana bubbles near each 1D block labeled by τ_1, and it can be deformed to double Majorana chains that can be trivialized because there is no on-site symmetry; near each 1D block labeled by τ_2/τ_3, it can also be deformed to double Majorana chains, nevertheless, these double Majorana chains cannot be trivialized because there is an on-site $\mathbb{Z}_2$ symmetry on each τ_2/τ_3. Equivalently, 1D FSPT state decorations on 1D blocks τ_2 and τ_3 can be deformed to a trivial state via 2D Majorana bubble equivalence. Furthermore, similar with the $p4m$ case, there is no effect on 0D blocks labeled by μ_2 and μ_3 by taking 2D Majorana bubble equivalence; nevertheless, similar with the $p2$ case, 2D Majorana bubble construction changes the fermion parity of each 0D block labeled by μ_1.

Subsequently we consider the 1D bubble equivalences. For instance, we decorate a pair of complex fermions [cf. Eq. (22)]: Near each 0D block μ_1, there are 2 complex fermions forming the following atomic insulator:

$$|\psi\rangle_{cmm}^{\mu_1} = c_1^\dagger c_2^\dagger|0\rangle \tag{112}$$

with rotation property

$$\boldsymbol{R}_{\mu_1}|\psi\rangle_{cmm}^{\mu_1} = c_2^\dagger c_1^\dagger|0\rangle = -|\psi\rangle_{cmm}^{\mu_1}, \tag{113}$$

i.e., 1D bubble construction on τ_1 changes the rotation eigenvalue at each 0D block μ_1. Near each 0D block μ_2, there are four complex fermions forming another atomic insulator,

$$|\psi\rangle_{cmm}^{\mu_2} = c_1'^\dagger c_2'^\dagger c_3'^\dagger c_4'^\dagger|0\rangle \tag{114}$$

with two independent reflection symmetry properties (D_2 symmetry at 0D block μ_2 can also be generated by two independent reflections $\boldsymbol{M}_{\tau_2}$ and $\boldsymbol{M}_{\tau_3}$)

$$\begin{aligned}\boldsymbol{M}_{\tau_2}|\psi\rangle_{cmm}^{\mu_2} &= c_3'^\dagger c_4'^\dagger c_1'^\dagger c_2'^\dagger|0\rangle = |\psi\rangle_{cmm}^{\mu_2},\\ \boldsymbol{M}_{\tau_3}|\psi\rangle_{cmm}^{\mu_2} &= c_4'^\dagger c_3'^\dagger c_2'^\dagger c_1'^\dagger|0\rangle = |\psi\rangle_{cmm}^{\mu_2},\end{aligned} \tag{115}$$

i.e., 1D bubble construction on τ_1 does not change anything on μ_2. Similar 1D bubble constructions can be held on 1D blocks τ_2 and τ_3, and we summarize the effects of 1D bubble constructions as following:

(1) 1D bubble construction on τ_1: changes the eigenvalue of $\boldsymbol{R}_{\mu_1}$ at μ_1;

(2) 1D bubble construction on τ_2: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_3}$ at μ_2 and μ_3;

(3) 1D bubble construction on τ_3: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_2}$ at μ_2 and μ_3;

With all possible trivializations, we are ready to study the trivial states. Start from the original trivial 0D block state (nothing is decorated on arbitrary 0D blocks),

$$[(+,+),(+,+,+),(+,+,+)].$$

If we take 2D Majorana bubble construction l_0 times, and take 1D bubble equivalences on τ_j with l_j times ($j=1,2,3$), above trivial state will be deformed to a new 0D block state labeled by

$$\begin{aligned}&[((-1)^{l_0},(-1)^{l_1}),(+,(-1)^{l_3},(-1)^{l_2}),\\ &\times(+,(-1)^{l_3},(-1)^{l_2})].\end{aligned} \tag{116}$$

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only four independent quantities ($l_j = 0,1,2,3$) in Eq. (116), hence all these trivial states form the following group:

$$\begin{aligned}\{\mathrm{TBS}\}_{cmm,0} &= \{\mathrm{TBS}\}_{cmm,0}^{1\mathrm{D}} \times \{\mathrm{TBS}\}_{cmm,0}^{0\mathrm{D}}\\ &= \mathbb{Z}_2 \times \mathbb{Z}_2^3 = \mathbb{Z}_2^4,\end{aligned} \tag{117}$$

here $\{\mathrm{TBS}\}_{cmm,0}^{1\mathrm{D}}$ represents the group of trivial states, and $\{\mathrm{TBS}\}_{cmm,0}^{0\mathrm{D}}$ represents the group of trivial states with nonvacuum 0D blocks.

Therefore, all independent nontrivial block states with different dimensions form the following groups:

$$\begin{aligned}E_{cmm,0}^{1\mathrm{D}} &= \{\mathrm{OFBS}\}_{cmm,0}^{1\mathrm{D}}/\{\mathrm{TBS}\}_{cmm,0}^{1\mathrm{D}} = \mathbb{Z}_1,\\ E_{cmm,0}^{0\mathrm{D}} &= \{\mathrm{OFBS}\}_{cmm,0}^{0\mathrm{D}}/\{\mathrm{TBS}\}_{cmm,0}^{0\mathrm{D}} = \mathbb{Z}_2^5,\end{aligned} \tag{118}$$

and together form the classification

$$\mathcal{G}_{cmm,0} = E_{cmm,0}^{0\mathrm{D}} = \mathbb{Z}_2^5, \tag{119}$$

here all $\mathbb{Z}_2$'s are from the nontrivial 0D block states. It is obvious that there is no nontrivial group extension because of the absence of nontrivial 1D block state, and the group structure of $\mathcal{G}_{cmm,0}$ has already been correct.

2. Spin-1/2 fermions

Now we turn to discuss systems with spin-1/2 fermions. First we consider the 0D block-state decorations: For 0D blocks labeled by μ_1, the twofold rotational symmetry acts on each of them internally, hence the total symmetry is $\mathbb{Z}_4^f$: nontrivial $\mathbb{Z}_2^f$ extension of on-site $\mathbb{Z}_2$ symmetry. And all different 0D block states, which can be characterized by different 1D irreducible representations of the corresponding symmetry group are

$$\mathcal{H}^1[\mathbb{Z}_4^f, U(1)] = \mathbb{Z}_4, \tag{120}$$

and there is no trivialization on them. Furthermore, for 0D blocks labeled by μ_2 and μ_3, the dihedral group symmetry D_2 acts on each of them internally, and similar with the $p4m$ case, the classification of corresponding 0D block states can be characterized by different 1D irreducible representations of the full symmetry group

$$\mathcal{H}^1\big[\mathbb{Z}_2^f\times_{\omega_2}(\mathbb{Z}_2\rtimes\mathbb{Z}_2),U(1)\big]=\mathbb{Z}_2^2. \tag{121}$$

Here different $\mathbb{Z}_2$'s represent the rotation and reflection eigenvalues at each D_2 center. As a consequence, all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{cmm,1/2}^{0\mathrm{D}}=\mathbb{Z}_4\times\mathbb{Z}_2^4, \tag{122}$$

and there is no trivialization on them. For spinless fermions, the rotation eigenvalue -1 at 0D block μ_1 is changed by atomic insulator $|\psi\rangle_{cmm}^{\mu_1}$ [cf. Eq. (112)]. Nevertheless, for spin-1/2 fermions, there is an additional minus sign under rotation to fulfill the condition $\boldsymbol{R}_{\mu_1}^2=-1$,

$$\boldsymbol{R}_{\mu_1}|\psi\rangle_{cmm}^{\mu_1}=c_2^\dagger(-c_1^\dagger)=|\psi\rangle_{cmm}^{\mu_1}, \tag{123}$$

i.e., $|\psi\rangle_{cmm}^{\mu_1}$ does not change the rotation eigenvalue -1 at 0D block μ_1. Similar for all other 0D blocks.

As a consequence, the classification attributed to 0D block-state decorations is

$$E_{cmm,1/2}^{0\mathrm{D}}=\mathbb{Z}_4\times\mathbb{Z}_2^4. \tag{124}$$

Subsequently we investigate the 1D block-state decoration. On τ_1, the unique possible 1D block state is Majorana chain because of the absence of the on-site symmetry; on τ_2 and τ_3, the total symmetry group is $\mathbb{Z}_4^f$, hence there is no candidate block state due to the trivial classification of the corresponding 1D FSPT phases. The Majorana chain decoration on τ_1 leaves two dangling Majorana modes at each μ_1, and four dangling Majorana modes at each μ_2. At μ_1, the two dangling Majorana modes, which can be gapped out by an entanglement pair without breaking any symmetry are

$$\boldsymbol{R}_{\mu_1}: i\gamma_1\gamma_2\mapsto -i\gamma_2\gamma_1=i\gamma_1\gamma_2, \tag{125}$$

at μ_2, the four Majorana modes have the following reflection symmetry properties (D_2 symmetry can also be generated by two independent reflections $\boldsymbol{M}_{\tau_2}$ and $\boldsymbol{M}_{\tau_3}$):

$$\begin{aligned}\boldsymbol{M}_{\tau_2}&:(\eta_1,\eta_2,\eta_3,\eta_4)\mapsto(\eta_2,-\eta_1,\eta_4,-\eta_3),\\ \boldsymbol{M}_{\tau_3}&:(\eta_1,\eta_2,\eta_3,\eta_4)\mapsto(\eta_4,\eta_3,-\eta_2,-\eta_1).\end{aligned} \tag{126}$$

Consider the following Hamiltonian containing two entanglement pairs of these four Majorana modes:

$$H_{\mu_2}=-i\eta_1\eta_3-i\eta_2\eta_4. \tag{127}$$

It is easy to verify that H_{μ_2} is invariant under the symmetry actions (126). As a consequence, all obstruction-free 1D block states form the following group:

$$\{\mathrm{OFBS}\}_{cmm,1/2}^{1\mathrm{D}}=\mathbb{Z}_2, \tag{128}$$

and it is easy to see that there is no trivialization (i.e., $\{\mathrm{TBS}\}_{cmm,1/2}^{0\mathrm{D}}=\mathbb{Z}_1$). So the classification attributed to 1D block-state decorations is

$$E_{cmm,1/2}^{1\mathrm{D}}=\mathbb{Z}_2. \tag{129}$$

With the classification data as Eqs. (124) and (129), we consider the group structure of the corresponding classification. Equivalently, we investigate if 1D block state extends 0D block state. The only possible case of stacking should happen on 1D blocks labeled by τ_1 because other 1D blocks have no nontrivial block state, similar with $p4m$ and $p2$ cases. We decorate two copies of Majorana chains on τ_1 that leave two dangling Majorana modes at each 0D block labeled by μ_1 and four dangling Majorana modes at each 0D block labeled by μ_2. At μ_1, these Majorana chains can be smoothly deformed to the state described by Eqs. (93) and (96), with the symmetry properties as Eq. (97). So similar with $p2$ case, near each 0D block labeled by μ_1, 1D block states extend 0D block state, and 0D block states at μ_1 have the group structure $\mathbb{Z}_8$ as a consequence. At μ_2, these Majorana chains can be smoothly deformed to two copies of the state described by Eqs. (93) and (96), and have eigenvalue -1 under twofold rotational symmetry. The classification data of 0D block states at μ_2 is determined by Eq. (121), hence if a 0D block state with eigenvalue -1 under twofold rotation is attached to each 1D block state near each 0D block labeled by μ_2, the rotation eigenvalue s of the obtained 0D block state becomes

$$s=(-1)\times(-1)=1. \tag{130}$$

Therefore, near 0D block μ_2 there is an appropriate 1D block state, which itself forms a $\mathbb{Z}_2$ structure under stacking, and there is no stacking between 1D and 0D block states at μ_2 as a consequence. Finally, the ultimate classification with accurate group structure is

$$\mathcal{G}_{cmm,1/2}=E_{cmm,1/2}^{0\mathrm{D}}\times_{\omega_2}E_{cmm,1/2}^{1\mathrm{D}}=\mathbb{Z}_8\times\mathbb{Z}_2^4, \tag{131}$$

here the symbol "$\times_{\omega_2}$" means that 1D and 0D block states $E_{cmm,1/2}^{1\mathrm{D}}$ and $E_{cmm,1/2}^{0\mathrm{D}}$ have nontrivial extension, and described by the following short exact sequence:

$$0\to E_{cmm,1/2}^{1\mathrm{D}}\to G_{cmm,1/2}\to E_{cmm,1/2}^{0\mathrm{D}}\to 0. \tag{132}$$

D. Rectangle lattice: *pgg*

For rectangle lattice, we demonstrate the crystalline TSC protected by pgg symmetry as an example. pgg is a nonsymmorphic wallpaper group and the corresponding point group is dihedral group. The corresponding point group for this case is twofold dihedral group D_2. For 2D blocks σ and 1D blocks τ_1 and τ_2, there is no on-site symmetry, and for 0D blocks μ_1 and μ_2, the on-site symmetry is $\mathbb{Z}_2$ because twofold rotational symmetry C_2 acts on the 0D blocks internally, as seen Fig. 9.

1. Spinless fermions

First we investigate the 0D block-state decoration. For an arbitrary 0D block, the total symmetry group is an on-site $\mathbb{Z}_2$ symmetry together with the fermion parity: $\mathbb{Z}_2^f\times\mathbb{Z}_2$, and the classification data can be characterized by different 1D irreducible representations of the symmetry group

$$\mathcal{H}^1\big[\mathbb{Z}_2^f\times\mathbb{Z}_2,U(1)\big]=\mathbb{Z}_2^2, \tag{133}$$

these two $\mathbb{Z}_2$'s represent the fermion parity and eigenvalues of twofold rotational symmetry on each 0D block, respectively. So at each 0D block, the block state can be labeled by $(\pm,\pm)$. According to this notation, the obstruction-free 0D

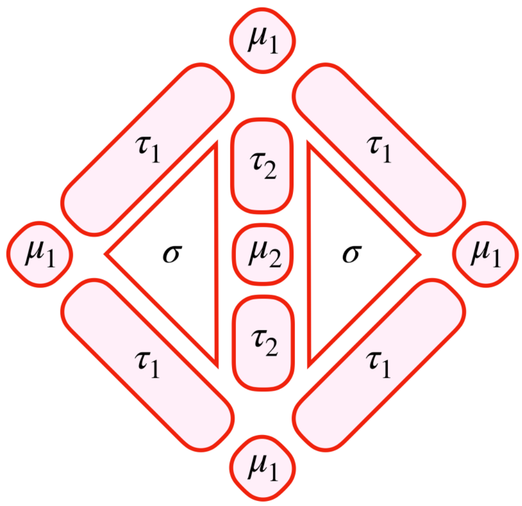


FIG. 9. #8 wallpaper group pgg and its cell decomposition.

block states form the following group:

$$\{\text{OFBS}\}^{0\text{D}}_{pgg,0} = \mathbb{Z}_2^4, \tag{134}$$

and the group elements can be labeled by (two brackets represent the block states at μ_1 and μ_2)

$$[(\pm,\pm),(\pm,\pm)].$$

Subsequently we investigate the 1D block-state decoration. Due to the absence of the on-site symmetry, the unique possible 1D block state is Majorana chain. So all 1D block states form a group

$$\{\text{BS}\}^{1\text{D}}_{pgg,0} = \mathbb{Z}_2^2. \tag{135}$$

Then we discuss the possible obstructions: we discuss the 1D block-state decorations on τ_1 and τ_2 separately.

a. Majorana chain decoration on τ_1. Majorana chain decoration on τ_1 leaves four dangling Majorana modes at each corresponding 0D blocks μ_1 with the following rotational symmetry properties:

$$\boldsymbol{R}_{\mu_1}: \gamma_j \mapsto \gamma_{j+2}. \tag{136}$$

Here all subscripts are taken with modulo 4 (i.e., γ_5 represents the Majorana mode labeled by γ_1). Consider the following Hamiltonian near each 0D block μ_1:

$$H = i\gamma_1\gamma_2 + i\gamma_3\gamma_4, \tag{137}$$

it is obvious that H is symmetric under (136), and it can gap out the four Majorana modes at each μ_1.

b. Majorana chain decoration on τ_2. Majorana chain decoration on τ_2 leaves two dangling Majorana modes at each corresponding 0D block, which can be gapped out by an entanglement pair. Nevertheless this entanglement pair breaks the fermion parity, and the no-open-edge condition is violated.

As a consequence, all obstruction-free 1D block states form the following group:

$$\{\text{OFBS}\}^{1\text{D}}_{pgg,0} = \mathbb{Z}_2, \tag{138}$$

and we have demonstrated that the 2D Majorana bubble cannot change the parity of Majorana chains of each 1D block in Sec. II, hence there is no trivialization (i.e., $\{\text{TBS}\}^{1\text{D}}_{pgg,0} = \mathbb{Z}_1$). Therefore, all independent nontrivial 1D block states are labeled by different group elements of the following group:

$$E^{1\text{D}}_{pgg,0} = \{\text{OFBS}\}^{1\text{D}}_{pgg,0}/\{\text{TBS}\}^{1\text{D}}_{pgg,0} = \mathbb{Z}_2. \tag{139}$$

With all obstruction-free block states, we consider possible trivializations via bubble construction. First of all, we consider the 2D bubble equivalence: We decorate a Majorana chain with anti-PBC on each 2D block that can be trivialized if it shrinks to a point. Similar with the $p2$ case, by some proper local unitary transformations, this assembly of bubbles can be deformed to an assembly of Majorana chains with odd fermion parity surrounding each of 0D block, and the fermion parities of all 0D blocks are changed simultaneously. Equivalently, the fermion parities of 0D blocks labeled by μ_1 and μ_2 are not independent.

Then we study the role of rotation symmetry. Consider the 1D bubble equivalence on τ_2: we decorate a pair of complex fermions [cf. Eq. (22)]: Near μ_2, there are two complex fermions, which form an atomic insulator,

$$|\psi\rangle^{\mu_2}_{pgg} = c_1^\dagger c_2^\dagger|0\rangle \tag{140}$$

with rotation property as ($\boldsymbol{R}_{\mu_2}$ represents the rotation operation centered at the 0D block labeled by μ_2)

$$\boldsymbol{R}_{\mu_2}|\psi\rangle^{\mu_2}_{pgg} = c_2^\dagger c_1^\dagger|0\rangle = -|\psi\rangle^{\mu_2}_{pgg}, \tag{141}$$

i.e., $|\psi\rangle^{\mu_2}_{pgg}$ can trivialize the rotation eigenvalue -1 at each 0D block labeled by μ_2, similar for the 0D block labeled by μ_1. Hence the rotation eigenvalues at μ_1 and μ_2 are not independent; and we further consider the 1D bubble equivalence on τ_1: Near each 0D block labeled by μ_1, there are four complex fermions, which form another atomic insulator

$$|\psi\rangle^{\mu_1}_{pgg} = c_1'^\dagger c_2'^\dagger c_3'^\dagger c_4'^\dagger|0\rangle \tag{142}$$

with rotation property as ($\boldsymbol{R}_{\mu_1}$ represents the rotation operation centered at the 0D block labeled by μ_1)

$$\boldsymbol{R}_{\mu_1}|\psi\rangle^{\mu_1}_{pgg} = c_3'^\dagger c_4'^\dagger c_1'^\dagger c_2'^\dagger = |\psi\rangle^{\mu_1}_{pgg}. \tag{143}$$

So there is no trivialization from this bubble construction.

With all possible bubble constructions, we are ready to study the trivial states. Start from the original trivial state,

$$[(+,+),(+,+)],$$

if we take 2D bubble construction l_0 times and 1D bubble construction on τ_2 with l_2 times, above trivial state will be deformed to a new 0D block state labeled by

$$[((-1)^{l_0},(-1)^{l_2}),((-1)^{l_0},(-1)^{l_2})]. \tag{144}$$

According to the definition of bubble equivalence, all these states should be trivial. It is easy to see that there are only two independent quantities in the state (144), hence all trivial states form the group

$$\{\text{TBS}\}^{0\text{D}}_{pgg,0} = \mathbb{Z}_2^2. \tag{145}$$

Therefore, all independent nontrivial 0D block states are labeled by different group elements of the following quotient group:

$$E^{0\text{D}}_{pgg,0} = \{\text{OFBS}\}^{0\text{D}}_{pgg,0}/\{\text{TBS}\}^{0\text{D}}_{pgg,0} = \mathbb{Z}_2^2. \tag{146}$$

It is straightforward to see that there is no stacking between 1D and 0D block states, and the ultimate classification with accurate group structure is

$$\mathcal{G}_{pgg,0}=E^{\mathrm{0D}}_{pgg,0}\times E^{\mathrm{1D}}_{pgg,0}=\mathbb{Z}_2^3. \tag{147}$$

2. *Spin-1/2 fermions*

First we investigate the 0D block-state decorations. All 0D blocks are twofold rotation centers, hence the total symmetry group of each 0D block is $\mathbb{Z}_4^f$, and different 0D block states, which can be characterized by different 1D irreducible representations of the corresponding symmetry group are

$$\mathcal{H}^1\big[\mathbb{Z}_4^f,U(1)\big]=\mathbb{Z}_4. \tag{148}$$

All root phases at each 0D block are characterized by group elements of $\{1,i,-1,-i\}$. So at each 0D block, the block state can be labeled by $\nu\in\{1,i,-1,-i\}$. According to this notation, the obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}^{\mathrm{0D}}_{pgg,1/2}=\mathbb{Z}_4^2, \tag{149}$$

and different group elements can be labeled by

$$[\nu_1,\nu_2]$$

where ν_1 and ν_2 label the 0D block state at μ_1 and μ_2. It is easy to see that there is no trivialization on 0D block states (i.e., $\{\mathrm{TBS}\}^{\mathrm{0D}}_{pgg,1/2}=\mathbb{Z}_1$), so the classification attributed to 0D block-state decorations is

$$E^{\mathrm{0D}}_{pgg,1/2}=\{\mathrm{OFBS}\}^{\mathrm{0D}}_{pgg,1/2}/\{\mathrm{TBS}\}^{\mathrm{0D}}_{pgg,1/2}=\mathbb{Z}_4^2. \tag{150}$$

Subsequently we investigate the 1D block-state decoration. The unique possible 1D block state is Majorana chain because of the absence of on-site symmetry.

a. Majorana chain deocration on τ_1. Majorana chain decoration on τ_1 leaves four dangling Majorana modes at each 0D blocks labeled by μ_1 with identical symmetry properties with the spinless fermions [cf. Eq. (136)], hence these four Majorana modes can be gapped out by some entanglement pairs in a symmetric way, and the no-open-edge condition is satisfied.

b. Majorana chain decoration on τ_2. Majorana chain decoration on τ_2 leaves two dangling Majorana modes at each 0D block μ_2, which can be gapped out by an entanglement pair in a symmetric way. Therefore the no-open-edge condition is satisfied. Consequently, all obstruction-free 1D block states form the following group:

$$\{\mathrm{OFBS}\}^{\mathrm{1D}}_{pgg,1/2}=\mathbb{Z}_2^2. \tag{151}$$

Then we demonstrate that there is no trivialization from bubble constructions: For spinless fermions, eigenvalue -1 of rotation $\boldsymbol{R}_{\mu_1}$ is changed by atomic insulator $|\psi\rangle^{\mu_1}_{pgg}$ [cf. Eq. (140)]. Nevertheless, for spin-1/2 fermions, there is an additional minus sign under rotation to fulfill the condition $\boldsymbol{R}_{\mu_1}^2=-1$,

$$\boldsymbol{R}_{\mu_1}|\psi\rangle^{\mu_1}_{pgg}=c_2^\dagger(-c_1^\dagger)|0\rangle=|\psi\rangle^{\mu_1}_{pgg}, \tag{152}$$

i.e., $|\psi\rangle^{\mu_1}_{pgg}$ does not change the eigenvalue of $\boldsymbol{R}_{\mu_1}$.

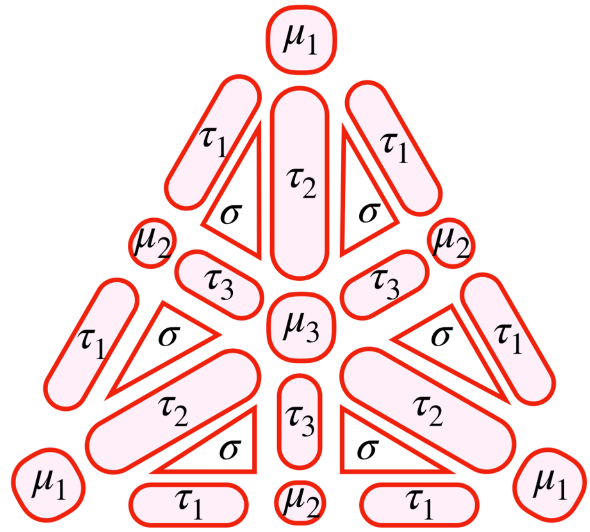


FIG. 10. #17 wallpaper group $p6m$ and its cell decomposition.

As the consequence, the classification of 2D FSPT phases with *pgg* symmetry attributed to 1D block-state decoration is

$$E^{\mathrm{1D}}_{pgg,1/2}=\{\mathrm{OFBS}\}^{\mathrm{1D}}_{pgg,1/2}/\{\mathrm{TBS}\}^{\mathrm{1D}}_{pgg,1/2}=\mathbb{Z}_2^2. \tag{153}$$

Then we study the possible stacking between 1D and 0D block states. If we decorate two Majorana chains on each 1D block labeled by τ_1, similar with *cmm* case, there is no stacking between 1D and 0D block states; if we decorate two Majorana chains on each 1D block labeled by τ_2, similar with *p*2 case, it can be smoothly deformed to an assembly of 0D root phases at 0D blocks μ_2. Therefore, the ultimate classification with accurate group structure is

$$\mathcal{G}_{pgg,1/2}=E^{\mathrm{1D}}_{pgg,1/2}\times_{\omega_2}E^{\mathrm{0D}}_{pgg,1/2}=\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_8, \tag{154}$$

here the symbol "$\times_{\omega_2}$" means that 1D and 0D block states $E^{\mathrm{1D}}_{cmm,1/2}$ and $E^{\mathrm{0D}}_{cmm,1/2}$ have nontrivial extension, described by the following short exact sequence:

$$0\to E^{\mathrm{1D}}_{pgg,1/2}\to G_{pgg,1/2}\to E^{\mathrm{0D}}_{pgg,1/2}\to 0. \tag{155}$$

E. Hexagonal lattice: *p6m*

For hexagonal lattice, we demonstrate the crystalline TSC protected by *p6m* symmetry as an example. The corresponding point group of *p6m* is dihedral group D_6, and for 2D blocks labeled by σ, there is no on-site symmetry; for arbitrary 1D block, the on-site symmetry is $\mathbb{Z}_2$, which is attributed to the reflection symmetry acting internally; for 0D blocks μ_1, the on-site symmetry group is $\mathbb{Z}_6\rtimes\mathbb{Z}_2$, which is attributed to the D_6 group acting internally; for 0D blocks μ_2, the on-site symmetry is $\mathbb{Z}_2\rtimes\mathbb{Z}_2$, which is attributed to the $D_2\subset D_6$ acting internally; for 0D blocks μ_3, the on-site symmetry is $\mathbb{Z}_3\rtimes\mathbb{Z}_2$, which is attributed to the $D_3\subset D_6$ acting internally. The cell decomposition is shown in Fig. 10.

1. *Spinless fermions*

Consider the 0D block-state decorations, for μ_j, $j=1,2,3$, the classification data can be characterized by different 1D irreducible representations of the full symmetry groups, respectively,

$$\mathcal{H}^1\left[\mathbb{Z}_2^f \times (\mathbb{Z}_6 \rtimes \mathbb{Z}_2), U(1)\right] = \mathbb{Z}_2^3,$$
$$\mathcal{H}^1\left[\mathbb{Z}_2^f \times (\mathbb{Z}_2 \rtimes \mathbb{Z}_2), U(1)\right] = \mathbb{Z}_2^3,$$
$$\mathcal{H}^1\left[\mathbb{Z}_2^f \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_2), U(1)\right] = \mathbb{Z}_2^2. \tag{156}$$

Similar to the $p4m$ case, the 0D block states at μ_1 and μ_2 can be labeled by $(\pm,\pm,\pm)$ where the three $\pm$'s represent the fermion parity and eigenvalues of two independent reflection generators $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_2}$, respectively. However, the 0D block states at μ_3 should be labeled by $(\pm,\pm)$ where the two $\pm$'s represent the fermion parity and reflection symmetry eigenvalues of $\boldsymbol{M}_{\tau_3}$.(We note that there is only one independent reflection eigenvalue for D_n symmetry group with odd n.) According to this notation, the obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{p6m}^{0\mathrm{D}} = \mathbb{Z}_2^8, \tag{157}$$

and the group elements can be labeled by (three brackets represent the block states at μ_1, μ_2, and μ_3)

$$[(\pm,\pm,\pm),(\pm,\pm,\pm),(\pm,\pm)].$$

Subsequently we investigate the 1D block-state decoration. For all 1D blocks, the total symmetry group is $\mathbb{Z}_2^f \times \mathbb{Z}_2$, and the candidate 1D block state is Majorana chain and 1D FSPT state. So all 1D block states form a group

$$\{\mathrm{BS}\}_{p6m,0}^{1\mathrm{D}} = \mathbb{Z}_2^6. \tag{158}$$

Then we discuss the decorations of these two root phases separately.

a. Majorana chain decoration. Consider Majorana chain decorations on 1D blocks labeled by τ_1, which leave six dangling Majorana modes at each μ_1 and two dangling Majorana modes at each μ_2. Near each 0D block μ_1, six dangling Majorana modes have the following rotational symmetry properties (all subscripts are taken with modulo 6):

$$\boldsymbol{R}_{\mu_1} : \gamma_j \mapsto \gamma_{j+1}, \ \ j = 1, ..., 6. \tag{159}$$

Then we consider the local fermion parity and its rotational symmetry property,

$$P_f = i\prod_{j=1}^{6}\gamma_j, \ \ \boldsymbol{R}_{\mu_1} : P_f \mapsto -P_f. \tag{160}$$

Thus these six dangling Majorana modes break fermion parity symmetry and a nondegenerate ground state is forbidden. The corresponding Majorana chain decoration on 1D blocks τ_1 is obstructed because of the violation of the no-open-edge condition. On τ_2, the Majorana chain decoration leaves six dangling Majorana modes at each μ_1 and three dangling Majorana mdoes at each μ_3. It is well-known that odd number of Majorana modes cannot be gapped out, hence Majorana chain decoration on τ_2 is obstructed. On τ_3, Majorana chain decoration leaves two dangling Majorana modes at each μ_2 and three dangling Majorana modes at each μ_3. Similar with the τ_2 case, Majorana chain decoration is obstructed. Note that if we consider all 1D blocks together and decorate a Majorana chain on each, it leaves 12 dangling Majorana modes at each μ_1, four dangling Majorana modes at each μ_2 and 6 dangling Majorana modes at each μ_3. Consider Majorana modes at each μ_2, with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 4):

$$\boldsymbol{R}_{\mu_3} : \gamma_j' \mapsto \gamma_{j+2}', \quad \boldsymbol{M}_{\tau_3} : \gamma_j' \mapsto \gamma_{6-j}'. \tag{161}$$

Then we consider the local fermion parity and its rotation and reflection symmetry properties,

$$P_f' = -\prod_{j=1}^{4}\gamma_j', \quad \begin{cases}\boldsymbol{R}_{\mu_3} : P_f' \mapsto P_f' \\ \boldsymbol{M}_{\tau_3} : P_f' \mapsto -P_f'\end{cases}. \tag{162}$$

Thus these Majorana modes cannot be gapped in a symmetric way, and Majorana chain decoration on all 1D blocks is obstructed. As a consequence, Majorana chain decoration does not contribute a nontrivial crystalline TSC.

b. 1D FSPT state decoration. 1D FSPT state decoration on τ_1 leaves 12 dangling Majorana modes at each μ_1 and four dangling Majorana modes at each μ_2. Similar with the $p4m$ and cmm cases, four Majorana modes at each μ_2 form a projective representation of D_2 symmetry group, and a nondegenerate ground-state is forbidden. Thus the 1D FSPT state decoration on τ_1 is *obstructed*.

1D FSPT state decoration on τ_2 leaves 12 dangling Majorana modes at each μ_1 and six dangling Majorana modes at each μ_3. Consider the Majorana modes at each μ_3, with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 3):

$$\begin{matrix}\boldsymbol{R}_{\mu_3} : & \eta_j \mapsto \eta_{j+1}, & \eta_j' \mapsto \eta_{j+1}' \\ \boldsymbol{M}_{\tau_3} : & \eta_j \mapsto -\eta_{5-j}, & \eta_j' \mapsto \eta_{5-j}'\end{matrix}, \quad j = 1,2,3. \tag{163}$$

Then we consider the local fermion parity with its rotation and reflection symmetry properties,

$$P_f^{\tau_2} = i\prod_{j=1}^{3}\eta_j\eta_j', \ \begin{cases}\boldsymbol{R}_{\mu_3} : P_f^{\tau_2} \mapsto P_f^{\tau_2} \\ \boldsymbol{M}_{\tau_3} : P_f^{\tau_2} \mapsto -P_f^{\tau_2}\end{cases}. \tag{164}$$

Hence these six Majorana modes break fermion parity symmetry and cannot be gapped out in a symmetric way. The corresponding 1D FSPT state decoration is *obstructed* because of the violation of the no-open-edge condition.

1D FSPT state decoration on τ_3 leaves four dangling Majorana modes at each μ_2 and six dangling Majorana modes at each μ_3. Similar with the 1D FSPT state decoration on τ_2 case, six Majorana modes at each μ_3 cannot be gapped out in a symmetric way: consider the Majorana modes as the edge modes of decorated Majorana chains on τ_3 at each μ_3, with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 3):

$$\boldsymbol{R}_{\mu_3} : \zeta_j \mapsto \zeta_{j+1}, \ \zeta_j' \mapsto \zeta_{j+1}', \tag{165}$$

$$\boldsymbol{M}_{\tau_3} : \zeta_j \mapsto -\zeta_{5-j}, \ \zeta_j' \mapsto \zeta_{5-j}', \tag{166}$$

with the local fermion parity and its rotation and reflection symmetry properties,

$$P_f^{\tau_3} = i\prod_{j=1}^{3}\zeta_j\zeta_j', \quad \begin{cases}\boldsymbol{R}_{\mu_3} : P_f^{\tau_3} \mapsto P_f^{\tau_3} \\ \boldsymbol{M}_{\tau_3} : P_f^{\tau_3} \mapsto -P_f^{\tau_3}\end{cases}. \tag{167}$$

Thus 1D FSPT state decoration on τ_3 is *obstructed*, and it does not contribute nontrivial crystalline TSC because of the violation of the no-open-edge condition.

Note that if we consider 1D blocks labeled by τ_2 and τ_3 together and decorate a 1D FSPT state on each of them, this decoration leaves 12 dangling Majorana modes at each μ_1 and μ_3, and four dangling Majorana modes at each μ_2. For the Majorana modes as the edge modes of the decorated 1D FSPT states at each μ_2, as aforementioned, they can be gapped out in a symmetric way. For Majorana modes as the edge modes of the decorated 1D FSPT states at each μ_1/μ_3, the local fermion parity is the product of $P_f^{\tau_2}$ and $P_f^{\tau_3}$, with the following symmetry properties:

$$P_f'' = P_f^{\tau_2} P_f^{\tau_3}, \quad \begin{cases} \boldsymbol{R}_{\mu_3} : P_f'' \mapsto P_f'' \\ \boldsymbol{M}_{\tau_3} : P_f'' \mapsto P_f'' \end{cases}. \tag{168}$$

Hence any symmetry operations commute with the fermion parity. Furthermore, there is no nontrivial projective representation of the D_3 group acting internally (identical with the internal symmetry group $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$), it can be obtained by calculating the following 2-cohomology of the symmetry group:

$$\mathcal{H}^2[\mathbb{Z}_3 \rtimes \mathbb{Z}_2, U(1)] = \mathbb{Z}_1. \tag{169}$$

Therefore, these 12 dangling Majorana modes form a linear representation of the symmetry group, and can be gapped out by some proper interactions in symmetry way. Nevertheless, four Majorana modes at each 0D block labeled by μ_2 form a projective representation of the D_2 symmetry group that forbids the nondegenerate ground state, so the 1D FSPT state decoration on τ_2 and τ_3 is still *obstructed* because of the violation of no-open-edge condition at each 0D block μ_2.

There is one exception: If we decorate a 1D FSPT phase on each 1D block (including τ_j, $j = 1, 2, 3$), the dangling Majorana modes at each 0D block can be gapped out in a symmetric way. In the aforementioned discussions we have elucidated that at each μ_3, there are 12 dangling Majorana modes via 1D FSPT state decorations that can be gapped in a symmetric way; and at each μ_2, there are eight dangling Majorana modes and similar with the $p4m$ and cmm case, they can be gapped out in a symmetric way because they form a linear representation of the corresponding symmetry group. Near each 0D block labeled by μ_1, this decoration leaves 24 dangling Majorana modes as the edge modes of decorated 1D FSPT phases. Consider half of them from 1D FSPT state decorations on τ_1 with the following rotation and reflection symmetry properties (all subscripts are taken with modulo 6):

$$\begin{aligned} &\boldsymbol{R}_{\mu_1} : \gamma_j \mapsto \gamma_{j+1},\ \gamma_j' \mapsto \gamma_{j+1}' \\ &\boldsymbol{M}_{\tau_1} : \gamma_j \mapsto \gamma_{8-j},\ \gamma_j' \mapsto \gamma_{8-j}' \end{aligned} \tag{170}$$

Then we consider the local fermion parity and its rotation and reflection symmetry properties:

$$P_f^{\tau_1} = -\prod_{j=1}^{6} \gamma_j \gamma_j', \quad \boldsymbol{R}_{\mu_1}, \boldsymbol{M}_{\tau_1} : P_f^{\tau_1} \mapsto P_f^{\tau_1}. \tag{171}$$

Hence arbitrary symmetry actions commute with the fermion parity of these 12 Majorana modes, and they form either a linear representation or a projective representation of the D_6 symmetry. Similar arguments can be held for other 12 Majorana modes. We should note that there is only one nontrivial projective representation of the D_6 symmetry group acting internally (i.e., $\mathbb{Z}_6 \rtimes \mathbb{Z}_2$ on-site symmetry) that can easily to be verified by the following 2-cohomology:

$$\mathcal{H}^2[\mathbb{Z}_6 \rtimes \mathbb{Z}_2, U(1)] = \mathbb{Z}_2. \tag{172}$$

So these 24 Majorana modes together can always form a linear representation of the D_6 symmetry at each 0D block labeled by μ_1, and they can be gapped out in a symmetric way. Thus the 1D FSPT state decorations on all 1D blocks simultaneously is *obstruction-free*, and all obstruction-free 1D block states form the following group:

$$\{\text{OFBS}\}_{p6m,0}^{\text{1D}} = \mathbb{Z}_2, \tag{173}$$

and the group elements can be labeled by $m_1 = m_2 = m_3$. Here $m_j = 0, 1$ ($j = 1, 2, 3$) represents the number of decorated 1D FSPT states on τ_j, respectively. According to aforementioned discussions, a necessary condition of an obstruction-free block state is $m_1 = m_2 = m_3$.

With all obstruction-free block states, subsequently we discuss all possible trivializations. First we consider the 2D bubble equivalence: Similar with the $p4m$ case, Majorana bubbles can be deformed to double Majorana chains at each nearby 1D block, and this is exactly the definition of the nontrivial 1D FSPT phase protected by on-site $\mathbb{Z}_2$ symmetry (by reflection symmetry acting internally). As a consequence, 1D FSPT state decorations on all 1D blocks can be deformed to a trivial state via 2D Majorana bubble equivalences. Furthermore, repeatedly similar with the $p4m$ case, Majorana bubble constructions have no effect on 0D blocks.

Subsequently we consider the 1D bubble equivalences. For example, on each 1D block labeled by τ_2, we decorate a pair of complex fermions [cf. Eq. (22) and Fig. 11(a)]: Near each 0D block labeled by μ_1, there are six complex fermions, which form an atomic insulator with even fermion parity,

$$|\psi\rangle_{p6m}^{\mu_1} = \prod_{j=1}^{6} c_j^\dagger |0\rangle, \tag{174}$$

hence $|\psi\rangle_{p6m}^{\mu_1}$ cannot change the fermion parity of the 0D block labeled by μ_1. Near each 0D block labeled by μ_3, there are three complex fermions, which form another atomic insulator with odd fermion parity

$$|\psi\rangle_{p6m}^{\mu_3} = c_1'^\dagger c_2'^\dagger c_3'^\dagger |0\rangle \tag{175}$$

and it can change the fermion parity at each 0D block labeled by μ_3. Then we consider the symmetry properties of these atomic insulators: the eigenvalues of $|\psi\rangle_{p6m}^{\mu_1}$ at μ_1 under two independent reflection operations are

$$\begin{aligned} \boldsymbol{M}_{\tau_1} |\psi\rangle_{p6m}^{\mu_1} &= c_6^\dagger c_5^\dagger c_4^\dagger c_3^\dagger c_2^\dagger c_1^\dagger = -|\psi\rangle_{p6m}^{\mu_1}, \\ \boldsymbol{M}_{\tau_2} |\psi\rangle_{p6m}^{\mu_1} &= c_1^\dagger c_6^\dagger c_5^\dagger c_4^\dagger c_3^\dagger c_2^\dagger = |\psi\rangle_{p6m}^{\mu_1}, \end{aligned} \tag{176}$$

i.e., 1D bubble construction on τ_2 can change the eigenvalue of $\boldsymbol{M}_{\tau_1}$ and leave the eigenvalue of $\boldsymbol{M}_{\tau_2}$ invariant. The eigenvalue of $|\psi\rangle_{p6m}^{\mu_3}$ at μ_3 under reflection $\boldsymbol{M}_{\tau_2}$ is

$$\boldsymbol{M}_{\tau_2} |\psi\rangle_{p6m}^{\mu_3} = c_1'^\dagger c_3'^\dagger c_2'^\dagger |0\rangle = -|\psi\rangle_{p6m}^{\mu_3}, \tag{177}$$

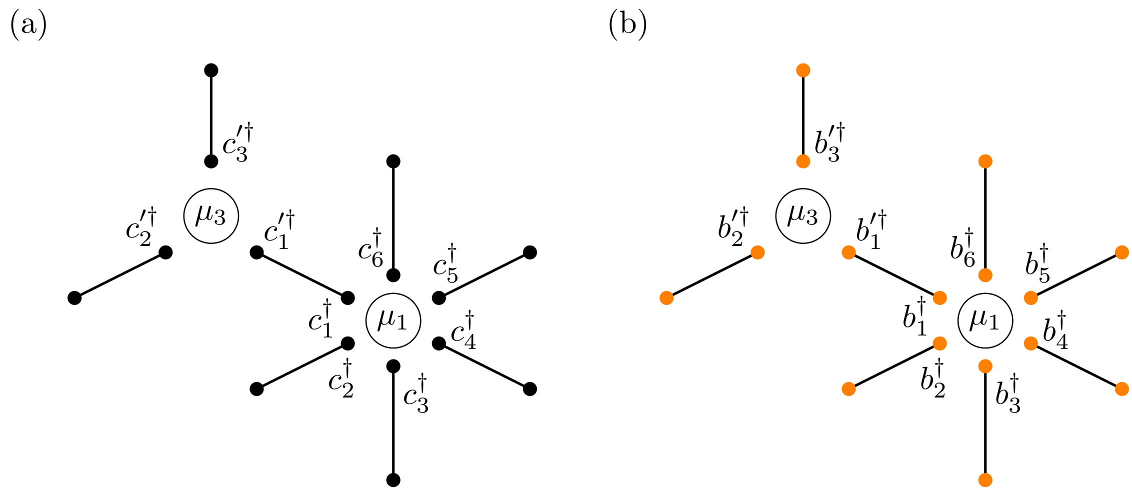


FIG. 11. 1D bubble construction on τ_2. (a) 1D type-I (fermionic) bubble construction, where complex fermions are indicated by solid black dots. Atomic insulators (174) and (175) are created by this procedure. (b) 1D type-II (bosonic) bubble construction, where bosonic modes are indicated by solid orange dots. Bosonic states (178) and (179) are created by this procedure.

i.e., 1D bubble construction on τ_2 can change the eigenvalue of $\boldsymbol{M}_{\tau_2}$. Similar 1D bubble constructions can be held on other 1D blocks, and we summarize the effects of 1D bubble constructions as following:

(1) 1D bubble construction on τ_1: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_2}$ at μ_1 and $\boldsymbol{M}_{\tau_3}$ at μ_2;

(2) 1D bubble construction on τ_2: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_1}$ at μ_1, $\boldsymbol{M}_{\tau_2}$ at μ_3 and the fermion parity of μ_3;

(3) 1D bubble construction on τ_3: simultaneously changes the eigenvalues of $\boldsymbol{M}_{\tau_1}$ at μ_2, $\boldsymbol{M}_{\tau_2}$ at μ_3 and the fermion parity of μ_3.

There is another type of 1D bubble construction on τ_2 and τ_3 (we denote the above "fermionic" 1D bubble construction by "type-I" and this "bosonic" 1D bubble construction by "type-II"): we decorate an bosonic 1D bubble [cf. Eq. (22) and Fig. 11(b)] on each τ_2 (here both yellow and red dots represent the 0D bosonic mode with reflection eigenvalue -1), near μ_1, there are six 0D bosonic modes, each of them carries reflection eigenvalue -1 (six bosonic modes changes nothing),

$$|\phi\rangle_{p6m}^{\mu_1}=b_1^{\dagger}b_2^{\dagger}b_3^{\dagger}b_4^{\dagger}b_5^{\dagger}b_6^{\dagger}|0\rangle. \tag{178}$$

Near μ_3, there are three 0D bosonic modes, each of them carries reflection eigenvalues -1,

$$|\phi\rangle_{p6m}^{\mu_3}=b_1'^{\dagger}b_2'^{\dagger}b_3'^{\dagger}|0\rangle. \tag{179}$$

Hence $|\phi\rangle_{p6m}^{\mu_3}$ changes the reflection eigenvalue by -1 at μ_3. Similar for 1D bubble constructions on τ_3.

With all possible bubble constructions, we are ready to investigate the trivial states. Start from the original 0D trivial block state (nothing is decorated on arbitrary 0D blocks):

$$[(+,+,+),(+,+,+),(+,+)],$$

if we take type-I 1D bubble constructions on τ_j with l_j times ($j=1,2,3$), and type-II 1D bubble constructions on τ_2 and τ_3 with l_2' and l_3' times, above trivial state will be deformed to a new block state labeled by

$$\begin{aligned}&[(+,(-1)^{l_2},(-1)^{l_1}),(+,(-1)^{l_3},(-1)^{l_1}),\\&\times((-1)^{l_2+l_3},(-1)^{l_2+l_3+l_2'+l_3'})].\end{aligned} \tag{180}$$

According to the definition of bubble equivalence, all these states should be trivial. Alternatively, all 0D block states can be viewed as vectors of an 8-dimensional $\mathbb{Z}_2$-valued vector space V, and all trivial 0D block states with the form as Eq. (180) can be viewed as vectors of the subspace of V. The dimension of this subspace is four because there are only four independent indices in l_1, l_2, l_3, and $l_2'+l_3'$. Together with the 2D bubble equivalence, all trivial states form the group

$$\begin{aligned}\{\mathrm{TBS}\}_{p6m,0}&=\{\mathrm{TBS}\}_{p6m,0}^{1\mathrm{D}}\times\{\mathrm{TBS}\}_{p6m,0}^{0\mathrm{D}}\\&=\mathbb{Z}_2\times\mathbb{Z}_2^4=\mathbb{Z}_2^5,\end{aligned} \tag{181}$$

here $\{\mathrm{TBS}\}_{p6m,0}^{1\mathrm{D}}$ represents the group of trivial states with nonvacuum 1D blocks (i.e., 1D FSPT phase decorations on all 1D blocks simultaneously), and $\{\mathrm{TBS}\}_{p6m,0}^{0\mathrm{D}}$ represents the group of trivial states with nonvacuum 0D blocks.

Therefore, all independent nontrivial block states are labeled by the group elements of the following quotient groups:

$$\begin{aligned}E_{p6m,0}^{1\mathrm{D}}&=\{\mathrm{OFBS}\}_{p6m,0}^{1\mathrm{D}}/\{\mathrm{TBS}\}_{p6m,0}^{1\mathrm{D}}=\mathbb{Z}_1,\\E_{p6m,0}^{0\mathrm{D}}&=\{\mathrm{OFBS}\}_{p6m,0}^{0\mathrm{D}}/\{\mathrm{TBS}\}_{p6m,0}^{0\mathrm{D}}=\mathbb{Z}_2^4,\end{aligned} \tag{182}$$

here all $\mathbb{Z}_2$'s are from the nontrivial 0D block states. It is obvious that there is no nontrivial group extension because of the absence of nontrivial 1D block state. Therefore, the ultimate classification of 2D crystalline FSPT phases with $p6m$ symmetry for spinless fermions is

$$\mathcal{G}_{p6m,0}=E_{p6m,0}^{1\mathrm{D}}\times E_{p6m,0}^{0\mathrm{D}}=\mathbb{Z}_2^4. \tag{183}$$

2. Spin-1/2 fermions

Consider the 0D block-state decoration, and similar with the $p4m$ case, the classification data can also be characterized by different 1D irreducible representations of alternative symmetry groups,

$$\begin{aligned}
&\mathcal{H}^1\big[\mathbb{Z}_2^f \times_{\omega_2} (\mathbb{Z}_6 \rtimes \mathbb{Z}_2), U(1)\big] = \mathbb{Z}_2^2,\\
&\mathcal{H}^1\big[\mathbb{Z}_2^f \times_{\omega_2} (\mathbb{Z}_2 \rtimes \mathbb{Z}_2), U(1)\big] = \mathbb{Z}_2^2,\\
&\mathcal{H}^1\big[\mathbb{Z}_2^f \times_{\omega_2} (\mathbb{Z}_3 \rtimes \mathbb{Z}_2), U(1)\big] = \mathbb{Z}_4.
\end{aligned} \tag{184}$$

For D_6 and D_2 centers, the physical meanings of two $\mathbb{Z}_2$'s in the classification data are rotation and reflection eigenvalues, respectively. Furthermore, the group structure of the classification of 0D FSPT phases protected by $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$ on-site symmetry for systems with spin-1/2 fermions is $\mathbb{Z}_4$. Equivalently, we can label different 0D block states by the group elements of the fourfold cyclic group,

$$\mathbb{Z}_4 = \{1, i, -1, -i\}. \tag{185}$$

So the 0D block states at μ_1 and μ_2 can be labeled by $(\pm, \pm)$, here these two $\pm$'s represent the twofold rotation and reflection symmetry eigenvalues (alternatively, they can also represent the eigenvalues of two independent reflection operations because even-fold dihedral group can also be generated by two independent reflections); the 0D block states at μ_3 can be labeled by $\nu \in \{1, i, -1, -i\}$ as the eigenvalues of $\mathbb{Z}_4^f$ symmetry. According to this notation, all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{p6m,1/2}^{0\mathrm{D}} = \mathbb{Z}_2^4 \times \mathbb{Z}_4, \tag{186}$$

and the group elements can be labeled by (three brackets represent the block states at μ_1, μ_2, and μ_3)

$$[(\pm, \pm), (\pm, \pm), \nu].$$

Then we investigate the possible trivializations. Consider the 1D bubble equivalence on 1D blocks labeled by τ_1: on each τ_1, the total on-site symmetry is $\mathbb{Z}_4^f$: nontrivial $\mathbb{Z}_2^f$ extension of the on-site symmetry $\mathbb{Z}_2$. Next we decorate an Eq. (22) onto each of them, here the yellow/red dots represent the 0D FSPT modes protected by $\mathbb{Z}_4^f$ symmetry, which are labeled by $i\,\&\,-i \in \mathbb{Z}_4$, cf. Eq. (185), and they can be trivialized if they shrink to a point. Near each 0D block labeled by μ_3, there are three 0D FSPT modes labeled by $i \in \mathbb{Z}_4$ and they can change the label of 0D block state decorated at each 0D block μ_3 by $-i \in \mathbb{Z}_4$. Therefore, the 0D block state on each μ_3 can be trivialized by this bubble construction. Near 0D block μ_1, this 1D bubble construction changes nothing because there is no $\mathbb{Z}_4^f$ on-site symmetry on μ_1. Similar 1D bubble construction can be held on τ_3.

With all possible bubble constructions, we are ready to investigate the trivial states. Start from the original trivial state (nothing decorated on arbitrary 0D block),

$$[(+, +), (+, +), 1],$$

if we take above 1D bubble constructions on τ_2 and τ_3 with l_2 and l_3 times, above trivial state will be deformed to a new 0D block state labeled by

$$[(+, +), (+, +), (-i)^{3(l_2+l_3)}]. \tag{187}$$

According to the definition of bubble equivalence, all these states should be trivial and all trivial states form the group

$$\{\mathrm{TBS}\}_{p6m,1/2}^{0\mathrm{D}} = \mathbb{Z}_4. \tag{188}$$

Therefore, all independent nontrivial 0D block states are labeled by different group elements of the following quotient group:

$$E_{p6m,1/2}^{0\mathrm{D}} = \{\mathrm{OFBS}\}_{p6m,1/2}^{0\mathrm{D}} \big/ \{\mathrm{TBS}\}_{p6m,1/2}^{0\mathrm{D}} = \mathbb{Z}_2^4. \tag{189}$$

Subsequently we consider the 1D block-state decoration. For arbitrary 1D blocks, the total on-site symmetry on them is $\mathbb{Z}_4^f$: nontrivial $\mathbb{Z}_2^f$ extension of $\mathbb{Z}_2$ on-site symmetry, hence there is no nontrivial 1D block state due to the trivial classification of the corresponding 1D FSPT phases, and the classification attributed to 1D block-state decorations is trivial,

$$E_{p6m,1/2}^{1\mathrm{D}} = \{\mathrm{OFBS}\}_{p6m,1/2}^{1\mathrm{D}} = \mathbb{Z}_1. \tag{190}$$

Therefore, it is obvious that there is no stacking between 1D and 0D block states, and the ultimate classification with accurate group structure is

$$\mathcal{G}_{p6m,1/2} = \mathbb{Z}_2^4. \tag{191}$$

IV. CONSTRUCTION AND CLASSIFICATION OF CRYSTALLINE TI

So far we have discussed the construction and classification of crystalline TSC in 2D interacting fermionic systems. In this section, we will discuss the crystalline TI with additional $U^f(1)$ symmetry by generalizing the real-space construction highlighted in Sec. II. In particular, all block-states decorations will admit an additional $U^f(1)$ internal symmetry. Below we demonstrate that 1D block-state decoration has no contribution and all nontrivial crystalline TI in 2D interacting fermionic systems can be constructed by 0D block-state decoration.

For 1D blocks, there are two different cases: symmetry group with/without the reflection symmetry operation. Since bosonic and fermionic systems can be mapped to each other by Jordan-Wigner transformation, the classification data of 1D SPT phases for bosonic and fermionic systems are identical: by calculating the different projective representations of the symmetry group. (However, the group structure of the classification data could be different in general as stacking operation has different physical meaning for boson and fermion systems.)

For symmetry groups without reflection symmetry operation, the on-site symmetry group of an arbitrary 1D blocks should be $U^f(1)$ charge conservation only, and the corresponding classification for 2D systems with spinless/spin-1/2 fermions can be calculated by the following group cohomology:

$$\mathcal{H}^2[U^f(1), U(1)] = \mathbb{Z}_1. \tag{192}$$

Thus, there is no nontrivial 1D block state for this case.

For the symmetry group with reflection symmetry operation, the on-site symmetry group of some 1D blocks should be $U^f(1)$ charge conservation and $\mathbb{Z}_2$ symmetry via reflection symmetry acting internally. The corresponding classification for 2D systems with spinless/spin-1/2 fermions can be calculated by the following group cohomology:

$$\mathcal{H}^2[U^f(1) \times \mathbb{Z}_2, U(1)] = \mathbb{Z}_1. \tag{193}$$

Again, there is also no nontrivial 1D block state for this case.

Below we will again study five representative cases belonging to different crystallographic systems:

(1) square lattice: $p4m$;
(2) parallelogrammatic lattice: $p2$.
(3) rhombic lattice: cmm;
(4) rectangle lattice: pgg;
(5) hexagonal lattice: $p6m$;

and all other cases are assigned in the Supplemental Material [88]. The classification results are summarized in Table III.

A. Square lattice: *p4m*

According to the cell decomposition (see Fig. 3), for 0D blocks labeled by μ_j ($j=1,2,3$), different 0D block states are characterized by different irreducible representations of the corresponding on-site symmetry group ($n=2,4$),

$$\mathcal{H}^1[U^f(1)\times(\mathbb{Z}_n\rtimes\mathbb{Z}_2),U(1)]=\mathbb{Z}\times\mathbb{Z}_2^2. \tag{194}$$

For systems with spinless fermions, the 0D block states at μ_j ($j=1,2,3$) can be labeled by $(n_j,\pm,\pm)$, where $n_j\in\mathbb{Z}$ represents the $U^f(1)$ charge carried by complex fermions decorated on μ_j and two $\pm$'s represent the eigenvalues of two independent reflection generators. According to this notation, the obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{p4m,0}^{U(1)}=\mathbb{Z}^3\times\mathbb{Z}_2^6, \tag{195}$$

and different group elements can be labeled by (three brackets represent the block states at μ_1, μ_2, and μ_3)

$$[(n_1,\pm,\pm),(n_2,\pm,\pm),(n_3,\pm,\pm)]. \tag{196}$$

Nevertheless, we should further consider possible trivializations. For systems with spinless fermions, we first consider the 1D bubble equivalence on 1D blocks labeled by τ_1: we decorate a 1D "particle-hole" bubble on each τ_1 that can be trivialized if we shrink them to a point. Near each 0D block labeled by μ_1, there are four particles forming the following atomic insulator,

$$|\phi\rangle_{p4m}^{\mu_1}=p_1^\dagger p_2^\dagger p_3^\dagger p_4^\dagger|0\rangle, \tag{197}$$

it has eigenvalues under independent reflections,

$$\boldsymbol{M}_{\tau_1}|\phi\rangle_{p4m}^{\mu_1}=p_1^\dagger p_4^\dagger p_3^\dagger p_2^\dagger=-|\phi\rangle_{p4m}^{\mu_1}, \tag{198}$$

$$\boldsymbol{M}_{\tau_3}|\phi\rangle_{p4m}^{\mu_1}=p_3^\dagger p_4^\dagger p_1^\dagger p_2^\dagger=|\phi\rangle_{p4m,}^{\mu_1} \tag{199}$$

i.e., eigenvalue -1 of $\boldsymbol{M}_{\tau_1}$ at each 0D block μ_1 can be trivialized by the atomic insulator $|\phi\rangle_{p4m}^{\mu_1}$. Near μ_2, there are two holes forming another atomic insulator,

$$|\phi\rangle_{p4m}^{\mu_2}=h_1^\dagger h_2^\dagger|0\rangle, \tag{200}$$

it has eigenvalues under independent reflections,

$$\boldsymbol{M}_{\tau_1}|\phi\rangle_{p4m}^{\mu_2}=h_1^\dagger h_2^\dagger|0\rangle=|\phi\rangle_{p4m}^{\mu_2}, \tag{201}$$

$$\boldsymbol{M}_{\tau_2}|\phi\rangle_{p4m}^{\mu_2}=h_2^\dagger h_1^\dagger|0\rangle=-|\phi\rangle_{p4m,}^{\mu_2} \tag{202}$$

i.e., eigenvalues -1 of the reflection $\boldsymbol{M}_{\tau_2}$ at each 0D block μ_2 can be trivialized by atomic insulator $|\phi\rangle_{p4m}^{\mu_2}$. Therefore, aforementioned 1D bubble construction leads to the dependence of reflection eigenvalues at μ_1 and μ_2 (can be changed simultaneously). Similar 1D bubble construction can be held on τ_2 and τ_3 as well.

Now we move to the $U^f(1)$ charge sector. As shown in Fig. 3, we note that within a specific unit cell, there is one 0D block labeled by μ_1 and μ_3, two 0D blocks labeled by μ_2. Consider the 1D bubble construction on τ_1: it adds four $U^f(1)$ charges at each 0D block μ_1 and removes two $U^f(1)$ charges at each 0D block μ_2, hence the $U^f(1)$ charge at μ_1 and μ_2 are not independent. Similar arguments are also applied to 1D blocks labeled by τ_2 and τ_3.

With the help of above discussions, we consider the 1D bubble equivalence. Start from the trivial state,

$$[(0,+,+),(0,+,+),(0,+,+)]. \tag{203}$$

Taking aforementioned 1D bubble constructions on τ_j with $l_j\in\mathbb{Z}$ times, it will lead to a new 0D block state labeled by

$$\begin{aligned}&[(4l_1+4l_3,(-1)^{l_1},(-1)^{l_3}),(-2l_1+2l_2,(-1)^{l_2},(-1)^{l_1}),\\&\times(-4l_2-4l_3,(-1)^{l_2},(-1)^{l_3})],\end{aligned} \tag{204}$$

which should be trivial. Alternatively, all 0D block states can be viewed as vectors of a 9-dimensional vector space V, where the $U^f(1)$ charge components are $\mathbb{Z}$-valued and all other components are $\mathbb{Z}_2$-valued attributed to rotation and reflection eigenvalues. Then all trivial 0D block states with the form as Eq. (204) can be viewed as a vector subspace V' of V. It is easy to see that there are only three independent quantities in Eq. (204): l_1, l_2 and l_3, so the dimension of the vector subspace V' should be three. For the $U^f(1)$ charge sector, we have the following relationship:

$$-(4l_1+4l_3)-2(-2l_1+2l_2)=-4l_2-4l_3, \tag{205}$$

i.e., there are only two independent quantities, which serves a $2\mathbb{Z}\times4\mathbb{Z}$ trivialization. The remaining one degree of freedom of the vector subspace V' should be attributed to the eigenvalues of point group symmetry action with $(-1)^{l_1}=(-1)^{l_2}=(-1)^{(-l_3)}$, which serve a $\mathbb{Z}_2$ trivialization. Therefore, all trivial states with the form as shown in Eq. (204) compose the following group:

$$\{\mathrm{TBS}\}_{p4m,0}^{U(1)}=2\mathbb{Z}\times4\mathbb{Z}\times\mathbb{Z}_2, \tag{206}$$

and different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{p4m,0}^{U(1)}&=\{\mathrm{OFBS}\}_{p4m,0}^{U(1)}/\{\mathrm{TBS}\}_{p4m,0}^{U(1)}\\&=\mathbb{Z}\times\mathbb{Z}_8\times\mathbb{Z}_4\times\mathbb{Z}_2^3.\end{aligned} \tag{207}$$

For systems with spin-1/2 fermions, the classification data of the corresponding 0D block states can be characterized by different irreducible representations of the corresponding on-site symmetry group ($n=2,4$),

$$\mathcal{H}^1[U^f(1)\times_{\omega_2}(\mathbb{Z}_n\rtimes\mathbb{Z}_2),U(1)]=2\mathbb{Z}\times\mathbb{Z}_2^2. \tag{208}$$

The precise meaning of ω_2 are refer to Sec. I B). To calculate this classification data, we should firstly calculate the following two cohomologies [88]:

$$\begin{aligned}&n_0\in\mathcal{H}^0(\mathbb{Z}_n\rtimes\mathbb{Z}_2,\mathbb{Z})=\mathbb{Z},\\&\nu_1\in\mathcal{H}^1[\mathbb{Z}_n\rtimes\mathbb{Z}_2,U(1)]=\mathbb{Z}_2^2.\end{aligned} \tag{209}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge carried by complex fermions, and two $\mathbb{Z}_2$'s represent the rotation and reflection eigenvalues. We demonstrate that the odd number of the $U^f(1)$ charge at each 0D block is not allowed: a specific n_0 is obstructed if and only if $(-1)^{\omega_2\smile n_0}\in\mathcal{H}^2[\mathbb{Z}_n\rtimes\mathbb{Z}_2,U(1)]$ is a nontrivial 2-cocycle with $U(1)$-coefficient. From Refs. [19] and [87] we know that for cases without $U^f(1)$ charge conservation, nontrivial 0-cocycle $n_0=1\in\mathcal{H}^0(\mathbb{Z}_n\rtimes\mathbb{Z}_2,\mathbb{Z}_2)$ leads to nontrivial 2-cocycle $(-1)^{\omega_2\smile n_0}\in\mathcal{H}^2[\mathbb{Z}_n\rtimes\mathbb{Z}_2,U(1)]$. So for $U^f(1)$ charge conserved cases, odd $n_0\in\mathcal{H}^0(\mathbb{Z}_n\rtimes\mathbb{Z}_2,\mathbb{Z})$ lead to nontrivial 2-cocycle $(-1)^{\omega_2\smile n_0}\in\mathcal{H}^2[\mathbb{Z}_n\rtimes\mathbb{Z}_2,U(1)]$. As a consequence, for systems with spin-$1/2$ fermions, we can only decorate even number of complex fermions on each 0D block and all obstruction-free block states form a group,

$$\{\mathrm{OFBS}\}_{p4m,1/2}^{U(1)}=(2\mathbb{Z})^3\times\mathbb{Z}_2^6. \tag{210}$$

Then we consider the possible trivializations via 1D bubble constructions. Similar to the TSC case, since the reflection properties of $|\phi\rangle_{p4m}^{\mu_1}$ and $|\phi\rangle_{p4m}^{\mu_2}$ at μ_1 and μ_2 are changed by an additional -1, there is no trivialization. The discussion of $U^f(1)$ charge sector is identical to the spinless case: start from the original trivial state (203), take above 1D bubble constructions on τ_j with $l_j\in\mathbb{Z}$ times, it leads to a new 0D block state labeled by

$$[(4l_1+4l_3,0,0),(-2l_1+2l_2,0,0),(-4l_2-4l_3,0,0)]. \tag{211}$$

Again, all states with the form (211) are trivial, forming the following group:

$$\{\mathrm{TBS}\}_{p4m,1/2}^{U(1)}=2\mathbb{Z}\times4\mathbb{Z}. \tag{212}$$

Different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{p4m,1/2}^{U(1)}&=\{\mathrm{OFBS}\}_{p4m,1/2}^{U(1)}/\{\mathrm{TBS}\}_{p4m,1/2}^{U(1)}\\&=2\mathbb{Z}\times\mathbb{Z}_2^7.\end{aligned} \tag{213}$$

Here $2\mathbb{Z}$ means that we can only decorate even number of complex fermions on each 0D block.

B. Parallelogrammatic lattice: *p*2

Similar to the $p4m$ case, different 0D block states are characterized by different irreducible representations of the symmetry group,

$$\mathcal{H}^1[U^f(1)\times\mathbb{Z}_2,U(1)]=\mathbb{Z}\times\mathbb{Z}_2. \tag{214}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge and $\mathbb{Z}_2$ represents the rotation eigenvalues. So 0D block states at μ_j ($j=1,2,3,4$) can be labeled by $(n_j,\pm)$, here $n_j\in\mathbb{Z}$ represents the $U^f(1)$ charge carried by complex fermions on μ_j and $\pm$ represents the eigenvalue of twofold rotation operation. According to this notation, all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{p2,0}^{U(1)}=\mathbb{Z}^4\times\mathbb{Z}_2^4. \tag{215}$$

We should further consider possible trivializations: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by τ_1: we decorate a 1D "particle-hole" bubble on each τ_1. Near each 0D block labeled by μ_1, there are two particles forming the following atomic insulator:

$$|\xi\rangle_{p2}^{\mu_1}=p_1^\dagger p_2^\dagger|0\rangle, \tag{216}$$

with following rotation property:

$$\boldsymbol{R}_{\mu_1}|\xi\rangle_{p2}^{\mu_1}=p_2^\dagger p_1^\dagger|0\rangle=-|\xi\rangle_{p2}^{\mu_1}, \tag{217}$$

i.e., rotation eigenvalue -1 at each 0D block μ_1 can be trivialized by atomic insulator $|\xi\rangle_{p2}^{\mu_1}$. Near μ_2, there are two holes forming another atomic insulator,

$$|\xi\rangle_{p2}^{\mu_2}=h_1^\dagger h_2^\dagger|0\rangle \tag{218}$$

with rotation property

$$\boldsymbol{R}_{\mu_2}|\xi\rangle_{p2}^{\mu_2}=h_2^\dagger h_1^\dagger|0\rangle=-|\xi\rangle_{p2}^{\mu_2}, \tag{219}$$

i.e., rotation eigenvalue -1 at each 0D block μ_2 can be trivialized by atomic insulator $|\xi\rangle_{p2}^{\mu_2}$. Therefore, aforementioned 1D bubble construction leads to the dependence of rotation eigenvalues at μ_1 and μ_2.

Now we move to the $U^f(1)$ charge sector. Repeatedly consider the aforementioned 1D bubble construction on τ_1: it adds two $U^f(1)$ charges at each 0D block μ_1 and removes two $U^f(1)$ charges at each 0D block μ_2, hence the $U^f(1)$ charge at μ_1 and μ_2 are not independent. We summarize effects of all possible 1D bubble constructions:

(1) 1D bubble construction on τ_1: Add two $U^f(1)$ charges on μ_1, eliminate two $U^f(1)$ charges on μ_2, and simultaneously change the rotation eigenvalues of μ_1 and μ_2;

(2) 1D bubble construction on τ_2: Add two $U^f(1)$ charges on μ_1, eliminate two $U^f(1)$ charges on μ_3, and simultaneously change the rotation eigenvalues of μ_1 and μ_3;

(3) 1D bubble construction on τ_3: Add two $U^f(1)$ charges on μ_2, eliminate two $U^f(1)$ charges on μ_4, and simultaneously change the rotation eigenvalues of μ_2 and μ_4;

With the help of above discussions, we consider the 1D bubble equivalence. Start from the original trivial state,

$$[(0,+),(0,+),(0,+),(0,+)]. \tag{220}$$

Taking aforementioned 1D bubble constructions on τ_j with $l_j\in\mathbb{Z}$ times ($j=1,2,3$), this trivial state will be deformed to a new 0D block state labeled by

$$\begin{aligned}&[(2l_1+2l_2,(-1)^{l_1+l_2}),(-2l_1+2l_3,(-1)^{l_1+l_3}),\\&\times(-2l_2,(-1)^{l_2}),(-2l_3,(-1)^{l_3})].\end{aligned} \tag{221}$$

According to the definition of bubble equivalence, this state should be trivial. Alternatively, all 0D block states can be viewed as vectors of an 8-dimensional vector space V, where the complex fermion components are $\mathbb{Z}$ valued, and all other components are $\mathbb{Z}_2$ valued. Then all trivial 0D block states with the form as Eq. (221) can be viewed as a vector space V' of V. It is easy to see that there are only three independent quantities in Eq. (221): l_1, l_2, and l_3. So the dimension of the vector subspace V' should be three. For the $U^f(1)$ charge sector, there are 3 independent quantities in the following four variables:

$$2l_1+2l_2,-2l_1+2l_3,-2l_2,-2l_3.$$

Thus all 1D bubble constructions serve a $(2\mathbb{Z})^3$ trivialization in $U^f(1)$ charge sector, and all trivial states form the following group:

$$\{\text{TBS}\}_{p2,0}^{U(1)} = (2\mathbb{Z})^3, \tag{222}$$

and different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{p2,0}^{U(1)} &= \{\text{OFBS}\}_{p2,0}^{U(1)}/\{\text{TBS}\}_{p2,0}^{U(1)}\\ &= \mathbb{Z}^4\times\mathbb{Z}_2^4/(2\mathbb{Z})^3 = \mathbb{Z}\times\mathbb{Z}_2^7.\end{aligned} \tag{223}$$

For systems with spin-1/2 fermions, 0D obstruction-free block states are identical with spinless case,

$$\{\text{OFBS}\}_{p2,1/2}^{U(1)} = \mathbb{Z}^4\times\mathbb{Z}_2^4, \tag{224}$$

then repeatedly consider the aforementioned 1D bubble constructions: rotation properties of $|\xi\rangle_{p2}^{\mu_1}$ and $|\xi\rangle_{p2}^{\mu_2}$ at μ_1 and μ_2 are changed by an additional -1 and it leads to no trivialization. Furthermore, it is easy to verify that the complex fermion decorations for spinless and spin-1/2 fermions are identical. So again we start from the original trivial state (220), take above 1D bubble constructions on τ_j with $l_j\in\mathbb{Z}$ times $(j=1,2,3)$, and it will lead to a new 0D block state labeled by

$$[(2l_1+2l_2,+),(-2l_1+2l_3,+),(-2l_2,+),(-2l_3,+)]. \tag{225}$$

Similar with the spinless case, all states with this form are trivial, forming the following group:

$$\{\text{TBS}\}_{p2,1/2}^{U(1)} = (2\mathbb{Z})^3, \tag{226}$$

and different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{p2,1/2}^{U(1)} &= \{\text{OFBS}\}_{p2,1/2}^{U(1)}/\{\text{TBS}\}_{p2,1/2}^{U(1)}\\ &= \mathbb{Z}^4\times\mathbb{Z}_2^4/(2\mathbb{Z})^3 = \mathbb{Z}\times\mathbb{Z}_4^3\times\mathbb{Z}_2.\end{aligned} \tag{227}$$

We notice that the classifications of 2D crystalline TI protected by $p2$ symmetry for both spinless and spin-1/2 fermions are identical. Now we give a comprehension of this issue: for both spinless and spin-1/2 fermions ($\boldsymbol{R}^2=1$ and $\boldsymbol{R}^2=-1$, respectively), the group structures of the total symmetry groups are identical: direct product of $U^f(1)$ charge conservation and twofold rotation symmetry: $U^f(1)\times C_2$. We explicitly formulate the $U^f(1)$ charge conservation and C_2 rotation symmetry as

$$C_2=\{E,\boldsymbol{R}\},\ \ U^f(1)=\{e^{i\theta}|\theta\in[0,2\pi)\}. \tag{228}$$

For systems with spinless fermions, $\boldsymbol{R}^2=1$. Nevertheless, we can twist the group elements of C_2 by a $U^f(1)$ phase factor as

$$\boldsymbol{R}'=\boldsymbol{R}e^{i\pi/2},\ \ e^{i\pi/2}\in U^f(1) \tag{229}$$

then we reformulate the total symmetry group with the twisted operators

$$C_2=\{E,\boldsymbol{R}'\},\ \ U^f(1)=\{e^{i\theta}|\theta\in[0,2\pi)\}. \tag{230}$$

But $\boldsymbol{R}'^2=-1$ for this case. Therefore, the symmetry groups for both spinless and spin-1/2 fermions are identical, and can be deformed to each other by Eq. (229). We stress that such a statement is true for all wallpaper group with a single reflection axis.

C. Rhombic lattice: *cmm*

Repeatedly consider the cell decomposition of *cmm* as illustrated in Fig. 8. For 0D blocks labeled by μ_1, different 0D block states are characterized by different irreducible representations of the symmetry group as

$$\mathcal{H}^1[U^f(1)\times\mathbb{Z}_2,U(1)]=\mathbb{Z}\times\mathbb{Z}_2. \tag{231}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge and $\mathbb{Z}_2$ represents the rotation eigenvalue -1. For 0D blocks labeled by μ_2 and μ_3, different 0D block states are characterized by different irreducible representations of the symmetry group as

$$\mathcal{H}^1[U^f(1)\times(\mathbb{Z}_2\rtimes\mathbb{Z}_2),U(1)]=\mathbb{Z}\times\mathbb{Z}_2^2. \tag{232}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge and two $\mathbb{Z}_2$'s represent the two independent reflection eigenvalue -1. Thus 0D block states at μ_1 can be labeled by $(n_1,\pm)$, here $n_1\in\mathbb{Z}$ represents the $U^f(1)$ charge at each μ_1 and $\pm$ represents the eigenvalues of twofold rotation operation $\boldsymbol{R}_{\mu_1}$; and 0D block states at μ_2/μ_3 can be labeled by $(n_2/n_3,\pm,\pm)$, here $n_2/n_3\in\mathbb{Z}$ represents the $U^f(1)$ charge at each μ_2/μ_3, and two $\pm$'s represent the eigenvalues of two independent reflection generators $\boldsymbol{M}_{\tau_2}$ and $\boldsymbol{M}_{\tau_3}$. According to this notation, all obstruction-free 0D block states form the following group:

$$\{\text{OFBS}\}_{cmm,0}^{U(1)}=\mathbb{Z}^3\times\mathbb{Z}_2^5. \tag{233}$$

We should further consider possible trivializations: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by τ_1: we decorate a 1D "particle-hole" bubble [cf. Eq. (22), here yellow and red dots represent particle and hole, respectively] on each τ_1, and they can be trivialized if we shrink them to a point. Near each 0D block labeled by μ_1, there are two particles forming atomic insulator,

$$|\xi\rangle_{cmm}^{\mu_1}=p_1^\dagger p_2^\dagger|0\rangle. \tag{234}$$

Near μ_2, there are four holes forming another atomic insulator,

$$|\xi\rangle_{cmm}^{\mu_2}=h_1^\dagger h_2^\dagger h_3^\dagger h_4^\dagger|0\rangle. \tag{235}$$

Similar with the crystalline TSC, rotation eigenvalue at each μ_1 can be changed by $|\xi\rangle_{cmm}^{\mu_1}$. Then we consider the 1D bubble equivalence on 1D blocks labeled by τ_2: we decorate an identical 1D "particle-hole" bubble as aforementioned on each τ_2. Near each 0D block labeled by μ_2, there are two particles forming the following atomic insulator:

$$|\eta\rangle_{cmm}^{\mu_2}=p_1'^\dagger p_2'^\dagger|0\rangle. \tag{236}$$

Near μ_3, there are two holes forming another atomic insulator,

$$|\eta\rangle_{cmm}^{\mu_3}=h_1'^\dagger h_2'^\dagger|0\rangle. \tag{237}$$

Similar with the crystalline TSC, eigenvalue of $\boldsymbol{M}_{\tau_3}$ at each μ_2/μ_3 can be changed by $|\eta\rangle_{cmm}^{\mu_2}/|\eta\rangle_{cmm}^{\mu_3}$. The bubble construction on τ_3 can be understood in a similar way.

Subsequently we consider the $U^f(1)$ charge sector. First of all, as shown in Fig. 8, we should identify that within a specific unit cell, there are two 0D blocks labeled by μ_1 and one 0D block labeled by μ_2/μ_3. Repeatedly consider above 1D bubble construction on τ_1: it adds two complex fermions on each 0D block μ_1 and removes four complex fermions at each 0D block μ_2 (by adding four holes), hence the numbers of complex fermions at μ_1 and μ_2 are not independent. Then we consider aforementioned 1D bubble equivalence on 1D blocks τ_2/τ_3: it adds two complex fermions at each 0D block μ_2 and adding two holes at each 0D block μ_3, hence the $U^f(1)$ charge at μ_2 and μ_3 are not independent.

With the help of above discussions, we consider the 0D block-state decorations. Start from the original trivial state (nothing is decorated on all blocks),

$$[(0,+),(0,+,+),(0,+,+)]. \tag{238}$$

Taking aforementioned 1D bubble construction on τ_j with $l_j\in\mathbb{Z}$ times ($j=1,2,3$), it will lead to a new 0D block state labeled by

$$\begin{aligned}&[(2l_1,(-1)^{l_1}),(-2l_1+2l_2+2l_3,(-1)^{l_3},(-1)^{l_2})\\&\times(-2l_2-2l_3,(-1)^{l_3},(-1)^{l_2})].\end{aligned} \tag{239}$$

According to the definition of bubble equivalence, all states with this form should be trivial. Alternatively, all 0D block states can be viewed as vectors of an 8-dimensional vector space V, where the complex fermion components are $\mathbb{Z}$ valued and all other components are $\mathbb{Z}_2$ valued. Then all trivial 0D block states with the form as Eq. (239) can be viewed as a vector subspace V' of V. It is easy to see that there are only three independent quantities in Eq. (239): l_1, l_2, and l_3. So the dimensionality of the vector subspace V' should be three. For the $U^f(1)$ charge sector, we have the following relationship:

$$-2l_1-(-2l_1+2l_2+2l_3)=-2l_2-2l_3, \tag{240}$$

i.e., there are only two independent quantities, which serve a $(2\mathbb{Z})^2$ trivialization. The remaining one degree of freedom of the vector subspace V' should be attributed to the eigenvalues of point group symmetry action with $(-1)^{l_2}=(-1)^{(-l_3)}$, which serves a $\mathbb{Z}_2$ trivialization. Therefore, all trivial states (239) form the following group:

$$\{\mathrm{TBS}\}_{cmm,0}^{U(1)}=(2\mathbb{Z})^2\times\mathbb{Z}_2, \tag{241}$$

and different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{cmm,0}^{U(1)}&=\{\mathrm{OFBS}\}_{cmm,0}^{U(1)}/\{\mathrm{TBS}\}_{cmm,0}^{U(1)}\\&=\mathbb{Z}^3\times\mathbb{Z}_2^5/(2\mathbb{Z})^2\times\mathbb{Z}_2=\mathbb{Z}\times\mathbb{Z}_4^2\times\mathbb{Z}_2^2.\end{aligned} \tag{242}$$

For systems with spin-1/2 fermions, like the cases without $U^f(1)$ charge conservation, the classification data of the 0D block states of 0D blocks labeled by μ_2 and μ_3 can be characterized by different irreducible representations of the corresponding on-site symmetry group,

$$\mathcal{H}^1[U^f(1)\times_{\omega_2}(\mathbb{Z}_2\rtimes\mathbb{Z}_2),U(1)]=2\mathbb{Z}\times\mathbb{Z}_2^2. \tag{243}$$

Here $2\mathbb{Z}$ represents the $U^f(1)$ charge carried by complex fermion, and two $\mathbb{Z}_2$'s represent the two independent reflection eigenvalues [similar with the $p4m$ case, we can only decorate even number of $U^f(1)$ charge on each 0D block]. and all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{cmm,1/2}^{U(1)}=\mathbb{Z}\times(2\mathbb{Z})^2\times\mathbb{Z}_2^5. \tag{244}$$

Then we discuss possible trivializations. Repeatedly consider aforementioned 1D bubble constructions, and now the rotation properties of $|\xi\rangle_{cmm}^{\mu_1}$, $|\xi\rangle_{cmm}^{\mu_2}$, $|\eta\rangle_{cmm}^{\mu_2}$ and $|\eta\rangle_{cmm}^{\mu_3}$ at μ_j, $j=1,2,3$ are changed by an additional -1; the reflection properties of $|\eta\rangle_{cmm}^{\mu_2}$ and $|\eta\rangle_{cmm}^{\mu_3}$ at μ_2 and μ_3 are also changed by an additional -1. All of them lead to no trivialization. Furthermore, it is easy to see that all arguments about the $U^f(1)$ charge sector are identical. Again we start from the original trivial state (238), and take above 1D bubble constructions on τ_j with l_j times ($j=1,2,3$), it will lead to a new 0D block state labeled by

$$[(2l_1,+),(-2l_1+2l_2+2l_3,+,+)(-2l_2-2l_3,+,+)]. \tag{245}$$

Similar with the spinless case, all states with this form are trivial, forming the following group:

$$\{\mathrm{TBS}\}_{cmm,1/2}^{U(1)}=(2\mathbb{Z})^2, \tag{246}$$

and all different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{cmm,1/2}^{U(1)}&=\{\mathrm{OFBS}\}_{cmm,1/2}^{U(1)}/\{\mathrm{TBS}\}_{cmm,1/2}^{U(1)}\\&=\mathbb{Z}\times(2\mathbb{Z})^2\times\mathbb{Z}_2^5/(2\mathbb{Z})^2=2\mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2^4.\end{aligned} \tag{247}$$

We should notice that the group structure of the classification should be $2\mathbb{Z}\times\mathbb{Z}_2^6$ rather than $\mathbb{Z}\times\mathbb{Z}_2^5$: two independent quantities are l_1 and l_2+l_3, hence the classification contributed from complex fermion decorations on μ_1 should be $\mathbb{Z}/2\mathbb{Z}=\mathbb{Z}_2$. Equivalently, 0D block state $(1,+)$ at μ_1 is nontrivial.

D. Rectangle lattice: *pgg*

Repeatedly consider the cell decomposition of *pgg* as illustrated in Fig. 9. For an arbitrary 0D block, different 0D block states are characterized by different irreducible representations of symmetry group as

$$\mathcal{H}^1[U^f(1)\times\mathbb{Z}_2,U(1)]=\mathbb{Z}\times\mathbb{Z}_2. \tag{248}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge and $\mathbb{Z}_2$ represents the eigenvalues of twofold rotational symmetry operation. So the 0D block state decorated on μ_j ($j=1,2$) can be labeled by $(n_j,\pm)$, where $n_j\in\mathbb{Z}$ represents the $U^f(1)$ charge carried by complex fermions on μ_j and $\pm$ represents the eigenvalues of twofold rotational symmetry on μ_j. According to this notation, all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{pgg,0}^{U(1)}=\mathbb{Z}^2\times\mathbb{Z}_2^2. \tag{249}$$

We should further consider the possible trivialization. For systems with spinless fermions, consider the 1D bubble equivalence on τ_2: we decorate a 1D "particle-hole" bubble [cf. Eq. (22)] on each τ_2 that can be trivialized if we shrink them to a point. Near each 0D block labeled by μ_1, there are two particles that form an atomic insulator,

$$|\phi\rangle_{pgg}^{\mu_1} = p_1^\dagger p_2^\dagger|0\rangle, \tag{250}$$

with rotation property as

$$\boldsymbol{R}_{\mu_1}|\phi\rangle_{pgg}^{\mu_1} = p_2^\dagger p_1^\dagger|0\rangle = -|\phi\rangle_{pgg}^{\mu_1}, \tag{251}$$

i.e., rotation eigenvalue -1 can be trivialized by the atomic insulator $|\phi\rangle_{pgg}^{\mu_1}$ at each 0D block labeled by μ_1. Near each 0D block labeled by μ_2, there are two holes that form another atomic insulator,

$$|\phi\rangle_{pgg}^{\mu_2} = h_1^\dagger h_2^\dagger|0\rangle, \tag{252}$$

with rotation property as

$$\boldsymbol{R}_{\mu_2}|\phi\rangle_{pgg}^{\mu_2} = h_2^\dagger h_1^\dagger|0\rangle = -|\phi\rangle_{pgg}^{\mu_2}, \tag{253}$$

i.e., rotation eigenvalue -1 can be trivialized by the atomic insulator $|\phi\rangle_{pgg}^{\mu_2}$ at each 0D block labeled by μ_2. Thus the 1D bubble construction on τ_2 can change the rotation eigenvalues of μ_1 and μ_2 simultaneously, which lead to the dependence of rotation eigenvalues of μ_1 and μ_2.

Subsequently we consider the $U^f(1)$ charge sector: consider 1D bubble equivalence on 1D blocks τ_2 [cf. Eq. (22)]: it adds two $U^f(1)$ charges at each 0D block μ_1 and removes two $U^f(1)$ charges at each 0D block μ_2, hence the numbers of $U^f(1)$ charges at μ_1 and μ_2 are not independent.

With the help of above discussions, we consider the 0D block-state decorations. Start from the trivial state,

$$[(0,+),(0,+)]. \tag{254}$$

Taking aforementioned 1D bubble construction on τ_2 with $n \in \mathbb{Z}$ times will obtain the group containing all trivial states,

$$\begin{aligned}\{\text{TBS}\}_{pgg,0}^{U(1)} &= \{[(2n,(-1)^n),(-2n,(-1)^n)]|n\in\mathbb{Z}\}\\ &= 2\mathbb{Z}.\end{aligned} \tag{255}$$

Therefore, the ultimate classification of crystalline TSC protected by pgg symmetry for 2D systems with spinless fermions is

$$\begin{aligned}\mathcal{G}_{pgg,0}^{U(1)} &= \{\text{OFBS}\}_{pgg,0}^{U(1)}/\{\text{TBS}\}_{pgg,0}^{U(1)}\\ &= \mathbb{Z}^2\times\mathbb{Z}_2^2/2\mathbb{Z} = \mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2.\end{aligned} \tag{256}$$

For systems with spin-1/2 fermions, 0D obstruction-free block states are identical with spinless case,

$$\{\text{OFBS}\}_{pgg,1/2}^{U(1)} = \mathbb{Z}^2\times\mathbb{Z}_2^2. \tag{257}$$

Then repeatedly consider the aforementioned 1D bubble constructions: The rotation properties of $|\phi\rangle_{pgg}^{\mu_1}$ and $|\phi\rangle_{pgg}^{\mu_2}$ are changed by an additional -1, which leads to no trivialization. It is easy to verify that the complex fermion decorations for spinless and spin-1/2 fermions are identical. Repeatedly consider the 1D bubble construction on τ_2 and it will lead to the following group containing all trivial states:

$$\{\text{TBS}\}_{pgg,1/2}^{U(1)} = \{[(2n,+),(-2n,+)]|n\in\mathbb{Z}\} = 2\mathbb{Z}. \tag{258}$$

Therefore, the ultimate classification of crystalline topological phases protected by pgg symmetry for 2D systems with spin-1/2 fermions is

$$\begin{aligned}\mathcal{G}_{pgg,1/2}^{U(1)} &= \{\text{OFBS}\}_{pgg,1/2}^{U(1)}/\{\text{TBS}\}_{pgg,1/2}^{U(1)}\\ &= \mathbb{Z}^2\times\mathbb{Z}_2^2/2\mathbb{Z} = \mathbb{Z}\times\mathbb{Z}_4\times\mathbb{Z}_2.\end{aligned} \tag{259}$$

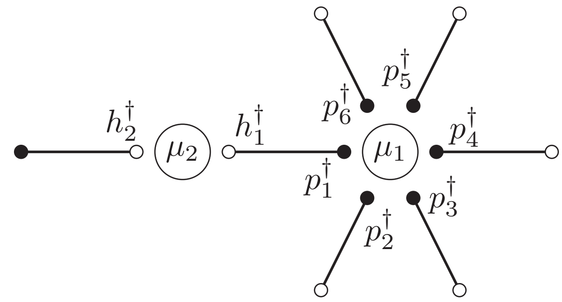


FIG. 12. 1D particle-hole bubble construction on τ_1, where particle/hole is labeled by solid/hollow dot. Atomic insulators (263) and (264) are created by this procedure.

E. Hexagonal lattice: *p6m*

Repeatedly consider the cell decomposition of $p6m$ as illustrated in Fig. 10. For 0D blocks labeled by μ_1 and μ_2, different 0D block states are characterized by different irreducible representations of the symmetry group as $n = 6, 2$,

$$\mathcal{H}^1[U^f(1)\times(\mathbb{Z}_n\rtimes\mathbb{Z}_2),U(1)] = \mathbb{Z}\times\mathbb{Z}_2^2. \tag{260}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge. For μ_1, two $\mathbb{Z}_2$'s represent the reflection eigenvalues of $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_2}$, respectively; for μ_2, two $\mathbb{Z}_2$'s represent the reflection eigenvalues of $\boldsymbol{M}_{\tau_1}$ and $\boldsymbol{M}_{\tau_3}$, respectively.

For 0D blocks labeled by μ_3, different 0D block states are characterized by different irreducible representations of the symmetry group as

$$\mathcal{H}^1[U^f(1)\times(\mathbb{Z}_3\rtimes\mathbb{Z}_2),U(1)] = \mathbb{Z}\times\mathbb{Z}_2. \tag{261}$$

Here $\mathbb{Z}$ represents the $U^f(1)$ charge and $\mathbb{Z}_2$ represents the eigenvalue -1 of the reflection $\boldsymbol{M}_{\tau_2}$.

Therefore, the 0D block states on μ_1 and μ_2 can be labeled by $(n_1/n_2,\pm,\pm)$, where n_1/n_2 represents the $U^f(1)$ charges on μ_1/μ_2 and two $\pm$'s represent the reflection eigenvalues; the 0D block states on μ_3 can be labeled by $(n_3,\pm)$, where n_3 represents the $U^f(1)$ charges on μ_3 and $\pm$ represents the eigenvalue of reflection operation. According to this notation, all obstruction-free 0D block states form the following group:

$$\{\text{OFBS}\}_{p6m,0}^{U(1)} = \mathbb{Z}^3\times\mathbb{Z}_2^5. \tag{262}$$

We should further consider possible trivializations: for systems with spinless fermions, consider the 1D bubble equivalence on 1D blocks labeled by τ_1: we decorate a 1D "particle-hole" bubble [cf. Eq. (22) and Fig. 12] at each τ_1. Near each 0D block labeled by μ_1, there are six particles forming the following atomic insulator:

$$|\xi\rangle_{p6m}^{\mu_1} = p_1^\dagger p_2^\dagger p_3^\dagger p_4^\dagger p_5^\dagger p_6^\dagger|0\rangle. \tag{263}$$

There are two holes forming another atomic insulator,

$$|\xi\rangle_{p6m}^{\mu_2} = h_1^\dagger h_2^\dagger|0\rangle. \tag{264}$$

Similar with the crystalline TSC, eigenvalue of $\boldsymbol{M}_{\tau_2}$ can be changed by Eq. (263) at each μ_1, and eigenvalue of $\boldsymbol{M}_{\tau_3}$ can be changed by Eq. (264) at each μ_2.

Then we consider the 1D bubble equivalence on 1D blocks labeled by τ_3: we decorate an identical 1D "particle-hole" bubble as aforementioned on each τ_3. Near each 0D block labeled by μ_2, there are two particles forming the following atomic insulator:

$$|\eta\rangle_{p6m}^{\mu_2} = p_1'^{\dagger} p_2'^{\dagger}|0\rangle. \tag{265}$$

Near each 0D block μ_3, there are three holes forming another atomic insulator,

$$|\eta\rangle_{p6m}^{\mu_3} = h_1'^{\dagger} h_2'^{\dagger} h_3'^{\dagger}|0\rangle. \tag{266}$$

Similar with the crystalline TSC, eigenvalue of $\boldsymbol{M}_{\tau_1}$ can be changed by $|\eta\rangle_{p6m}^{\mu_2}$ at each μ_2, and eigenvalue of $\boldsymbol{M}_{\tau_2}$ can be changed by $|\eta\rangle_{p6m}^{\mu_3}$ at each μ_3.

1D bubble construction on τ_2 can be understood in a similar way, which changes the eigenvalue of $\boldsymbol{M}_{\tau_1}$ at μ_1 and the eigenvalue of $\boldsymbol{M}_{\tau_2}$ at μ_3. In addition, we also need to consider an alternative 1D bubble equivalence on 1D blocks τ_2 (we label above 1D "particle-hole" bubble construction by "type-I", and label this 1D "bosonic" bubble construction by "type-II"): we decorate an alternative 1D bubble on each 1D block labeled by τ_2 [cf. Eq. (22) and Fig. 11(b)], here both yellow and red dots represent the 0D bosonic modes carry eigenvalues -1 of reflection symmetry. According to this 1D bubble construction, the reflection eigenvalue at each 0D block μ_3 is changed by -1 while the reflection eigenvalue at each 0D block μ_2 remains invariant. Another type-II 1D bubble construction can also be constructed on τ_3.

Subsequently we consider the $U^f(1)$ charge sector. First of all, we should identify that within a specific unit cell, there is one 0D block labeled by μ_1, two 0D blocks labeled by μ_3 and three 0D blocks labeled by μ_2. Repeatedly consider the aforementioned 1D bubble construction on τ_1: it adds six $U^f(1)$ charges at each 0D block μ_1 and removes two $U^f(1)$ charges at each 0D block μ_2, hence the number of $U^f(1)$ charges at μ_1 and μ_2 are not independent. Similar argument can also be applied for τ_2 and τ_3.

With the help of above discussions, we consider the 0D block-state decorations. Start from the original trivial state,

$$[(0,+,+),(0,+,+),(0,+)]. \tag{267}$$

Taking aforementioned type-I 1D bubble constructions on τ_j with l_j times ($j = 1, 2, 3$), and type-II 1D bubble constructions on τ_2/τ_3 with l_2'/l_3' times, it will lead to a new 0D block state labeled by

$$\begin{aligned}&[(6l_1+6l_2,(-1)^{l_2},(-1)^{l_1}),(-2l_1+2l_3,(-1)^{l_3},(-1)^{l_1}),\\ &\times(-3l_2-3l_3,(-1)^{l_2+l_3+l_2'+l_3'})].\end{aligned} \tag{268}$$

According to the definition of bubble equivalence, all states with this form should be trivial. Alternatively, all 0D block states can be viewed as vectors of an 8-dimensional vector space V, where the complex fermion components are $\mathbb{Z}$ valued and all other components are $\mathbb{Z}_2$ valued. Then all trivial 0D block states with the form as Eq. (268) can be viewed as a vector subspace V' of V. As a consequence, there are only four independent quantities in Eq. (268): l_1, l_2, l_3, and $l_2'+l_3'$. So the dimensionality of the vector subspace V' should be four. For the $U^f(1)$ charge sector, we have the following relationship:

$$-(6l_1+6l_2)-3(-2l_1+2l_3) = 2(-3l_2-3l_3), \tag{269}$$

i.e., there are only two independent quantities that serve a $2\mathbb{Z}\times3\mathbb{Z}$ trivialization. The remaining two degrees of freedom of the vector subspace V' should be attributed to the eigenvalues of point group symmetry actions labeled by $(-1)^{l_1} = (-1)^{(-l_2)} = (-1)^{l_3}$ and $(-1)^{l_2'+l_3'}$, which serve a $\mathbb{Z}_2^2$ trivialization. Therefore, all trivial states with form as shown in Eq. (268) compose the group

$$\{\mathrm{TBS}\}_{p6m,0}^{U(1)} = 2\mathbb{Z}\times3\mathbb{Z}\times\mathbb{Z}_2^2, \tag{270}$$

hence different independent nontrivial 0D block states can be labeled by different group elements of the following quotient group:

$$\begin{aligned}\mathcal{G}_{p6m,0}^{U(1)} &= \{\mathrm{OFBS}\}_{p6m,0}^{U(1)}/\{\mathrm{TBS}\}_{p6m,0}^{U(1)}\\ &= \mathbb{Z}^3\times\mathbb{Z}_2^5/2\mathbb{Z}\times3\mathbb{Z}\times\mathbb{Z}_2^2 = \mathbb{Z}\times\mathbb{Z}_{12}\times\mathbb{Z}_2^2.\end{aligned} \tag{271}$$

For systems with spin-1/2 fermions, like the cases without $U^f(1)$ charge conservation, the classification data of the corresponding 0D block states on μ_1, μ_2, and μ_3 can be characterized by different irreducible representations of the corresponding on-site symmetry group,

$$\begin{aligned}&\mathcal{H}^1[U^f(1)\times_{\omega_2}(\mathbb{Z}_6\rtimes\mathbb{Z}_2),U(1)] = 2\mathbb{Z}\times\mathbb{Z}_2^2,\\ &\mathcal{H}^1[U^f(1)\times_{\omega_2}(\mathbb{Z}_2\rtimes\mathbb{Z}_2),U(1)] = 2\mathbb{Z}\times\mathbb{Z}_2^2.\end{aligned} \tag{272}$$

Here each $2\mathbb{Z}$ represents the $U^f(1)$ charge carried by complex fermion, and different $\mathbb{Z}_2$'s represent the rotation and reflection eigenvalues at each 0D block labeled by μ_1 and μ_2 (similar with the $p4m$ case, we can only decorate even number of $U^f(1)$ charge on each 0D block). and all obstruction-free 0D block states form the following group:

$$\{\mathrm{OFBS}\}_{p6m,1/2}^{U(1)} = \mathbb{Z}\times(2\mathbb{Z})^2\times\mathbb{Z}_2^5, \tag{273}$$

then repeatedly consider the aforementioned 1D bubble constructions, the reflection properties of the atomic insulators: $|\xi\rangle_{p6m}^{\mu_1}$, $|\xi\rangle_{p6m}^{\mu_2}$, $|\eta\rangle_{p6m}^{\mu_1}$, $|\eta\rangle_{p6m}^{\mu_2}$, and $|\eta\rangle_{p6m}^{\mu_3}$ are changed by an additional -1, and all of them lead to no trivialization. Other 1D bubble constructions are identical. So again we start from the original trivial state (267), take above type-I 1D bubble constructions on τ_j with l_j times ($j = 1, 2, 3$), and type-II 1D bubble constructions on τ_2/τ_3 with l_2'/l_3' times, it will lead to a new 0D block state labeled by

$$\begin{aligned}&[(6l_1+6l_2,+,+),(-2l_1+2l_3,+,+),\\ &\times(-3l_2-3l_3,(-1)^{l_2'+l_3'})].\end{aligned} \tag{274}$$

The $U^f(1)$ charge sector is identical with spinless case, and there is one independent nonzero reflection eigenvalue $(-1)^{l_2'+l_3'}$. Therefore, all trivial states with form as shown in Eq. (274) compose the following group:

$$\{\mathrm{TBS}\}_{p6m,1/2}^{U(1)} = 2\mathbb{Z}\times3\mathbb{Z}\times\mathbb{Z}_2, \tag{275}$$

and different independent nontrivial 0D block states can be labeled by different group elements of the following group:

$$\begin{aligned}\mathcal{G}_{p6m,1/2}^{U(1)} &= \{\text{OFBS}\}_{p6m,1/2}^{U(1)} \big/ \{\text{TBS}\}_{p6m,1/2}^{U(1)} \\ &= \mathbb{Z} \times (2\mathbb{Z})^2 \times \mathbb{Z}_2^5/(2\mathbb{Z} \times 3\mathbb{Z} \times \mathbb{Z}_2) \\ &= 2\mathbb{Z} \times \mathbb{Z}_6 \times \mathbb{Z}_2^3. \end{aligned} \tag{276}$$

V. GENERALIZED CRYSTALLINE EQUIVALENCE PRINCIPLE

In this section, we discuss how to generalize the crystalline equivalence principle that is rigorously proven for interacting bosonic systems [51]. By comparing the classification results of the topological crystalline TSC summarized in Table I, Table II and the classification results of crystalline TI summarized in Table III with the classification results of the 2D FSPT phases protected by the corresponding on-site symmetry [89,90], we verify the fermionic crystalline equivalence principle for all TSC and TI (for both spinless and spin-$1/2$ cases) constructed in this paper.

In particular, we should map the space group symmetry to on-site symmetry according to the following rules:

(1) Subgroup of translational symmetry along a particular direction should be mapped to the on-site symmetry group $\mathbb{Z}$. Equivalently, the total translational subgroup should be mapped to the on-site symmetry group $\mathbb{Z}^2$;

(2) n-fold rotational symmetry subgroup should be mapped to the on-site symmetry group $\mathbb{Z}_n$;

(3) Reflection symmetry subgroup should be mapped to the time-reversal symmetry group $\mathbb{Z}_2^T$, which is antiunitary; and

(4) Spinless (spin-$1/2$) fermionic systems should be mapped into spin-$1/2$ (spinless) fermionic systems.

The additional twist on spinless and spin-$1/2$ fermions can be naturally interpreted as the spin rotation of fermions: a 2π rotation of a fermion around a specific axis results in a -1 phase factor [88].

Apparently, bosonic/fermionic crystalline SPT phases will be mapped to the corresponding on-site symmetry bosonic/fermionic SPT phases. In fact, there is even a more precise one to one mapping between crystalline SPT phases and the corresponding on-site symmetry SPT phases. It is well known that the 2D fermionic SPT states protected by on-site symmetry have a layered structure: They can be constructed by decorating (subject to certain obstructions) 1D Majorana chains to 1D symmetry domain walls, 0D complex fermion modes to intersection points of domain walls, in addition to the bosonic SPT layer. We find that all crystalline SPT states constructed via 1D block state with Majorana chain decorations will be mapped to on-site symmetry SPT states with Majorana chain decorations to 1D symmetry domain wall, while all crystalline SPT states constructed via 1D block state with FSPT decorations or 0D complex fermion decoration will be mapped to on-site symmetry SPT states with 0D complex fermion modes decoration to intersection points of domain walls. For crystalline topological insulator with an additional $U^f(1)$ charge conservation symmetry, there would be no Majorana chain decoration for the corresponding on-site symmetry SPT phases, that is why there is also no 1D block state in our crystalline SPT states construction.

VI. CONCLUSIONS AND DISCUSSION

In this paper, we derive the classification of crystalline TSC and TI in 2D interacting fermionic systems by using the explicit real-space constructions. For a 2D system with a specific wallpaper group symmetry, we first decompose the system into an assembly of unit cells. Then according to the so-called *extensive trivialization* scheme, we can further decompose each unit cell into an assembly of lower-dimensional blocks. After cell decompositions, we can decorate some lower-dimensional block states on them, and investigate the *obstruction* and *trivialization* for all block states by checking the no-open-edge condition and bubble equivalence. An obstruction/trivialization free decoration corresponds to a nontrivial crystalline SPT phase. We further investigated the group structures of the classification data by considering the possible stacking between 1D and 0D block states. Finally, with the complete classification results, we compare our results with classification of 2D FSPT phases protected by the corresponding on-site symmetry, we verify the crystalline equivalence principle for generic 2D interacting fermionic systems.

We believe that the real-space construction scheme for crystalline SPT is also applicable to 3D interacting fermionic systems, with similar procedures discussed in this work. We conjecture that the crystalline equivalence principle is also correct for 3D crystalline FSPT phases as well. In future works, we will try to construct and fully classify the crystalline TSC/TI in 3D interacting fermionic systems.

We stress that the method in this paper can also be applied to cases with mixture of internal and space group symmetries, i.e., when considering about the lower-dimensional block states, we should also include the internal symmetry together with the space group symmetry acting internally that leads to different lower-dimensional root phases and bubbles. Then based on these root phases, we can further discuss possible obstructions and trivializations by using the general paradigms highlighted in Sec. II.

Moreover, we also predict an intriguing fermionic crystalline TSC (that cannot be realized in both free-fermion and interacting bosonic systems) with $p4m$ wall paper group symmetry. The iron-based superconductor could be a natural strongly correlation electron system to realize such a new phase, especially the monolayer iron selenide/pnictide [91]. Since the spin-orbit interaction in FeSe is relatively small [distinct from Fe(Se,Te) because of the absence of tellurium], we can effectively treat fermions in this system as spinless.

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