Power laws and symmetries in a minimal model of financial market economy

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A financial market is a system resulting from the complex interaction between participants in a closed economy. We propose a minimal microscopic model of the financial market economy based on the real economy's symmetry constraint and minimality requirement. We solve the proposed model analytically in the mean-field regime, which shows that various kinds of universal power-law-like behaviors in the financial market may depend on one another, just like the critical exponents in physics. We then discuss the parameters in the proposed model and we show that each parameter in our model can be related to measurable quantities in the real market, which enables us to discuss the cause of a few kinds of social and economic phenomena.

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I. INTRODUCTION

Modeling the real financial market has been a major challenge in the field of traditional economics [1,2] and in the emergent field of econophysics [3–14]. The financial market can be modeled at a phenomenological level by writing down a stochastic differential equation for the price change; this approach was first taken by Bachelier [15], a few years before Einstein's investigation of Brownian motion [16]. The primary difficulty lies in modeling the interaction and behavior of individual participants of a financial market at the microscopic level [1,8]. The classical economics approach assumes the rationality of human beings and that they maximize the predefined utility functions with some given information. The rationality assumption results in the efficient market hypothesis, which predicts that the price of the stock market (or any market in general) follows a random walk and that ultimately the driving force of price change is exogenous, i.e., caused by an injection of new information [17]. However, these predictions deviate far from what we observe in reality. For example, large price jumps occur about seven to eight times per stock per day on average, but only one report is released regarding each of the stocks every three days [18]; this suggests that external information is not sufficient to explain the market dynamics.

A series of universal statistical relations that could not be explained by classical economics is known to hold [19] (known as the stylized facts); one of the fundamental problems of socioeconomic modeling is to explain the existence of these stylized facts [20]. For example, let S_t denote the price of a stock at time t; then the return is defined as $r_t := \ln S_t/S_{t-1}$. The return is known to be heavy tailed with kurtosis roughly equal to 4. If one measures the. cumulative distribution function of the largest returns of a stock, it is known to obey a power-law distribution with exponent roughly -3 to -5; the daily traded number of stock shares (called volumes) are known to have a power-law distribution with exponent -3 [19]. What is more surprising about these facts is that these stylized facts appear almost universally across different nations, markets, and time [21,22]; they even hold for the newly established bitcoin market [23]. This signature of universality calls for an explanation, while no consistent model exists yet to unify these phenomena. In fact, it might even be a question of whether such a unifying model could exist. This situation is somewhat similar to the situation of turbulence in fluid dynamics. The first-principle theories following directly from the Navier-Stokes equation cannot explain the emergence of turbulence yet. However, it is widely expected that an ultimate correct theory needs to explain the well-known observed Kolmogorov 5/3 power law [24,25] (along with a few other universal empirical facts).

In contrast to the classical economic models, the econophysics approaches to the problem are often microscopically oriented and with phenomenology as high-level guidance. The line of work closest to physics is the Ising spin-based models in [26,27]. These models are based on the classical Ising model and the main results are obtained by relabeling the physical objects in the relevant economic terms. For example, a spin is interpreted as a single investor and the pairwise interaction between two spins is interpreted as the tendency of people to be influenced by other people around them. This line of work has been further developed to model even more complicated interactions between the investors [28]. However, the major criticism of these models is that they are unlikely to be realistic models of the markets and the agents constituting them. For example, the action of an agent is unlikely to behave like a spin, which only purchases $+1$ or -1 unit of stock at any given time. (For detailed reviews and comparisons of the above-discussed models see [29–31].) There are also microscopically founded models in the standard economics literature [31–34], often called the heterogeneous agent-based models. However, these works have a vastly different purpose from the present work; they mostly focus on studying the agents' behavior in the specified model of the market and thereby explaining the market crashes and boom. The present work's goal is to explain the universality of many stylized facts.

We argue that one major limitation of previous works is the lack of first-principles modeling of the cash flow. In a real economy, the stock price (or the price of any commercial goods) must be decided by the cash flow into and out of this stock. This cash flows out of the stock to the hands of the individual participants and they must then reinvest based on their present wealth. A key component of first-principle modeling of the financial market must then be the modeling of cash flow, and the total wealth of the economy should be conserved if there is no explicit money-printing process. We therefore explicitly model the cash flow between the wealth of the individuals to the market, with partial conservation of money. Moreover, inspired by the scaling phenomena in physics, we argue that the existence of the universal relations in finance and economics may be understood through a similar mechanism in physics, where symmetries may lead to universalities. Therefore, we propose to establish a simple model of a financial economy, motivated by (i) symmetry constraints and (ii) the requirement of minimality.

In summary, this work proposes a minimal model of a closed economy where the participants trade in a financial market, motivated by the real economic system's symmetries and microscopic necessities. From this minimal model, we see that the commonly observed universal scaling laws appear naturally and robust to initial conditions changes. The model's parameters can be related to measurable statistics of the real economy, increasing the explanatory and predictive power of the proposed theory.

The organization of this paper is as follows. In the next section we introduce the model most directly related to the proposed model, the limitation of which we use to motivate the present work. In Sec. III we establish the proposed model. In Sec. IV we analyze our model in the mean-field regime. In Sec. V we link the model to the real economy and discuss the insights the model may provide to enhance our understanding of the economy. In the Supplemental Material we also demonstrate the usefulness of the proposed model by showing that it may be used as a computation model for simulating realistic financial data [35].

II. BACKGROUND AND BASIC MODELS

A. Dynamics of price

Finance is the study of the market. Two core concepts in finance are time and uncertainty. Time relates to the fact that the market is dynamic, i.e., changing with time. For example, the average price of a stock is different today from next month. The uncertainty aspect refers to the fact that the future price of anything cannot be determined for sure, i.e., knowing all the past prices and all the relevant information does not suffice to predict the price tomorrow with 100% certainty. The combination of these two fundamental facts results in a (conjectured) consequence that the price of any product in the market obeys some stochastic differential equation

$$dS_t = f(S_t)dt + dB_t \tag{1}$$

for some random process dB_t. Alternatively, the price may obey a stochastic difference equation

$$\Delta S_t = f(S_t)\Delta t + \eta_t \tag{2}$$

for some random noise η_t. One picture assumes that the fundamental process of price change is continuous in time; the other assumes discreteness. So far, there is no clear favoring of one over another in the literature since it is unclear whether the fundamental price formation process is continuous time or discrete time (or neither). This indeterminacy is a consequence of the following observation: While the transactions may take place at any point in time, the transactions need to occur in a discrete manner. One cannot make a "continuous" money transaction.

Two most commonly adopted equations to describe the motion of a stock price is the geometric Brownian motion

$$S_{t+1} = (1 + r_s)S_t + \sigma_t S_t \eta_t, \tag{3}$$

where η_t is drawn from a Gaussian distribution. Here the word "geometric" refers to the fact that the noise term $\sigma_t S_t \eta_t$ is proportional to the price S_t itself. The other model is more recent and is often called the Heston model [36],

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t, \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dU_t, \end{aligned} \tag{4}$$

where ν_t is the instantaneous volatility. Unlike the geometric Brownian motion model, the Heston model assumes that the variance ν_t of the noise is also random and follows a random walk of its own. These two models are the widely adopted phenomenological models of price motion. The limitation of these two equations are obvious: (i) The investors are coarse grained into the noise term and do not appear in the equations explicitly and (ii) the deterministic part of the equations is a simple linear dynamics, which cannot reflect the complicated nonequilibrium dynamics that the market is going through.

B. Dynamics of wealth

There has also been strong interest in modeling the wealth dynamics of individuals. There is a strong sense of how physics in general and thermodynamics in particular should be relevant to this problem because the wealth, like energy, should be conserved in total despite complicated microscopic exchanges of money between the investors [37,38]. Two basic phenomenological models of the evolution of wealth in an economy were proposed in [39,40]. Reference [40] starts from the generalized Lotka-Volterra model, while Ref. [39] models the dynamics through an Ising spin model. Both approaches run into the problem that the full model is not analytically solvable and the simplifying approach assumes that all the participants in the economy are identical and that they feel a static force; this approximation is in essence a mean-field approach, as is also used in [39]. In the mean-field limit, these

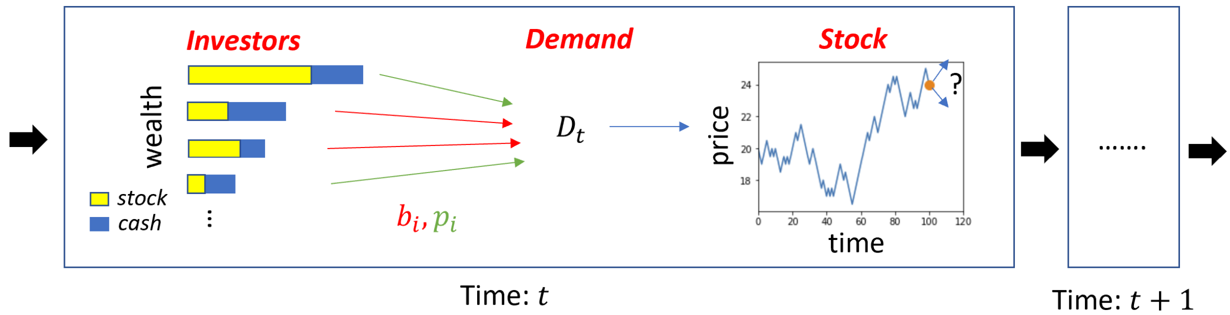

FIG. 1. Illustration of the dynamics of a market. The market consists of the price of a stock and a number of investors (also called agents and participants in this work). At every time step t, the investors make financial decisions based on their wealth and the observed price. These decisions, taken together, create a net demand. The net demand creates motion in the stock price. Previously, the dynamics of the wealth and the dynamics of the price were separately modeled in a phenomenological way; the main contribution of this work is to propose a minimal model that connects the two kinds of dynamics.

two models reduce to a similar form

$$\frac{dW_i}{dt} = -a_0 W_i + a_1 + \sigma W_i \epsilon(t), \tag{5}$$

where W_i denotes the wealth of the ith participant in the economy. In [39], a_0 is proportional to the rate of monetary exchange J between two different individuals and a_1 is proportional to J times the average wealth in the economy; equivalently, $a_0 W_i$ models the amount of money that the ith participant pays to other participants, while a_1 models the payment that W_i receives from others. From this view, the connectivity of the underlying Ising model and the rate of exchange J is crucial since a phase transition from an egalitarian society to a highly unequal society happens in the same way that a phase transition occurs in the standard Ising model when the disordered phase transitions to an ordered phase as the temperature is reduced. In [40] the interpretation is similar: a_1 is interpreted as the regular income received by a person and a_0 is the redistribution of wealth due to taxation, subsidies, and other fixed economic activities.

Given the Langevin equation above, the way to proceed is to write its equivalent Fokker-Planck equation and solve the stable distribution. Treating a_0 and a_1 as constants,[1] it is easy to show that the stationary distribution is

$$P(W_i) \sim W_i^{-1+a_0/\sigma^2} \exp\left(-\frac{a_1}{\sigma^2 W_i}\right) \tag{6}$$

and the exponent a is the desired exponent of wealth. One limitation of these approaches is that the key parameter a_0 is not an observable; a_0 is proportional to the average rate of economic contact between participants in the market, but there is no way to measure this degree of contact objectively. When interpreted as wealth redistribution by government policies, it is far from clear how a_0 may be calculated unambiguously. The problem is exacerbated when one recognizes that the effect of fiscal policy on the economy is highly non-linear and that a first-order expansion in W_i is insufficient to model such policies [41,42]. We attribute the difficulty of making sense of the parameter a_0 to the lack of the specification of the underlying economic model in these models. One cannot model economic exchanges correctly without specifying how the transactions between investors occur. Therefore, we argue that one crucial step in the development of a first-principle theory of the market dynamics is the development of a model that consistently connects the dynamics of the price and the dynamics of the wealth. See Fig. 1 for an illustration of the market dynamics we propose to understand.

We propose to model the investors' wealth and the dynamics of the market in a consistent framework. Some other attempts in this direction also exist [43,44], where the price and the wealth dynamics are modeled simultaneously; however, these models often have a large number of parameters, and the goal is to understand the population dynamics of the investors in market crashes and booms. In contrast, this work aims at establishing a minimal model to identify symmetries' role in forming generic and universal patterns in a financial market economy.

III. PROPOSED MODEL

This section establishes our model, justified by empirical facts and fundamental symmetries in the problem. Let there be a single stock with price $S(t)$ and N investors (agents) in the market. The complete state of the ith agent is given by a pair of scalars $M_i(t)$ and $\Pi_i(t)$, where M_i is the amount of money the agent holds and $\Pi_i(t)$ the number of shares. Note that standard models such as the New-Keynesian models often assume a setting where a risky stock exists alongside a riskless fixed-return government bond. However, this two-stock model is only meaningful when one incorporates a utility maximization procedure done in the standard economic models [43,45,46]. Our framework can be extended to treat this setting, but this adds unjustified complication to the model, reducing the analytical tractability and interpretive power that we are attempting to achieve in the discussion.

At any time $t \in \mathbb{Z}^+$, there are two possible actions for each agent $b_i(t), p_i(t) \in [0, 1]$, which denote the percentage of money the agent will invest in the stock and the percentage of share the agent will sell, respectively. Note that b and p

[1]The calculation in the original work is actually about the relative wealth and is slightly more complicated in computation, but the ultimate interpretation does not change. To make these works more easily compared with the present work, we solve this slightly simplified version of the original.

are unitless. A key feature that differentiates our model from the previous models is that we parametrize agents' actions by the unitless variables b and p that denote the willingness of the buying and selling, independent of how much money the agent has. This choice of b and p is natural to investment since it does not cause any income effect.[2] After the transaction at time t, the change in the cash that the ith agent holds is the return due to the stock sale minus the money spent: $\Delta M_i = p_i(t)\Pi_i(t)S(t) - b_i(t)M_i(t)$; likewise, one can write down the change in one's stock holding $\Delta\Pi_i$. Therefore, at a given time t, the change in money and stock holding for the ith agent is then

$$\Delta M_i(t) = p_i(t)\Pi_i(t)S(t) - b_i(t)M_i(t),$$
$$\Delta\Pi_i(t) = -p_i(t)\Pi_i(t) + \frac{b_i(t)M_i(t)}{S(t)}. \quad (7)$$

Note that the above equations are correct by definition and what is hard to model is the price change to close the system of equations above. Also, there is strong evidence that setting b_i and p_i to be random variables may be appropriate here. It was discovered that when incorrect financial operations (such as spending more money than one has) are forbidden, computer agents making random financial decisions produce a price trajectory that is very close to a trajectory produced by real human beings [47]. In the classical economics literature, such choices are said to be following the algorithm of a zero-intelligence agent [47,48]. Therefore, a model based on zero-intelligence traders offers strong explanatory power because it allows one to identify whether an observed socioeconomic phenomenon is caused by human rationality or by the fundamental properties of the market mechanisms.

We assume that the change in price ΔS_t is an analytical function of all the observables in the market, such as the price and order histories. The price change thus takes the general functional form

$$\Delta S(t) = g\big[\{p(t')\Pi(t'), b(t')M(t'), S(t')\}_{t'=1}^{t}\big], \quad (8)$$

i.e., $S(t)$ can be a general functional. It follows from standard classical economic arguments that the price is determined by the supply and demand curve of the market [49]. We therefore expand $S(t)$ to the first order in the immediate average demand $D(t) := \frac{1}{N}\sum_{i=1}^{N}\Delta\Pi_i(t)$, $\Delta S(t) = c_0 - \frac{1}{\lambda}D(t) + O(D^2)$, for some positive constant λ, which can be called the market depth [26] and is the susceptibility of the price of a perturbative external demand. Moreover, the price should not change when there is no net excessive demand in the market, and so $c_0 = 0$, and we arrive at our price change function

$$\Delta S(t) = -\frac{1}{\lambda}D(t). \quad (9)$$

Note that λ may still be a function of the observables in the market; for example, they may both depend on price.

While the price change function we use is linear, it generates nontrivial dynamics. Similar linear price impact functions can be derived from classical economic arguments [49] and have been used in the Ising spin-based market model [26]. An increasing amount of research studies the exact functional form of the price impact [6]; these interesting price impact functions can be studied in our framework by simply redefining the price impact function above. In this work we use the linear price impact function. Therefore, the following set of equations determines our model of the market:

$$\Delta M_i(t) = p_i(t)\Pi_i(t)S(t) - b_i(t)M_i(t), \quad (10a)$$

$$\Delta\Pi_i(t) = -p_i(t)\Pi_i(t) + \frac{b_i(t)M_i(t)}{S(t)}, \quad (10b)$$

$$\Delta S(t) = \frac{1}{\lambda}\sum_{i=1}^{N}\Delta\Pi_i(t). \quad (10c)$$

Note that, in this model, total wealth is only partially conserved. Equations (10a) and (10b) conserve the total wealth, while (10c) breaks such conservation.[3] However, we do not think this is a significant problem because, in the real economy, it is never the case that the wealth and the financial market constitute a closed system and part of the wealth may be distributed to other nonfinancial objects. In physics terms, one might imagine the existence of a conceptual heat bath of money. While the total money of the system and the bath should be conserved, it is not the case that the wealth in the system is conserved. This also reflects the difficulty of determining the price impact function and that better modeling of ΔS will be crucial for future research.

Despite the formal simplicity of Eqs. (10), λ needs to depend on the price and other parameters. In this work we assume that both the cash and the shares are infinitesimally divisible, which is a standard assumption in theoretical finance. Under this assumption, two fundamental symmetries exist in a financial market.[4] (i) Our decisions and the market should be unaffected by a rescaling of the unit of money $M \to kM$ and $S \to kS$ for some $k > 0$ and therefore the dynamical equations should be invariant to such rescaling. This rescaling symmetry was used previously in [39,40] to derive the equation of motion of the wealth distribution but has not been applied to market modeling. (ii) Likewise, the financial market should be invariant to a redefinition of the unit of the shares $\Pi \to z\Pi$ and $S \to S/z$ for some $z > 0$. While Eqs. (10a) and

[2]For example, it does not seem reasonable to assume that richer people will invest a higher fraction of their wealth in the market in comparison to poorer people.

[3]One popular way to enforce total wealth conservation is to define the price impact function through a Walrasian auction, but the Walrasian auction is in itself a highly nonlinear mechanism and involves making unjustifiable the price-demand curve of each agent.

[4]On the other hand, we note that these two symmetries are broken if either the money or the share is not infinitesimally divisible, which is the case in reality. For example, in any monetary system, the smallest amount of usable cash is lower bounded. Also, in real financial markets, the smallest number of buyable shares is also lower bounded. Our assumption amounts to assuming that these two bounding effects are negligible. This assumption is also empirically justified. Taking the stock price of Apple as an example, the current price of Apple per stock is roughly \$200 per share, while the smallest unit of USD is \$0.01. Also, the daily average traded volume of Apple is roughly 10^8, orders of magnitudes larger than the minimum tradeable volume.

(10b) are invariant to such rescaling, Eq. (10c) is not for an arbitrary λ. Therefore, these two facts present additional functional constraints in the form of λ. One way to impose such a constraint is by defining

$$\frac{1}{\lambda} = \frac{S(t)}{\lambda_0 \Pi(t)}, \tag{11}$$

where $\Pi(t) := \frac{1}{N}\sum_i^N \Pi_i(t)$ is the average holding of the stock share at time t. Additionally, we are also also interested in the wealth $W_i(t) := M_i(t) + S(t)\Pi_i(t)$ of the individuals and its stationary distribution $p(W_i)$.

IV. MEAN-FIELD ANALYSIS

It is well known that coupled sets of differential equations are difficult to analyze and this is also a major challenge in the field of financial modeling; in some sense, the lack of analytical tractability is a major limitation of many of the financial models in the field [44]. The lack of an analytical solution limits the models' explanatory power. This problem also exists in the model we proposed in Eqs. (10). We thus limit our theoretical study to the following minimalistic mean-field limit. We will see that, even in this simple mean-field analysis, the model already exhibits rich behavior.

The agents defined in the preceding section take arbitrary strategies $b_i(t)$ and $p_i(t)$, which may result in arbitrarily complex interaction and price dynamics. The simplest choice for such strategies is a time-independent strategy. We also set b_i and p_i to be random variables (i.e., we take the zero-intelligence limit). We proceed further by taking the mean-field limit, where $b_i(t) = b$ and $p_i(t) = p$ for all i; we also require the agents to start from the same initial condition such that $M_i(0) = M_0$ and $\Pi_i(0) = \Pi_0$. Now we take the continuous-time limit, which is equivalent to assuming that the market depth λ is sufficiently large so that the price cannot change too drastically in a short period of time. For an arbitrary quantity X, $\Delta X \to \dot{X}dt$, $p \to pdt$, and $b \to bdt$; the set of equations becomes

$$\begin{aligned}\dot{M}(t) &= p\Pi(t)S(t) - bM(t),\\ \dot{\Pi}(t) &= -p\Pi(t) + \frac{bM(t)}{S(t)},\\ \dot{S}(t) &= \frac{1}{\lambda}\dot{\Pi}(t),\end{aligned} \tag{12}$$

where we have assumed the factor of N in the definition of λ and removed the dependence on the subscript i because the agents are identical in the mean-field limit.

In general, $\lambda = \lambda(S)$ can be a function of the price and the set of differential equations may be written as a single differential equation for S (see [35]),

$$\begin{aligned}\dot{S}(t) = {}& \frac{b\big[M_0 - \int_0^t dt\,\lambda \frac{dS}{dt} S(t)\big]}{\lambda S(t)}\\ & - \frac{p}{\lambda}\left(\Pi_0 + \int_0^t dt\,\lambda\frac{dS}{dt}\right) + \sigma\epsilon(t),\end{aligned} \tag{13}$$

where, to account for the random nature of the price, we add a noise term $\sigma\epsilon(t)$ to the right-hand side and $\epsilon(t) \sim \mathcal{N}(0, 1)$. In addition, σ is the volatility of the price and may also be a function of price. This stochastic term models some fundamental uncertainty in the determination of the price and may be derived by treating b and p as random variables in the zero-intelligence limit (or it may be a result of injection of external information into the market).

Theoretically, we focus on studying the theoretical properties of (13) with different choices of λ and σ, determined through symmetry constraints. In this section we also use the integrals of the deterministic equations as the definition of the other relevant variables; for example, $\Pi(t) := \Pi(0) + \int_0^t dt\,\dot{S}/\lambda$. The distribution for S can be solved by first writing out the Fokker-Planck equation and finding the stationary distribution. We then take the set of equations (12) as the definition for other quantities to find the distribution of wealth and return. We present the calculation in [35]. We show the solutions for the most representative examples in Table I. We consider four different choices. (i) In the case of $\lambda = \lambda_0$ and $\sigma = \sigma_0$, no symmetry exists in the system. We see that it results in an unrealistic distribution for the price and wealth. The exponent for the power-law part has a positive exponent, while the real-world distributions always have negative exponents. (ii) In the case of $\lambda = \lambda_0/S(t)$ and $\sigma = \sigma_0 S(t)$, the system is invariant to a rescaling of the unit of money (referred to as the $S \to kS$ symmetry) and no explicit power-law behavior emerges in this case. (iii) In the case of $\lambda = \lambda_0\Pi(t)$ and $\sigma = \sigma_0$, the system is symmetric to redefining the unit of stock shares; again, one does not see the emergence of universal scaling behavior. (iv) The case of $\lambda = \lambda_0\Pi(t)/S(t)$ and $\sigma = \sigma_0 S(t)$ is the simplest kind of model that is invariant to the redefinition of both the unit of money and the share. We see that, perhaps surprisingly, unitless scaling laws emerge for both the price and wealth distributions. More importantly, the exponents do not depend on the initial condition of the markets; this suggests the universality of these distributions. We discuss the meaning of the derived exponent in detail in the next section. Note that, when $\lambda = c_0 S^{c_1}$ (the fourth case), Eq. (14) simplifies to

$$\begin{aligned}\dot{S}(t) = {}& \frac{b\big[2M_0 + \lambda' S_0^2/(c_1+2)\big]}{2\lambda S(t)} - \left(\frac{b}{c_1+2} + \frac{p}{c_1+1}\right)S(t)\\ & - \frac{p}{\lambda}\left(\Pi_0 - \frac{\lambda' S_0}{c_1+1}\right) + \sigma\epsilon(t),\end{aligned} \tag{14}$$

where we have defined a constant $\lambda' := c_0 S_0^{c_1}$.

It is worth exploring the fourth model with both symmetries deeper. Besides the price and wealth distributions, stylized facts are also known to exist for the return $R_t := \ln(S_t/S_{t-1}) \approx \delta t \frac{d}{dt}\ln(S)$ and the traded volume $V_t := \Delta\Pi_t \approx \delta t\,\dot{\Pi}(t)$. See Table II for a summary of these quantities in our theory. We see that the price, wealth, and volume obey heavy-tailed distributions with unitless exponents. The mean-field theory not only predicts the formula for each of the quantities, but it also predicts two relations between them, which we list in the fourth column. This relation is reminiscent of the scaling relations in critical phenomena. Moreover, our theory predicts that the Pareto exponent of wealth β is always smaller than the exponents of volume and price.

TABLE I. Distributions of price and wealth when (14) obeys different kinds of symmetry. When both rescaling symmetries are satisfied, one obtains meaningful predictions for the price and wealth; universal power-law scaling that is initial-condition independent only emerges when both symmetries are modeled in the dynamical equation. Here b is the average tendency of buying, p is the average tendency of selling, and λ_0 is the susceptibility of price to an excessive demand.

λ, σ	Symmetry	Price distribution	Wealth distribution ($W \gg 1$)
$\lambda = \lambda_0, \sigma = \sigma_0$		$S^{b(2M_0+\lambda S_0^2)/\lambda\sigma^2} e^{f_1(S)}$	$W^{b(2M_0+\lambda S_0^2)/\lambda\sigma^2 - 1/2}$
$\lambda = \frac{\lambda_0}{S}, \sigma = \sigma_0 S$	$S \to kS$	not power law	not power law
$\lambda = \lambda_0 \Pi, \sigma = \sigma_0$	$\Pi \to k\Pi$	not power law	not power law
$\lambda = \frac{\lambda_0 \Pi}{S}, \sigma = \sigma_0 S$	$S \to kS, \Pi \to k\Pi$	$S^{-1-(2/\sigma^2)[b/(\lambda_0+1)+p/\lambda_0]} e^{f_4(S)}$	$W^{-1-[2/\sigma^2(\lambda_0+1)][b/(\lambda_0+1)+p/\lambda_0]}$

V. IMPLICATIONS

While we explicitly referred to each agent as a person, one agent may also be interpreted as a collection of people who share the propensity for investment, e.g., an institution, a fund, or even the economy as a whole (when the mean-field limit is taken). Of particular interest here is when the agents are interpreted as representative of the whole economy. In this interpretation, b_i and p_i become the economy's average tendency to buy and sell. This interpretation is especially appropriate for the mean-field model in Eq. (14) because all the agents are assumed to be identical. In this light, we discuss the implications of the proposed model.

We first link the model's parameters to real-world measurable quantities. Four unitless quantities σ_0, λ_0, b, and p exist in our theory and the scaling exponents predicted by our theory are dependent on them. The market depth λ_0 may be measured by measuring β and γ:

$$\lambda_0 = \frac{\gamma}{\gamma - \beta}. \tag{15}$$

Since the market depth must be positive, we have that $\gamma \geqslant \beta$; this means that the tail of he trading volume cannot be heavier than the tail of the wealth. This condition agrees with the intuition that the investment one makes cannot be larger than the amount of available wealth. For an economy, this is also true. At a macroscopic level, one expects γ to be very close to its lower limit, i.e., $\gamma \approx \beta$ due to borrowing and leveraging money from the bank for investment. This agrees with the measured value in real economies where $\beta \approx 1.36$ and $\gamma \approx 1.40$ [50,51]. Plugging in these values, we can estimate the value $\lambda_0 \approx 35 \sim 10^1$, or $1/\lambda_0 = 0.029$.

TABLE II. Pareto exponents of price, wealth, and traded stock volume and the relation between different exponents. We note that the model predicts the power-law exponents to be dependent on one another, which is reminiscent of the scaling relations in critical phenomena. Here b is the average tendency of buying, p is the average tendency of selling, and λ_0 is the susceptibility of price to an excessive demand.

Observable	Pareto exponent	Predicted formula	Scaling relation
price $S(t)$	α	$\alpha = \frac{2}{\sigma^2}(\frac{b}{\lambda_0+1} + \frac{p}{\lambda_0})$	α
wealth $W(t)$	β	$\beta = \frac{2}{\sigma_0^2(\lambda_0+1)}(\frac{b}{\lambda_0+1} + \frac{p}{\lambda_0})$	$\frac{\alpha\gamma}{\alpha+\gamma}$
volume $V(t)$	γ	$\gamma = \frac{2}{\sigma^2\lambda_0}(\frac{b}{\lambda_0+1} + \frac{p}{\lambda_0})$	$\frac{\alpha}{\alpha-\beta}$

The parameters b and p can also be related to measurable quantities. We define the total market capitalization $\mu := N\Pi S$ and the total wealth of the society $W_{\rm tot} := N(M + \Pi S)$. Close to a stationary state, the injection of money into the economy must be equal to that leaving the market. We thus have

$$bNM = pN\Pi S \to b = \frac{\mu}{W_{\rm tot} - \mu} p := \kappa p, \tag{16}$$

which relates b to p through the measurable quantities μ and $W_{\rm tot}$, and the quantity κ may be called the market activity index: The higher the κ, the more active the market. Data show that [52], across different countries, κ is a peaked distribution, centered around 0.25. It is now useful to consider the meaning of the parameter p. Here p is defined as the amount of stock sold through a unit time Δt; therefore, p is directly linked to the stock's liquidity. When Δt is taken to be a day, p is the stock's daily turnover rate (or the daily equivalent of the annual average turnover rate), a well-measured quantity. We denote the daily turnover rate by p^* from now on. The annual turnover rate for most countries is centered around 1.30, which translates to a daily rate of 0.0036. This means that one may rewrite the factor as $b/(\lambda_0 + 1) + p/\lambda_0 = [\kappa/(\lambda_0 + 1) + 1/\lambda_0]p^*$, which is approximately equal to 0.0080 on average in the world. One may obtain a different estimate of the market depth value from this analysis to check the consistency of the theory. Here

$$\beta = \frac{2}{\sigma^2(\lambda_0 + 1)}\left(\frac{\kappa}{\lambda_0 + 1} + \frac{1}{\lambda_0}\right)p^*,$$

which implies

$$\lambda_0 \approx \sqrt{\frac{2(1+\kappa)p^*}{\beta\sigma^2}} \approx 12 \sim 10^1,$$

where we have approximated σ_0 by the daily stock price volatility, which is of order 0.01 for the U.S. stocks; this estimation of λ_0 agrees in the order of magnitude with the independent estimate of the market depth in the preceding paragraph. Therefore, every parameter of our proposed model can be measured in a real market and economies in principle and more importantly, the theory gives consistent predictions regarding the market depth and the empirical power-law exponents.

Now it is interesting to compare with the result on the distribution of wealth in [39,40]. In [39] the Pareto exponent takes the form a_0/σ^2, where a_0 is proportional to the rate of monetary exchange in the economy and σ^2 is the degree of randomness in the wealth acquisition process. In comparison, this work predicts an exponent of $\frac{b+2p}{\sigma_0^2(\lambda_0+1)}$. One can naturally see that the σ^2 terms are analogous and the term $\frac{b+2p}{\lambda_0+1}$ translates directly to the term a_0 in the Cont-Bouchaud model; as discussed, the term $\frac{b+2p}{\lambda_0+1}$ is proportional to the market activity weighted by the sensitivity of the market to external stimulus, which may be called the rate of monetary exchange. In this sense, this work gives the parameters in the Cont-Bouchaud model a precise meaning. Reference [40] avoids interpreting the a_0/σ^2 term directly, but links this term to the average number of members in a household L and argues that the Pareto exponent is equal to $\alpha = \frac{L}{L-1}$. Further, the Pareto index is a monotonically decreasing function of L; therefore, the larger the average number of family members is, the more inequality exists in society. In our theory, we showed that $\alpha \sim \kappa p$, which is a measure of market activity. One interpretation is that the market activity is inversely proportional to the size of a household, which corroborates the long observed fact that child birth rate is inversely correlated with the average income in society [53].

One might also discuss the effect of economic growth or inflation on society within the framework. For example, we can model the effect of economic growth or inflation by adding a term $f(S)$ to the right-hand side of Eq. (14). We consider and compare two kinds of growth. One is constant $f(S) = k_0$, which only shifts the equilibrium value of the price and does not affect the distribution's power-law exponents. Alternatively, we might consider a growth proportional to price $f(S) = k_0 S$, which directly affects the exponents and changes the term $\frac{b}{\lambda_0+1} + \frac{p}{\lambda_0}$ to $\frac{b}{\lambda_0+1} + \frac{p}{\lambda_0} - k_0$. Note that, in economic growth (or inflation), $k_0 > 0$ and thus growth increases inequality by lowering the Pareto exponent; on the other hand, economic decay reduces inequality. This implication agrees with the intuition that a growing market tends to create extremely rich people either by chance or through their better investment skills and rich people may become richer through some self-reinforcing mechanism, for example, by becoming more influential in society.

VI. CONCLUSION

In this work we have argued from basic principles what the simplest form a financial market model should take: It needs to connect the price dynamics to the wealth dynamics. The dynamics we derived is based on fundamental symmetry constraints; when analyzed in a mean-field regime, the dynamics leads to interesting universal behaviors that mimic the real market. One insight that this work provides is that the universality of the empirically observed power laws in the real market might emerge due to the relevant symmetries in the market. Furthermore, as in the standard physical critical phenomena, the scaling exponents are dependent on one another; this corroborates the success of dimensional analysis in understanding economic and financial systems [54].

However, the simple model and our analysis of it are minimal. For example, the analysis gives no sensible prediction about the statistics related to the return, nor does it answer what heterogeneous agents and their interaction may provide to the financial system. Nevertheless, this work may pave the way for further understanding of the cause and the nature of the commonly observed stylized facts in economics and it may also serve as a baseline model for modeling financial processes in the industry. Possible future works include finding more connections between more power-law indices and possibly predicting the existence of yet unnoticed power laws. One might also investigate the influence of various economic policies on the financial system. The price dynamics may also be investigated under the assumption of the existence of a fundamental price. As mentioned in this work, the thermodynamics of the proposed microscopic model should also be interesting to investigate, for example, how to define entropy, whether fundamental relations exist such as the second law of thermodynamics, how to define work and heat, and how work and heat interact in financial systems? These are beyond the scope of this work but will be important problems to explore in the future.

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