

Irreversible efficiency and Carnot theorem for heat engines operating with multiple heat baths in linear response regime

Yuki Izumida ^{*}

Department of Complexity Science and Engineering, Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa 277-8561, Japan



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The Carnot theorem, one expression of the second law of thermodynamics, places a fundamental upper bound on the efficiency of heat engines operating between two heat baths. The Carnot theorem can be stated in a more generalized form for heat engines operating with multiple heat baths, where the maximum efficiency is achieved for reversible heat engines operating quasistatically between two heat baths. In this study, we determine the irreversible efficiency of heat engines operating with multiple heat baths in a linear response regime, i.e., under small temperature differences and a slow variation of the control parameters, by quantifying the impact of the dissipation by irreversible operations. The Carnot theorem is derived as a natural consequence of it. Because the result obtained is based on the linear response relation and fluctuation-dissipation theorem in the universal framework of linear response theory, it has wide applicability to irreversible heat engines operating in the linear response regime.

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About two centuries ago, Carnot in his celebrated study [1] revealed the existence of the fundamental upper bound of the efficiency of heat engines determined by the absolute temperatures of two heat baths, which led to the discovery of entropy by Clausius and provided the foundation for the formulation of thermodynamics. For heat engines operating with two heat baths, the Carnot theorem is formulated as [2]

$$\eta = \frac{W}{Q_H} \leq 1 - \frac{T_C}{T_H} \equiv \eta_C \left(\frac{T_C}{T_H} \right), \quad (1)$$

where T_H and T_C ($T_C < T_H$) are the absolute temperatures of the hot and cold heat baths, respectively, Q_H is the heat input from the hot heat bath, W is the work output, and η is the efficiency of the heat energy conversion defined as the ratio of the work output to heat input. The maximum value η_C is the Carnot efficiency, which is expressed as a function of the ratio of the absolute temperatures of the heat baths and is achieved for reversible heat engines such as a Carnot cycle.

A heat engine operating between two heat baths is a special cycle. In general, we may consider general heat engine cycles operating with multiple heat baths; we may even consider heat engines with an infinite number of heat baths labeled with continuously changeable temperatures along the cycle. The widely known Carnot theorem, Eq. (1), is then generalized

as [3,4]

$$\eta = \frac{W}{Q_{\text{in}}} \leq 1 - \frac{T_{\min}}{T_{\max}} = \eta_C \left(\frac{T_{\min}}{T_{\max}} \right), \quad (2)$$

where Q_{in} represents the total heat input from the heat baths, and $T_{\min}$ and $T_{\max}$ represent the minimum and maximum values of the temperatures of the heat baths along the cycle, respectively. The equality of Eq. (2) is achieved for the reversible heat engines operating between two heat baths, in which case $T_{\min} = T_C$ and $T_{\max} = T_H$ hold [4]. It should be noted that the reversibility condition alone does not yield the equality, and the condition of the two heat baths is necessary. This highlights the importance of the role of reversible heat engines operating under the isothermal condition.

Power, which is defined as the work per unit time, also characterizes practical heat engines [2]. It is defined by

$$P \equiv \frac{W}{\mathcal{T}}, \quad (3)$$

where $\mathcal{T}$ is the time required to complete one cycle. Because the reversible heat engines operating quasistatically ($\mathcal{T} \rightarrow \infty$) produce vanishing power, they are of no practical use. It is to be determined to what extent the efficiency of the irreversible heat engines operating in finite cycle time is reduced. The Carnot theorem formulated in terms of the inequalities such as those presented in Eqs. (1) and (2) does not provide any answers. By quantifying the effect of the irreversible dissipation intrinsic in the finite-time operation of the heat engines based on nonequilibrium theories or specific models, various efficiency formulas for heat engines operating at maximum power [5–25], tradeoff relations between efficiency and power [26–34], and tighter bounds for efficiency than that provided by Eq. (1) [35–42] have been obtained. So far, however, most of the studies have focused on the heat engines with two heat

^{*}izumida@k.u-tokyo.ac.jp

baths [i.e., the case represented in Eq. (1)] or used different definitions of efficiency [26,27,35–41], and less is known about the irreversible efficiency of the general heat engines to which the Carnot theorem represented in Eq. (2) can be applied.

In this study, we formulate the irreversible efficiency of general heat engines operating with multiple heat baths in a linear response regime with explicit inclusion of the effect of dissipation. The Carnot theorem presented in Eq. (2) as inequality is naturally derived from it. The linear response regime refers to a regime with small temperature differences (i.e., small $\Delta T \equiv T_{\max} - T_{\min}$) and a slow variation of the control parameters operating the heat engines. The linear response theory is a universal framework for describing systems in nonequilibrium states in the vicinity of the equilibrium states [43,44]. It is established on the linear response relations, which connect the external forces applied to a system and their conjugate displacements in terms of response functions. The response function is related to the equilibrium time-correlation function by the fluctuation-dissipation theorem [43,44]. The result in our study is derived based only on the linear response theory applied to the heat engine cycles. Because we do not assume any specific dynamics, working substance, and heat engine cycle, the formula derived has wide applicability to heat engines operating in the linear response regime.

Let Γ be the phase-space variables that specify the state of a working substance (system) of a heat engine. The heat engine is operated by a time-dependent parameter $\lambda(t)$ with a period $\mathcal{T}$, which is included in the Hamiltonian $H(\Gamma; \lambda(t))$ of the system. The heat engine is in contact with the heat baths labeled with the absolute temperature $T(t)$, which changes periodically in time between $T_{\min}$ and $T_{\max}$ with the period $\mathcal{T}$. Let $\mathcal{P}(\Gamma; \Lambda(t))$ be a probability distribution function of Γ at time t , where $\Lambda(t) \equiv (\lambda(t), T(t))$ denote the control parameters.

The heat engine undergoes a periodic change along the cycle. For a sufficiently slow driving such that the quasistatic limit becomes a good approximation, at each instant of time along the cycle, the probability distribution is approximated by the canonical distribution described below:

$$\mathcal{P}(\Gamma; \Lambda(t)) \simeq \mathcal{P}_{\text{eq}}(\Gamma; \Lambda(t)) \equiv e^{-\frac{H(\Gamma; \lambda(t)) - F(\Lambda(t))}{k_B T(t)}}, \quad (4)$$

where $F(\Lambda(t)) \equiv -k_B T(t) \ln \int d\Gamma e^{-H(\Gamma; \lambda(t))/k_B T(t)}$ and k_B are the Helmholtz free energy and Boltzmann constant, respectively. In this description, the time t merely denotes the label that differentiates the different equilibrium states.

For the energetics of the heat engine, we define the average energy of the system as

$$E(t) \equiv \langle H(\Gamma; \lambda(t)) \rangle_{\Lambda(t)} = \int d\Gamma H(\Gamma; \lambda(t)) \mathcal{P}(\Gamma; \Lambda(t)), \quad (5)$$

where $\langle \cdot \rangle_{\Lambda(t)}$ denotes an average with respect to $\mathcal{P}(\Gamma; \Lambda(t))$. By differentiating $E(t)$ with respect to t as $\dot{E} = \frac{d}{dt} \int d\Gamma H(\Gamma; \lambda(t)) \mathcal{P}(\Gamma; \Lambda(t))$, where the overdot denotes the time derivative, we can decompose $E(t)$ into the sum of work and heat fluxes $j_w(t)$ and $j_q(t)$, described as

follows:

$$\begin{aligned} \dot{E} &= \int d\Gamma \frac{\partial H}{\partial t} \mathcal{P}(\Gamma; \Lambda(t)) + \int d\Gamma H(\Gamma; \lambda(t)) \frac{\partial \mathcal{P}(\Gamma; \lambda(t))}{\partial t} \\ &= \left\langle \frac{\partial H(\Gamma; \lambda(t))}{\partial \lambda} \right\rangle_{\Lambda(t)} \dot{\lambda} + \int d\Gamma H(\Gamma; \lambda(t)) \frac{\partial \mathcal{P}(\Gamma; \Lambda(t))}{\partial \Lambda} \cdot \dot{\Lambda} \\ &\equiv -j_w(t) + j_q(t). \end{aligned} \quad (6)$$

The work flux $j_w(t)$ can be written in terms of thermodynamic variables as

$$j_w(t) = \langle p \rangle_{\Lambda(t)} \dot{\lambda}, \quad (7)$$

where we have defined the generalized pressure $p \equiv -\partial H / \partial \lambda$. We can also represent $j_q(t)$ in terms of thermodynamic variables. We introduce the Shannon entropy of the system:

$$\langle s \rangle_{\Lambda(t)} = -k_B \int d\Gamma \mathcal{P}(\Gamma; \Lambda(t)) \ln \mathcal{P}(\Gamma; \Lambda(t)), \quad (8)$$

where $s \equiv -k_B \ln \mathcal{P}(\Gamma; \Lambda(t))$ is the stochastic entropy [45,46]. For a sufficiently slow driving where Eq. (4) holds, we can represent the heat flux $j_q(t)$ in terms of the equilibrium entropy $S \equiv \langle s \rangle_{\text{eq}, \Lambda(t)}$ and the temperature $T(t)$ as

$$j_q(t) \simeq T(t) \dot{S}, \quad (9)$$

where $\langle \cdot \rangle_{\text{eq}, \Lambda(t)}$ is the average with respect to $\mathcal{P}_{\text{eq}}(\Gamma; \Lambda(t))$. It should be noted that the expression of $j_q(t)$ in terms of S is valid only for the near-equilibrium condition in contrast to Eq. (7). The expressions (7) and (9) as the product of the conjugate thermodynamic variables are convenient for the application of the linear response theory to the heat engine cycle below.

For further evaluation of $j_w(t)$ and $j_q(t)$ toward the formulation of the efficiency of the heat engine cycle, we recall the linear response theory. Let $\Lambda = (\lambda, T)$ be the control parameters of any point on the cycle as a reference state whose small variation can be considered as a small perturbation to the system. The entire cycle is constructed as an accumulation of such small variations.

First, we consider the static response of thermodynamic variables when a perturbation is added quasistatically. Upon the introduction of a small variation in the control parameters $\Lambda \rightarrow \Lambda + \delta\Lambda$, where $\delta\Lambda \equiv (\delta\lambda, \delta T)$ serve as the external forces, the canonical distribution Eq. (4) changes as [43]

$$\mathcal{P}_{\text{eq}}(\Gamma; \Lambda + \delta\Lambda) \simeq \mathcal{P}_{\text{eq}}(\Gamma; \Lambda) \left(1 + \frac{1}{k_B T} \Delta\mathbf{A} \cdot \delta\Lambda \right). \quad (10)$$

Here, $\mathbf{A}$ is the conjugate displacement to $\delta\Lambda$ represented as

$$\mathbf{A} = (A_\lambda, A_T) \equiv (p, s) = \left(-\frac{\partial H}{\partial \lambda}, -k_B \ln \mathcal{P} \right), \quad (11)$$

where $\Delta\mathbf{A} \equiv \mathbf{A} - \langle \mathbf{A} \rangle_{\text{eq}, \Lambda}$, and we used $\Delta s \simeq \Delta H / T$ for the derivation of Eq. (10), which is shown from Eqs. (4), (5), and (8). It should be noted that $\langle \Delta\mathbf{A} \rangle_{\text{eq}, \Lambda} = \mathbf{0}$ from the definition. By using Eqs. (10) and (11), we can derive $(\mu, \nu = \lambda, T)$

$$\langle A_\mu \rangle_{\text{eq}, \Lambda + \delta\Lambda} = \langle A_\mu \rangle_{\text{eq}, \Lambda} + \chi_{\mu\nu} \delta\Lambda_\nu, \quad (12)$$

where $\langle \cdot \rangle_{\text{eq}, \Lambda + \delta \Lambda}$ is the average with respect to $\mathcal{P}_{\text{eq}}(\Gamma; \Lambda + \delta \Lambda)$, and

$$\chi_{\mu\nu} \equiv \frac{\partial \langle A_\mu \rangle_{\text{eq}, \Lambda}}{\partial \Lambda_\nu} \quad (13)$$

denotes the static response coefficient. The symmetry of the static response coefficients can be shown from the Maxwell relation:

$$\chi_{\mu\nu} = \frac{\partial \langle A_\mu \rangle_{\text{eq}, \Lambda}}{\partial \Lambda_\nu} = -\frac{\partial^2 F}{\partial \Lambda_\mu \partial \Lambda_\nu} = \frac{\partial \langle A_\nu \rangle_{\text{eq}, \Lambda}}{\partial \Lambda_\mu} = \chi_{\nu\mu}, \quad (14)$$

where we used $\langle A_\mu \rangle_{\text{eq}, \Lambda} = -\partial F / \partial \Lambda_\mu$.

In general, a perturbation is added dynamically in a time-dependent manner. We generalize the above static response of Λ so that it is included as a special case. Let us consider that the system in the equilibrium state with $\Lambda(t - \delta t) = (\lambda(t - \delta t), T(t - \delta t))$ at time $t - \delta t$ is perturbed by the time-dependent variation $\delta \Lambda(t') \equiv \Lambda(t') - \Lambda(t - \delta t)$ until $t' = t$, where $\delta \Lambda(t - \delta t) = \mathbf{0}$. Here, the time increment δt is chosen such that it is sufficiently small in a macroscopic timescale while it is sufficiently large compared to a microscopic timescale, such as the correlation time of the system. Following the standard linear response theory [43], we can describe the linear response of the thermodynamic variables A of the system upon the addition of this perturbation as

$$\begin{aligned} \langle \Delta A_\mu \rangle_{\Lambda(t)} &= \langle A_\mu \rangle_{\Lambda(t)} - \langle A_\mu \rangle_{\text{eq}, \Lambda(t - \delta t)} \\ &= \int_{t - \delta t}^t \Phi_{\mu\nu}(t - t') \delta \Lambda_\nu(t') dt', \end{aligned} \quad (15)$$

where $\Phi_{\mu\nu}$ denotes the response function satisfying the causality $\Phi_{\mu\nu}(t - t') = 0$ for $t < t'$. The response function is related to the equilibrium time-correlation function by the fluctuation-dissipation theorem:

$$\Phi_{\mu\nu}(\tau) = -\frac{\theta(\tau)}{k_B T} \frac{d}{d\tau} C_{\mu\nu}(\tau), \quad (16)$$

where $\theta(\tau)$ is the Heaviside step function and $C_{\mu\nu}(\tau)$ is the equilibrium time-correlation function between ΔA_μ and ΔA_ν defined by

$$C_{\mu\nu, \Lambda(t - \delta t)}(\tau) \equiv \langle \Delta A_\mu(\tau) \Delta A_\nu(0) \rangle_0. \quad (17)$$

Here, $\langle \cdot \rangle_0$ denotes the average with respect to the unperturbed equilibrium dynamics with Λ being held fixed as $\Lambda(t - \delta t)$. In particular, $C_{\mu\nu, \Lambda(t - \delta t)}(0) = \langle \Delta A_\mu(0) \Delta A_\nu(0) \rangle_0 = \langle \Delta A_\mu \Delta A_\nu \rangle_{\text{eq}, \Lambda(t - \delta t)}$ is the equilibrium correlation function. By changing the variable from t' to $\tau = t - t'$ in Eq. (15), we have $\int_{t - \delta t}^t \Phi_{\mu\nu}(t - t') \delta \Lambda_\nu(t') dt' = \int_0^{\delta t} \Phi_{\mu\nu}(\tau) \delta \Lambda_\nu(t - \tau) d\tau$. The range of τ that contributes to the integral is less than the correlation time that characterizes $C_{\mu\nu, \Lambda(t - \delta t)}(\tau)$ [43], which is sufficiently smaller than δt . We can thus replace the upper limit of the integral with infinity as $\int_0^{\delta t} \Phi_{\mu\nu}(\tau) \delta \Lambda_\nu(t - \tau) d\tau \simeq \int_0^\infty \Phi_{\mu\nu}(\tau) \delta \Lambda_\nu(t - \tau) d\tau$. Moreover, because we are considering a sufficiently slow change of the control parameters, the change in $\delta \Lambda_\nu$ during the correlation time is sufficiently small, and we can expand $\Lambda_\nu(t - \tau)$ in terms of τ as $\delta \Lambda_\nu(t - \tau) = \delta \Lambda_\nu(t) - \dot{\Lambda}_\nu(t) \tau + O(\tau^2)$, where we used $\delta \dot{\Lambda}_\nu(t') = \dot{\Lambda}_\nu(t')$. By using this approximation of $\delta \Lambda_\nu(t - \tau)$

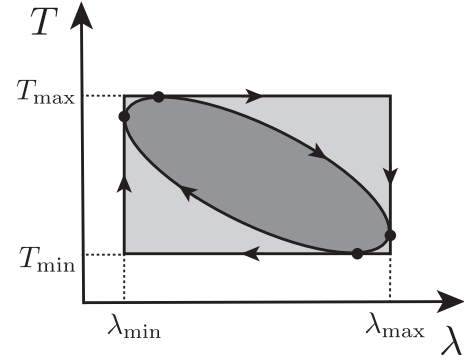


FIG. 1. Schematic illustration of the heat engine cycle in the space of the control parameters $\Lambda = (\lambda, T)$. The outer rectangular-shaped cycle represents an idealized cycle, which attains the upper bound in Eq. (27) (see the main text).

and Eq. (16), we obtain

$$\begin{aligned} \int_0^\infty \Phi_{\mu\nu}(\tau) \delta \Lambda_\nu(t - \tau) d\tau \\ \simeq \chi_{\mu\nu, \Lambda(t - \delta t)} \delta \Lambda_\nu(t) - R_{\mu\nu, \Lambda(t - \delta t)} \dot{\Lambda}_\nu(t), \end{aligned} \quad (18)$$

where we used partial integration with respect to τ and $C_{\mu\nu, \Lambda(t - \delta t)}(\infty) = 0$, and we assumed $\lim_{\tau \rightarrow \infty} C_{\mu\nu, \Lambda(t - \delta t)}(\tau) \tau = 0$. Here, the static response coefficient $\chi_{\mu\nu}$ in Eq. (13) is expressed in terms of the correlation function (Kirkwood relation):

$$\chi_{\mu\nu, \Lambda(t - \delta t)} = \frac{1}{k_B T} C_{\mu\nu, \Lambda(t - \delta t)}(0), \quad (19)$$

where it is a positive-semidefinite symmetric matrix, as evidenced from their construction using the covariance matrix. We also defined the generalized friction coefficient [46]

$$R_{\mu\nu, \Lambda(t - \delta t)} \equiv \frac{1}{k_B T} \int_0^\infty C_{\mu\nu, \Lambda(t - \delta t)}(\tau) d\tau, \quad (20)$$

which is a positive-semidefinite matrix, reflecting the stability of equilibrium dynamics. The symmetry is also derived from the time-reversal and time-translation invariance of the equilibrium dynamics for time-reversible symmetric observables at equilibrium, such as the thermodynamic variables under consideration [46]. From Eqs. (15) and (18), we have

$$\begin{aligned} \langle A_\mu \rangle_{\Lambda(t)} &\simeq \langle A_\mu \rangle_{\text{eq}, \Lambda(t - \delta t)} + \chi_{\mu\nu, \Lambda(t - \delta t)} \delta \Lambda_\nu(t) \\ &\quad - R_{\mu\nu, \Lambda(t - \delta t)} \dot{\Lambda}_\nu(t) \\ &= \langle A_\mu \rangle_{\text{eq}, \Lambda(t)} - R_{\mu\nu, \Lambda(t - \delta t)} \dot{\Lambda}_\nu(t) \\ &\simeq \langle A_\mu \rangle_{\text{eq}, \Lambda(t)} - R_{\mu\nu, \Lambda(t)} \dot{\Lambda}_\nu(t), \end{aligned} \quad (21)$$

where we used Eq. (12) in the second equality and approximated as $R_{\mu\nu, \Lambda(t - \delta t)} \simeq R_{\mu\nu, \Lambda(t)}$ in the third equality. By using Eq. (21), we can construct the heat engine cycle (Fig. 1) and formulate its efficiency. We assume the following specific form as $T(t)$:

$$T(t) \equiv T_{\min} + \gamma_q(t) \Delta T, \quad (22)$$

where $\gamma_q(t)$ is an arbitrary periodic function satisfying $0 \leq \gamma_q(t) \leq 1$ and $\Delta T = T_{\max} - T_{\min}$. Meanwhile, we assume that

the protocol $\lambda(t)$ satisfies the following condition:

$$\lambda(0) = \lambda(\mathcal{T}) = \lambda_{\min}, \quad \lambda(\mathcal{T}_{\text{in}}) = \lambda_{\max}, \quad (23)$$

where the working substance absorbs heat during $0 \leq t \leq \mathcal{T}_{\text{in}}$ and rejects heat during $\mathcal{T}_{\text{in}} < t \leq \mathcal{T}$. It should be noted that $T(0) \neq T_{\min}$ [$\gamma_q(0) \neq 0$] and $T(\mathcal{T}_{\text{in}}) \neq T_{\max}$ [$\gamma_q(\mathcal{T}_{\text{in}}) \neq 1$] in general.

The efficiency of the cycle is obtained as

$$\eta = \frac{W}{Q_{\text{in}}} = \frac{\int_0^{\mathcal{T}} j_w(t) dt}{\int_0^{\mathcal{T}_{\text{in}}} j_q(t) dt}. \quad (24)$$

We wish to evaluate this quantity up to the lowest order of the small quantities. By using Eqs. (7), (21), and (22), we can approximate W in Eq. (24) as

$$W \simeq \int_0^{\mathcal{T}} \chi_{\lambda T, (\lambda(t), T_{\min})} \gamma_q(t) \dot{\lambda} dt \Delta T - \int_0^{\mathcal{T}} R_{\lambda \lambda, (\lambda(t), T_{\min})} \dot{\lambda}^2 dt. \quad (25)$$

Here, we used the approximation $\langle p \rangle_{\text{eq}, \Lambda(t)} \simeq \langle p \rangle_{\text{eq}, (\lambda(t), T_{\min})} + \chi_{\lambda T, (\lambda(t), T_{\min})} \gamma_q(t) \Delta T$, where $\oint \langle p \rangle_{\text{eq}, (\lambda, T_{\min})} d\lambda = -\oint \partial F(\lambda, T_{\min}) / \partial \lambda d\lambda = 0$, and we neglected the higher-order term of $O(\dot{\lambda} \gamma_q \Delta T, \dot{\lambda}^2 \Delta T)$. Because W in the numerator is constituted by the small quantities, it is sufficient to evaluate Q_{in} in the denominator up to $O(1)$:

$$Q_{\text{in}} \simeq T_{\min} [S(\lambda_{\max}, T_{\min}) - S(\lambda_{\min}, T_{\min})], \quad (26)$$

where we used Eqs. (9) and (22). Using Eqs. (24)–(26) and the Maxwell relation $\chi_{\lambda T, (\lambda(t), T_{\min})} = \chi_{T \lambda, (\lambda(t), T_{\min})} = \partial S / \partial \lambda|_{\Lambda=(\lambda(t), T_{\min})}$ and reparametrizing the time-dependent quantities as $\lambda(t) = \tilde{\lambda}(\theta)$, $\gamma_q(t) = \tilde{\gamma}_q(\theta)$, $S(\Lambda(t)) = \tilde{S}(\tilde{\Lambda}(\theta))$, and $R_{\lambda \lambda, \Lambda(t)} = \tilde{R}_{\lambda \lambda, \tilde{\Lambda}(\theta)}$ in terms of $\theta \equiv t/\mathcal{T}$ ($0 \leq \theta \leq 1$), we finally obtain

$$\eta = \eta_{\text{rev}} - \frac{\int_0^1 \tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2 d\theta}{T_{\min} [S(\lambda_{\max}, T_{\min}) - S(\lambda_{\min}, T_{\min})] \mathcal{T}}. \quad (27)$$

Here, the first term

$$\eta_{\text{rev}} \equiv \frac{\int_0^1 \frac{\partial \tilde{S}(\tilde{\lambda}(\theta), T_{\min})}{\partial \theta} \tilde{\gamma}_q(\theta) d\theta}{S(\lambda_{\max}, T_{\min}) - S(\lambda_{\min}, T_{\min})} \eta_C \left(\frac{T_{\min}}{T_{\max}} \right) \quad (28)$$

denotes the reversible efficiency with $\eta_C(T_{\min}/T_{\max}) \simeq \Delta T/T_{\min}$, and the nonpositive second term denotes the dissipation effect due to the finite-time operation of heat engines. η_{rev} is achieved for the quasistatic limit $\mathcal{T} \rightarrow \infty$. As $0 \leq \tilde{\gamma}_q \leq 1$ and $\partial \tilde{S} / \partial \theta \geq 0$ for $0 \leq \theta \leq \theta_{\text{in}} \equiv \mathcal{T}_{\text{in}}/\mathcal{T}$ and $\partial \tilde{S} / \partial \theta \leq 0$ for $\theta_{\text{in}} < \theta \leq 1$, we have $\eta_{\text{rev}} \leq \eta_C(T_{\min}/T_{\max})$. The equality is achieved for the heat engine cycles, in which the heat input upon the variation of λ is operated under the isothermal condition. In the space of the control parameters $\Lambda = (\lambda, T)$, such an idealized cycle is given, e.g., as a rectangular-shaped cycle constituted by the isothermal processes with $\lambda_{\min} \rightarrow \lambda_{\max}$ at $T = T_{\max}$ ($\gamma_q = 1$) and $\lambda_{\max} \rightarrow \lambda_{\min}$ at $T = T_{\min}$ ($\gamma_q = 0$), connected by the isochoric processes with $T_{\max} \rightarrow T_{\min}$ at $\lambda = \lambda_{\max}$ and $T_{\min} \rightarrow T_{\max}$ at $\lambda = \lambda_{\min}$ (Fig. 1) [47]. Equation (27) together with Eq. (28) constitute our main result.

A few remarks are in order with respect to the main result, Eqs. (27) and (28). First, the Carnot theorem for heat engines operating with multiple heat baths in Eq. (2) is derived from Eq. (27): As the second term of the right-hand side

of Eq. (27) is nonpositive, we obtain $\eta \leq \eta_{\text{rev}}$. Moreover, as $\eta_{\text{rev}} \leq \eta_C(T_{\min}/T_{\max})$, as stated above, Eq. (2) can be derived.

Furthermore, by choosing the optimal protocol for the parameter $\lambda(t)$ for a fixed cycle time, we can obtain another bound for η as

$$\eta \leq \eta_{\text{rev}} - \frac{\mathcal{L}^2}{T_{\min} [S(\lambda_{\max}) - S(\lambda_{\min})] \mathcal{T}}, \quad (29)$$

where $\mathcal{L}$ is the minimum value of $[\int_0^1 \tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2 d\theta]^{1/2}$ and is called the thermodynamic length [46,48]:

$$\begin{aligned} \mathcal{L} &\equiv \left| \int_0^1 \sqrt{\tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2} d\theta \right| \\ &\leq \sqrt{\int_0^1 \tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2 d\theta}. \end{aligned} \quad (30)$$

Here, we used the Cauchy-Schwartz inequality, and the equality is achieved for the optimal protocol, which makes the integrand a constant without any dependence on θ . The thermodynamic length is used to describe the minimum dissipation during near-equilibrium transitions [48–58] and is invariant under reparametrization. Equation (27) is tighter than $\eta \leq \eta_{\text{rev}}$ as it includes the extent of dissipation and cycle time explicitly. Similar forms of Eq. (29) have been obtained recently for adiabatically driven heat engines using different definitions of efficiency [35–41].

We can also obtain the efficiency at maximum power by using Eq. (27). The power in Eq. (3) is obtained by

$$P \simeq \frac{\int_0^1 \frac{\partial \tilde{S}(\tilde{\lambda}(\theta), T_{\min})}{\partial \theta} \tilde{\gamma}_q(\theta) d\theta \Delta T}{\mathcal{T}} - \frac{\int_0^1 \tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2 d\theta}{\mathcal{T}^2}. \quad (31)$$

By maximizing P with respect to $\mathcal{T}$, we obtain the maximum power as

$$P^* = \frac{\left[\int_0^1 \frac{\partial \tilde{S}(\tilde{\lambda}(\theta), T_{\min})}{\partial \theta} \tilde{\gamma}_q(\theta) d\theta \right]^2}{4 \int_0^1 \tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2 d\theta} \Delta T^2. \quad (32)$$

Here, the cycle time at maximum power is $\mathcal{T}^* = 2 \int_0^1 \tilde{R}_{\lambda \lambda, (\tilde{\lambda}(\theta), T_{\min})} \left(\frac{d\tilde{\lambda}}{d\theta} \right)^2 d\theta / \int_0^1 \frac{\partial \tilde{S}(\tilde{\lambda}(\theta), T_{\min})}{\partial \theta} \tilde{\gamma}_q(\theta) d\theta \Delta T$. By using Eq. (27) and $\mathcal{T}^*$, we obtain the efficiency at maximum power η^* :

$$\eta^* = \frac{\eta_{\text{rev}}}{2} \leq \frac{1}{2} \eta_C \left(\frac{T_{\min}}{T_{\max}} \right). \quad (33)$$

Equation (33) states that the efficiency of any heat engine cycle operating at maximum power in the linear response regime is bounded from above by one-half of the Carnot efficiency, where the equality condition agrees with that of $\eta_{\text{rev}} \leq \eta_C(T_{\min}/T_{\max})$. The inequality (33) shows the advantages of heat engines operating with two heat baths in the context of powerful heat engines.

As a final remark, we note that the reversible efficiency η_{rev} can be derived in terms of equilibrium thermodynamics. A reversible heat engine cycle can be approximated as an infinite sum of infinitesimally small reversible Carnot

cycles. For each infinitesimally small Carnot cycle operating between $T_{\min}$ and $T_{\min} + \delta T$, the infinitesimal work and heat are approximated as $\delta W \simeq \delta p \delta \lambda = (\partial p / \partial T)_\lambda \delta T \delta \lambda$ and $\delta Q \simeq (T_{\min} + \delta T)(\partial S / \partial \lambda)_T \delta \lambda$, respectively. The efficiency of this infinitesimally small cycle is the Carnot efficiency: $\eta = \delta W / \delta Q = \delta T / (T_{\min} + \delta T) \simeq \delta T / T_{\min}$, where we used the Maxwell relation $(\partial p / \partial T)_\lambda = (\partial S / \partial \lambda)_T$. By stretching this cycle into the direction of λ and placing $\delta T = \delta \tilde{T}(\theta) = \tilde{\gamma}_q(\theta) \Delta T$ parametrized by θ , we obtain

$$\eta = \frac{\oint \delta W}{\oint \delta Q} \simeq \frac{\int_0^1 \frac{\partial \tilde{S}(\tilde{\lambda}(\theta), T_{\min})}{\partial \theta} \tilde{\gamma}_q(\theta) d\theta}{S(\lambda_{\max}, T_{\min}) - S(\lambda_{\min}, T_{\min})} \frac{\Delta T}{T_{\min}}, \quad (34)$$

where we used $[\partial \tilde{S}(\tilde{\lambda}, T_{\min}) / \partial \tilde{\lambda}]_T (d\tilde{\lambda} / d\theta) = \partial \tilde{S}(\tilde{\lambda}(\theta), T_{\min}) / \partial \theta$. Thus, we recover the reversible efficiency of a heat engine cycle characterized by γ_q , as shown in Eq. (28). Meanwhile, we derived the irreversible efficiency in Eq. (27) as the sum of the reversible efficiency and the correction representing the effect of dissipation in a unified manner. This suggests that our result serves as a natural extension of the efficiency of the reversible heat engines to that of irreversible heat engines based on the linear response theory.

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