

Charge ambiguity and splitting of monopoles

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This paper focuses on topological defects, appearing in mean-field-theory treatments of physical systems such as ultracold atomic gases and gauge field theories. We begin by investigating the ambiguity and addition of topological charges using the mathematical formalism of covering spaces, which clarifies many aspects these phenomena. Subsequently, we classify topological-defect configurations consisting of several monopoles and unknotted ring defects in terms of homotopy groups and fundamental-group actions on them, thus generalizing the previous classifications of a single monopole and a single unknotted ring defect. Finally, we examine the decay of multiply charged topological monopoles under small perturbations of the physical system, and analyze the conditions under which multiply charged monopoles are inclined to split into several singly charged monopoles.

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I. INTRODUCTION

For the past century, the theory of topology has been fruitfully applied [1] in a variety of computational [2] and physical problems [3]. For example, it has been used in the study of graph colorings [4,5] and computational electromagnetics [6], and has provided ways to classify materials and their properties in terms of the lattice structure [7] or the structure of electronic wave functions in conductors [8]. Curiously, topological defects in an ordered medium, i.e., nontrivial structures that cannot be fully removed by local and continuous deformations of the medium, have attracted persistent scientific fascination [7,9–12], including investigations of the compatibility of cosmic microwave background with the existence of topological defects in the early universe [13,14] and the recent spatially resolved observations of monopoles [15], knots [16], and three-dimensional skyrmions [17] in quantum fields. Usually, topological defects are treated as classical objects, but their quantum nature has attracted some attention recently [18].

It has been understood since the late 1970s that topological defects can be classified using the methods of homotopy theory [7,11,19,20], and that a fundamental role in the classification is played by the homotopy groups $\pi_n(X)$ of the order parameter space X , which is the space of all physically distinct local configurations of the system. For example, in a three-dimensional system, a nontrivial fundamental group $\pi_1(X)$ permits topologically distinct line defects, *vortices*, and a nontrivial

second homotopy group $\pi_2(X)$ permits topologically distinct point defects, *monopoles*. However, in order to fully classify topological-defect configurations and to study interactions between multiple defects, the homotopy groups of X are not enough. An immediate manifestation of this is the behavior of monopoles around line defects: depending on the type of the monopole and the vortex, it may be possible that the charge of the monopole is altered as it travels around the singular line, or through a ring [10,21–23]. There exist condensed-matter realizations of these types of line defects, namely, Alice strings and Alice rings that can appear in certain gauge field theories [10,23,24] and are thought to be a potential solution to the baryogenesis problem [25]. Remarkably, Alice rings have recently been observed in condensed-matter systems [26], which may allow in the future experimental studies of the topology of these peculiar structures.

However, the existing literature is currently lacking clarity in the formulation of topological-charge ambiguity and topological-charge addition. If the order parameter space is the real projective plane $\mathbb{R}P^2$, which is also referred to as the space of nematic vectors, these phenomena are well understood [9]. Namely, isolated charges are not classified by their integer charge $\pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$ but rather by its absolute value, due to the action of the fundamental group on π_2 . However, in the presence of multiple charges, also the relative signs are well defined, leaving only the total sign of the system ambiguous. Moreover, the combined charge of several monopoles can be computed by locally orienting the directors around the coalescence path, leading to more refined information than charge combination using the orbit group of Trebin [20], which loses all information of the charges except their values modulo 2.

In this article, we propose a conceptually clear description of charge ambiguity and the charge addition process using a generalization of the above picture. The mathematical framework of covering spaces and lifting results [27] is well suited for this purpose, the case of nematic fields and their orientations corresponding to the two-sheeted covering of $\mathbb{R}P^2$ by the

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sphere S^2 . As with nematics, this approach to charge addition allows us to retain more information than when employing the orbit group [20]. The idea is to replace the order parameter space X with the source of a particularly symmetric covering space $\tilde{X} \rightarrow X$ having the property that the topological charges of monopoles in fields taking values in \tilde{X} are unambiguous, and can be added using the group law of $\pi_2(\tilde{X})$.

In addition, we classify the topological-defect configurations consisting of ring defects and monopoles, up to continuous deformations that leave the cores fixed. This is mathematically equivalent to classification up to continuous deformations that, roughly speaking, do not alter the topology of the core configuration [28]. The result is a generalization of the well-known classification of monopole configurations [9] and isolated ring defects [29]. Classification in the presence of knotted cores is more involved [30], and we leave general investigations of knotted defects for future work. Instead, we advance the study of multiply charged monopoles by showing that such a monopole can be split into several monopoles of charge ± 1 using an arbitrarily small local modification. Moreover, under certain physically realistic hypotheses, almost any small perturbation will have such an effect.

Throughout the work, we denote by (X, x) the order parameter space considered as a *pointed space*, i.e., X is a topological space and $x \in X$ is a distinguished point, the *basepoint*. In Sec. II we study the phenomenon of charge ambiguity and the process of charge addition using a particular covering space of X as the main tool. In Sec. III we classify topological-defect configurations consisting of monopoles and ring defects by the means of elementary homotopy theory, and in Sec. IV we study how multiply charged monopoles split using topology and differential geometry of manifolds. In Appendix B, we summarize the theory of covering spaces and deck transformations, and in Appendix C we define mapping spaces and recall their basic properties. When referring to the Appendixes, we refer to the exact position where the result being used is presented. For example, the computation of the set of homotopy classes of maps in terms of the set of basepoint-preserving homotopy classes of pointed maps will be referred to as Corollary C.1. The reader who is not interested in the mathematical details is suggested to ignore these references.

II. CHARGE AMBIGUITY AND CHARGE ADDITION

In this section, we study charge ambiguity and charge addition using a particularly well-behaved covering space of the order parameter space X . This covering space allows us to enhance order parameter fields into closely related fields without charge ambiguity and where the combined charge of several monopoles can be computed using the group operation of the second homotopy group. Section II A treats systems containing monopoles and Sec. II B expands these studies to systems containing monopoles and ring defects.

Recall that a covering space $q: \tilde{Y} \rightarrow Y$ is a particularly simple finite-to-one map. For the rest of this section, we denote by $p: (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ the covering space of smallest degree which has the property that the $\pi_1(\tilde{X}, \tilde{x})$ action on $\pi_2(\tilde{X}, \tilde{x})$ (Definition B.6) is trivial. We refer to such a covering space as the *charge ambiguity resolving cover* (CARC). The

existence of such a covering space is guaranteed by Proposition B.3. For example, in the case of a real projective plane $\mathbb{R}P^2$, the CARC is provided by the map $S^2 \rightarrow \mathbb{R}P^2$ that identifies antipodal points.

A more interesting example arises as the CARC of the order parameter space of an SO(10) grand unified theory (GUT) equipped with a Higgs field that breaks the SO(10) symmetry down to $SO(2)^5 \times S_5$ [31]. Hence, the order parameter space is homeomorphic $SO(10)/[SO(2)^5 \times S_5]$, and the CARC is given by the 120-fold covering space $SO(10)/SO(2)^5 \rightarrow SO(10)/[SO(2)^5 \times S_5]$. This is a toy model: its purpose is to illustrate interesting topological behavior by means of a simple example, rather than provide a physically realistic example of a symmetry-breaking pattern potentially useful in describing our universe. See Table I to find other relevant examples of CARCs.

We model the spatial region of our physical system, the *cloud*, with the solid ball $\mathbb{B}^3 := \{y \in \mathbb{R}^3 : \|y\| \leq 1\}$, and denote by $C \subset \mathbb{B}^3$ the the points where the order parameter is well defined. Concretely, \mathbb{B}^3 can be the extent of a spinor Bose-Einstein condensate, and C the complement of the cores of the various singular defects contained in the cloud. The *order parameter field* is modeled by a continuous map $\Psi: C \rightarrow X$. Given topological spaces Y and Z , the set of homotopy classes of continuous maps $Y \rightarrow Z$ is denoted by $[Y, Z]$.

A. The simply connected case

In this section, we restrict to the case where the order parameter field has finitely many monopoles and no other defects. In other words, C is a solid ball \mathbb{B}^3 with finitely many points removed from its interior. Thus C is simply connected. Given an embedded sphere $\Omega \subset C$, the finest homotopy theoretic invariant of the order parameter field Ψ one can associate to Ω is the free homotopy class $[\Psi|_\Omega] \in [S^2, X]$, which by definition classifies the the winding of Ψ along Ω up to homotopy. We call this element the type of the topological charge contained inside Ω . As the free homotopy classes of maps from S^2 to X correspond to $\pi_1(X, x)$ orbits in $\pi_2(X, x)$ [7], it makes sense to define the topological charge inside Ω as an element of $\pi_2(X, x)$ belonging to the $\pi_1(X, x)$ orbit corresponding to $[\Psi|_\Omega]$. The topological charge is by definition ambiguous up to the π_1 action on $\pi_2(X, x)$. If Ω encloses a single monopole at $y \in \mathbb{B}^3 \setminus C$, then the topological charge inside Ω , and its type, are referred to as the topological charge of the monopole at y , and the type of the monopole at y , respectively. More generally, if Ω encloses the monopoles at $y_1, \dots, y_n \in \mathbb{B}^3 \setminus C$ and no others, then the topological charge inside Ω is the combined topological charge of the monopoles at y_1, \dots, y_n . The type of the combined charge is a well-defined element of $\pi_2(X, x)/\pi_1(X, x)$ rather than just of the orbit group [20], and therefore our approach retains more information than the one employing the orbit group.

In order to study these phenomena in more detail, we choose a continuous lift $\tilde{\Psi}: C \rightarrow \tilde{X}$ of Ψ , taking values in the CARC, the existence of which is guaranteed by Lemma B.2. Because the π_1 action on $\pi_2(\tilde{X}, \tilde{x}) \cong \pi_2(X, x)$ is trivial, it follows that $[S^2, \tilde{X}]$ and $\pi_2(X, x)$ are isomorphic sets. Importantly, every monopole of the lifted field $\tilde{\Psi}$ has an unambiguous topological charge, and the charge addition

TABLE I. Examples of physically relevant order parameter spaces, their first and second homotopy groups, and the π_1 action on π_2 . The target of the CARC is the order parameter space of the system. The π_1 action on π_2 is denoted by $\alpha.\beta$, where $\alpha \in \pi_1$ and $\beta \in \pi_2$.

System	Phase	CARC	π_1	π_2	π_1 action
Liquid crystal	UN [21]	$S^2 \rightarrow \mathbb{RP}^2$	\mathbb{Z}_2	\mathbb{Z}	$[n].m = (-1)^n m$
Gaseous BEC	Spin-1 polar [21]	$S^1 \times S^2 \rightarrow (S^1 \times S^2)/\mathbb{Z}_2$	\mathbb{Z}	\mathbb{Z}	$n.m = (-1)^n m$
	Spin-2 UN [21]	$S^1 \times S^2 \rightarrow S^1 \times \mathbb{RP}^2$	$\mathbb{Z} \times \mathbb{Z}_2$	\mathbb{Z}	$(n_1, [n_2]).m = (-1)^{n_2} m$
$^3\text{He-A}$	Dipole free [21]	$S^2 \times \text{SO}(3) \rightarrow (S^2 \times \text{SO}(3))/\mathbb{Z}_2$	\mathbb{Z}_4	\mathbb{Z}	$[n].m = (-1)^n m$
SO(10) GUT	Various	$\text{SO}(10)/\text{SO}(2)^5 \rightarrow \text{SO}(10)/[\text{SO}(2)^5 \rtimes S_5]$	S_5	$\mathbb{Z}_{\text{even}}^5$	$(\sigma, [n]).(m_1, \dots, m_5) = (m_{\sigma_1}, \dots, m_{\sigma_5})$

process described in the previous paragraph reduces to the group law of $\pi_2(\tilde{X}, \tilde{x}) \cong \pi_2(X, x)$. The lift $\tilde{\Psi}$ is not unique; however, the different lifts are related to each other via symmetries of the CARC \tilde{X} relative to X (Definition B.4). The charge of a single monopole, computed in various lifts $\tilde{\Psi}$, ranges through all the elements in the corresponding π_1 orbit of $\pi_2(X, x)$ (Remark B.2). Another subtle point is that, even if two monopoles of the system are of the same type, they might lift to monopoles of different charges. If two monopoles lift to monopoles of different charge in one lift, then this is true for all lifts. This behavior is illuminated by a concrete example in Fig. 1.

B. Beyond the simply connected case

Here, we consider an order parameter field that has, in addition to monopoles, singularities in the form of unknotted circles in the interior of the cloud. Such systems occur in condensed-matter physics, for example, if monopoles decay into Alice rings [26,32]. This situation creates a new challenge: the field may no longer admit a lift taking values in the CARC, as is illustrated by Fig. 2. However, one can define the π_1 -equivalence class of topological charge of the field Ψ inside an embedded sphere $\Omega \subset C$ as above, generalizing the definition of topological charge of an isolated monopole to this situation. We would like to define the topological charge of a ring defect in a similar fashion, but here a problem arises: there may be multiple homotopically inequivalent embedded spheres $S^2 \subset C$ that enclose only a single ring defect and no other defects [see Figs. 3(a) and 3(b)]. Hence, without any further choices, the topological charge of a ring defect is ill defined, even as an element of $\pi_2(X, x)/\pi_1(X, x)$.

A conceptually clear solution to all of the problems mentioned above is provided by equipping each singular ring with a membrane, that is, a two-dimensional disk, possibly deformed, closing the loop. The complement C' of the membranes in C is simply connected. Moreover, the ring-shaped core has a unique homotopy class of enclosing spheres that do not cross the membrane, and the topological charge inside such a sphere is referred to as the Cheshire charge of the ring defect [10,23]. Importantly, the restriction Ψ' of Ψ to C' admits a lift $\tilde{\Psi}' : C' \rightarrow \tilde{X}$ taking values in the CARC by Lemma B.2.

The lifted field $\tilde{\Psi}'$ behaves in an interesting fashion near the membranes: there are two limiting configurations on the membrane obtained by approaching it from opposite sides (Fig. 4). This behavior is analogous to the gauge discontinuities that appear in the study of Alice rings in cosmology [10,23] and branch cuts that appear when investigating

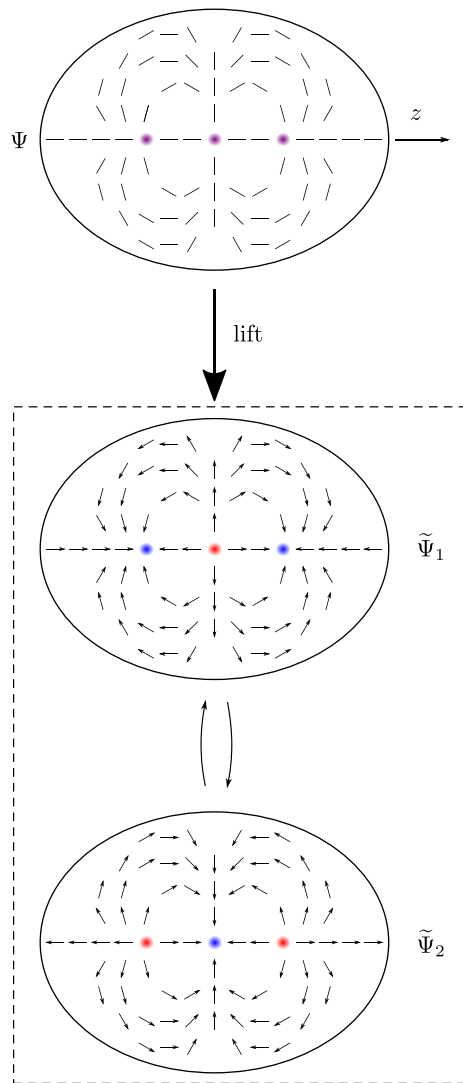


FIG. 1. Charge ambiguity and addition for a nematic vector field. A nematic field Ψ taking values in \mathbb{RP}^2 and its lifts $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$. All configurations have z as an axis of rotational symmetry. The field Ψ is visualized as a field of arrows without heads, or rods, and the lifted fields $\tilde{\Psi}_1$ and $\tilde{\Psi}_2$, taking values in the universal cover S^2 , are illustrated as the oriented versions of Ψ . Topological charges of the monopoles are well defined in the lifted fields $\tilde{\Psi}_i$, red and blue indicating charge $+1$ and -1 , respectively. The two lifts are interchanged by the symmetry of the CARC that reverses the orientation of the arrows. Any two monopoles will have either identical or opposite signs regardless of the chosen lift, and the absolute value of the total charge inside the cloud is always $+1$.

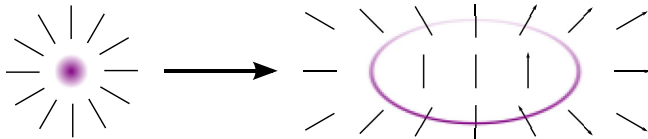


FIG. 2. Once a monopole in the nematic field (dot on the left) decays into an Alice ring (ring on the right), the field may no longer admit a lift along a covering space. This failure is caused by the existence of closed loops that would have to flip the orientation, as illustrated in the figure. Mathematically, this is the content of Lemma B.2 and Remark B.1.

knotted defects in nematic liquid crystals [30]. A convenient, although not physically distinguishable, way to mathematically describe this behavior is to consider every ring with a membrane as a portal that operates on fields passing through by applying to them a discrete symmetry of the CARC \tilde{X} relative to X .

The framework described in the previous paragraph also provides some insight on charge transfer, which has been studied previously [12,21,22]. The order parameter field near a monopole which travels through a membrane is transformed

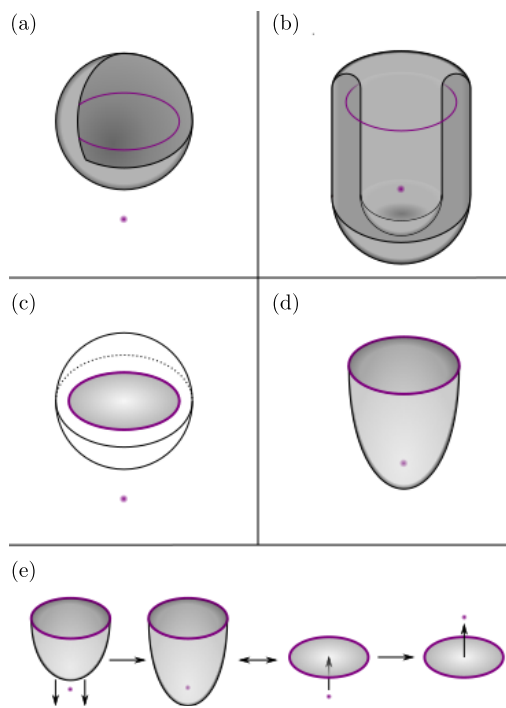


FIG. 3. [(a), (b)] Two homotopically nonequivalent ways of enclosing the ring-shaped singularity inside a sphere while leaving the monopole outside. If the ring is equipped with a membrane, as in (c), there exists a unique homotopy class of embedded spheres that do not cross the membrane and enclose only the ring defect. [(c), (d)] Additional homotopically nonequivalent choices of membranes. (e) The following two processes are topologically equivalent: the membrane is moved past a monopole and the monopole is moved through the membrane. Note that the Cheshire charge of the ring after the monopole has passed through it is equal to the topological charge inside the sphere in (b).

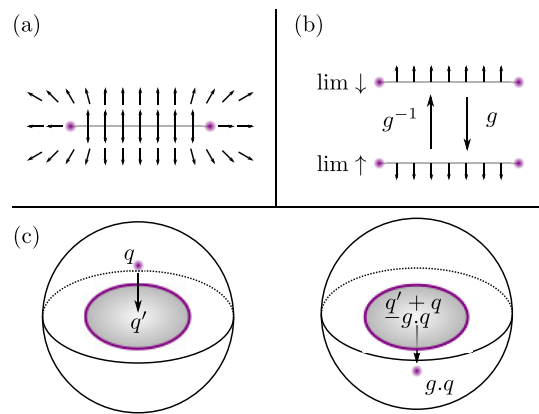


FIG. 4. (a) Slice of a ring-shaped singularity (purple dots) with a membrane (gray line) together with the lifted order parameter field on the complement of the membrane. (b) The lifted order parameter field has two limiting configurations on the membrane, which in the picture are related to each other by the antipodal symmetry of S^2 . It is possible to show that, in general, the two limiting configurations are related to each other by a symmetry of the CARC \tilde{X} relative to X (Definition B.4). (c) If a monopole passes through such a membrane, one has to apply the symmetry g to the order parameter field around it, altering the topological charge of the monopole from q to $g.q$. Since the total charge inside the sphere remains unchanged, the Cheshire charge of the ring must change by $q - g.q$ [10,22].

by a symmetry g of the CARC \tilde{X} . This can affect the charge of the traveling defect, and consequently charge conservation dictates that the Cheshire charge of the ring must also be affected, as shown in Fig. 4.

We end this section by proposing a mathematical definition of an Alice ring which captures the essential topological features. The proposed definition generalizes cosmic Alice rings [10,23] and half-quantum vortex rings [21,22].

Definition II.1. Let $D \subset \mathbb{B}^3$ be an unknotted ring-shaped singularity, and let γ be a small loop that winds about the singularity once. We define D to be an *Alice ring (for monopoles)* if any element, or equivalently every element, of the conjugacy class of $\pi_1(X, x)$ corresponding to $\Psi \circ \gamma : S^1 \rightarrow X$ acts nontrivially on $\pi_2(X, x)$. In physical terms, D is an Alice ring if there exists a type of monopoles, the charge of which would be altered by traveling about D .

III. CONFIGURATIONS OF MONOPOLES AND RING DEFECTS

In this section, we study and classify defect configurations consisting of monopoles and unknotted ring defects, generalizing the well-known classification of monopole configurations [9] and of isolated ring defects [29]. Our method is to decompose the complement of the cores into simple pieces, and then invoke standard properties of mapping spaces reviewed in Appendix C.

More precisely, we aim to find the elements of the set $[C, X]$, where C —as in Sec. II B—is the complement of isolated points and unknotted circles inside a solid ball \mathbb{B}^3 . Note that the elements of $[C, X]$ are the topological-defect configurations of the system with a prespecified core configuration,

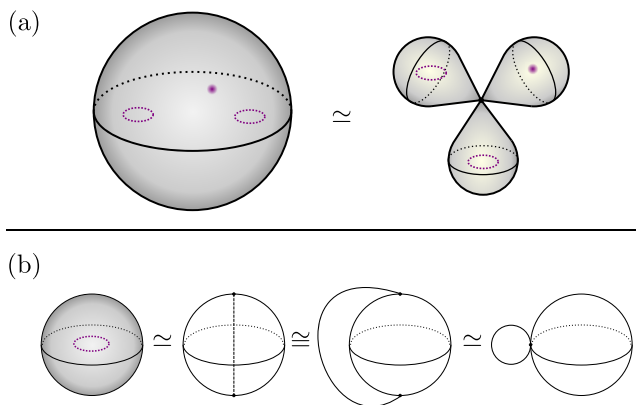


FIG. 5. (a) Solid ball containing two unknotted ring defects and a monopole (purple dashed circle, purple dot) is equivalent to the space obtained from several balls containing a single defect by attaching them together at a point. (b) The complement of an unknotted circle inside a solid ball is equivalent to the space that is obtained by attaching a circle and a sphere at a point. The intermediate spaces are a hollow sphere with a chord running through it, and a homeomorphic (abstract) topological space presented by a different three-dimensional embedding.

considered up to continuous deformations leaving the cores fixed. However, as any smooth isotopy of the configuration of cores—roughly speaking, deformation in which defect cores are not allowed to be pinched, to cross each other, or to be pierced by the order parameter field—can be canceled by an ambient isotopy of \mathbb{B}^3 [28], the set $[C, X]$ may be regarded as classifying topological-defect configurations up to continuous deformations that change the core configuration up to smooth isotopy.

As illustrated in Fig. 5, C is homotopically equivalent to a space that is obtained from several clouds, each of which contains a single defect, by attaching them at a single point. Moreover, each of these pieces is equivalent to either S^2 if the cloud contains a monopole or the space obtained from attaching a circle S^1 and a two-sphere S^2 at a single point. Such a decomposition allows us to compute $[C, X]$ using the methods of Appendix C (Proposition C.2 and Corollary C.1): $[C, X]$ is isomorphic to

$$\left[\prod_y \pi_2(X, x)_y \right] \times \left[\prod_d \pi_1(X, x)_d \times \pi_2(X, x)_d \right] / \pi_1(X, x), \tag{1}$$

where y and d range over all the monopoles and the ring defects of the system, and where $\pi_1(X, x)$ acts on each factor of the product by the usual π_1 action on π_i (Definition B.6). In other words, the equivalence class of the configuration is completely determined by the topological charges of all the defects and the topological vorticities of all the ring defects, and these data are ambiguous up to the simultaneous action of $\pi_1(X, x)$.

IV. INSTABILITY OF MULTIPLY CHARGED MONOPOLES

In this section, we study the stability properties of isolated monopoles. We will proceed by identifying the topological charge with the degree of a proper map of manifolds and then invoking well-known transversality theorems. The physical interpretation of this is that a multiply charged monopole may be split into multiple singly charged monopoles.

As in Sec. II A, C is the complement of finitely many points in the interior of a solid ball. Furthermore, we assume that the order parameter space is such that the CARC is of the form $S^2 \times Y$, where Y is a space with trivial $\pi_2(Y)$. The first four rows of Table I are examples of such order parameter spaces. An order parameter field $\Psi : C \rightarrow X$ admits a lift $\tilde{\Psi} : C \rightarrow S^2 \times Y$ by Lemma B.2. Moreover, by neglecting the Y component of $\tilde{\Psi}$, and accounting for the magnitude of the order parameter, we obtain a map $\Phi : \mathbb{B}^3 \rightarrow \mathbb{R}^3$. Note that C is the preimage of $\mathbb{R}^3 \setminus \{\bar{0}\}$, where $\bar{0}$ denotes the origin, and the cores of the monopoles are exactly the points in the preimage of $\bar{0}$. A monopole at $y \in \mathbb{B}^3$ is regular if Φ is a local homeomorphism at y ; i.e., there exists an open neighborhood of y mapping homeomorphically—in a way that admits a continuous inverse—onto an open neighborhood of $\bar{0}$. Such a monopole has topological charge ± 1 . We assume that none of the monopoles is situated at the boundary $\partial\mathbb{B}^3$ of the cloud; in other words, the image $\Phi(\partial\mathbb{B}^3)$ is disjoint from $\bar{0}$.

Let U be an open ball around $\bar{0} \in \mathbb{R}^3$ that is small enough to be contained in the complement of $\Phi(\partial\mathbb{B}^3)$. The preimage $V := \Phi^{-1}U \subset \mathbb{B}^3$ consists of the volume of the cloud on which the order parameter is “small” in magnitude. We assume for simplicity that V is homeomorphic to the disjoint union $\coprod_{i \in I} V_i$, where each V_i contains a single monopole, located at y_i , the degree of which is denoted by n_i . It turns out that the restricted maps $\Phi_i : V_i \rightarrow U$ have degree n_i [33], respectively. Well-known results in the topology of manifolds [34] imply that the field Φ can be modified, locally around the points y_i (i.e., leaving Φ unchanged outside V), up to homotopy, in a way that splits the monopole at y_i into exactly $|n_i|$ regular monopoles contained in the near vicinity of y_i (i.e., inside V_i). Moreover, these local modifications can be chosen to be arbitrarily small.

In fact, multiply charged monopoles are extremely unstable in the following sense. Let us call $v \in \mathbb{B}^3$ a regular point of Φ if Φ is a local homeomorphism at v , and $u \in \mathbb{R}^3$ a regular value of Φ if the preimage $\Phi^{-1}\{u\}$ consists of finitely many regular points. A point or a value that is not regular is called *critical*. Under the assumption that $\Phi_i : V_i \rightarrow U$ is open and discrete [35], sets of critical points and critical values have a codimension of 2 inside U [35,36]. It follows that for every regular value u , i.e., every point of U outside a subset of codimension 2, the preimage of u consists of $|n_i|$ regular points. Under the alternative hypothesis that the Φ_i are once continuously differentiable, by Sard’s theorem [37,38] the set of critical values in U has measure 0, and so the preimage of almost any $u \in U$ consists of regular points, whose degrees add up to n_i . See Fig. 6 for illustration of this behavior in one dimension. Hence, in either of the two cases considered above, applying almost any small perturbation by a constant function to the field Φ will split all the monopoles into monopoles of charges ± 1 in a way that preserves the total charge.

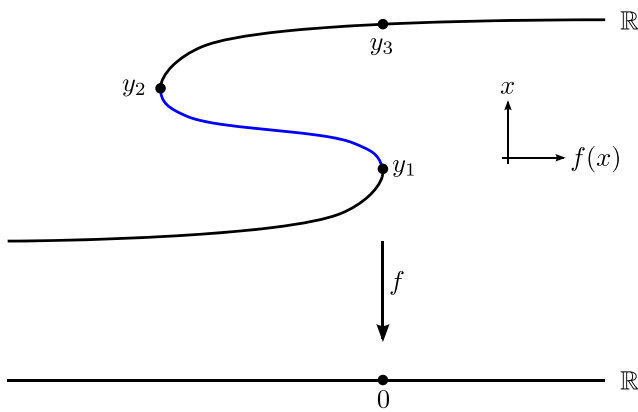


FIG. 6. Transposed graph of a proper and continuously differentiable map $f : \mathbb{R} \rightarrow \mathbb{R}$ is projected to the value $f(x)$. The map f is orientation reversing on the segment between points y_1 and y_2 (marked in blue), and orientation preserving elsewhere. It is not open at y_1 and y_2 where it changes orientation, as the images of small open neighborhoods of these points are half-open intervals. If we regard f as an order parameter field, there exists a “monopole” of degree 1 at y_3 and a monopole of degree 0 at y_2 . A small perturbation either completely destroys the monopole at y_1 or creates a ± 1 monopole-antimonopole pair in its place, depending on the direction of the perturbation. In either case, the total charge is conserved.

V. CONCLUSIONS

We studied charges of topological defects using a particularly symmetric covering space of the order parameter space X . Even though the charge of an isolated monopole is well defined only up to the π_1 action on $\pi_2(X)$, the configuration of n charges is well defined up to a simultaneous π_1 action on $\prod_{i=1}^n \pi_2(X)$, allowing for the computation of the combined charge of several monopoles as an element of $\pi_2(X)/\pi_1(X)$. This is an improvement over the orbit group approach [20], which may be used to compute the combined charge only modulo the subgroup of $\pi_2(X)$ generated by elements of form $q - \alpha \cdot q$, where $q \in \pi_2(X)$ and $\alpha \in \pi_1(X)$. Our method also illuminates the influence of ring-shaped defects on monopoles and the phenomenon of Cheshire charge. We also suggested a definition for an Alice ring in purely topological terms: it is a ring defect the vortex type of which, considered as an element of π_1 , has a nontrivial action on π_2 .

We classified defect configurations consisting of monopoles and ring defects, up to continuous deformations that leave the cores fixed. The configurations are classified by the topological charges of the defects and by the topological vortex types of the ring defects, up to simultaneous π_1 actions. We also studied the stability of multiply charged monopoles and showed that it is possible to split a multiply charged monopole into monopoles of degrees ± 1 using only small local modifications to the order parameter field. In physical terms this implies that a multiply charged monopole is topologically equivalent to a defect configuration consisting of multiple singly charged monopoles.

Our results provide a “geometric” means to understand topological charge amalgamation and the topological influence of vortices on topological charges, generalizing the

method of orienting fields of nematic vectors. They can be used as tools in the theoretical investigation of a wide variety of condensed-matter systems containing monopoles and ring defects. Moreover, many interesting questions about condensed-matter systems containing more complicated defect configurations, such as knotted or linked vortex loops, remain open.

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APPENDIX A: MATHEMATICAL NOTATION

Here, we explain common mathematical notation used throughout the article.

Symbol	Meaning
\cong	Isomorphism or a homeomorphism.
\simeq	Homotopy equivalence.
$[x]$	Equivalence class of x .
$\{x\}$	One-element set containing x .
$X \rightarrow Y$	Map from X to Y .
$X \hookrightarrow Y$	Injective map $X \rightarrow Y$.
$x \mapsto f(x)$	Formulaic description of a map $f : X \rightarrow Y$. Also known as an anonymous function.
$f^{-1}S$	Inverse image of $S \subset Y$ in a map $f : X \rightarrow Y$.
$[X, Y]$	Homotopy classes of continuous maps $X \rightarrow Y$.
$[X, Y]_*$	Basepoint-preserving homotopy classes of pointed maps $(X, x) \rightarrow (Y, y)$. The basepoints $x \in X$ and $y \in Y$ are usually omitted from notation.
$\pi_n(X, x)$	n th homotopy group of X based at $x \in X$. Isomorphic as a set to $[S^n, X]_*$.
$\text{Deck}(\tilde{X}/X)$	Group of deck transformations of a covering space $\tilde{X} \rightarrow X$.
$X \coprod Y$	Disjoint union.
$X \vee Y$	Wedge sum of pointed spaces (X, x) and (Y, y) .
$g \cdot x$	Given an action of a group G on a space or a set X , this denotes the result of acting by $g \in G$ on $x \in X$.
X/G	Quotient space or set. Also called the set or the space of orbits.

APPENDIX B: COVERING SPACES

In this section, we recall the basics of the theory of covering spaces. A basic reference with many pictures is Chap. 1.3 of Hatcher’s book [27]. As a general rule, we give full statements of the results, but we do not explain the technical terminology unless absolutely necessary. We stress here that all the technical assumptions appearing in this section hold

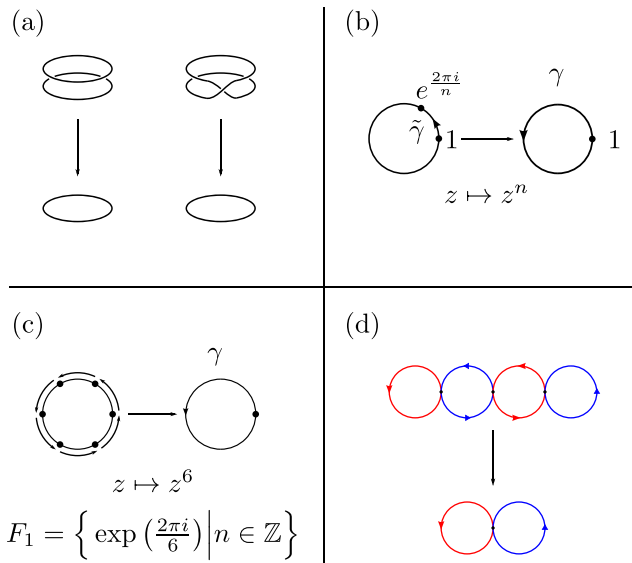


FIG. 7. (a) Examples of a trivial and nontrivial covering space (of degree 2) of the circle S^1 . (b) Nontrivial covering of degree n of S^1 , regarded as the unit circle in the complex plane. The unique lift $\tilde{\gamma}$ of the path γ starting at 1 is indicated in the picture. The group of deck transformations is given by rotations by $2\pi/n$ rad. Since there are exactly n such rotations this is a Galois covering. (c) The fiber F_1 over one of the six-fold cover. The monodromy action of $[\gamma]$ cyclically permutes the elements in the fiber. Note that for each deck transformation ψ , there exists a homotopy class of loops whose monodromy action on F_1 coincides with the action of ψ on F_1 (cf. Proposition B.1). (d) Example of a non-Galois covering space: the only deck transformation is the identity, but the covering has degree 3.

for manifolds, which should be connected if the result asks for connectivity. We do not repeat the statement of this assumption below.

Definition B.1. Let X be a topological space. Then a *covering space* of X is a continuous morphism $p : \tilde{X} \rightarrow X$ having the property that for any $x \in X$, there exists an open neighborhood U of x and a homeomorphism $\phi : \coprod_{i \in I} U_i \rightarrow p^{-1}U$ —each U_i being a copy of U —such that the composition

$$\coprod_{i \in I} U_i \xrightarrow{\phi} p^{-1}U \xrightarrow{p|_{p^{-1}U}} U$$

restricts to the identity map on each U_i . If X is path connected, then the *degree* $\deg(p)$ of p is the cardinality of any set I that appears as above. See Fig. 7 for examples.

Lemma B.1 (Path lifting lemma). Let $p : \tilde{X} \rightarrow X$ be a covering space, and let $\gamma : [0, 1] \rightarrow X$ be a continuous path in X . Let $\tilde{x}_0 \in \tilde{X}$ be such that $p(\tilde{x}_0) = \gamma(0)$. Then there exists a unique continuous map $\tilde{\gamma}$ such that $\gamma = p \circ \tilde{\gamma}$ and $\tilde{\gamma}(0) = \tilde{x}_0$. The path $\tilde{\gamma}$ is called the *lift of γ starting at \tilde{x}_0* .

Analogously, for each \tilde{x}_1 satisfying $p(\tilde{x}_1) = \gamma(1)$, there exists a unique *lift of γ ending at \tilde{x}_1* .

Definition B.2. Let $p : \tilde{X} \rightarrow X$ be a covering space, $x \in X$, and consider the *fiber* over x , $F_x := p^{-1}\{x\}$. The *monodromy action* of $\pi_1(X, x)$ on F_x is defined as follows: given $[\gamma] \in \pi_1(X, x)$ and $a \in F_x$, we define $[\gamma].a$ as $\tilde{\gamma}(0)$, where $\tilde{\gamma}$ is the

unique lift of γ ending at a . The result does not depend on the chosen representative γ .

The path lifting lemma can be used to prove a more general lifting result. Before this, we need the following definition.

Definition B.3. A *pointed space* is a pair (X, x) , where X is a topological space and $x \in X$. A *morphism* or a *map* $f : (X, x) \rightarrow (Y, y)$ of pointed spaces is a continuous map $f : X \rightarrow Y$ satisfying $f(x) = y$.

Lemma B.2 (General lifting lemma). Let $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ be a (pointed) covering map, and let $f : (Z, z) \rightarrow (X, x)$ be a morphism of pointed spaces, where Z is connected and locally path connected. Then there exists a unique *lift* $\tilde{f} : (Z, z) \rightarrow (\tilde{X}, \tilde{x})$, i.e., $f = p \circ \tilde{f}$, if and only if the image of $f_* : \pi_1(Z, z) \rightarrow \pi_1(X, x)$ is contained in the image of the injective homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}) \hookrightarrow \pi_1(X, x)$.

Remark B.1. Consider the fiber sequence

$$(F_x, \tilde{x}) \xhookrightarrow{i} (\tilde{X}, \tilde{x}) \xrightarrow{p} (X, x),$$

where i is the inclusion of the fiber over x . The long exact homotopy sequence gives the following sequence of maps,

$$\{e\} \rightarrow \pi_1(\tilde{X}, \tilde{x}) \xrightarrow{p_*} \pi_1(X, x) \xrightarrow{\delta} F_x,$$

proving immediately that p_* is injective. The map δ is closely related to the monodromy action as δ sends $[\gamma]$ to $[\gamma].\tilde{x}$. The class $[\gamma]$ lies in the image of p_* if and only if it lifts to a closed loop starting at \tilde{x} . If \tilde{X} is path connected, δ is a surjection, and hence F_x —equipped with the monodromy action—is isomorphic to the $\pi_1(X, x)$ set $\pi_1(\tilde{X}, \tilde{x}) \setminus \pi_1(X, x)$ (left cosets).

Definition B.4. The group of the covering-space automorphisms of $p : \tilde{X} \rightarrow X$, i.e., homeomorphisms $\psi : \tilde{X} \rightarrow \tilde{X}$ such that $p = p \circ \psi$, is called the group of *deck transformations* of p , and it is denoted by $\text{Deck}(\tilde{X}/X)$. The canonical group action of $\text{Deck}(\tilde{X}/X)$ on \tilde{X} is compatible with the projection p by definition. Hence, we obtain an induced morphism $\bar{p} : \tilde{X}/\text{Deck}(\tilde{X}/X) \rightarrow X$.

Definition B.5. If both \tilde{X} and X are path connected and \bar{p} is a homeomorphism, then p is called a *Galois covering*. Note that by Lemma B.2, the action of $\text{Deck}(\tilde{X}/X)$ restricts to a free action on the fibers F_x . It is clear that p is Galois if and only if the action is transitive as well. Moreover, if $\deg(p)$ is finite, then p being a Galois covering is equivalent to the equality $|\text{Deck}(\tilde{X}/X)| = \deg(p)$.

Example B.1 (Finite Galois coverings arise as quotients).

If X is a connected Hausdorff topological space and a finite group G acts on it freely, then the quotient map $X \rightarrow X/G$ is a Galois covering with $\text{Deck}(X/(X/G)) = G$. Conversely, if $p : \tilde{X} \rightarrow X$ is a Galois covering, then p is equivalent to $\tilde{X} \rightarrow \tilde{X}/\text{Deck}(\tilde{X}/X)$.

Proposition B.1 (Characterization of Galois covers). Let X be connected and locally path connected, and suppose $p : \tilde{X} \rightarrow X$ is a connected covering. Then p is a Galois covering if and only if there exists such a $\tilde{x} \in \tilde{X}$ that the subgroup

$$p_* : \pi_1(\tilde{X}, \tilde{x}) \hookrightarrow \pi_1(X, p(\tilde{x}))$$

is normal. If this is the case, then the above holds for all $\tilde{x} \in \tilde{X}$.

Proposition B.2 (Presentation of the Galois group). Let $(\tilde{X}, \tilde{x}) \rightarrow (X, x)$ be a pointed Galois covering. Since

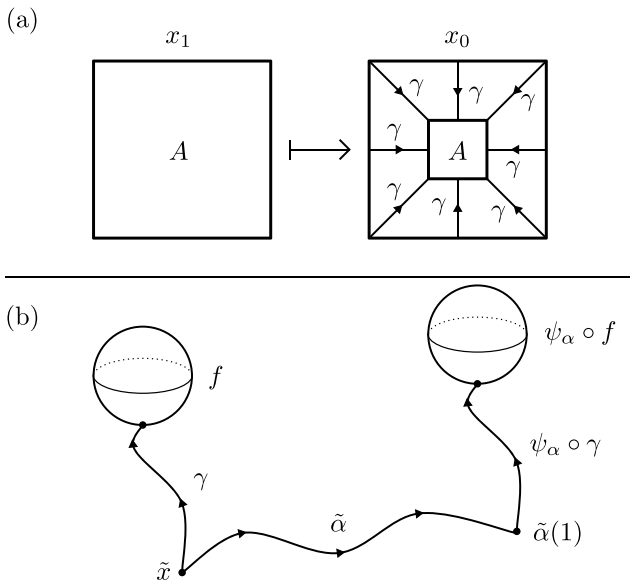


FIG. 8. (a) Action of a path γ from x_0 to x_1 on higher homotopy groups. Left: An element $[A] \in \pi_n(X, x_1)$ is presented by depicting A as a continuous map from the n hypercube to X taking constant value x_1 at the boundary. Right: The class $\gamma_*[A]$ is presented in a similar fashion. (b) Path $\tilde{\alpha}$ in \tilde{X} is the unique lift of a closed loop α in X starting at \tilde{x} . If there exists a deck transformation ψ_α satisfying $\psi_\alpha(\tilde{x}) = \tilde{\alpha}(1)$ (unique such deck transformation exists if $\tilde{X} \rightarrow X$ is a Galois covering; see Proposition B.2), then it transforms a sphere $f : S^2 \rightarrow \tilde{X}$ attached to \tilde{x} by a path γ to a sphere that is attached in a similar manner to $\tilde{\alpha}(1)$.

$\text{Deck}(\tilde{X}/X)$ acts freely and transitively on F_x , the function

$$\phi : g \in \text{Deck}(\tilde{X}/X) \mapsto g.\tilde{x} \in F_x$$

is a bijection of sets. Thus the composition

$$\phi^{-1} \circ \delta : \pi_1(X, x) \rightarrow \text{Deck}(\tilde{X}/X),$$

where δ is as in Remark B.1, is a surjective group homomorphism with kernel $\pi_1(\tilde{X}, \tilde{x})$. In particular

$$\text{Deck}(\tilde{X}/X) \cong \pi_1(X, x)/\pi_1(\tilde{X}, \tilde{x}).$$

Definition B.6. Let X be a topological space. Then, a path γ in X with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ induces group homomorphisms $\gamma_* : \pi_i(X, x_1) \rightarrow \pi_i(X, x_0)$ for all $i \geq 1$ (see Fig. 8 for details). The homomorphism γ_* depends only on the path homotopy class of γ ; i.e., it does not change if γ is deformed while keeping the endpoints fixed. In particular, for every $x \in X$, we obtain a group action of $\pi_1(X, x)$ on $\pi_i(X, x)$, where each $[\gamma] \in \pi_1(X, x)$ operates as γ_* on $\pi_i(X, x)$. This action is referred to as the π_1 action on the homotopy groups.

Remark B.2 (Action of $\text{Deck}(\tilde{X}/X)$ is the π_1 action). Let $p : (\tilde{X}, \tilde{x}) \rightarrow (X, x)$ be a pointed covering space, $i \geq 2$, and let (S^i, e) be the i sphere pointed at the north pole e . If the $\pi_1(\tilde{X}, \tilde{x})$ action on $\pi_i(\tilde{X}, \tilde{x})$ is trivial, the set $[S^i, \tilde{X}]$ of free homotopy classes of spheres is isomorphic to $\pi_i(\tilde{X}, \tilde{x})$. Indeed, given a continuous map $f : S^2 \rightarrow \tilde{X}$, choose a path γ from \tilde{x} to $f(e)$ and let $A_f := \gamma_*([f]) \in \pi_i(X, \tilde{x})$. The class A_f does not depend on the choice of γ : given another path γ' from \tilde{x} to $f(e)$, then $\gamma_*([f])$ and $\gamma'_*([f])$ differ by the action

of a closed loop on $\pi_i(\tilde{X}, \tilde{x})$. Since the π_1 action was assumed to be trivial, A_f does not depend on the choice of γ . We have described a bijection $v : [S^i, \tilde{X}] \rightarrow \pi_i(X, x)$.

Suppose then that p is a Galois covering, and consider the deck transformation ψ_α associated to $[\alpha] \in \pi_1(X, x)$ as in Proposition B.2. Then, as in Fig. 8(b), $\psi_\alpha \circ f : S^i \rightarrow \tilde{X}$ can be connected to \tilde{x} by composing $\tilde{\alpha}$ with $\psi_\alpha \circ \gamma$, where $\tilde{\alpha}$ is the lift of α starting at \tilde{x} . If we denote by $B_f \in \pi_i(\tilde{X}, \tilde{x})$ the element thus obtained, then

$$p_*(B_f) = \alpha_*[p_*(A_f)] \in \pi_i(X, x),$$

where p_* is the homomorphism $\pi_i(\tilde{X}, \tilde{x}) \rightarrow \pi_i(X, x)$. Hence,

$$(p_* \circ v)(\psi_\alpha[f]) = \alpha_*[(p_* \circ v)[f]],$$

identifying the $\text{Deck}(\tilde{X}/X)$ action on $[S^i, \tilde{X}]$ with the π_1 action on $\pi_i(X, x)$.

Proposition B.3. Let (X, x) be connected, locally path connected, and semilocally simply connected, and let $I \subset \mathbb{N}_{\geq 1}$. The π_1 action on the homotopy groups provides a group homomorphism

$$\pi_1(X, x) \rightarrow \prod_{i \in I} \text{Aut}[\pi_i(X, x)],$$

where $\text{Aut}(G)$ denotes the automorphism group of G . Let us denote the kernel of the above homomorphism by K . There exists a connected pointed covering space (\tilde{X}, \tilde{x}) with $\pi_1(\tilde{X}, \tilde{x}) = K$, and this covering space is unique up to unique isomorphism of pointed covering spaces. Moreover, this is a Galois covering space as kernels are normal.

The map $p : \tilde{X} \rightarrow X$ is the smallest covering of X , on which the π_1 action on $\pi_i(\tilde{X}, \tilde{x})$ is trivial for all $i \in I$: given another covering space $q : \tilde{X}' \rightarrow X$ with this property, there exists a continuous map $\psi : \tilde{X}' \rightarrow \tilde{X}$ satisfying $q = p \circ \psi$.

APPENDIX C: MAPPING SPACES

In this section, we recall the basics of the theory of mapping spaces. A basic reference is Chap. 5 of tom Dieck's book on algebraic topology [39].

Definition C.1. Let X and Y be compactly generated spaces (e.g., unions of manifolds). The mapping space $\text{Map}(X, Y)$ is the set of all continuous maps $X \rightarrow Y$ equipped with the compact-open topology. If (X, x) and (Y, y) are compactly generated pointed spaces, then the pointed mapping space $\text{Map}_*(X, Y)$ is the subspace of $\text{Map}(X, Y)$ consisting of those continuous morphisms $f : X \rightarrow Y$ satisfying $f(x) = y$.

Proposition C.1. Let I denote the unit interval, and let (X, x) and (Y, y) be compactly generated pointed spaces. Then there exists a natural homeomorphism

$$\text{Map}[I, \text{Map}(X, Y)] \cong \text{Map}(I \times X, Y)$$

that identifies the paths in $\text{Map}(X, Y)$ with homotopies of continuous maps $X \rightarrow Y$. Similarly, the composition

$$\begin{aligned} \text{Map}[I, \text{Map}_*(X, Y)] &\rightarrow \text{Map}[I, \text{Map}(X, Y)] \\ &\cong \text{Map}(I \times X, Y) \end{aligned}$$

identifies the paths in $\text{Map}_*(X, Y)$ with basepoint-preserving homotopies of morphisms $(X, x) \rightarrow (Y, y)$.

The previous result identifies the set of path components of $\text{Map}(X, Y)$ with the set $[X, Y]$ of homotopy classes of continuous maps $X \rightarrow Y$. Similarly, the path components of $\text{Map}_*(X, Y)$ are identified with $[X, Y]_*$, that is, the set of morphisms $(X, x) \rightarrow (Y, y)$ up to basepoint-preserving homotopies.

Theorem C.1. If (X, x) and (Y, y) are pointed spaces, and (X, x) is well pointed (e.g., X is a manifold or X admits a triangulation with x as a vertex), then the evaluation at x , $\text{ev}_x : \text{Map}(X, Y) \rightarrow Y$, is a fibration with fiber $\text{ev}_x^{-1}\{y\} = \text{Map}_*(X, Y)$.

Fibrations $p : E \rightarrow B$ admit a path-lifting result analogous to Lemma B.1, but instead of the lift being unique, it is only unique up to homotopies preserving the starting point and the composition with p . Similarly to Definition B.2, this gives rise to the *monodromy action* of $\pi_1(B, b)$ on the path components of $p^{-1}\{b\}$. As the path components of E are precisely the orbits of the monodromy action, we obtain the following result.

Corollary C.1. If (X, x) and (Y, y) are compactly generated pointed spaces and (X, x) is well pointed, then there exists a natural isomorphism

$$[X, Y] \cong [X, Y]_* / \pi_1(Y, y).$$

Example C.1. If (S^n, e_0) is the n sphere pointed at the north pole and (X, x) is a compactly generated pointed space, then $[S^n, X]_* \cong \pi_n(X, x)$, and the monodromy action coincides with the usual π_1 action on the n th homotopy group.

Next, we recall a decomposition result that is useful for computing mapping spaces.

Definition C.2. Let (X, x) and (Y, y) be pointed topological spaces. Then their *wedge summation* is the pointed space $(Y \vee Z, *)$, where $Y \vee Z$ is the space obtained by attaching Y and Z along y and z , and $*$ $\in Y \vee Z$ is the point of attachment.

The defining feature of the wedge sum is that, for any third pointed space (Z, z) , there exists a canonical homeomorphism

$$\text{Map}_*(X \vee Y, Z) \cong \text{Map}_*(X, Z) \times \text{Map}_*(Y, Z). \quad (\text{C1})$$

This has the following immediate consequence.

Proposition C.2. Let (X, x) , (Y, y) , and (Z, z) be pointed topological spaces. Then

$$[Y \vee Z, W]_* \cong [Y, W]_* \times [Z, W]_*. \quad (\text{C2})$$

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