Spectral analysis of current fluctuations in periodically driven stochastic systems

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Probability current fluctuations play an important role in nonequilibrium statistical mechanics, and are a key object of interest in both theoretical studies and in practical applications. So far, most of the studies were devoted to the fluctuations of the time-averaged probability current, the zero-frequency Fourier component of the time-dependent current. However, in many practical applications the fluctuations at other frequencies are of equal importance. Here we study the statistics of all the probability current's Fourier component in periodically driven stochastic systems. We restrict our study to "trapped" systems where the degrees of freedom of the system cannot achieve arbitrarily large values as time becomes large, in contrast to, e.g., diffusing systems. First, we discuss possible methods to calculate the current statistics, valid even when the current's Fourier frequency is incommensurate with the driving frequency, breaking the time periodicity of the system. Somewhat surprisingly, we find that the cumulant generating function (CGF), that encodes all the statistics of the current, is composed of a continuous background at any frequency accompanied by either positive or negative discontinuities at current's frequencies commensurate with the driving frequency. We show that cumulants of increasing orders display discontinuities at an increasing number of locations but with decreasing amplitudes that depend on the rational frequency ratio. All these discontinuities are then transcribed in the behavior of the CGF. As the measurement time increases, these discontinuities become sharper but keep the same amplitude and eventually lead to discontinuities of the CGF at all the frequencies that are commensurate with the driving frequency in the limit of infinitely long measurement. We demonstrate our formalism and its consequences on three types of models: an underdamped Brownian particle in a periodically driven harmonic potential; a periodically driven run-and-tumble particle; and a two-state system. Our results show a rich and interesting structure in experimentally accessible and important objects: the fluctuations of alternating currents as a function of their frequency.

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I. INTRODUCTION

Equilibrium statistical mechanics is a mature framework that provides a calculation scheme for the ensemble averages and the fluctuations of many quantities of interest, with only mild assumptions on the system, e.g., thermodynamic equilibrium and short-range interactions. In practice, however, equilibrium systems are the exception rather than the norm. Despite many efforts, no equivalent general framework is known for systems which are far from thermal equilibrium. Characterizing the mean value and fluctuations for most quantities of interest remains a challenging task, even in the relatively simple cases of a system in a nonequilibrium steady state [1–3] or a periodically driven system [4–6].

Important progress has been achieved in the 1950s for systems which are close to thermal equilibrium, in the form of the fluctuation-dissipation relations [7]. These relate the equilibrium fluctuations of a system to the rate of dissipation when the system is weakly driven away from equilibrium. The fluctuation-dissipation relations were later found to be the near-equilibrium limit of the Gallavotti-Cohen fluctuation relations derived in the 1990s, which constitute rare exact results holding arbitrarily far from equilibrium. They imply a fundamental symmetry on the distributions of the entropy production [8,9] and of the probability currents [10] in the system and naturally extend the fluctuation-dissipation relations. While they hold both for entropy production and currents for nonequilibrium steady-state systems [8,10,11], in periodically driven systems they hold only for entropy production [12] but generically not for the currents [13].

Even more recently, thermodynamic uncertainty relations have been derived and proven to hold arbitrarily far from equilibrium [14–16]. These relations bound the probability current fluctuations in the system through explicit relations between these fluctuations and the mean entropy production. They imply that decreasing the current fluctuations in the steady state comes at a cost in terms of the entropy production. Recently, these relations were extended to periodically driven systems too [17–21]. Both the Gallavotti-Cohen and the thermodynamic uncertainty principle hold for arbitrary systems, but are of most practical interest in small systems where large fluctuations occur with a nonvanishing probability.

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In this study we consider the fluctuations of the probability current's Fourier components in stochastic nonequilibrium systems that are subject to time-periodic driving. The fluctuations of the zero-frequency probability current (commonly referred to as the "dc current") are tightly related to both the Gallavotti-Cohen and the thermodynamic uncertainty relations described above. They were thus studied intensively for systems which are subject to a time-independent thermodynamic forcing that breaks detailed balance, driving them into a nonequilibrium steady state [22–30]. However, only few works were dedicated to other Fourier components of the currents. To the best of our knowledge, only the zero-frequency current fluctuations [31,32] and the fluctuations of the current's Fourier component corresponding to the frequency of the periodic forcing [33] have been investigated in periodically driven stochastic systems.

The motivation to study the statistics of the probability current's Fourier components is natural from a theoretical perspective since these fluctuations are a simple generalization of the fruitful zero-frequency Fourier component. In addition to the theoretical motivation, probability current fluctuations are of interest from a practical point of view too. Often, these fluctuations have important implications. This is commonly the case when the currents in a periodically driven system, e.g., the ac electric currents in any device, can affect a system which is sensitive to a different frequency range, e.g., a measuring device that is sensitive to the electric fields in some frequency range. In such cases, the electromagnetic radiation generated by the ac electric currents is detected in the measuring device. If there were no current fluctuations in the driven system, then the currents at the driving frequency would not affect the measuring device, as they are out of the frequency range of the measuring device. However, due to the thermal environment, the driven system generically does have current fluctuations on a very broad range of frequencies. Current fluctuations in this frequency range are thus a source of noise impacting the measuring device. Hence, the frequency dependence of the current fluctuations in the periodically driven system is of great importance.

An interesting example that demonstrates the importance of current fluctuations is the Paul trap, commonly used to trap ions [34]. In this system, a high-frequency alternating electric current is driven through the trap's electrodes. This electric current generates a time-dependent electromagnetic field that traps the ions. However, due to the thermal noise in the electrodes, the electric current is fluctuating. Fluctuations of the current at the trapping frequency (which is not at the same frequency as the driving frequency) heat the ions, and limit many applications. Commonly, the heating rate of the ions is orders of magnitude higher than expected by standard thermal noise considerations [35]. The exact reason for this anomalous heating is yet poorly understood, and understanding the frequency dependence of the current fluctuations might be useful to better understand this phenomenon. We note that although in the case of ion traps the current is an electric current of charged particles, it is mainly the current fluctuations, not density fluctuations, that are affecting the ions. Therefore, although our results were derived for systems of noninteracting particles, they can be applied for current fluctuations in electric circuits in systems where the electron-electron interactions can be neglected.

In this paper we study the statistics of all the Fourier components of the probability current when a stochastic Markovian system is periodically driven at a given driving frequency. Our main object of interest is the *cumulant generating function* (CGF) of the current Fourier components, which encodes the full information on the current statistics. First, we use the Oseledets theorem to extend the framework of the current cumulant generating function from the case where the driving frequency is commensurate with the Fourier's frequency of the current to the case of incommensurate frequency as well. We then discuss two methods for calculating the probability current CGF: For commensurate frequencies, the CGF can be calculated directly from the Floquet spectrum of the tilted operator, with the same method used for the zero-frequency current CGF in periodically driven systems [31–33]. However, for incommensurate frequencies this method cannot be applied since the tilted operator depends on both frequencies and is hence aperiodic. A different method is thus used: it exploits the Floquet representation of the density propagator in terms of the driving frequency together with the specific relations between the density and the current's Fourier frequencies to calculate all the cumulants (i.e., the connected moments) of the probability current distribution.

Surprisingly, we find that the CGF has a structure similar to the famous Thomae's function [36], namely, it is composed of a continuous "background" for frequencies which are incommensurate with respect to the periodic driving, accompanied by (positive or negative) discontinuities at commensurate frequencies. The integer denominator of the rational ratio between the commensurate frequencies at which the discontinuities appear can be directly related to the order of the cumulant that has a discontinuous behavior at this frequency. Moreover, we explain how our results are modified for a finite time experiment, where the discontinuities are smoothed into continuous peaks that get sharper as the duration of the measurement increases. This general behavior is demonstrated on a Brownian particle in a periodically driven harmonic potential and a periodically driven trapped run-and-tumble particle, where analytical results for the first two cumulants can be obtained, and in a two-state system where the cumulant generating function can be numerically evaluated.

The setup studied in this paper seems at first sight similar to the *stochastic resonance* setup, where a bistable system coupled to a noise source is weakly perturbed by time-periodic forcing (see [37] for a review on stochastic resonance). However, we mention some key discrepancies to note between these setups. The main object of interest in stochastic resonance is the response of the system to the driving as a function of the noise intensity, where the response of the system is usually the transitions between the two metastable states at the driving frequency. In other words, the interest in stochastic resonance is in the case where the driving frequency coincides with the *fixed Fourier frequency* of the current between the two metastable states. In contrast, we consider a quite generic periodic driving, with a *fixed noise intensity* that is not necessarily weak and does not necessarily have two metastable states. We are interested in the frequency dependence of the fluctuations and not on the noise intensity dependence. Finally, we consider the full current statistics, and not only the average current as in stochastic resonance.

The paper is organized as follows: In Sec. II we detail the general setting that we consider and give the main technical background needed to obtain our results. In Sec. III we provide a short summary of our main results and their implications. In Sec. IV we derive a general expression for the time-averaged ac current and its variance and discuss briefly the behavior of higher-order cumulants. In Sec. V we show how to apply our results to simple exactly solvable systems. Finally, in Sec. VI we conclude and give some insights on new unexplored directions. Some technical details of the computations and complicated expressions are relegated to the Appendixes. In particular, in Appendix A we derive the expression of the cumulants for arbitrary order.

II. SETUP: PERIODICALLY DRIVEN MARKOVIAN SYSTEMS

A. Markov propagator

In this paper we consider the statistics of the current's Fourier components in a periodically driven stochastic system. To this end, we assume that the evolution of the probability distribution associated with the system is Markovian, as detailed in what follows. The microstate of the system, for example, the position of a particle in space, is denoted by the vector $\mathbf{x}$. We assume that the d-dimensional space of all $\mathbf{x}$ is simply connected.[1] The probability distribution of all the microstates at time t is uniquely characterized by the initial state of the system, which for simplicity we assume to be a specific microstate $\mathbf{x_0}$, as well as by the propagator $G(\mathbf{x}, t|\mathbf{x}_0, t_0)$, which is the transition probability density to be in a final state $\mathbf{x}$ at time t, provided that the system was at $\mathbf{x}_0$ at time t_0. It satisfies the Fokker-Planck equation [38]

$$\begin{aligned}\partial_t G &= \mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t)G,\\ \mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t) &= -\sum_\beta \partial_{x_\beta} F_\beta(\mathbf{x}, t) + \sum_{\beta,\gamma} \partial_{x_\beta,x_\gamma} D_{\beta\gamma}(\mathbf{x}, t),\end{aligned} \tag{1}$$

where here and in the following we use greek letters for the indices of the microstate space. The initial condition is given by

$$G(\mathbf{x}, t_0|\mathbf{x}_0, t_0) = \delta^d(\mathbf{x} - \mathbf{x}_0), \tag{2}$$

where $\delta^d(\mathbf{x}) = \prod_{\beta=1}^d \delta(x_\beta)$ is the d-dimensional Dirac delta function. By virtue of probability conservation, the propagator is normalized to unity,

$$\forall\, t \geqslant t_0, \quad \int d\mathbf{x}\, G(\mathbf{x}, t|\mathbf{x}_0, t_0) = 1. \tag{3}$$

Throughout the paper we assume that the drift $F_\beta(\mathbf{x}, t)$ and diffusion $D_{\beta\gamma}(\mathbf{x}, t)$ in Eq. (1) are both explicitly time dependent, nonsingular, and time periodic with cycle time T_d. We denote the corresponding fundamental angular frequency by $\omega_d = (2\pi)/T_d$, and we refer to it as the *driving frequency*, as this time dependence is what drives the system out of equilibrium. We will only consider cases where the system has a unique well-defined time-periodic stationary state in the limit $t \to \infty$ which is independent on the initial position $\mathbf{x}_0$. An important class of systems that are not encompassed by the framework of this paper are diffusing systems.

Equation (1) can be formally solved by

$$G(\mathbf{x}, t|\mathbf{x}_0, t_0) = \mathcal{T} e^{\int_{t_0}^t d\tau\, \mathcal{L}(\mathbf{x},\nabla_\mathbf{x},\tau)} \delta^d(\mathbf{x} - \mathbf{x}_0), \tag{4}$$

where $\mathcal{T}$ is the time-ordering operator. Exploiting the discrete time-translation symmetry of the system $\mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t + T_d) = \mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t)$ for any time t, we expand the propagator using Floquet theory as [39,40]

$$G(\mathbf{x}, t|\mathbf{x}_0, t_0) = \sum_{k=0}^\infty e^{-\lambda_k(t-t_0)} f_k(\mathbf{x}, t) g_k(\mathbf{x}_0, t_0), \tag{5}$$

where the functions f_k and g_k are the eigenvectors $U_{T_d} f_k = e^{-\lambda_k T_d} f_k$ and $U_{T_d}^\dagger g_k = e^{\lambda_k T_d} g_k$ of the operators

$$U_{T_d}(t) = \mathcal{T} e^{\int_t^{t+T_d} d\tau\, \mathcal{L}(\mathbf{x},\nabla_\mathbf{x},\tau)}, \tag{6}$$

$$U_{T_d}^\dagger(t_0) = \mathcal{T} e^{-\int_{t_0}^{t_0+T_d} d\tau\, \mathcal{L}^\dagger(\mathbf{x}_0,\nabla_{\mathbf{x}_0},\tau)}, \tag{7}$$

and where

$$\mathcal{L}^\dagger(\mathbf{x}, \nabla_\mathbf{x}, t_0) = \sum_\beta F_\beta(\mathbf{x}, t_0)\partial_{x_\beta} + \sum_{\beta,\gamma} D_{\beta\gamma}(\mathbf{x}, t_0)\partial_{x_\beta,x_\gamma} \tag{8}$$

is the adjoint of the evolution operator $\mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t)$. The Floquet eigenvalues λ_k's are ordered in the following way:

$$\mathrm{Re}(\lambda_0) = 0 < \mathrm{Re}(\lambda_1) \leqslant \mathrm{Re}(\lambda_2) \leqslant \cdots \tag{9}$$

and thus all but λ_0 have positive real part. Under our assumptions, the periodic solution is unique [12] and the eigenvalue $\lambda_0 = 0$ is nondegenerate. On the other hand, as $(t - t_0) \to \infty$, we obtain that

$$G(\mathbf{x}, t|\mathbf{x}_0, t_0) \xrightarrow[t\to\infty]{} f_0(\mathbf{x}, t) g_0(\mathbf{x}_0, t_0). \tag{10}$$

As this stationary state is independent of $\mathbf{x}_0$, the eigenfunction $g_0(\mathbf{x}_0, t_0)$ must be independent of both $\mathbf{x}_0$ and t_0. It is convenient to choose $g_0(\mathbf{x}_0, t_0) = 1$, so that the associated eigenfunction $f_0(\mathbf{x}, t)$ is positive and normalized to unity. Note that we assume this steady state to be nonsingular, i.e., $f_0(\mathbf{x}, t) > 0$ for $\mathbf{x}$ in a finite domain. More generally, exploiting the identities

$$G(\mathbf{x}, t|\mathbf{x}_0, t) = \delta^d(\mathbf{x} - \mathbf{x}_0), \tag{11}$$

$$\int d\mathbf{y}\, G(\mathbf{x}, t|\mathbf{y}, \tau) G(\mathbf{y}, \tau|\mathbf{x}_0, t_0) = G(\mathbf{x}, t|\mathbf{x}_0, t_0) \tag{12}$$

one can show that the eigenfunctions $f_k(\mathbf{x}, t)$ and $g_k(\mathbf{x}_0, t)$ are biorthogonal and satisfy a completeness relation

$$\sum_k f_k(\mathbf{x}, t) g_k(\mathbf{x}_0, t) = \delta^d(\mathbf{x} - \mathbf{x}_0), \tag{13}$$

$$\int d\mathbf{x}\, f_k(\mathbf{x}, t) g_l(\mathbf{x}, t) = \delta_{k,l}, \tag{14}$$

where in the last equation $\delta_{k,l}$ is the Kronecker delta, $\delta_{k,l} = 1$ if $k = l$ and $\delta_{k,l} = 0$ otherwise. As the propagator has a unique time-periodic expression $\lim_{n\to\infty} G(\mathbf{x}, nT_d +$

[1]This assumption is necessary to obtain our results. Although a generalization to nonsimply connected domains is possible, we do not consider these cases here.

$t|\mathbf{x}_0, t) = f_0(\mathbf{x}, t)$, the fluctuations of the microstates have a finite large-t limit, i.e.,

$$\forall\, \alpha, \quad \forall\, p \geqslant 0,$$
$$\lim_{n\to\infty} |\langle x_\alpha(nT_d + \tau)^p\rangle| = \left|\int d^d\mathbf{x}\, x_\alpha^p\, f_0(\mathbf{x}, \tau)\right| < \infty, \quad (15)$$

which clearly discards diffusive systems where $\mathrm{Var}(x_\alpha(t)) \propto t$ from our study.

B. Currents, alternating currents, and their statistics

So far we discussed the probability propagator in periodically driven systems. Next, let us discuss the probability current and its fluctuations. The probability current vector at time t and position $\mathbf{x}$, denoted by $\mathbf{J}(\mathbf{x}, t)$, is defined through the continuity equation [38]

$$\partial_t G + \nabla_{\mathbf{x}} \cdot \mathbf{J} = 0. \quad (16)$$

Using Eq. (1), it is given by

$$J_\alpha(\mathbf{x}, t) = F_\alpha(\mathbf{x}, t)G(\mathbf{x}, t) - \sum_\beta \partial_{x_\beta} D_{\alpha\beta}(\mathbf{x}, t)G(\mathbf{x}, t),$$

where here and in what follows $J_\alpha(\mathbf{x}, t)$ is the αth component of $\mathbf{J}_\alpha(\mathbf{x}, t)$, x_α is the αth component of $\mathbf{x}$, and $\dot{x}_\alpha$ is the αth component of $\dot{\mathbf{x}}$, the time derivative of the variable $\mathbf{x}(t)$.

For simplicity, we focus our attention on the state-space integral of the above current, given by

$$J_\alpha(t) = \int d\mathbf{x}\, J_\alpha(\mathbf{x}, t). \quad (17)$$

The generalization of our results for the position-dependent current is straightforward. As we are interested in all the Fourier components of the current, it is useful to write them explicitly,

$$J_\alpha(\omega_c) \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau\, \cos(\omega_c\tau) J_\alpha(\tau), \quad (18)$$

where for simplicity we used the cos transform.[2]

Whereas the equations above can be used to define and calculate the average current and its Fourier components, it cannot be used to calculate their fluctuations. To define and calculate current fluctuations, one has to consider a quantity associated with a single realization of the system, and whose average over the ensemble of realizations coincides with the above current. Let us first demonstrate the standard way this is done for the zero-frequency Fourier component, before generalizing this method to all other Fourier components. For any specific trajectory $\mathbf{x}(t)$, we associate an *empirical zero-frequency probability current*, using

$$\tilde{J}_\alpha(t) = \frac{1}{t}\int_0^t \dot{x}_\alpha(\tau) d\tau \equiv \frac{Q_\alpha(t)}{t}, \quad (19)$$

where we have also introduced $Q_\alpha(t)$, the *fluctuating charge* along the αth component [41], which is useful in what follows. Note that while the vocabulary used is similar to electrical systems, we do not assume that the system is an electrical system (although this can be the case); the term "charge" simply refers to the time integral of the probability current. The above definition of the empirical current is useful since the average empirical current $\langle\tilde{\mathbf{J}}(t)\rangle$, where averaging is done over noise realizations, equals the space integral of the Fokker-Planck current, defined in Eq. (17) [42,43]. As $t \to \infty$, this average also coincides with the zero-frequency component $\omega_c = 0$ of Eq. (18). As this empirical current is defined per realization, it is a stochastic quantity and it is possible to calculate not only its average value, but also its fluctuations. An additional important point is that ergodicity implies that in the limit $t \to \infty$, the empirical current of a specific realization converges to its ensemble average, i.e., to the Fokker-Planck current as well. Therefore, no ensemble averaging is required in this limit.

The above definitions for the zero Fourier frequency empirical current and fluctuating charge can be easily generalized to any other Fourier frequency in the following way. For a given realization, the empirical alternating current at frequency ω_c is defined as

$$\tilde{J}_{\omega_c;\alpha}(t) = \frac{1}{t}\int_0^t d\tau\, \dot{x}_\alpha(\tau)\, \cos(\omega_c\tau), \quad (20)$$

and in the $t \to \infty$ it converges to the ω_c Fourier coefficient of the current, namely, to $J_\alpha(\omega_c)$ defined in Eq. (18).

The corresponding empirical (fluctuating) alternating charge at frequency ω_c is defined as

$$\begin{aligned} Q_{\omega_c;\alpha}(t) = t\tilde{J}_{\omega_c;\alpha}(t) &= \int_0^t d\tau\, \dot{x}_\alpha(\tau)\, \cos(\omega_c\tau) \\ &= [x_\alpha(\tau)\cos(\omega_c\tau)]_{\tau=0}^{\tau=t} + \omega_c\int_0^t d\tau\, x_\alpha(\tau)\, \sin(\omega_c\tau). \end{aligned} \quad (21)$$

Note that the term "fluctuating" corresponds to a random quantity that differs from one realization to the next and should not be confused with the term "alternating." The random variable $Q_{\omega_c;\alpha}(t)$ is a functional of the whole path $\mathbf{x}(\tau)$ for all $\tau \in [0, t]$. However, $Q_{\omega_c;\alpha}(t)$ can be calculated for any specific realization of the system using the above integral.

Taking $\omega_c = 0$ for the empirical alternating current and charge, the empirical direct charge defined in Eq. (19) is recovered. As we consider a simply connected space, this direct charge is path independent and is only a function of the initial and final positions, namely, $Q_{\omega_c=0;\alpha}(t) = x_\alpha(t) - x_\alpha(0)$. The probability distribution function (PDF) $P_{\omega_c=0,\alpha}(Q; t)$ of this zero-frequency (or direct) charge $Q_{0,\alpha}(t)$ can be obtained from the large-time limit of the propagator $G(\mathbf{x}, t|\mathbf{x}_0, 0) \to f_0(\mathbf{x}, t)$ and is given in the latter by

$$P_{\omega_c=0,\alpha}(Q; t) \to \int d\mathbf{x}\, f_0(\mathbf{x}, t)\delta(x_\alpha - x_{0,\alpha} - Q). \quad (22)$$

For all other values of the frequency ω_c, the empirical alternating charge is path dependent, and therefore computing the probabilty distribution function $P_{\omega_c,\alpha}(Q; t)$ of the alternating charge is a nontrivial task. It turns out to be simpler in this case to consider instead its moment-generating function (MGF)

$$\langle e^{\mu Q_{\omega_c;\alpha}(t)}\rangle_{\mathbf{x}_0} \equiv \int_{-\infty}^{\infty} dQ\, e^{\mu Q} P_{\omega_c,\alpha}(Q; t), \quad (23)$$

[2] A similar quantity can be defined with a sin function instead of the cos.

where the average $\langle \dots \rangle_{\mathbf{x}_0}$ is taken over all the stochastic trajectories starting from the same initial state $\mathbf{x}(0) = \mathbf{x}_0$. This generating function can also be written as

$$\langle e^{\mu Q_{\omega_c;\alpha}(t)} \rangle_{\mathbf{x}_0} = \int d\mathbf{x}\, G_\mu(\mathbf{x}, t|\mathbf{x}_0), \tag{24}$$

where the function $G_\mu(\mathbf{x}, t|\mathbf{x}_0, 0)$ evolves in time according to the so-called "tilted" operator [44]

$$\partial_t G_\mu = \mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t) G_\mu, \tag{25}$$

$$G_\mu(\mathbf{x}, 0|\mathbf{x}_0) = \delta^d(\mathbf{x} - \mathbf{x}_0). \tag{26}$$

The tilted operator can be written as

$$\mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t) = \sum_{k=0}^{2} [\mu \cos(\omega_c t)]^k \mathcal{L}^k(\mathbf{x}, \nabla_\mathbf{x}, t), \tag{27}$$

where

$$\begin{aligned} \mathcal{L}^0 &= \mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t), \\ \mathcal{L}^1 &= F_\alpha(\mathbf{x}, t) - \sum_\beta \partial_{x_\beta}[D_{\alpha\beta}(\mathbf{x}, t) + D_{\beta\alpha}(\mathbf{x}, t)], \\ \mathcal{L}^2 &= D_{\alpha\alpha}(\mathbf{x}, t). \end{aligned} \tag{28}$$

Similarly to the case of the propagator, one can formally express the solution as

$$G_\mu(\mathbf{x}, t|\mathbf{x}_0) = \mathcal{T} e^{\int_0^t d\tau\, \mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t)} \delta^d(\mathbf{x} - \mathbf{x}_0). \tag{29}$$

We note, however, that in contrast to $\mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t)$ which is periodic by construction, $\mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t)$ is not necessarily time periodic, as it has periodic components varying with both frequencies ω_c and ω_d. Only in the case where these frequencies are commensurate, does $\mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t)$ become periodic. In what follows, we first address the commensurate case, and then the more challenging case of incommensurate frequencies.

C. Commensurate frequencies

Let us first consider the case where the frequencies of the drive ω_d and of the charge ω_c are commensurate. In this case, the corresponding cycle times T_d and T_c are also commensurate, and there exist two natural numbers $n, m \in \mathbb{N}$, such that

$$T = nT_d = mT_c < \infty, \tag{30}$$

and consequently the tilted operator is periodic with a finite period T, namely, $\mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t + T) = \mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, t)$. Utilizing this periodicity, it is possible to apply the same method commonly used for calculating the zero-frequency current fluctuation in periodically driven systems [31–33]. This is done using the Floquet theory. To this end, we define the operators

$$U_{T;\mu}(t) = \mathcal{T} e^{\int_t^{t+T} d\tau\, \mathcal{L}_\mu(\mathbf{x}, \nabla_\mathbf{x}, \tau)}, \tag{31}$$

$$U^\dagger_{T;\mu}(t_0) = \mathcal{T} e^{-\int_{t_0}^{t_0+T} d\tau\, \mathcal{L}^\dagger_\mu(\mathbf{x}_0, \nabla_{\mathbf{x}_0}, \tau)}. \tag{32}$$

Introducing the common ordered set of Floquet eigenvalues of these operators $\{\lambda_k(\mu),\ \mathrm{Re}[\lambda_0(\mu)] < \mathrm{Re}[\lambda_1(\mu)] \leqslant \cdots\}$ and their associated respective eigenfunctions $f_k^\mu(\mathbf{x}, t)$ and $g_k^\mu(\mathbf{x}_0, t)$, we may express the MGF as

$$\langle e^{\mu Q_{\omega_c;\alpha}(t)} \rangle_{\mathbf{x}_0} = \sum_k e^{-\lambda_k(\mu) t} a_k(t; \mathbf{x}_0), \tag{33}$$

where the functions $a_k(t; \mathbf{x}_0) = \int d\mathbf{x}\, f_k^\mu(\mathbf{x}, t) g_k^\mu(\mathbf{x}_0, 0)$ are periodic in t with period T. In contrast to the propagator $G(\mathbf{x}, t|\mathbf{x}_0)$, the tilted propagator $G_\mu(\mathbf{x}, t|\mathbf{x}_0)$ does not propagate a probability distribution, and its eigenvalue with lowest real part $\lambda_0(\mu)$ is nonzero in general, apart for $\mu = 0$ where $\mathcal{L}_{\mu=0}(\mathbf{x}, \nabla_\mathbf{x}, t) = \mathcal{L}(\mathbf{x}, \nabla_\mathbf{x}, t)$. Using the Floquet theory, the *cumulant generating function* (CGF) in the large-time limit reads as

$$\begin{aligned} \chi_\mu(\omega_c) &\equiv \lim_{t\to\infty} \frac{1}{t} \ln\langle e^{\mu Q_{\omega_c;\alpha}(t)} \rangle_{\mathbf{x}_0} = -\lambda_0(\mu) \\ &= \sum_{p=1}^{\infty} \mathcal{Q}_p^\alpha(\omega_c)\, \mu^p, \end{aligned} \tag{34}$$

where $\mathcal{Q}_p^\alpha(\omega_c)$ corresponds to the large-time limit of the rescaled cumulant of the alternating charge

$$\mathcal{Q}_p^\alpha(\omega_c) \equiv \lim_{t\to\infty} \frac{\langle Q^p_{\omega_c;\alpha}(t) \rangle_{\mathrm{co}}}{t} = -\, \partial^p_\mu \lambda_0(\mu)\big|_{\mu=0}. \tag{35}$$

Here, the subscript co refers to a connected correlation function, and p is the order of the cumulant. The limit in this equation is well defined as the cumulants scale at most linearly with time. The specific form of the CGF obtained here is consistent with the probability distribution function (PDF) $P_{\omega_c}(Q; t|\mathbf{x}_0)$ of the alternating charge $Q_{\omega_c;\alpha}(t)$ taking a large deviation scaling form in the large-time limit. By definition, the MGF

$$\langle e^{\mu Q_{\omega_c;\alpha}(t)} \rangle_{\mathbf{x}_0} = \tilde{P}_{\omega_c}(Q; t|\mathbf{x}_0), \tag{36}$$

where $\tilde{P}_{\omega_c}(s; t|\mathbf{x}_0)$ is the (two-sided) Laplace transform of the PDF $P_{\omega_c}(Q; t|\mathbf{x}_0)$ at the specific value $s = -\mu$. Taking the inverse Laplace transform, we obtain, using a saddle-point approximation in the large-time-t limit, that the *atypical fluctuations* of the alternating charge are described by the large deviation form

$$P_{\omega_c}(Q; t|\mathbf{x}_0) = \int_\mathcal{C} \frac{ds}{2i\pi} \langle e^{s(Q - Q_{\omega_c;\alpha}(t))} \rangle_{\mathbf{x}_0} \asymp e^{-t\, \Phi_{\omega_c}(\frac{Q}{t})}, \tag{37}$$

where $\mathcal{C}$ is the Bromwich contour and we have used the notation (common in large deviation theory)

$$A(x, t) \asymp e^{-tB(\frac{x}{t})} \Leftrightarrow -\lim_{t\to\infty} \frac{1}{t} \ln A(at, t) = B(a).$$

The Gärtner-Ellis theorem states rigorously that the large deviation function $\Phi_{\omega_c}(J)$ is given by the Legendre transform of the CGF, i.e.,

$$\Phi_{\omega_c}(J) = \max_\mu [\mu J - \chi_{\omega_c}(\mu)], \tag{38}$$

which can simply be understood as a result of a saddle-point approximation for large Q and t with $J = Q/t = O(1)$. Using that all the nonzero cumulants have the same time scaling $O(t)$, we also expect the central limit theorem to hold for the long-time *typical fluctuations* of the alternating charge. The corresponding distribution is Gaussian with fluctuations of order $\sqrt{\mathrm{Var}(Q_{\omega_c;\alpha}(t))} = O(\sqrt{t})$ around the average value

$$P_{\omega_c,\alpha}(Q; t) \approx \frac{\exp\left(-\frac{(Q - \langle Q_{\omega_c;\alpha}(t)\rangle)^2}{2\,\mathrm{Var}(Q_{\omega_c;\alpha}(t))}\right)}{\sqrt{2\pi\, \mathrm{Var}\big(Q_{\omega_c;\alpha}(t)\big)}}. \tag{39}$$

D. Incommensurate frequencies

We now consider the situation where the frequency ω_d of the drive and ω_c of the current are incommensurate. As the tilted operator $\mathcal{L}_\mu(\mathbf{x},\nabla_\mathbf{x},t)$ is *not* periodic, we cannot use a Floquet expansion to describe the tilted propagator $G_\mu(\mathbf{x},t|\mathbf{x}_0)$ and it is not obvious how to compute the limit in Eq. (34). However, one can use Oseledets' multiplicative ergodic theorem [45] to show, nevertheless, that this limit is well defined for any frequency ω_c. This implies in particular that all the cumulants scale at most linearly in time in this more generic case.

A naive way to compute the CGF in this case is to consider a series of frequencies $\omega_c(n)$, commensurate with ω_d for any finite n but that converges to the desired incommensurate value of ω_c as $n\to\infty$. The CGF can then be calculated for any n using the technique described in Sec. II C for commensurate frequencies, and in the limit $n\to\infty$ one can hope to estimate $\chi_\mu(\omega_c)$. However, this naive approach only works if the CGF $\chi_\mu(\omega_c)$ is a continuous function of ω_c (at a fixed value of μ). Surprisingly, we prove below that for commensurate frequencies this is not the case, though it is a continuous function for incommensurate values.

III. MAIN RESULTS

Let us summarize our main results and their consequences. We first state the exact conditions under which these results hold: the system is any periodically driven system on a simply connected state space, with a fundamental frequency ω_d and a gapped Floquet spectrum, i.e., with a nonzero gap between the zero and first nonzero Floquet eigenvalues $\mathrm{Re}(\lambda_0)=0<\mathrm{Re}(\lambda_1)$. For these systems, the moments of the microstates are finite (15). We show analytically that under these conditions, the cumulant generating function (CGF) $\chi_\mu(\omega_c)$ of the alternating current in direction α is *not* a continuous function of the current's frequency ω_c in the large-time limit. The CGF varies smoothly for any current frequency ω_c incommensurate with the frequency ω_d of the drive, but displays additional discontinuities at commensurate frequencies. This feature can be explained by considering the behavior of the rescaled cumulants $\mathcal{Q}_p^\alpha(\omega_c)$ of order $p\geqslant 1$ as a function of the current frequency ω_c. Let us first mention the explicit results, derived in Sec. IV, for the average alternating charge

$$\mathcal{Q}_1^\alpha(\omega_c)=\lim_{t\to\infty}\frac{\langle Q_{\omega_c;\alpha}(t)\rangle}{t}=-\omega_c\sum_{n=0}^{\infty}\mathrm{Im}\big[C_0^n\big]\delta_{\omega_c,n\omega_d},\tag{40}$$

where C_0 is the Fourier coefficient defined through Eqs. (48) and (49). For the variance of the alternating charge, we obtain

$$\mathcal{Q}_2^\alpha(\omega_c)=\lim_{t\to\infty}\frac{\mathrm{Var}\big(Q_{\omega_c;\alpha}(t)\big)}{t}=\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)+\mathcal{Q}_2^{\mathrm{d},\alpha}(\omega_c),\tag{41}$$

where $\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)$ is the continuous "background" cumulant (hence the superscript b) which is nonzero at any frequency ω_c, and is given by

$$\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)=\sum_{l_2=1}^{\infty}\sum_{k=-\infty}^{\infty}\frac{\omega_c^2(\lambda_{l_2}-ik\omega_d)\,C_{0,l_2}^{-k,k}}{(\lambda_{l_2}-ik\omega_d)^2+\omega_c^2},\tag{42}$$

whereas $\mathcal{Q}_2^{\mathrm{d},\alpha}(\omega_c)$ is a zero almost everywhere discontinuous function (hence the superscript d) that is nonzero only for $\omega_c=n\omega_d/2$ with integer n, given by

$$\mathcal{Q}_2^{\mathrm{d},\alpha}(\omega_c)=-\sum_{n=1}^{\infty}\sum_{l_2=1}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{\sigma=\pm1}\frac{\omega_c^2\,C_{0,l_2}^{\sigma n-k,k}\,\delta_{2\omega_c,n\omega_d}}{2[\lambda_{l_2}+i(\sigma\omega_c-k\omega_d)]},$$

and in both equations the coefficients $C_{l_1,l_2}^{k_1,k_2}$ are defined through Eqs. (54) and (58).

More generally, we show analytically that higher-order cumulants of even order, namely, with $p=2r$ and $r\in\mathbb{N}^*$, are the sum of a generically nonzero continuous background $\mathcal{Q}_{2r}^{\mathrm{b},\alpha}(\omega_c)$, and of a discontinuous part $\mathcal{Q}_{2r}^{\mathrm{d},\alpha}(\omega_c)$ displaying discontinuities at frequencies

$$\omega_c=\frac{n}{2z}\omega_d,\quad n\in\mathbb{N}^*,\quad 1\leqslant z\in\mathbb{N}\leqslant r.\tag{43}$$

Similarly, we show analytically that the cumulants of odd order, with $p=2r+1$ and $r\in\mathbb{N}$, have zero continuous background $\mathcal{Q}_{2r+1}^{\mathrm{b},\alpha}(\omega_c)=0$ and their discontinuous part $\mathcal{Q}_{2r+1}^{\alpha}(\omega_c)=\mathcal{Q}_{2r+1}^{\mathrm{d},\alpha}(\omega_c)$ displays discontinuities at frequencies

$$\omega_c=\frac{n}{2z+1}\omega_d,\quad n\in\mathbb{N}^*,\quad 0\leqslant z\in\mathbb{N}\leqslant r.\tag{44}$$

Note that the number of discontinuities generically increases with the order p of the cumulant. However, the height of these discontinuities decreases with the order p, and goes to zero at the $p\to\infty$ limit, making $\chi_\mu(\omega_c)$ a continuous function at incommensurate ω_c. Depending on the specific periodic driving in the system, it may well be that some Fourier coefficients of $G(\mathbf{x},t|\mathbf{x}_0,0)$ are not present and consequently some of the discontinuities in Eqs. (43) and (44) will not appear in the pth-order cumulant $\mathcal{Q}_p^{\mathrm{d},\alpha}(\omega_c)$. However, no additional discontinuities than those in Eqs. (43) and (44) can appear.

The discontinuous behavior of the cumulant generating function is a result of the long-time limit: one clearly does not expect any discontinuity of the cumulant generating function at finite time. We show that at large, but finite time t, the behavior of the cumulant generating function is qualitatively the same, but as expected the discontinuities in the spectrum of the cumulants are smeared over a typical frequency scale of order $\sim t^{-1}$. The presence of these peaks of nonzero width for finite t can have important impact on measurements in experiments: a device with a working frequency ω_c placed in an environment that has driving frequency ω_d is subject to dissipation that depends nontrivially on the ratio ω_c/ω_d. In particular, choosing a value of this ratio within a window of order $\sim t^{-1}$ to a rational value p/q with q relatively small can lead to a substantial increase (or decrease) of the dissipation, which can be very useful experimentally.

IV. DERIVATION OF THE FIRST TWO CUMULANTS AND DISCUSSION ON HIGHER ORDER

In this section we derive explicitly the analytical expression of the first two cumulants in the long-time limit and discuss the structure of cumulants of arbitrary order $p \geqslant 1$. We relegate the detailed derivation of these cumulants to Appendix A.

A. Average alternating current

Let us first compute the finite time average alternating charge $\langle Q_{\omega_c;\alpha}(t)\rangle$. For $x(0)=0$, it reads as

$$\langle Q_{\omega_c;\alpha}(t)\rangle = \omega_c \sum_{l_1=0}^{\infty}\int_0^t dt_1\, e^{-\lambda l_1 t_1} C_{l_1}(t_1)\sin(\omega_c t_1) + \omega_c \sum_{l_1=0}^{\infty} e^{-\lambda l_1 t} C_{l_1}(t)\cos(\omega_c t), \tag{45}$$

where we have used

$$\langle x_\alpha(t)\rangle = \int d\mathbf{x}\, x_\alpha G(\mathbf{x},t|\mathbf{0},0) \tag{46}$$

$$= \sum_{l_1=0}^{\infty} e^{-\lambda_{l_1} t}\int d\mathbf{x}\, x_\alpha f_{l_1}(\mathbf{x},t) g_{l_1}(\mathbf{0},0).$$

$$= \sum_{l_1=0}^{\infty} e^{-\lambda_1 t} C_{l_1}(t). \tag{47}$$

We use the index l_1 to ease the generalization to higher-order cumulants in what follows. The second line in the above equation was obtained by applying Eq. (5) to express $G(\mathbf{x},t|\mathbf{0},0)$ in terms of f_{l_1} and g_{l_1}. Finally, in the third line we have defined the coefficients

$$C_{l_1}(t) = \int d\mathbf{x}\, x_\alpha f_{l_1}(\mathbf{x},t) g_{l_1}(\mathbf{0},0) \tag{48}$$

$$= \sum_{k=-\infty}^{\infty} C_{l_1}^k e^{ik\omega_d t} \tag{49}$$

which are by construction periodic functions of t with a fundamental frequency ω_d and can thus be expanded as in the second line (49). Note that $\langle x_\alpha(t)\rangle$ is a real function of t. Therefore, if the Floquet eigenvalue λ_{l_1} is real, the associated function $C_{l_1}(t)$ must also be real and $C_{l_1}^{-n} = C_{l_1}^{n*}$ (we denote by z^* here and in the following the complex conjugate of z). If the Floquet eigenvalue λ_{l_1} is complex, it comes in a complex-conjugate pair, say $\lambda_{l_1}^* = \lambda_{l_1+1}$, and correspondingly one must have $C_{l_1}(t) = C_{l_1+1}(t)^*$, which yields $C_{l_1}^{-n} = C_{l_1+1}^{n}{}^*$. Inserting this expansion into Eq. (45), we obtain in the regime where $t \gg \lambda_1^{-1}$

$$\langle Q_{\omega_c;\alpha}(t)\rangle \approx \sum_{k=-\infty}^{\infty}\Bigg[\frac{C_0^k}{\omega_c^2 - k^2\omega_d^2}\big[\omega_c^2 - k^2\omega_d^2\cos(\omega_c t)e^{ik\omega_d t} + ik\omega_d\omega_c e^{ik\omega_d t}\sin(\omega_c t)\big] + \sum_{l_1=1}^{\infty}\frac{C_{l_1}^k\,\omega_c^2}{\omega_c^2 + (\lambda_{l_1} - ik\omega_d)^2}\Bigg]. \tag{50}$$

As $\lambda_0 = 0$ we know that $C_0^{k*} = C_0^{-k}$, thus the rescaled alternating charge takes the following scaling form in the double scaling limit $t\to\infty$, $\omega_c - n\omega_d \to 0$ with $\phi = (\omega_c - n\omega_d)t = O(1)$ fixed:

$$\frac{\langle Q_{\omega_c;\alpha}(t)\rangle}{t} \approx -\omega_c\, \mathcal{J}_n((\omega_c - n\omega_d)t),$$
$$\mathcal{J}_n(\phi) = -\mathrm{Re}\big[C_0^n\big]\frac{2}{\phi}\sin\left(\frac{\phi}{2}\right)^2 + \mathrm{Im}\big[C_0^n\big]\frac{\sin(\phi)}{\phi}. \tag{51}$$

Taking the limit in $\mathcal{Q}_1^\alpha(\omega_c) = \lim_{t\to\infty}\langle Q_{\omega_c;\alpha}(t)\rangle/t$, the expression above can be simplified to obtain

$$\mathcal{Q}_1^\alpha(\omega_c) = -\omega_c\sum_{n=0}^{\infty}\mathcal{J}_n(0)\delta_{n\omega_d,\omega_c} = -\omega_c\sum_{n=0}^{\infty}\mathrm{Im}\big[C_0^n\big]\delta_{\omega_c,n\omega_d}. \tag{52}$$

This structure is clearly discontinuous in the $t\to\infty$ limit as any term for $\omega_c \neq n\omega_d$ decays as $1/t$. The average alternating charge $\mathcal{Q}_1^\alpha(\omega_c)$ is zero for any frequency $\omega_c \neq n\omega_d$ that is not a harmonic of the driving frequency ω_d as one could have naively guessed: there is no average probability current in periodically driven systems, except at frequencies which are integer multiples of the driving frequency.

B. Variance of the alternating charge

Next, we turn to the computation of the variance of the alternating charge. To this end, we first compute the connected two-time correlation function:

$$\langle x_\alpha(t_1)x_\alpha(t_2)\rangle_{\mathrm{co}} = \int d\mathbf{x}_1 d\mathbf{x}_2 x_{1,\alpha}x_{2,\alpha}[G(\mathbf{x}_2,t_2|\mathbf{x}_1,t_1) - G(\mathbf{x}_2,t_2|\mathbf{0},0)]G(\mathbf{x}_1,t_1|\mathbf{0},0)$$
$$= \sum_{l_1,l_2=0}^{\infty}\int d\mathbf{x}_1 d\mathbf{x}_2 x_{1,\alpha}x_{2,\alpha}\, e^{-(\lambda_{l_1}t_1 + \lambda_{l_2}t_2)} f_{l_1}(\mathbf{x},t_1)g_{l_1}(\mathbf{0},0)\big[e^{\lambda_{l_2}t_1} f_{l_2}(\mathbf{x}_2,t_2)g_{l_2}(\mathbf{x}_1,t_1) - f_{l_2}(\mathbf{x}_2,t_2)g_{l_2}(\mathbf{0},0)\big]. \tag{53}$$

In analogy to Eq. (48), we define the time-periodic coefficients

$$C_{l_1,l_2}(t_1,t_2) = \int d\mathbf{x}_1 d\mathbf{x}_2 x_{1,\alpha}x_{2,\alpha} f_{l_2}(\mathbf{x}_2,t_2)g_{l_2}(\mathbf{x}_1,t_1)f_{l_1}(\mathbf{x},t_1)g_{l_1}(\mathbf{0},0), \tag{54}$$

with which we can write

$$\langle x_\alpha(t_1)x_\alpha(t_2)\rangle_{\rm co}=\sum_{l_1,l_2=0}^{\infty}e^{-(\lambda_{l_2}t_2+\lambda_{l_1}t_1)}[e^{\lambda_{l_2}t_1}C_{l_1,l_2}(t_1,t_2)-C_{l_1}(t_1)C_{l_2}(t_2)]. \tag{55}$$

Using that $g_0(\mathbf{x},t)=1$, one can check from Eqs. (54) and (48) that $C_{l_1,l_2=0}(t_1,t_2)=C_{l_2=0}(t_2)C_{l_1}(t_1)$. For $t_2>t_1\gg\lambda_1^{-1}$, we can then approximate the connected correlation as

$$\langle x_\alpha(t_1)x_\alpha(t_2)\rangle_{\rm co}\approx\sum_{l_2=1}^{\infty}e^{-\lambda_{l_2}(t_2-t_1)}C_{0,l_2}(t_1,t_2). \tag{56}$$

The variance of the alternating charge is obtained at finite but large time $t\gg\lambda_1^{-1}$ as

$$\begin{aligned}\mathrm{Var}\big(Q_{\omega_c;\alpha}(t)\big)=&\,2\omega_c^2\sum_{l_2=1}^{\infty}\int_0^t dt_2\,\sin(\omega_c t_2)\int_0^{t_2}d\tau\,e^{-\lambda_{l_2}\tau}C_{0,l_2}(t_2-\tau,t_2)\sin(\omega_c t_2-\tau)\\&+2\omega_c\sum_{l_2=1}^{\infty}\cos(\omega_c t)\int_0^t d\tau\,e^{-\lambda_{l_2}\tau}C_{0,l_2}(t-\tau,t)\sin(\omega_c t-\tau)+\mathrm{Var}(x_\alpha(t))\cos^2(\omega_c t).\end{aligned} \tag{57}$$

Inserting the Fourier expansion of the coefficients

$$C_{0,l_2}(t-\tau,t)=\sum_{k_1,k_2=-\infty}^{\infty}C_{0,l_2}^{k_1,k_2}e^{i\omega_d[(k_1+k_2)t-k_1\tau]}, \tag{58}$$

it yields the following expression:

$$\begin{aligned}\mathrm{Var}\big(Q_{\omega_c;\alpha}(t)\big)\approx&-\sum_{k_1,k_2=-\infty}^{\infty}\sum_{\sigma_1,\sigma_2=\pm1}\sum_{l_2=1}^{\infty}\frac{\sigma_1\sigma_2C_{0,l_2}^{k_1,k_2}\,\omega_c^2\,e^{i[\omega_d(k_1+k_2)-(\sigma_1+\sigma_2)\omega_c]\frac{t}{2}}}{2[\lambda_{l_2}+i(k_1\omega_d-\sigma_1\omega_c)]}\frac{2\,\sin\left\{[\omega_d(k_1+k_2)-(\sigma_1+\sigma_2)\omega_c]\frac{t}{2}\right\}}{\omega_d(k_1+k_2)-(\sigma_1+\sigma_2)\omega_c}\\&+\sum_{k_1,k_2=-\infty}^{\infty}\sum_{\sigma_1,\sigma_2=\pm1}\sum_{l_2=1}^{\infty}\frac{\omega_c}{2i}\sigma_1C_{0,l_2}^{k_1,k_2}\frac{e^{i[\omega_d(k_1+k_2)-\omega_c(\sigma_1+\sigma_2)]t}}{\lambda_{l_2}+i(k_1\omega_d-\sigma_1\omega_c)}+\mathrm{Var}(x_\alpha(t))\cos^2(\omega_c t),\end{aligned} \tag{59}$$

where we have used the representation

$$\sin(\omega_c t)=\sum_{\sigma=\pm1}\frac{(-\sigma)}{2i}e^{-i\sigma\omega_c t}. \tag{60}$$

We consider systems for which the cumulants of the microstates have a finite large-t limit (15), such that the term proportional to $\mathrm{Var}(x_\alpha(t))$ in this expression can be discarded from the computation of $\mathcal{Q}_2^\alpha(\omega_c)=\lim_{t\to\infty}\mathrm{Var}(Q_{\omega_c;\alpha}(t))/t$. Note that this is quite different from diffusive systems, which however do not fulfill our hypotheses and thus do not enter the class of systems considered in this paper. Similarly, the first term in the second line of (59) remains finite in the limit $t\to\infty$ and does not contribute to $\mathcal{Q}_2^\alpha(\omega_c)$. Taking the limit $\mathcal{Q}_2^\alpha(\omega_c)=\lim_{t\to\infty}\mathrm{Var}(Q_{\omega_c;\alpha}(t))/t$ of (59) gives

$$\mathcal{Q}_2^\alpha(\omega_c)=\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)+\mathcal{Q}_2^{\mathrm{d},\alpha}(\omega_c), \tag{61}$$

where the continuous background $\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)$ corresponds to the term for which $\sigma_1=-\sigma_2$ in the first line of Eq. (59) and reads as

$$\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)=\sum_{l_2=1}^{\infty}\sum_{k=-\infty}^{\infty}\frac{\omega_c^2(\lambda_{l_2}-ik\omega_d)C_{0,l_2}^{-k,k}}{(\lambda_{l_2}-ik\omega_d)^2+\omega_c^2}, \tag{62}$$

and the discontinuous part corresponds to the term for which $\sigma_1=\sigma_2$ in the first line of Eq. (59) and reads as

$$\mathcal{Q}_2^{\mathrm{d},\alpha}(\omega_c)=-\sum_{n=1}^{\infty}\sum_{l_2=1}^{\infty}\sum_{k=-\infty}^{\infty}\sum_{\sigma=\pm1}\frac{\omega_c^2\,C_{0,l_2}^{\sigma n-k,k}\,\delta_{2\omega_c,n\omega_d}}{2[\lambda_{l_2}+i(\sigma\omega_c-k\omega_d)]}. \tag{63}$$

It generally presents discontinuities for any integer or half-integer ratio of ω_c by ω_d but some discontinuities might not appear if the corresponding coefficients $C_{0,l_2}^{\sigma n-k,k}$ are zero. In contrast to $\mathcal{Q}_1^\alpha(\omega_c)$, which is zero at all frequencies except those that satisfy $\omega_c=n\omega_d$, the rescaled variance $\mathcal{Q}_2^\alpha(\omega_c)$ is generically nonzero for any frequency ω_c and equal to $\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c)$ away from the integer and half-integer multiplications of ω_d. Therefore, there are current fluctuations even at frequency with no average current. This is expected, as there are current fluctuations even in equilibrium systems.

C. Higher-order cumulants and the continuity of the CGF

The expressions for higher-order cumulants can be obtained explicitly too, but are quite cumbersome. In Appendix A we derive the general expression for the cumulants and in Appendixes B and C we, respectively, give explicit expressions for the third and fourth cumulants of the alternating charge.

To nevertheless justify our conclusion that the CGF is a discontinuous function at ω_c which is commensurate with the driving frequency ω_d, we next consider the structure of the p-order cumulant (the exact expression is given in Appendix A). Similarly to Eq. (59) for $p=2$, the general expression of $\mathcal{Q}_p^\alpha(\omega_c)$ is a sum over the p Fourier modes $k_1,\dots,k_p$ associated with the driving frequency ω_d coming from the p-times connected correlation function $\langle\prod_{k=1}^p x_\alpha(t_k)\rangle_{\rm co}$ and a sum over $\sigma_1,\dots,\sigma_p$ coming from the mode representation of $\sin(\omega_c t_k)$ in Eq. (60) for $k=1,\dots,p$. As we show in Appendix A, the leading time-dependent contribution to $t^{-1}Q_{\omega_c;\alpha}(t)$ of the strings $\{k_1,\dots,k_p\};\{\sigma_1,\dots,\sigma_p\}$ is given at finite but large time by

$$t^{-1}I_p(\{k_1,\dots,k_p\};\{\sigma_1,\dots,\sigma_p\};t)=\int_0^t\frac{d\tau}{t}e^{i\Omega_p\tau}, \tag{64}$$

where

$$\Omega_p=\omega_d\sum_{j=1}^p k_j-\omega_c\sum_{j=1}^p\sigma_j. \tag{65}$$

In the infinite-time limit $t\to\infty$, the only terms in the summation over the strings $\{k_1,\dots,k_p\};\{\sigma_1,\dots,\sigma_p\}$ that have nonzero contribution to $\mathcal{Q}_p(\omega_c)$, are the terms for which Eq. (64) has a nonzero limit. This is not the case for most strings $\{k_1,\dots,k_p\}$ and $\{\sigma_1,\dots,\sigma_p\}$ and only when $\Omega_p=0$ does Eq. (64) has a nonzero limit (equal to 1), namely, when the right-hand side of Eq. (65) vanishes.

There are several important consequences of the above observation. Let us therefore focus on the equation for the resonance condition of the pth cumulant

$$\omega_d\sum_{j=1}^p k_j-\omega_c\sum_{j=1}^p\sigma_j=0. \tag{66}$$

This equality holds when both sums are zero independently

$$\sum_{j=1}^p k_j=0,\quad \sum_{j=1}^p\sigma_j=0 \tag{67}$$

or when ω_c and ω_d reach some commensurate values

$$\frac{\omega_c}{\omega_d}=\frac{\sum_{j=1}^p k_j}{\sum_{j=1}^p\sigma_j}. \tag{68}$$

For incommensurate ω_d and ω_c, Eq. (68) cannot hold since both the numerator and denominator in the right-hand side of the equation are integers. Therefore, the only valid resonance condition corresponds to the case where both quantities in Eq. (67) are zero. One would thus expect in general the cumulants $\mathcal{Q}_p^\alpha(\omega_c)$ to have both a continuous background $\mathcal{Q}_p^{\mathrm{b},\alpha}(\omega_c)$ coming from the resonance condition (67) and a discontinuous part $\mathcal{Q}_p^{\mathrm{d},\alpha}(\omega_c)$ coming from the resonance condition (68).

However, remembering that $\sigma_j=\pm1$, the second line of Eq. (67) cannot vanish for odd values of p and the continuous background is therefore zero for odd cumulants. Thus, for incommensurate frequencies all the odd cumulants are exactly zero. For even values of p, there are strings such that each of the sums in Eq. (67) vanish independently, and therefore generically even cumulants have a nonzero continuous background $\mathcal{Q}_p^{\mathrm{b},\alpha}(\omega_c)$.

Next, let us consider commensurate frequencies. In this case, there are two options for the condition in Eq. (65) to hold: either Eq. (67) holds, or instead Eq. (68) can hold. For even values of p, the latter case is possible whenever the relation between ω_c and ω_d is of the form

$$\omega_c=\frac{n}{2z}\omega_d,\quad n\in\mathbb{N}^*,\quad 1\leqslant z\in\mathbb{N}\leqslant\frac{p}{2} \tag{69}$$

since $\sum_j k_j$ can take any integer value, whereas $\sum_j\sigma_j$ is bounded between $-p$ and p (skipping every other value). Similarly, for odd values of p, Eq. (65) can hold when

$$\omega_c=\frac{n}{2z+1}\omega_d,\quad n\in\mathbb{N}^*,\quad 0\leqslant z\in\mathbb{N}\leqslant\frac{p-1}{2}. \tag{70}$$

In both the even and odd cases we expect contributions to the cumulants in addition to those that appear in the incommensurate cases. Therefore, generically they have discontinuities at these specific commensurate frequencies. Note that depending on the specific periodic driving that one imposes, some of the discontinuities in Eqs. (69) and (70) may very well not appear in the pth-order cumulant $\mathcal{Q}_p^{\mathrm{d},\alpha}(\omega_c)$ but no additional discontinuities can appear.

At this point, one can rightfully suspect that the cumulant generating function $\chi_\mu(\omega_c)$, which holds the discontinuity of every cumulant, is for fixed μ an everywhere discontinuous function of the alternating charge's frequency ω_c. However, as we show in Appendix A 4, it is a continuous function for frequencies ω_c incommensurate with ω_d, similarly to the Thomae's function [36], as the magnitudes of the discontinuities decrease with p and eventually vanish as $p\to\infty$.

Before turning to some specific exactly solvable examples, we next show how to obtain the expression of the CGF in the limit $\omega_c\gg\omega_d$.

D. Large-ω_c limit of the CGF

In the case $\omega_c\gg\omega_d$, a simplified expression for the tilted operator $\mathcal{L}_\mu(\mathbf{x},\nabla_\mathbf{x},t)$ can be obtained. The bare dynamics that evolves over the period T_d is not fast enough to change over a period $T_c\ll T_d$, such that the terms depending on ω_c can be replaced in practice by their cycle average value over the period T_c, i.e.,

$$\mathcal{L}_\mu(\mathbf{x},\nabla_\mathbf{x},t)G_\mu(\mathbf{x},t|\mathbf{x}_0)\approx\overline{\mathcal{L}_\mu(\mathbf{x},\nabla_\mathbf{x},t)}^{\omega_c}G_\mu(\mathbf{x},t|\mathbf{x}_0),$$
$$\overline{\mathcal{L}_\mu(\mathbf{x},\nabla_\mathbf{x},t)}^{\omega_c}=\mathcal{L}(\mathbf{x},\nabla_\mathbf{x},t)+\frac{\mu^2}{2}D_{\alpha\alpha}(\mathbf{x},t), \tag{71}$$

where $\overline{A}^{\omega_c}=T_c^{-1}\int_0^{T_c}A(t)dt$. In the specific case where $D_{\alpha\alpha}(\mathbf{x},t)=D_{\alpha\alpha}(t)$ is independent of $\mathbf{x}$, one can show that the tilted propagator can be simply expressed as a function of the bare propagator as

$$G_\mu(\mathbf{x},t)\approx G(\mathbf{x},t|\mathbf{x}_0,0)e^{\frac{\mu^2}{2}\int_0^t D_{\alpha\alpha}(\tau)d\tau}. \tag{72}$$

From this result, one obtains that as $\omega_c\to\infty$

$$\chi_\mu(\omega_c)=\lim_{t\to\infty}\frac{1}{t}\ln\int d\mathbf{x}\,G_\mu(\mathbf{x},t)\approx\frac{\mu^2}{2}\overline{D_{\alpha\alpha}}^{\omega_d}, \tag{73}$$

$$\mathcal{Q}_p(\omega_c)=\partial_\mu^p\chi_\mu(\omega_c)\big|_{\mu=0}\approx\delta_{p,2}\overline{D_{\alpha\alpha}}^{\omega_d}, \tag{74}$$

where $\overline{D_{\alpha\alpha}}^{\omega_d}=T_d^{-1}\int_0^{T_d}D_{\alpha\alpha}(\tau)d\tau$. In particular, it implies that both the typical and atypical fluctuations of the alternating charge are Gaussian in the $\omega_c \to \infty$ limit. On the other hand, taking the same limit $\omega_c \gg \omega_d$ in Eq. (62), one obtains

$$\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c \to \infty) = \sum_{l_2=1}^{\infty} \sum_{k=-\infty}^{\infty} (\lambda_{l_2} - ik\omega_d)\, C_{0,l_2}^{-k,k} = \lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \langle x_\alpha(t_2)\dot{x}_\alpha(\tau)\rangle_{\mathrm{co}}\Big|_{t_2=\tau} = \frac{\overline{\mathrm{Var}(\dot{x}_\alpha)}^{\omega_d}}{2}. \tag{75}$$

The variance of the alternating charge can thus expressed in this limit as

$$\mathcal{Q}_2^{\mathrm{b},\alpha}(\omega_c) \approx \frac{\overline{\mathrm{Var}(\dot{x}_\alpha)}^{\omega_d}}{2} = \overline{D_{\alpha\alpha}}^{\omega_d}, \quad \omega_c \gg \omega_d. \tag{76}$$

The last equality is a consequence of the fluctuation-dissipation theorem for the time-averaged quantities for time-periodic quantities [46], which holds arbitrarily far from equilibrium.

V. SPECIFIC EXAMPLES

We have derived general expressions for the first two cumulants of the alternating charge. Next, we demonstrate these results in specific examples, which are commonly used in statistical mechanics and where a full analytical computation is possible.

A. Underdamped Ornstein-Uhlenbeck

As a first example, we consider the following dynamics:

$$\dot{v}(t) = \dot{\mu}(t) - \Gamma(t)[v(t) - \mu(t) + \Omega(t)x(t) - \sqrt{2D(t)}\eta(t)],$$
$$\dot{x}(t) = v(t), \tag{77}$$

where $\eta(t)$ is a Gaussian white noise with zero mean $\langle\eta(t)\rangle = 0$ and variance $\langle\eta(t_1)\eta(t_2)\rangle = \delta(t_1 - t_2)$. The damping $\Gamma(t)$, trapping frequency $\Omega(t)$, diffusion coefficient $D(t)$, and drift $\mu(t)$ are all time-periodic functions with fundamental angular frequency ω_d. Keeping track of both (x, v) at time t, the system is Markovian and the general theory described in the sections above applies to this case.

By linearity of the above equation, both the position $x(t)$ and the speed $v(t)$ are expressed as (infinite) linear combinations of the Gaussian random variables $\eta(\tau)$ for $\tau \in [0, t]$ and thus also have Gaussian fluctuations. The same reasoning applies to the empirical alternating charge $Q_{\omega_c;x}(t)$, which is expressed as (infinite) sums of $x(\tau)$ for $\tau \in [0, t]$. It yields that all the cumulants beyond the variance are zero, i.e.,

$$\mathcal{Q}_p^x(\omega_c) = 0, \quad p > 2 \tag{78}$$

$$\chi_\mu(\omega_c) = \mu\mathcal{Q}_1^x(\omega_c) + \frac{\mu^2}{2}\mathcal{Q}_2^x(\omega_c). \tag{79}$$

A direct consequence is that the large deviation function takes the quadratic form

$$\Phi_{\omega_c}(J) = \frac{\left[J - \mathcal{Q}_1^x(\omega_c)\right]^2}{2\mathcal{Q}_2^x(\omega_c)}. \tag{80}$$

The joint bare propagator $G(v, x, t|v_0, x_0, t_0)$ of the speed v and position x satisfies the equation

$$\partial_t G = -[v + \mu(t)]\partial_x G + \Gamma(t)\partial_v\{[v + \Omega(t)x]G\} + \Gamma(t)D(t)\partial_v^2 G. \tag{81}$$

While this problem can be studied in the most general case where $\Gamma(t)$, $\Omega(t)$, $\mu(t)$, and $D(t)$ are all time dependent, we consider a simpler analytically tractable example with constant damping $\Gamma(t) = \Gamma_0$ and trapping frequency $\Omega(t) = \Omega_0$ and introduce the Fourier series

$$\mu(t) = \sum_{k=-\infty}^{\infty} \mu_k\, e^{ik\omega_d t}, \quad D(t) = \sum_{k=-\infty}^{\infty} D_k\, e^{ik\omega_d t}, \tag{82}$$

which correspond to periodic forcing of the drift and the diffusion coefficients. We suppose that the functions $\mu(t)$ and $D(t)$ are real, such that for all k, $\mu_k = \mu_{R,|k|} + i\,\mathrm{sgn}[k]\mu_{I,|k|}$ and similarly $D_k = D_{R,|k|} + i\,\mathrm{sgn}[k]D_{I,|k|}$. In this case, the position at time t, starting with the initial condition $(x(0), v(0)) = (0, 0)$ is given by

$$x(t) = \sqrt{\frac{\Gamma_0}{\Gamma_0 - 4\Omega_0}} \sum_{\alpha=\pm 1} \alpha \int_0^t d\tau\, e^{-\frac{(t-\tau)}{2}[\Gamma_0 - \alpha\sqrt{\Gamma_0^2 - 4\Gamma_0\Omega_0}]} \times \left[\mu(\tau) + \frac{\dot{\mu}(\tau)}{\Gamma_0} - \eta(\tau)\right]. \tag{83}$$

This expression allows to identify the Floquet spectrum $\lambda_0 = 0 < \mathrm{Re}(\lambda_1) \leqslant \mathrm{Re}(\lambda_2)$ and $\lambda_k = +\infty$ for $k \geqslant 3$ with

$$\lambda_1 = \frac{\Gamma_0}{2} - \frac{\sqrt{\Gamma_0^2 - 4\Gamma_0\Omega_0}}{2},$$
$$\lambda_2 = \frac{\Gamma_0}{2} + \frac{\sqrt{\Gamma_0^2 - 4\Gamma_0\Omega_0}}{2}. \tag{84}$$

Using the expression of the position, one can show that

$$\langle x(t)\rangle = \sqrt{\frac{4\Gamma_0}{\Gamma_0 - 4\Omega_0}} \int_0^t d\tau\, e^{-\frac{\Gamma_0}{2}\tau} \times \sinh\left(\sqrt{\Gamma_0^2 - 4\Gamma_0\Omega_0}\frac{\tau}{2}\right)\left[\mu(t-\tau) + \frac{\dot{\mu}(t-\tau)}{\Gamma_0}\right]. \tag{85}$$

Inserting the Fourier series of $\mu(t)$ and taking the long-time limit $t \gg \lambda_1^{-1}$, one can identify this expression with

$$\langle x(t)\rangle \approx \sum_{k=-\infty}^{\infty} C_0^k e^{ik\omega_d t}, \tag{86}$$

which allows to obtain the expression of the coefficients

$$C_0^k = \frac{(\Gamma_0 + ik\omega_d)\mu_k}{\Gamma_0(\Omega_0 + ik\omega_d) - k^2\omega_d^2}. \tag{87}$$

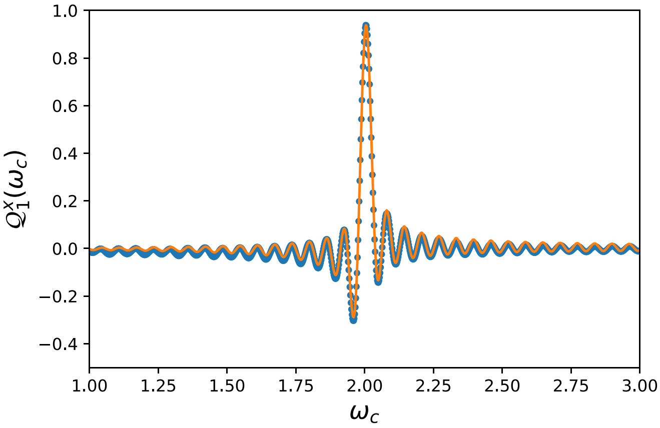

FIG. 1. Comparison between the time-averaged alternating current $\langle Q_{\omega_c}(t)\rangle/t$ (blue dots) computed numerically for a measurement time $t = 10^2 \gg \mathrm{Re}(\lambda_1)^{-1} = \frac{1}{2}$ for a periodically driven underdamped Brownian particle with $\Gamma_0 = 2$, $\Omega_0 = 1$, $\omega_d = 1$ and for $\mu(t) = 3/2\cos(2\omega_d t)$ and the analytical expression at finite time (orange line) given by inserting the coefficient C_0^k given in (87) into Eq. (51). The agreement is excellent.

Using this expression, we obtain the expression of the first cumulant

$$\mathcal{Q}_1^x(\omega_c) = \sum_{n=1}^{\infty} Q_{1,n}\delta_{n\omega_d,\omega_c},$$
$$Q_{1,n} = \frac{\mu_{R,n}\Gamma_0 n^2\omega_d^2\left[\Gamma_0(\Gamma_0-\Omega_0)+n^2\omega_d^2\right] - \mu_{I,n}\Gamma_0^2\Omega_0 n\omega_d}{\Gamma_0^2\Omega_0^2+\Gamma_0(\Gamma_0-2\Omega_0)n^2\omega_d^2+n^4\omega_d^4}. \tag{88}$$

In Fig. 1, a comparison between the analytical result for the rescaled average alternating charge $\langle Q_{\omega_c}(t)\rangle/t$ and results from a numerical simulations are plotted for a finite but large time $t = 10^2 \gg \mathrm{Re}(\lambda_1)^{-1} = \frac{1}{2}$. The numerical simulation is conducted by computing the evolution of random paths by discretization of the Langevin equation (77) with a time step of $dt = 10^{-2}$ and averaging over the Gaussian noise. The agreement between the finite-time expression and the numerical simulations is excellent.

One can similarly compute the two-time correlation function

$$\langle x(t_1)x(t_2)\rangle_{\mathrm{co}} = \langle x(t_1)x(t_2)\rangle - \langle x(t_1)\rangle\langle x(t_2)\rangle$$
$$= \sum_{\alpha_1,\alpha_2=\pm1}\frac{2\Gamma_0\alpha_1\alpha_2}{\Gamma_0-4\Omega_0}e^{-\frac{(t_2-t_1)}{2}\left[\Gamma_0-\alpha_2\sqrt{\Gamma_0^2-4\Gamma_0\Omega_0}\right]}$$
$$\times\int_0^{t_1}d\tau\, e^{-\frac{(t_1-\tau)}{2}\left[2\Gamma_0-(\alpha_1+\alpha_2)\sqrt{\Gamma_0^2-4\Gamma_0\Omega_0}\right]}D(\tau). \tag{89}$$

From this expression, it is possible to identify the coefficients

$$C_{0,l_2}^{k_1,k_2} = \frac{2\Gamma_0^{3/2}D_{k_1}}{\sqrt{\Gamma_0-4\Omega_0}}\frac{(\delta_{l_2,1}-\delta_{l_2,2})\delta_{k_2,0}}{[\Gamma_0+ik_1\omega_d][2\lambda_{l_2}+ik_1\omega_d]}. \tag{90}$$

One can check that for $\Gamma_0 < 4\Omega_0$, we have that $\mathrm{Im}[\lambda_1] = \mathrm{Im}[\lambda_2] = 0$ and $\mathrm{Im}[C_{0,l_2}^{0,0}] = 0$ while for $\Gamma_0 > 4\Omega_0$ we have $\lambda_1 = \lambda_2^*$ such that $C_{0,1}^{0,0} = C_{0,2}^{0,0*}$. After simplifications, one can obtain explicitly the value of the rescaled variance of the alternating charge by inserting the expression of the coefficients (90) and of the spectrum (84) into (41) and (42):

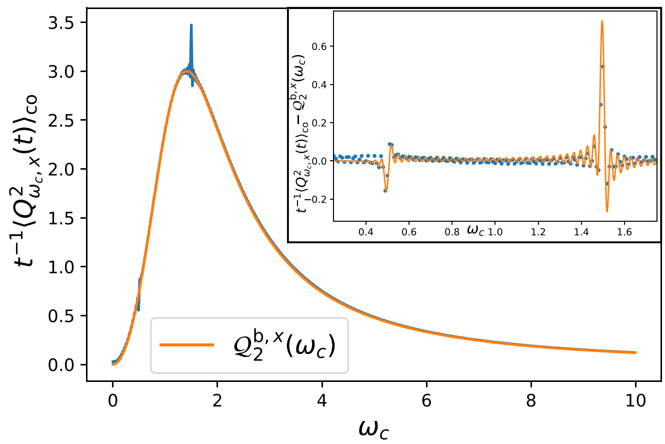

FIG. 2. Comparison between the rescaled variance of the alternating charge $\langle Q^2_{\omega_c,x}(t)\rangle_{\mathrm{co}}/t$ (in blue) computed numerically for a measurement time $t = 10^2$ for a periodically driven underdamped Brownian particle with $\Gamma_0 = 2$, $\Omega_0 = 1$, $\omega_d = 1$ and for $D(t) = 3[1 + 1/2\cos(\omega_d t) + 1/3\cos(3\omega_d t)]$ and the background value $\mathcal{Q}_2^{\mathrm{b},x}(\omega_c)$ (in orange) given in Eq. (92). In the inset we have plotted a comparison between the difference between the numerical result and the analytical expression for the background (in blue dots), highlighting the presence of discontinuities at values of ω_c corresponding to half the Fourier components of $D(t)$, i.e., for $\omega_c/\omega_d = \frac{1}{2}, \frac{3}{2}$ and the finite-time analytical prediction for the discontinuities (in orange). The discrepancy between the numerical simulation and analytical results is a consequence of the finite-time effects.

$$\mathcal{Q}_2^x(\omega_c)$$
$$= \frac{\Gamma_0^2\omega_c^2 D_0}{\Gamma_0^2\Omega_0^2+\Gamma_0(\Gamma_0-2\Omega_0)\omega_c^2+\omega_c^4}$$
$$-\sum_{n=1}^{\infty}\frac{\Gamma_0^2\omega_c^2 D_{R,n}\left[\Gamma_0^2\Omega_0^2-\Gamma_0(\Gamma_0+2\Omega_0)\omega_c^2+\omega_c^4\right]}{\left[\Gamma_0^2\Omega_0^2+\Gamma_0(\Gamma_0-2\Omega_0)\omega_c^2+\omega_c^4\right]^2}\delta_{n\omega_d,2\omega_c}$$
$$-\sum_{n=1}^{\infty}\frac{2\Gamma_0^3\omega_c^3 D_{I,n}\left(\Gamma_0\Omega_0-\omega_c^2\right)}{\left[\Gamma_0^2\Omega_0^2+\Gamma_0(\Gamma_0-2\Omega_0)\omega_c^2+\omega_c^4\right]^2}\delta_{n\omega_d,2\omega_c}. \tag{91}$$

Note that the continuous background

$$\mathcal{Q}_2^{\mathrm{b},x}(\omega_c) = \frac{\Gamma_0^2\omega_c^2 D_0}{\Gamma_0^2\Omega_0^2+\Gamma_0(\Gamma_0-2\Omega_0)\omega_c^2+\omega_c^4} \tag{92}$$

is identical to the variance of the fluctuating charge for a system with constant diffusion coefficient D_0, i.e., in the absence of any periodic drive. The only effect of the periodic drive is thus the emergence of the discontinuities at frequencies $\omega_c = n\omega_d/2$ for integer n. The continuous background of the spectrum of $\mathcal{Q}_2^{\mathrm{b},x}(\omega_c)$ presents a local maximum $\mathcal{Q}_2^{\mathrm{b},x}(\omega_c = \sqrt{\Gamma_0\Omega_0}) = D_0$ for $\sqrt{\Gamma_0\Omega_0}$. In Fig. 2, we show a comparison between our analytical computation for the rescaled variance $\mathcal{Q}_2^x(\omega_c)$ as a function of ω_c. The agreement is excellent for the background $\mathcal{Q}_2^{\mathrm{b},x}(\omega_c)$ and fairly good for the discontinuities (the deviations stem from neglecting some finite-time effects).

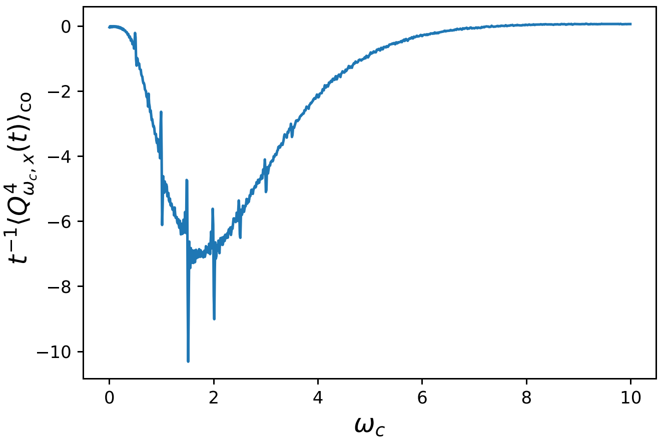

FIG. 3. Plot of the rescaled fourth-order cumulant of the alternating charge $\langle Q^4_{\omega_c,x}(t)\rangle_{\rm co}/t$ (in blue) computed numerically for a measurement time $t = 10^2$ computed numerically for a periodically driven run-and-tumble particle with $\gamma = 2$, $\Omega_0 = 1$, $\omega_d = 1$ and for $v(t) = 3[1 + 1/2\cos(\omega_d t) + 1/3\cos(3\omega_d t)]$. This even cumulant displays both a continuous background as well as discontinuities for ω_c corresponding to values equal to half or a quarter of the Fourier frequencies of $v(t)^4$.

While the average alternating charge $\mathcal{Q}_1^x(\omega_c)$ is independent of the diffusion coefficient $D(t)$, the variance $\mathcal{Q}_2^x(\omega_c)$ is independent of the drift $\mu(t)$. For a finite value of Γ_0, one has that $\mathcal{Q}_2^x(\omega_c) \to 0$ as $\omega_c \to \infty$ which is consistent with the absence of a term of the form $\partial_x^2 G$ in Eq. (81). Finally, the overdamped limit can easily be obtained in this example by taking the limit $\Gamma_0 \to \infty$. In this limit the microstate is reduced to the only position x and the Fokker-Planck equation for the effective propagator $G(x, t|x_0, t_0)$ reads as

$$\partial_t G = \Omega(t)\partial_x\{[x - \mu(t)]G\} + D(t)\partial_x^2 G. \qquad (93)$$

The background variance reads as $\mathcal{Q}_2^{\rm b}(\omega_c) = D_0\omega_c^2/(\Omega_0^2 + \omega_c^2)$. In the limit $\omega_c \to \infty$, one has that $\mathcal{Q}_2(\omega_c) \to D_0$, which is consistent with the prefactor of $\partial_x^2 G$ being $D(t)$ and $\overline{D(t)}^{\omega_d} = D_0$.

B. Periodically driven run-and-tumble particle

We next consider a simple extension of the model discussed in the previous example to characterize a particle subject to telegraphic noise confined in a time-varying potential, i.e.,

$$\dot{x}(t) = -\Omega(t)x(t) + v(t)\sigma(t), \quad x(0) = 0, \qquad (94)$$

where $\sigma(t) = \pm 1$ is a telegraphic noise with $\langle\sigma(t)\rangle = 0$ and $\langle\sigma(t_1)\sigma(t_2)\rangle = e^{-2\gamma|t_2-t_1|}$. The noise $\sigma(t)$ is not Gaussian in this framework, which in turn renders the alternating charge $Q_{\omega_c;x}(t)$ not Gaussian either. Hence,

$$\mathcal{Q}_p^x(\omega_c) \neq 0, \quad p > 2. \qquad (95)$$

In Fig. 3, we have plotted the fourth-order rescaled cumulant $\mathcal{Q}_4^{{\rm b},x}(\omega_c)$ for this specific model, showing indeed that it converges to a nonzero value for any finite ω_c. Note also that this cumulant presents discontinuities at general frequencies of the form $\omega_c = n\omega_d/4$ with n an integer number.

At any time t, the process (x, σ) is Markovian and this problem is therefore covered by the general theory exposed in the previous sections. The propagator $G_{\sigma,\sigma_0}(x, t|x_0, t_0)$ satisfies the equation

$$\begin{aligned}\partial_t G_{\sigma,\sigma_0} = &- \sigma v(t)\partial_x G_{\sigma,\sigma_0} + \Omega(t)\partial_x(xG_{\sigma,\sigma_0})\\ &+ \gamma(G_{-\sigma,\sigma_0} - G_{\sigma,\sigma_0}).\end{aligned} \qquad (96)$$

One can again obtain the value of the position at time t as

$$x(t) = \int_0^t d\tau\, e^{-\int_\tau^t \Omega(\tau')d\tau'} v(\tau)\sigma(\tau). \qquad (97)$$

Let us consider the special case where $\Omega(t) = \Omega_0$ to keep the model analytically tractable. We introduce the Fourier series

$$v(t) = \sum_{k=-\infty}^{\infty} v_k\, e^{ik\omega_d t}. \qquad (98)$$

As $\langle x(t)\rangle = 0$, it yields immediately $\mathcal{Q}_1(\omega_c) = 0$. The two-time connected correlation function reads as instead

$$\begin{aligned}\langle x(t_2)x(t_1)\rangle_{\rm co} = e^{-\Omega_0(t_1+t_2)} &\int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2\\ &\times e^{\Omega_0(\tau_1+\tau_2)-2\gamma|\tau_1-\tau_2|}v(\tau_1)v(\tau_2).\end{aligned} \qquad (99)$$

Inserting the Fourier expansion of $v(t)$ and computing explicitly the integrals, we identify this equation for large but finite time with

$$\langle x(t_2)x(t_1)\rangle_{\rm co} \approx \sum_{l_2=1}^{\infty} e^{-\lambda_{l_2}(t_2-t_1)} \sum_{k_1,k_2=-\infty}^{\infty} C_{0,l_2}^{k_1,k_2} e^{i\omega_d(k_1t_1+k_2t_2)}. \qquad (100)$$

We thus obtain the gapped Floquet spectrum for this model with

$$\lambda_0 = 0 < \lambda_1 = \min(\Omega_0, 2\gamma) \leqslant \lambda_2 = \max(\Omega_0, 2\gamma), \qquad (101)$$

as well as the coefficients

$$C_{0,2\gamma}^{k_1,k_2} = \frac{v_{k_1}v_{k_2}}{[\Omega_0 + 2\gamma + ik_1\omega_d][\Omega_0 - 2\gamma + ik_2\omega_d]}, \qquad (102)$$

$$C_{0,\Omega_0}^{k_1,k_2} = \sum_{n=-\infty}^{\infty} \frac{v_n v_{k_1-n}\delta_{k_2,0}}{\Omega_0 + 2\gamma + i(k_1-n)\omega_d} \times \left(\frac{2}{2\Omega_0 + ik_1\omega_d} - \frac{1}{\Omega_0 - 2\gamma + in\omega_d}\right). \qquad (103)$$

Note that for this model, we expect a dynamical phase transition for $\Omega_0 = 2\gamma$ where the two Floquet eigenvalues cross each other. In this example, the background rescaled variance of the alternating charge $\mathcal{Q}_2^{\rm b}(\omega_c)$ depends on the full time oscillations of $v(t)$ and reads as

$$\begin{aligned}\mathcal{Q}_2^{{\rm b},x}(\omega_c) &= \frac{2\gamma|v_0|^2\omega_c^2}{(\Omega_0^2+\omega_c^2)(4\gamma^2+\omega_c^2)} + \frac{\omega_c^2}{\Omega_0^2+\omega_c^2}\\ &\times \sum_{n=1}^{\infty} \frac{4\gamma|v_n|^2(4\gamma^2+n^2\omega_d^2+\omega_c^2)}{(4\gamma^2+n^2\omega_d^2)^2 + 2(4\gamma^2-n^2\omega_d^2)\omega_c^2+\omega_c^4}.\end{aligned} \qquad (104)$$

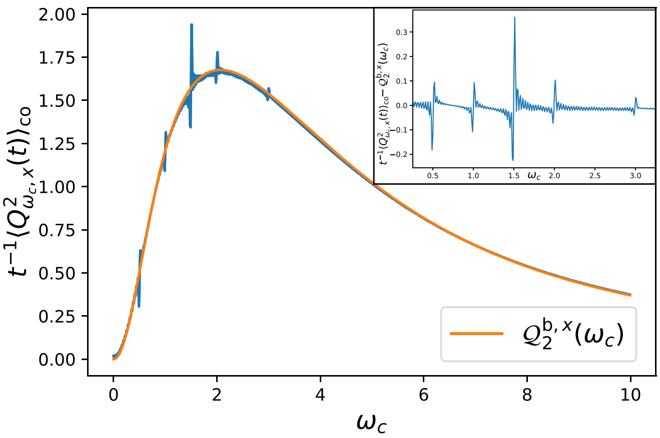

FIG. 4. Comparison between the rescaled variance of the alternating charge $\langle Q^2_{\omega_c,x}(t)\rangle_{\rm co}/t$ (in blue) computed numerically for a measurement time $t = 10^2$ for a periodically driven run-and-tumble particle with $\gamma = 2$, $\Omega_0 = 1$, $\omega_d = 1$ and for $v(t) = 3[1 + 1/2\cos(\omega_d t) + 1/3\cos(3\omega_d t)]$ and the background value $\mathcal{Q}_2^{\rm b,x}(\omega_c)$ (in orange) given in Eq. (104). In the inset we have plotted the difference between these two results showing that the agreement is excellent apart from the presence of discontinuities at values of ω_c corresponding to half the Fourier components of $v(t)^2$, i.e., for $\omega_c/\omega_d = \frac{1}{2}, 1, \frac{3}{2}, 2, 3$. The measurement time is $t = 10^2$.

In Fig. 4 we show a comparison between this analytical result and the rescaled variance of the alternating charge for the RTP model described above. The agreement is excellent apart for values of ω_c in close proximity to half integer values of the Fourier components of $v(t)^2$. Taking the limit $\omega_c \to \infty$ for finite γ, one obtains that $\mathcal{Q}_2^{\rm b,x}(\omega_c) \to 0$, which is consistent with the absence of term of the form $\partial_x^2 G$ in Eq. (96). On the other hand, taking the limit $\gamma, v_n \to \infty$ with $|v_n|^2/\gamma = O(1)$, one obtains the result for the overdamped Brownian motion $\mathcal{Q}_2^{\rm b}(\omega_c) = D_0\omega_c^2/(\Omega_0^2 + \omega_c^2)$ where $D_0 = (|v_0|^2 + 2\sum_{n>1}|v_n|^2)/(2\gamma)$.

C. Two-level system

In the previous two examples, we were able to compute the first few cumulants of the alternating charge. Achieving a numerical computation of the fourth-order cumulant with enough precision was a hard task that required $\sim 10^9$ realizations. However, computing the cumulant generating function itself beyond Gaussian models, even numerically, seems a formidable task. We therefore consider next a different example, where the cumulant generating function can actually be evaluated numerically by a different method. Specifically, we consider a two-level system, with a probability vector $|P(t)\rangle = (p_1(t), p_2(t))^{\rm T}$, where $p_i(t)$ is the probability for the system to be in state i at time t. $|P(t)\rangle$ evolves in time according to a master equation

$$|\dot{P}(t)\rangle = R(t)|P(t)\rangle, \quad R(t) = \begin{pmatrix} -k_1(t) & k_2(t) \\ k_1(t) & -k_2(t) \end{pmatrix}, \qquad (105)$$

where $R(t)$ is periodic with angular frequency ω_d. A given realization of the process is a sequence of successive states of the system $i_1 \to i_2 \to \cdots \to i_n$ and the corresponding sequence of times t_k at which the system jumped from state i_k to i_{k+1}. For any realization, we define its *empirical alternating charge* at frequency ω_c as

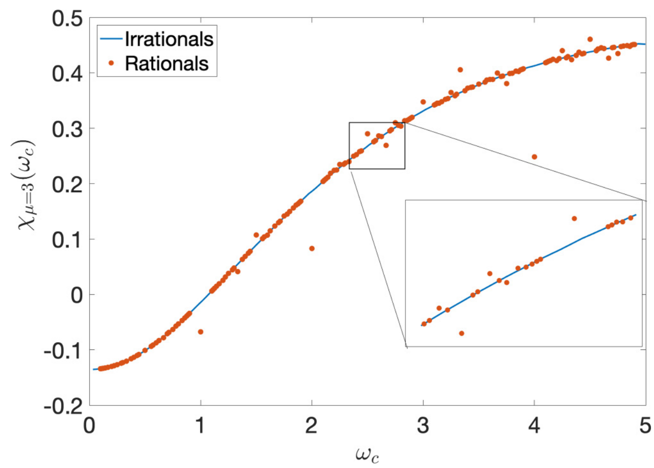

FIG. 5. Plot of the CGF $\chi_\mu(\omega_c)$ for the two-state system, with $\mu = 3$ and $\omega_d = 1$. The specific periodic driving $k_1(t)$ and $k_2(t)$ are described in Eq. (109). The blue line is the continuous background, calculated for incommensurate values of ω_c, and the red dots were calculated at frequencies commensurate with $\omega_d = 1$. The inset shows a blowup of a small segment, where one can observe that the function $\chi_\mu(\omega_c)$ for rational ω_c is always away from the value of $\chi_\mu(\omega_c)$ at irrational ω_c.

$$Q_{\omega_c}(t) = \sum_k (i_{k+1} - i_k)\cos(\omega_c t_k), \qquad (106)$$

where the sum is over the times $t_k < t$ where the system jumps from one state to the other of the specific realization. As in the continuous state systems, we are interested in the fluctuations of this quantity as $t \to \infty$. As the system is only composed of two states, there is no cycle and therefore the direct current (dc) corresponding to $\omega_c = 0$ is identically zero. Recent progress has been made on this particular system [47] and in particular an exact representation of the CGF has been obtained for the dc currents. Adapting the method presented in [47] to the alternating charge (see details in Appendix D), we obtain

$$\chi_\mu(\omega_c) = -\lim_{t\to\infty}\int_0^t \frac{d\tau}{t}[k_1(\tau) + k_2(\tau)e^{-\mu\cos(\omega_c\tau)}y_\mu(\tau)]. \qquad (107)$$

In that expression, the function $y_\mu(t)$ satisfies a first-order nonlinear differential equation for fixed μ, which reads as [47]

$$\dot{y}_\mu = (y_\mu - e^{\mu\cos(\omega_c t)})(k_1 + k_2 e^{-\mu\cos(\omega_c t)}y_\mu). \qquad (108)$$

Using this particular representation, we show in Fig. 5 the CGF $\chi_\mu(\omega_c)$ as a function of ω_c, calculated for $\mu = 3$ and the following rates:

$$k_1(t) = \exp(\cos(\omega_d t) - 1),$$
$$k_2(t) = \exp(-4\sin(5\omega_d t) + 3\sin(7\omega_d t) - 2). \qquad (109)$$

These rates are simply chosen as an illustration.

The blue line was calculated for the series of frequencies $\omega_c = \frac{n}{10\pi}$ for integer values of n, incommensurate with $\omega_d = 1$. The red dots were calculated for the series of frequencies $\omega_c = \frac{n}{m}$ for $m \in \mathbb{N}^* \leqslant 50$ and all $n \in \mathbb{N} \leqslant 6m$, commensurate with ω_d. Both the commensurate and incommensurate cases were estimated by first solving numerically the differential equation (108) and then evaluating the integral in Eq. (107) up to $t = 2 \times 10^4$.

VI. CONCLUSION

In this paper we have computed explicitly the large-time fluctuations of the alternating charge $Q_{\omega_c;\alpha}(t)$ (defined as the time-integrated instantaneous alternating current in direction α) for a periodically driven stochastic system. We have shown that the cumulant generating function of this quantity (and as a consequence its large deviation function) is not a continuous function of the frequency ω_c of the charge. In particular, we have shown that there exists both a continuous background for any value of the charge and driving frequency, respectively ω_c and ω_d, and additional peaks for commensurate pair of frequencies. This general result has been confirmed by considering some exactly solvable models.

It would be interesting to test these fairly general results experimentally, especially on systems where current fluctuations play an important role, e.g., trapped ion setups. For this, it is important to consider the finite-time version of the results.

We have focused our interest in this paper on systems with a discrete gapped Floquet spectrum on a simply connected domain, where the position of the particle is confined to a finite portion of space. It would be interesting to consider similar results for systems on a finite loop, where there might be long-lasting average currents. Some simple exactly solvable cases can be considered, hinting that the cumulants of the alternating charge behave quite differently in this case, but obtaining a general result remains a challenge at the moment.

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APPENDIX A: DERIVATION OF THE CUMULANTS OF ARBITRARY ORDER

We consider in this Appendix the cumulants of the alternating charge $Q_{\omega_c;\alpha}(t)$ in the large-time limit for a gapped system, i.e., $\mathrm{Re}(\lambda_1) > \lambda_0 = 0$ where λ_0, λ_1 are the two lowest Floquet eigenvalues associated to the "bare" evolution operator $\mathcal{L}(\mathbf{x}, \nabla_{\mathbf{x}}, t)$, i.e., in the absence of tilting $\mu = 0$ (in this case the operator is periodic by definition and therefore the Floquet theory can be applied). As seen in Sec. II, we expect that the rescaled cumulants

$$\mathcal{Q}_p^{\alpha}(\omega_c) = \lim_{t\to\infty} \frac{\langle Q_{\omega_c;\alpha}^p(t)\rangle_{\mathrm{co}}}{t} \tag{A1}$$

are of order $O(1)$ at most [we recall that a function of time $A(t)$ is defined to be of order $O(t^{\alpha})$ if $0 < \lim_{t\to\infty} t^{-\alpha}A(t) < \infty$]. In the following, we derive systematically for any value of p its expression as a function of the parameters ω_c, ω_d and the Floquet spectrum $\{\lambda_k, k \in \mathbb{N}\}$. Before deriving this expression, we first use Eq. (21) to rewrite

$$Q_{\omega_c;\alpha}(t) = [x_\alpha(\tau)\cos(\omega_c\tau)]_0^t + q_{\omega_c;\alpha}(t),$$
$$q_{\omega_c;\alpha}(t) = \omega_c\int_0^t d\tau\, x_\alpha(\tau)\sin(\omega_c\tau). \tag{A2}$$

For the systems that we are considering, we expect the average value and fluctuations of $x_\alpha(t)$ remain finite as $t \to \infty$ [Eq. (15)]. We start by showing the following identity for the infinite-time limit:

$$\mathcal{Q}_p^{\alpha}(\omega_c) = \lim_{t\to\infty} \frac{\langle Q_{\omega_c;\alpha}^p(t)\rangle_{\mathrm{co}}}{t} = \lim_{t\to\infty} \frac{\langle q_{\omega_c;\alpha}^p(t)\rangle_{\mathrm{co}}}{t}. \tag{A3}$$

1. Proof of identity (A3)

Let us first show that the CGF $\chi_\mu(\omega_c)$ as defined in Eq. (34) is identical to the CGF

$$\tilde{\chi}_\mu(\omega_c) = \lim_{t\to\infty} \frac{1}{t}\ln\langle e^{\mu q_{\omega_c;\alpha}(t)}\rangle, \tag{A4}$$

where we remind that

$$q_{\omega_c;\alpha}(t) = \omega_c\int_0^t d\tau\, x_\alpha(\tau)\sin(\omega_c\tau). \tag{A5}$$

To show this identity, we use that

$$Q_{\omega_c;\alpha}(t) = q_{\omega_c;\alpha}(t) + x_\alpha(t)\cos(\omega_c t) - x_\alpha(0), \tag{A6}$$

where $x_\alpha(0) = x_0$ is fixed. We introduce the joint PDF $P_t(q, \mathbf{x})$ for the respective random variables $q_{\omega_c}(t)$ and the final position $\mathbf{x}(t)$. We use Bayes' theorem as

$$P_t(q, \mathbf{x}) = P_t(q|\mathbf{x})P_t(\mathbf{x}), \tag{A7}$$

where $P_t(\mathbf{x}) = G(\mathbf{x}, t|\mathbf{x}_0, 0)$ is the PDF of the final position $\mathbf{x}(t)$ and $P_t(q|\mathbf{x})$ is the PDF of $q_{\omega_c;\alpha}(t)$ for a fixed final position $\mathbf{x}(t) = \mathbf{x}$. In the large-time limit, the final position $\mathbf{x}(t)$ remains finite for the trapped systems that we consider [Eq. (15)] while one naturally expects that the random variable $q_{\omega_c;\alpha}(t) = O(t)$. We thus expect the large deviation form

$$P_t(q|\mathbf{x}) \asymp e^{-t\,\varphi(\frac{q}{t},\mathbf{x})}, \tag{A8}$$

while there is no large deviation form for the final position

$$-\lim_{t\to\infty}\frac{1}{t}\ln G(\mathbf{x}, t|\mathbf{x}_0, 0) = \lambda_0 = 0. \tag{A9}$$

We may then show that on the one hand the MGF for $Q_{\omega_c;\alpha}(t)$ reads as, in the large-time limit,

$$\begin{aligned}\langle e^{\mu Q_{\omega_c;\alpha}(t)}\rangle &= e^{-\mu x_0}\int d\mathbf{x}\, G(\mathbf{x}, t|\mathbf{x}_0, 0)e^{\mu x_\alpha\cos(\omega_c t)}\\ &\quad\times\int dq\, e^{\mu q}P_t(q|\mathbf{x})\\ &\asymp e^{-\mu x_0}\int d\mathbf{x}\, f_0(\mathbf{x}, t)e^{\mu x_\alpha\cos(\omega_c t)}\\ &\quad\times\int \frac{dQ}{t}\, e^{t[\mu Q - \varphi(Q,\mathbf{x})]},\end{aligned} \tag{A10}$$

while on the other hand the MGF for $q_{\omega_c;\alpha}(t)$ reads as

$$\begin{aligned}\langle e^{\mu\, q_{\omega_c;\alpha}(t)}\rangle &= e^{-\mu x_0}\int d\mathbf{x}\, G(\mathbf{x}, t|\mathbf{x}_0, 0)\int dq\, e^{\mu q}P_t(q|\mathbf{x})\\ &\asymp e^{-\mu x_0}\int d\mathbf{x}\, f_0(\mathbf{x}, t)\int\frac{dQ}{t}\, e^{t[\mu Q - \varphi(Q,\mathbf{x})]}.\end{aligned} \tag{A11}$$

Using a saddle point-approximation then naturally yields the result

$$\chi_\mu(\omega_c) = \tilde{\chi}_\mu(\omega_c) = \max_{Q,\mathbf{x}} [\mu Q - \varphi(Q, \mathbf{x})], \tag{A12}$$

showing that the additional term $\mu x_\alpha \cos(\omega_c t)$ is irrelevant in this large-t limit. As the CGF for the two random variables $Q_{\omega_c;\alpha}(t)$ and $q_{\omega_c;\alpha}(t)$ are identical, it yields immediately the identity for the rescaled cumulants

$$\mathcal{Q}_p^\alpha(\omega_c) = \lim_{t\to\infty} \frac{\langle Q^p_{\omega_c;\alpha}(t)\rangle_{\text{co}}}{t} = \lim_{t\to\infty} \frac{\langle q^p_{\omega_c;\alpha}(t)\rangle_{\text{co}}}{t} \tag{A13}$$

of arbitrary order p. From this identification, we already obtain that the system does not have a direct charge $\mathcal{Q}_p^\alpha(\omega_c = 0) = 0$ for any $p \geqslant 0$. This is indeed expected, as nonzero cumulants of the charge would imply some fluctuations of the steady-state current, which are incompatible with our assumption of simply connected microstate space. We note, however, that in a compact domain and a nontrivial topology, e.g., a particle on a ring, the empirical alternating and direct charges are not defined as per the second line of Eq. (21). Indeed, the fluctuation of the direct charge might be nonzero for such systems [6].

2. General expression for the connected cumulants

In order to obtain the expression of the cumulant of order p of $q_{\omega_c;\alpha}(t)$ defined as

$$\langle q^p_{\omega_c;\alpha}(t)\rangle_{\text{co}} = \left\langle \prod_{j=1}^p \int_0^t dt_j\, x_\alpha(t_j)\,\omega_c\, \sin(\omega_c t_j)\right\rangle_{\text{co}} = p!\,\omega_c^p \int_0^t dt_p \cdots \int_0^{t_2} dt_1 \left\langle \prod_{j=1}^p x_\alpha(t_j)\right\rangle_{\text{co}} \prod_{j=1}^p \sin(\omega_c t_j), \tag{A14}$$

where the times $t \geqslant t_p \geqslant t_{p-1} \geqslant \cdots \geqslant t_2 \geqslant t_1 \geqslant 0$ in the second line are ordered, one needs to use the general expression of the p-times connected correlation functions $\langle \prod_{j=1}^p x_\alpha(t_j)\rangle_{\text{co}}$. This term is conveniently expressed in terms of the lower-order ($n \leqslant p$)-times (disconnected) correlation functions $\langle \prod_{j=1}^n x_\alpha(t_j)\rangle$ as follows [48]:

$$\left\langle \prod_{k=1}^p x_\alpha(t_k)\right\rangle_{\text{co}} = \sum_{k=1}^p (k-1)!(-1)^{k+1} \sum_{\pi\in P_k(p)} \prod_{B\in\pi} \prod_{i=1}^k \left\langle \prod_{j\in B_i} x(t_j)\right\rangle, \tag{A15}$$

where $\pi \in P_k(p)$ is an element of the groups of partitions of $\{1,\ldots,p\}$ into k blocks B_i's with $i = 1,\ldots,k$. We denote n_i the number of elements in block i of the partition π, with $\sum_{i=1}^k n_i = p$, and each element within a given block is ordered, i.e .,$B_i(1) < \cdots < B_i(n_i)$.

The n-times (disconnected) correlation functions $\langle \prod_{j=1}^n x_\alpha(t_j)\rangle$ can be computed explicitly by introducing the propagator between each of the times t_j's, which results in

$$\left\langle \prod_{j=1}^p x_\alpha(t_j)\right\rangle = \int d\mathbf{x}_1 \cdots \int d\mathbf{x}_p \prod_{j=1}^p [x_{j,\alpha}\, G(\mathbf{x}_j, t_j|\mathbf{x}_{j-1}, t_{j-1})] = \sum_{l_1,\ldots,l_p=0}^\infty \left[\prod_{j=1}^p e^{-\lambda_{l_j}(t_j - t_{j-1})}\right] C_{l_1,\ldots,l_n}(t_1,\ldots,t_p), \tag{A16}$$

where $t_0 = 0$. The third line is obtained by inserting the Floquet expansion (5) of the propagator where $l_1,\ldots,l_p$ refer to indices in the Floquet spectrum and introducing the function

$$C_{l_1,\ldots,l_p}(t_1,\ldots,t_n) = g_{l_1}(\mathbf{x}_0, 0) \int d\mathbf{x}_p\, x_{p,\alpha}\, f_{l_p}(\mathbf{x}_p, t_p) \prod_{j=1}^{p-1} \int d\mathbf{x}_j\, x_{j,\alpha}\, f_{l_j}(\mathbf{x}_j, t_j) g_{l_{j+1}}(\mathbf{x}_j, t_j). \tag{A17}$$

This function is periodic with fundamental frequency ω_d in all its variables and can be expanded in Fourier series as

$$C_{l_1,\ldots,l_p}(t_1,\ldots,t_p) = \sum_{k_1,\ldots,k_p=-\infty}^\infty C^{k_1,\ldots,k_p}_{l_1,\ldots,l_p} e^{i\sum_{j=1}^n k_j \omega_d t}. \tag{A18}$$

Notice that as $g_0(\mathbf{x}, t) = 1$ for all $\mathbf{x}$ and t, the term $\int d\mathbf{x}_j\, x_{j,\alpha} f_{l_j}(\mathbf{x}_j, t_j) g_{l_{j+1}}(\mathbf{x}_j, t_j)$ connecting l_j with l_{j+1} effectively becomes independent of l_{j+1} for $l_{j+1} = 0$. Thus, this term simplifies as a product of smaller-order correlation function

$$C_{l_1,\ldots,l_p}(t_1,\ldots,t_n) = C_{l_1,\ldots,l_j}(t_1,\ldots,t_j) \times C_{0,\ldots,l_p}(t_{j+1},\ldots,t_n), \quad l_{j+1} = 0. \tag{A19}$$

Note also that as $\lambda_0 = 0$ in Eq. (A21) for $l_{j+1} = 0$, there exists a similar decoupling

$$\prod_{m=1}^p e^{-\lambda_{l_m}(t_m - t_{m-1})} = \prod_{m=1}^j e^{-\lambda_{l_m}(t_m - t_{m-1})} \prod_{n=j+2}^p e^{-\lambda_{l_n}(t_n - t_{n-1})}, \quad l_{j+1} = 0, \tag{A20}$$

effectively disconnecting the correlations before t_{j+1} from the correlations after this time.

Using the above and inserting (A16) and (A17) into Eq. (A21), taking the limit where $t_1 \gg \lambda_1^{-1}$ and $\delta t_i = t_i - t_{i-1} = O(1)$ for $i = 2, \dots, p$, the connected correlation function simplifies to

$$\left\langle \prod_{j=1}^{p} x_\alpha(t_j) \right\rangle_{\rm co} \approx \sum_{k=1}^{p}(k-1)!(-1)^{k+1} \sum_{\pi\in P_k(p)} \prod_{B\in\pi} \sum_{l_1,\dots,l_p=0}^{\infty} (1-\delta_{l_p,0})\delta_{l_1,0} \prod_{j=2}^{p} e^{-\mu_j(l_2,\dots,l_p;\pi)\delta t_j} \prod_{i=1}^{k} \left[\tilde{C}_{l_{B_i(1)},\dots,l_{B_i(n_i)}}(t_{B_i(1)},\dots,t_{B_i(n_i)})\right], \tag{A21}$$

where, making explicit use of relation (A19), we have introduced the functions $\tilde{C}_{l_1}(t_1) = \delta_{l_1,0}C_0(t_1)$ and for $p > 1$

$$\tilde{C}_{l_1,\dots,l_p}(t_1,\dots,t_p) = \delta_{l_1,0} \prod_{k=2}^{p}(1-\delta_{l_k,0}) C_{l_1,\dots,l_p}(t_1,\dots,t_p). \tag{A22}$$

The values of the strictly positive rates $\mu_j(l_2,\dots,l_p;\pi) > 0$ for $2 \leqslant j \leqslant p$ depend on the specific partition π and read as, supposing that $j \in B_i$,

$$\mu_j = \lambda_{l_j} + \lambda_{l_{j+1}}(1-\delta_{l_j,0}) + \sum_{r=j+1}^{p} \lambda_{l_r} \prod_{m\neq i} [\delta_{r\in B_m}\Theta(j - \max\{s\in B_m, s<r\})], \tag{A23}$$

with $\delta_{r\in B_m} = \sum_{s\in B_m}\delta_{r,s}$. The values of λ_r's for $r > j$ are included in μ_j if the times t_r and t_j do not belong to the same block and t_j is larger than all the times $t_s < t_r$ belonging to the same block as r.

In particular, Eq. (A21) shows that the p-time connected correlation decays exponentially with the associated rate μ_j as soon as one of the time differences $\delta t_j = t_j - t_{j-1}$ becomes large.

3. Proof of the general result

We are now able to prove our main claim by computing the general expression of $\mathcal{Q}_p^\alpha(\omega_c)$. Changing variables in the integrals from the times $t_1, \dots, t_p$ into $\delta t_2 = t_2 - t_1, \delta t_3, \dots, \delta t_p = t_p - t_{p-1}$ and t_p, we can rewrite Eq. (A14) as

$$\left\langle q^p_{\omega_c;\alpha}(t)\right\rangle_{\rm co} = p!\,\omega_c^p \int_0^t dt_p \int_0^{t_p} d\delta t_p \cdots \int_0^{t_p-\sum_{k=3}^p \delta t_k} d\delta t_2 \left\langle \prod_{j=0}^{p-1} x_\alpha\left(t_p - \sum_{k=0}^{j-1}\delta t_{p-k}\right)\right\rangle_{\rm co} \prod_{j=0}^{p-1} \sin\left[\omega_c\left(t_p - \sum_{k=0}^{j-1}\delta t_{p-k}\right)\right]. \tag{A24}$$

It is clear that the variable t_p is of order $O(t)$ in the large-t limit. The p-times connected correlation term in the integrand $\langle\prod_{j=1}^p x_\alpha(t_j)\rangle_{\rm co} = \langle\prod_{j=0}^{p-1} x_\alpha(t_p - \sum_{k=0}^{j-1}\delta t_{p-k})\rangle_{\rm co}$ is given in Eq. (A21). This function does not decay with the final time t_p for fixed values of the δt_j's but rather is a periodic function of this variable. On the other hand, we have seen that this function decays exponentially with all the variables δt_j's for $2 \leqslant j \leqslant p$. We thus expect that the dominating contribution to the integral will come from $\delta t_j = O(1)$ while $t_p = O(t)$ in the large-t limit and we may safely replace the upper bounds $t_p - \sum_{k=j+1}^p \delta t_k$ in the integral over δt_j by $+\infty$ in this limit.

Introducing the Fourier expansion of $C_{l_1,\dots,l_p}(t_1,\dots,t_p)$ defined in Eq. (A18) and using the representation

$$\sin(\omega_c t) = \sum_{\sigma=\pm1} \frac{(-\sigma)}{2i} e^{-i\sigma\omega_c t}, \tag{A25}$$

one can express Eq. (A24) in the long-time limit as

$$\begin{aligned}
\left\langle q^p_{\omega_c;\alpha}(t)\right\rangle_{\rm co} \approx{}& p!\,\omega_c^p \sum_{k=1}^{p}(k-1)!(-1)^{k+1} \sum_{\pi\in P_k(p)} \prod_{B\in\pi} \sum_{l_1,\dots,l_p=0}^{\infty} (1-\delta_{l_p,0})\delta_{l_1,0} \\
&\times \sum_{k_1,\dots,k_p=-\infty}^{\infty} \sum_{\sigma_1,\dots,\sigma_p=\pm1} \prod_{m=1}^{p}\left(\frac{-\sigma_m}{2i}\right) \prod_{i=1}^{k}\left[\delta_{l_{B_i(1)},0}\prod_{k=2}^{n_i}(1-\delta_{l_{B_i(k)},0}) C^{k_{B_i(1)},\dots,k_{B_i(n_i)}}_{l_{B_i(1)},\dots,l_{B_i(n_i)}}\right] \\
&\times \int_0^t dt_p \int_0^\infty d\delta t_p \cdots \int_0^\infty d\delta t_2 \prod_{j=2}^{p} e^{-\mu_j(l_2,\dots,l_p;\pi)\delta t_j} \prod_{m=1}^{p} e^{i(k_m\omega_d - \omega_c\sigma_m)[t_p - \sum_{n=0}^{m-1}\delta t_{p-n}]} \\
\approx{}& p!\,\omega_c^p \sum_{k=1}^{p}(k-1)!(-1)^{k+1} \sum_{\pi\in P_k(p)} \prod_{B\in\pi} \sum_{l_1,\dots,l_p=0}^{\infty} (1-\delta_{l_p,0})\delta_{l_1,0} \sum_{k_1,\dots,k_p=-\infty}^{\infty} \sum_{\sigma_1,\dots,\sigma_p=\pm1} \prod_{m=1}^{p}\left(\frac{-\sigma_m}{2i}\right) \\
&\times \frac{\prod_{i=1}^{k}\left[\delta_{l_{B_i(1)},0}\prod_{k=2}^{n_i}(1-\delta_{l_{B_i(k)},0}) C^{k_{B_i(1)},\dots,k_{B_i(n_i)}}_{l_{B_i(1)},\dots,l_{B_i(n_i)}}\right]}{\prod_{j=2}^{p}\left[\mu_j(l_2,\dots,l_p;\pi) + i\sum_{l=j}^{p}(\sigma_l\omega_c - k_l\omega_d)\right]} I_p(\{k_1,\dots,k_p\};\{\sigma_1,\dots,\sigma_p\};t),
\end{aligned} \tag{A26}$$

where the function $I_p(\{k_1,\dots,k_p\};\{\sigma_1,\dots,\sigma_p\};t)$ is defined in Eq. (64) that we reproduce here:

$$t^{-1}I_p(\{k_1,\dots,k_p\};\{\sigma_1,\dots,\sigma_p\};t)=\int_0^t\frac{d\tau}{t}e^{i\Omega_p\tau}. \tag{A27}$$

4. General expression of $\mathcal{Q}_p^\alpha(\omega_c)$

Using the tools presented above it is possible to obtain the general expression for the cumulant of arbitrary order p. It reads as

$$\mathcal{Q}_p^\alpha(\omega_c)=p!\sum_{k=1}^{p}(k-1)!(-1)^{k+1}\sum_{\pi\in P_k(p)}\prod_{B\in\pi}\sum_{k_1,\dots,k_p=-\infty}^{\infty}\sum_{\sigma_1,\dots,\sigma_p=\pm1}\sum_{l_1,\dots,l_p=0}^{\infty}(1-\delta_{l_p,0})\delta_{l_1,0}$$
$$\times\prod_{j=1}^{p}\left(\frac{-\sigma_j}{2i}\right)\frac{\prod_{i=1}^{k}\left[\delta_{l_{B_i(1)},0}\prod_{k=2}^{n_i}(1-\delta_{l_{B_i(k)},0})C^{k_{B_i(1)},\dots,k_{B_i(n_i)}}_{l_{B_i(1)},\dots,l_{B_i(n_i)}}\right]}{\prod_{j=2}^{p}\left[\mu_j(l_2,\dots,l_p;\pi)+i\sum_{l=j}^{p}(\sigma_l\omega_c-k_l\omega_d)\right]}\omega_c^p\,\delta_{\sum_{j=1}^{p}(\omega_dk_j-\omega_c\sigma_j),0}. \tag{A28}$$

The Kronecker delta function in the second line of this expression ensures that the resonance condition is fulfilled. For incommensurate frequencies, this happens only when Eq. (67) holds but for commensurate frequencies, this also happens when Eq. (68) holds.

We now explore, for a fixed μ, the continuity of the cumulant generating function

$$\chi_\mu(\omega_c)=\sum_{p=1}^{\infty}\frac{\mu^p}{p!}\mathcal{Q}_p^\alpha(\omega_c). \tag{A29}$$

As discussed above, each term of this series has specific discontinuities as detailed in Eqs. (69) and (70). For a sequence of rational ratios that convergence to an irrational ratio, namely, a sequence of (n_i,m_i) and a corresponding ω_c^i

$$\omega_c^i=\omega_d\frac{n_i}{m_i}, \tag{A30}$$

such that $a=\lim_{i\to\infty}n_i/m_i$ is an irrational number, the resonance condition in Eq. (68) imposes that the cumulants which display a discontinuity must be of increasing order p_i. To this end, let us examine carefully the denominator of Eq. (A28). For any value of p_i, the resonance condition imposes

$$\frac{\sum_{j=1}^{p_i}k_j}{\sum_{j=1}^{p_i}\sigma_j}=\frac{n_i}{m_i}. \tag{A31}$$

For random strings $\{\sigma_j,\ j=1,\dots,p\}$ and $\{k_j,\ j=1,\dots,p\}$ with the constraint of fulfilling the latter equation, the resonance condition imposes the following equality:

$$\left|\sum_{l=j}^{p_i}(\sigma_l\omega_c-k_l\omega_d)\right|=\left|\sum_{l=1}^{j-1}(\sigma_l\omega_c-k_l\omega_d)\right|. \tag{A32}$$

The left-hand side of this equation scales, in the large-j limit, as $O(\sqrt{j})$. Thus, we expect the product appearing in the denominator of Eq. (A28),

$$\prod_{j=2}^{p_i}\left[\mu_j+i\sum_{l=j}^{p_i}(\sigma_l\omega_c-k_l\omega_d)\right], \tag{A33}$$

to grow in modulus rather rapidly as a function of p_i [as we expect $\mu_j=O(1)$, this term will be of order $\sqrt{p_i!}$]. Therefore, we expect that $\mathcal{Q}_{p_i}^\alpha(\omega_c)/(p_i!)$ goes to zero as $p_i\to\infty$. The discontinuous part of the CGF at frequency ω_c^i is provided by the terms $\mathcal{Q}_p^\alpha(\omega_c)/(p!)$ of order $p>p_i$ and these rescaled cumulants are getting smaller and smaller as p_i is increased, we expect that in the limit where $i\to\infty$, the discontinuities vanish completely from the CGF.

APPENDIX B: EXPRESSION OF THE THIRD-ORDER CUMULANT $\mathcal{Q}_3^\alpha(\omega_c)$

Let us now consider the cumulant of order $p=3$. To obtain its value, we first consider the three times connected correlation function for $t_3>t_2>t_1\gg1$:

$$\langle x_\alpha(t_1)x_\alpha(t_2)x_\alpha(t_3)\rangle_{\rm co}=\sum_{l_2,l_3=1}^{\infty}\prod_{j=2}^{3}e^{-\lambda_{l_j}(t_j-t_{j-1})}C_{0,l_2,l_3}(t_1,t_2,t_3)-\sum_{l_3=1}^{\infty}\prod_{j=2}^{3}e^{-\lambda_{l_3}(t_j-t_{j-1})}C_{0,l_3}(t_1,t_3)C_0(t_2). \tag{B1}$$

Note that $p=3$ is the first term for which there is more than one partition of $\{1,\dots,p\}$ that gives a nonzero contribution to this connected correlation, namely, $\{1,2,3\}$ and $\{1,3\}\{2\}$:

$$\mathcal{Q}_3^\alpha(\omega_c)=\frac{3\omega_c^3}{4i}\sum_{k_1,k_2,k_3=-\infty}^{\infty}\sum_{\sigma_1,\sigma_2,\sigma_3=\pm1}\sum_{l_3=1}^{\infty}\frac{\sigma_1\sigma_2\sigma_3}{\lambda_{l_3}+i(\sigma_3\omega_c-k_3\omega_d)}\delta_{\sum_{j=1}^{3}k_j\omega_d,\sum_{j=1}^{3}\sigma_j\omega_c}$$
$$\times\left[\sum_{l_2=1}^{\infty}\frac{C^{k_1,k_2,k_3}_{0,l_2,l_3}}{\lambda_{l_2}+i[(\sigma_2+\sigma_3)\omega_c-(k_2+k_3)\omega_d]}-\frac{C^{k_1,k_3}_{0,l_3}C^{k_2}_0}{\lambda_{l_3}+i[(\sigma_2+\sigma_3)\omega_c-(k_2+k_3)\omega_d]}\right]. \tag{B2}$$

This third-order cumulant is always zero if the frequencies ω_c and ω_d are incommensurate. It is only nonzero for frequencies of the form $\omega_c = n\omega_d$ and $n\omega_d/3$ for $n \in \mathbb{N}^*$.

APPENDIX C: EXPRESSION OF THE FOURTH-ORDER CUMULANT $\mathcal{Q}_4^\alpha(\omega_c)$

One can use the expression of the connected four-point functions for $t_4 > t_3 > t_2 > t_1 \gg 1$ with $\delta t_j = t_j - t_{j-1}$ for $j = 2, 3, 4$, which reads as

$$\langle x_\alpha(t_1)x_\alpha(t_2)x_\alpha(t_3)x_\alpha(t_4)\rangle_{\text{co}} \tag{C1}$$

$$= \sum_{l_2,l_3,l_4=1}^{\infty}\prod_{j=2}^{4} e^{-\lambda_{l_j}\delta t_j} C_{0,l_2,l_3,l_4}(t_1,t_2,t_3,t_4) - \sum_{l_2,l_4=1}^{\infty} e^{-\lambda_{l_4}(\delta t_4+\delta t_3)-\lambda_{l_2}\delta t_2} C_{0,l_2,l_4}(t_1,t_2,t_4)C_0(t_3)$$
$$- \sum_{l_3,l_4=1}^{\infty} e^{-\lambda_{l_4}\delta t_4-\lambda_{l_3}\delta t_3-\lambda_{l_3}\delta t_2} C_{0,l_3,l_4}(t_1,t_3,t_4)C_0(t_2) - \sum_{l_3,l_4=1}^{\infty} e^{-\lambda_{l_4}\delta t_4-(\lambda_{l_4}+\lambda_{l_3})\delta t_3-\lambda_{l_3}\delta t_2}$$
$$\times[C_{0,l_4}(t_1,t_4)C_{0,l_3}(t_2,t_3) + C_{0,l_4}(t_2,t_4)C_{0,l_3}(t_1,t_3)] + \sum_{l_4=1}^{\infty}\prod_{j=2}^{4} e^{-\lambda_{l_4}\delta t_j} C_{0,l_4}(t_1,t_4)C_0(t_2)C_0(t_3). \tag{C2}$$

The partition of $\{1,2,3,4\}$ that contributes besides the identity are for two blocks $\{1,2,4\}\{3\}$, $\{1,3,4\}\{2\}$, $\{1,3\}\{2,4\}$, $\{2,4\}\{1,3\}$ and for three blocks $\{1,4\}\{2\}\{3\}$. Using the explicit expressions for the rates μ_j's corresponding to each partition, we obtain

$$\mathcal{Q}_4^\alpha(\omega_c) = \frac{3\omega_c^4}{2}\sum_{k_1,\ldots,k_4=-\infty}^{\infty}\sum_{\sigma_1,\ldots,\sigma_4=\pm1}\sum_{l_4=1}^{\infty}\frac{\prod_{j=1}^4\sigma_j}{\lambda_{l_4}+i(\sigma_4\omega_c-k_4\omega_d)}\delta_{\sum_{j=1}^4 k_j\omega_d,\sum_{j=1}^4\sigma_j\omega_c}$$
$$\times\left(\sum_{l_2,l_3=1}^{\infty}\frac{C_{0,l_2,l_3,l_4}^{k_1,k_2,k_3,k_4}}{\prod_{n=2}^3\left[\lambda_{l_n}+i\sum_{j=n}^4(\sigma_j\omega_c-k_j\omega_d)\right]} + \frac{C_{0,l_4}^{k_1,k_4}C_0^{k_2}C_0^{k_3}}{\prod_{n=2}^3\left[\lambda_{l_4}+i\sum_{j=n}^4(\sigma_j\omega_c-k_j\omega_d)\right]}\right.$$
$$-\sum_{l_3=1}^{\infty}\frac{C_{0,l_3,l_4}^{k_1,k_3,k_4}C_0^{k_2}}{\prod_{n=2}^3\left[\lambda_{l_3}+i\sum_{j=n}^4(\sigma_j\omega_c-k_j\omega_d)\right]} - \sum_{l_2=1}^{\infty}\frac{C_{0,l_2,l_4}^{k_1,k_2,k_4}C_0^{k_3}}{\left[\lambda_{l_4}+i\sum_{j=3}^4(\sigma_j\omega_c-k_j\omega_d)\right]\left[\lambda_{l_2}+i\sum_{j=2}^4(\sigma_j\omega_c-k_j\omega_d)\right]}$$
$$\left.-\sum_{l_3=1}^{\infty}\frac{1}{\sum_{j=3}^4\left[\lambda_{l_j}+i(\sigma_j\omega_c-k_j\omega_d)\right]}\left[\frac{C_{0,l_4}^{k_1,k_4}C_{0,l_3}^{k_2,k_3}}{\lambda_{l_4}+i\sum_{j=2}^4(\sigma_j\omega_c-k_j\omega_d)} + \frac{C_{0,l_4}^{k_2,k_4}C_{0,l_3}^{k_1,k_3}}{\lambda_{l_3}+i\sum_{j=2}^4(\sigma_j\omega_c-k_j\omega_d)}\right]\right). \tag{C3}$$

APPENDIX D: CUMULANT GENERATING FUNCTION FOR THE ALTERNATING CHARGE OF THE TWO-LEVEL SYSTEM

A convenient way to compute the CGF $\chi_\mu(\omega_c)$ is to introduce a counting vector $|P_s(t)\rangle$ (see [49] for details about this method), such that the ith component of $|P_s(t)\rangle$ is the ensemble average of $e^{\mu Q_{\omega_c}(t)}$ on trajectories that are in the state i at time t. The initial condition is taken such that $|P_\mu(0)\rangle = |P(0)\rangle$ at $t=0$, and it evolves with time according to

$$|\dot{P}_\mu(t)\rangle = R_\mu(t)|P_\mu(t)\rangle, \tag{D1}$$

$$R_\mu(t) = \begin{pmatrix} -k_1(t) & k_2(t)e^{-\mu\cos(\omega_c t)} \\ k_1(t)e^{\mu\cos(\omega_c t)} & -k_2(t)\end{pmatrix}, \tag{D2}$$

and $R_\mu(t)$ is called the "tilted rate matrix." The vector $|P_\mu(t)\rangle$ can be (formally) expressed at any time t as

$$|P_\mu(t)\rangle = \mathcal{T}e^{\int_0^t R_\mu(\tau)d\tau}|P_\mu(0)\rangle, \tag{D3}$$

where $\mathcal{T}$ is the time-ordering operator. Using this counting vector method, the cumulant generating function is given by [11]

$$\chi_\mu(\omega_c) = \ln\langle e^{\mu Q_{\omega_c}(t)}\rangle = \lim_{t\to\infty}\frac{1}{t}\ln\langle 1|P_\mu(t)\rangle$$
$$= \lim_{t\to\infty}\frac{1}{t}\ln\langle 1|\mathcal{T}e^{\int_0^t R_\mu(\tau)d\tau}|P(0)\rangle, \tag{D4}$$

where $\langle 1| = (1,1)$. While this analytical formula is exact, it is not very convenient as the time-ordered exponential in Eq. (D3) is a complicated object and hard to compute in practice. To simplify the computation of the counting vector, we first introduce a dynamical invariant of the process. We then obtain an explicit expression for the counting vector $|P_\mu(t)\rangle$ in terms of this dynamical invariant and finally compute the CGF.

The derivation below follows the derivation provided in [47], but generalizes it for all the current's Fourier components.

1. Dynamical invariant

A dynamical invariant corresponds to a matrix $F_\mu(t)$ which is diagonalizable, has time-independent eigenvalues that we set here to ±1 for convenience, and satisfies for any time the differential equation

$$\dot{F}_\mu(t) = R_\mu(t)F_\mu(t) - F_\mu(t)R_\mu(t). \tag{D5}$$

A convenient way to parametrize a matrix satisfying the first two properties is [47]

$$F_\mu(t) = \begin{pmatrix} z_\mu(t) & [1+z_\mu(t)]y_\mu(t)^{-1} \\ [1-z_\mu(t)]y_\mu(t) & -z_\mu(t) \end{pmatrix}, \tag{D6}$$

with left and right time-dependent orthonormal eigenvectors $\langle r_\sigma(t)|l'_\sigma(t)\rangle = \delta_{\sigma,\sigma'}$ with $\sigma, \sigma' = \pm$ that read as [47]

$$|l_+(t)\rangle = \begin{pmatrix} 1 \\ \frac{1-z_\mu(t)}{1+z_\mu(t)}y_\mu(t) \end{pmatrix}, \tag{D7}$$
$$\langle r_+(t)| = \frac{1}{2}\begin{pmatrix} 1+z_\mu(t) & \frac{1+z_\mu(t)}{y_\mu(t)} \end{pmatrix},$$
$$|l_-(t)\rangle = \frac{1}{2}\begin{pmatrix} 1-z_\mu(t) \\ -[1-z_\mu(t)]y_\mu(t) \end{pmatrix},$$
$$\langle r_-(t)| = \begin{pmatrix} 1 & -\frac{1+z_\mu(t)}{1-z_\mu(t)}\frac{1}{y_\mu(t)} \end{pmatrix}.$$

For the matrix $F_\mu(t)$ to satisfy Eq. (D5), the functions $y_\mu(t)$ and $z_\mu(t)$ need to satisfy the differential equations

$$\dot{y}_\mu = (y_\mu - e^{\mu\cos(\omega_c t)})(k_1 + k_2 e^{-\mu\cos(\omega_c t)}y_\mu),$$
$$\dot{z}_\mu = k_2 e^{-\mu\cos(\omega_c t)}(1-z_\mu)y_\mu - k_1 e^{\mu\cos(\omega_c t)}\frac{(1+z_\mu)}{y_\mu}. \tag{D8}$$

Any pair of initial values $(y_\mu(0), z_\mu(0))$ gives a valid dynamical invariant. Thus, choosing such a pair defines a valid dynamical invariant. We focus here on the long-time behavior such that the initial condition is not relevant and choose for convenience $y_\mu(0) = 0$ and $z_\mu(0) = -1$. The equation for $y_\mu(t)$ has two fixed points: a stable fixed point $-k_1(t)/k_2(t)e^{\mu\cos(\omega_c t)} < 0$ and an unstable fixed point $e^{\mu\cos(\omega_c t)} > 0$. Note that starting from $y_\mu(0) \leqslant 0$, the solution $y_\mu(t)$ oscillates around the stable fixed point and remains negative at all time t. The equation for $z_\mu(t)$ is linear and can be solved exactly. In particular, it is easy to realize that the function $z_\mu(t)$ grows exponentially with time t as $y_\mu(\tau) < 0$ for any $\tau \in [0, t]$:

$$z_\mu(t) \asymp e^{-\int_0^t d\tau[y_\mu(\tau)k_2(\tau)e^{-\mu\cos(\omega_c\tau)} + \frac{k_1(\tau)}{y_\mu(\tau)}e^{\mu\cos(\omega_c\tau)}]}. \tag{D9}$$

In the large-time limit, the expressions of the eigenvectors in Eq. (D10) simplify as

$$|l_+(t)\rangle \approx \begin{pmatrix} 1 \\ y_\mu(t) \end{pmatrix}, \quad \langle r_+(t)| \approx \frac{z_\mu(t)}{2}\begin{pmatrix} 1 & y_\mu^{-1}(t) \end{pmatrix},$$
$$|l_-(t)\rangle \approx \frac{z_\mu(t)}{2}\begin{pmatrix} -1 \\ y_\mu(t) \end{pmatrix}, \quad \langle r_-(t)| \approx \begin{pmatrix} 1 & -y_\mu^{-1}(t) \end{pmatrix}. \tag{D10}$$

2. Expressing the counting vector and the CGF

Coming back to the problem of finding an expression for the counting vector $|P_\mu(t)\rangle$, one can check by using Eq. (D1) together with Eq. (D5) that the following identity holds:

$$\partial_t[F_\mu(t)|P_\mu(t)\rangle] = R_\mu(t)F_\mu(t)|P_\mu(t)\rangle. \tag{D11}$$

It is then possible to express the counting vector $|P_\mu(t)\rangle$ in the basis of the eigenvectors $|l_\pm(t)\rangle$ of $F_\mu(t)$. In this basis, one has that

$$|P_\mu(t)\rangle = \sum_{\sigma=\pm} a_\sigma(t)|l_\sigma(t)\rangle, \quad a_\sigma(t) = \langle r_\sigma(t)|P_\mu(t)\rangle. \tag{D12}$$

Inserting this equation into Eqs. (D1) and (D11), one obtains that for $\sigma, \sigma' = \pm$

$$[\langle r_{\sigma'}(t)|R_\mu(t)|l_\sigma(t)\rangle - \langle r_{\sigma'}(t)|\dot{l}_\sigma(t)\rangle]a_\sigma(t) = \delta_{\sigma,\sigma'}\dot{a}_\sigma(t). \tag{D13}$$

Inserting the expressions of the eigenvectors in Eq. (D7), using the expression of the tilted rate matrix and the differential equations (D8), one can then check that $\langle r_{\sigma'}(t)|R_\mu(t)|l_\sigma(t)\rangle = \langle r_{\sigma'}(t)|\dot{l}_\sigma(t)\rangle$ for $\sigma \neq \sigma'$. On the other hand, Eq. (D13) yields

$$|P_\mu(t)\rangle = \sum_{\sigma=\pm} a_\sigma(0)e^{\int_0^t d\tau\phi_\sigma(\tau)}|l_\sigma(t)\rangle, \tag{D14}$$

$$\phi_\pm(\tau) = \langle r_\pm(\tau)|R_\mu(\tau)|l_\pm(\tau)\rangle - \langle r_\pm(\tau)|\dot{l}_\pm(\tau)\rangle. \tag{D15}$$

We can now express $|P_\mu(t)\rangle$ in the large-time limit as a function of the two functions $z_\mu(t)$ and $y_\mu(t)$. Inserting the expressions of the eigenvectors in Eq. (D10), using the expression of the tilted matrix $R_\mu(t)$ and the differential equations in Eq. (D8), one obtains at large time

$$|P_\mu(t)\rangle \asymp e^{-\int_0^t d\tau[k_1(\tau)+k_2(\tau)e^{-\mu\cos(\omega_c\tau)}y_\mu(\tau)]}. \tag{D16}$$

Finally, using Eq. (D4), we obtain the explicit expression of the CGF as given in Eq. (107) of the main text.

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