Geometrically robust linear optics from non-Abelian geometric phases

Julien Pinske and Stefan Scheel*
Institut für Physik, Universität Rostock, Albert-Einstein–Straße 23-24, D-18059 Rostock, Germany

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We construct a unified operator framework for quantum holonomies generated from bosonic systems. For a system whose Hamiltonian is bilinear in the creation and annihilation operators, we find a holonomy group determined only by a set of selected orthonormal modes obeying a stronger version of the adiabatic theorem. This photon-number independent description offers deeper insight as well as a computational advantage when compared to the standard formalism on geometric phases. In particular, a strong analogy between quantum holonomies and linear optical networks can be drawn. This relation provides an explicit recipe of how any linear optical quantum computation can be made geometrically robust in terms of adiabatic or nonadiabatic geometric phases.

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I. INTRODUCTION

Recent years have witnessed an increased interest in the notions of Abelian and non-Abelian geometric phases (quantum holonomies) beyond the gauge theories of elementary particles. The experimental and theoretical study of such phase factors is relevant to the simulation of lattice gauge theories [1,2], the understanding of symmetry groups [3,4], as well as much of modern mathematics [5]. Moreover, their topological properties give rise to quantum evolutions that are inherently fault tolerant and might therefore be highly desirable assets for quantum information processing [6–8].

These purely geometric transformations arise when a state vector is parallel transported along a closed loop in a suitable subspace. For adiabatic holonomies this subspace is usually some (non)degenerate ground-state subspace [9,10]. In the case of a nonadiabatic quantum holonomy, evolution takes place in a subspace on which the mean energy vanishes, thus making the evolution purely geometric [11]. Physical implementations rely, for instance, on atomic transitions [12,13], the manipulation of trapped ions [14,15], or superconducting qubits [16,17], as well as controlled guiding of coherent light [18,19] or photons [20]. The latter implementation is of particular interest as the dimension of the relevant subspace increases with a rising number of photons [21]. This hints at a formulation of geometric phases that is independent of the overall particle number to which the system is subjected.

In this article, we present an operator-based formalism for the unified treatment of adiabatic and nonadiabatic quantum holonomies in terms of a holonomic Heisenberg picture that generalizes the concept of an adiabatic Heisenberg picture first introduced in Ref. [22]. It bears some resemblance to (projective adiabatic) elimination procedures [23] and relies on the usage of effective Hamiltonians [24,25] to describe the geometric phase of a system. The holonomic Heisenberg picture developed here employs a generalized parallel transport condition that imposes the disappearance of Hamiltonian dynamics on a specific set of solutions. Remarkably, in a linear optical setting we find a quantum holonomy on the level of superoperators that only depends on a set of selected orthonormal modes of the underlying system. Subsequently, this eliminates the need for an explicit calculation of projectors onto the relevant subspace, which becomes quickly unfeasible for large bosonic systems. Besides offering a major computational advantage over the standard formalism, it also provides deeper insight into the emergence of geometric phases in second quantization. From our general argument it follows that linear optical setups based on photonic holonomies can be described equally by a holonomic scattering matrix in analogy to the Reck-Zeilinger scheme [26]. Thus, it is possible to make any linear optical computation geometrically robust by referring to auxiliary modes while undergoing cyclic evolution. We benchmark our findings by studying a number of examples which are important to modern quantum optics.

II. OPERATOR FORMULATION OF QUANTUM HOLONOMIES

The treatment of bosonic many-particle systems can be pursued elegantly by referring to a Fock representation. Here, the Hilbert space of a system consisting of M modes is the Fock space $\mathscr{H}=\bigotimes_{k=1}^{M}\mathscr{H}_k$ defined as a finite tensor product of single-mode Fock spaces $\mathscr{H}_k=\mathrm{Span}\{|n\rangle_k \mid n\in\mathbb{N}_0\}$. Starting from the multimode vacuum state $|\mathbf{0}\rangle=|0,\dots,0\rangle$, the Fock space $\mathscr{H}$ can be viewed as being generated from the (unital $*$-) algebra $\mathscr{A}=\bigotimes_{k=1}^{M}\mathscr{A}_k$ containing analytic functions of creation and annihilation operators $\{\hat{a}_k,\hat{a}_k^\dagger\}_k$

*stefan.scheel@uni-rostock.de

satisfying the canonical commutation relations $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and $[\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0$. We express their relation as $\mathscr{H} \cong \mathscr{A}\,|\mathbf{0}\rangle$, which says that any multimode state $|\Psi\rangle \in \mathscr{H}$ can be expanded as $|\Psi\rangle = \hat{F}(\hat{a}_1^\dagger, \dots, \hat{a}_M^\dagger)\,|\mathbf{0}\rangle$, where

$$\hat{F} = \sum_{p_1,\dots,p_M \in \mathbb{N}_0} c_{p_1,\dots,p_M} \frac{(\hat{a}_1^\dagger)^{p_1}}{\sqrt{p_1!}} \cdots \frac{(\hat{a}_M^\dagger)^{p_M}}{\sqrt{p_M!}},$$

with $\sum_{p_1,\dots,p_M} |c_{p_1,\dots,p_M}|^2 = 1$.

Let us consider a time-dependent Hamiltonian $\hat{H}(t)$ belonging to $\mathscr{A}$. While the general evolution of a quantum state $|\Psi(t)\rangle$ is subject to Schrödinger's equation $i\frac{d}{dt}|\Psi\rangle = \hat{H}\,|\Psi\rangle$, a quantum holonomy transforms the state in a manner that is independent of the spectrum of $\hat{H}(t)$ or runtime T. Certainly not any solution to Schrödinger's equation will be of such nature, but only those belonging to a subset of solutions $\mathscr{K}(t) = \{|\eta_j(t)\rangle\}_j$. As these states evolve purely geometrically, they shall satisfy $\langle\eta_j|\dot{\eta}_k\rangle = 0$, thus being independent of the systems dynamics. More formally, we will summarize the latter as $\langle\frac{d}{dt}\rangle_{\mathscr{K}} = 0$ being a condition for the parallel transport of a quantum state. Expanding the solutions in $\mathscr{K}$ in terms of an orthonormal basis $\{|\psi_j\rangle\}_j$ of a subspace $\mathscr{H}_\psi$ as $|\eta_k(t)\rangle = \sum_j U_{jk}(t)\,|\psi_j(t)\rangle$ with initial condition $|\eta_k(0)\rangle = |\psi_k(0)\rangle$, and employing the law for parallel transport leads to the first-order differential equation $(\hat{U}^{-1}\dot{\hat{U}})_{jk} = (\hat{A}_t)_{kj} = \langle\psi_j|\dot{\psi}_k\rangle$. A formal solution is given by the time-ordered matrix exponential $\hat{U}(T) = \mathcal{T}e^{\int_0^T \hat{A}_t dt}$. In this context, and throughout the article, $\mathscr{H}_\psi$ plays the role of a geometrically protected subspace. Here we are not interested in arbitrary evolutions but in those that represent a loop $\gamma(t)$ through the subspace, that is, $\mathscr{H}_\psi(0) = \mathscr{H}_\psi(T)$. The unitary is then given by a path-ordered matrix exponential $\hat{U}(\gamma) = \mathcal{P}e^{\oint_\gamma \hat{A}}$ known as a quantum holonomy [10,11]. The connection might then be expressed through its anti-Hermitian components $(\hat{A})_{jk} = \langle\psi_k|\,d\,|\psi_j\rangle$ mediating parallel transport along the loop γ. In order for expectation values to coincide on the subspace $\mathscr{H}_\psi$, any mode $\hat{a}_k$ in $\mathscr{A}$ must evolve according to $\hat{a}_k \mapsto \hat{U}^\dagger(\gamma)\hat{a}_k\hat{U}(\gamma)$. This implies that the Heisenberg equation of motion for a purely holonomic evolution reads

$$\langle\dot{\hat{a}}_k\rangle_{\mathscr{H}_\psi} = \langle[\hat{a}_k, \hat{A}_t(\gamma)]\rangle_{\mathscr{H}_\psi} + \langle\partial_t\hat{a}_k\rangle_{\mathscr{H}_\psi}, \qquad (1)$$

which means that the generator of dynamics is now given by a parallel transport map $\hat{A}_t(\gamma) = \hat{U}^\dagger(\gamma)\hat{A}_t\hat{U}(\gamma)$ instead of a general Hamiltonian. Having determined the evolution of modes $\hat{a}_k$ from Eq. (1), then characterizes adiabatic changes of any function $\hat{F}(\hat{a}_k, \hat{a}_k^\dagger)$ in the algebra $\mathscr{A}$.

For concreteness, if we consider $\mathscr{K}$ to be the set of adiabatic solutions, the states $\{|\psi_j(t)\rangle\}_j$ form an orthonormal basis for the degenerate ground-state subspace $\mathscr{H}_\psi = \mathscr{H}_0$ of the Hamiltonian $\hat{H}(t)$. When traversing this loop slowly, by which we mean the $|\psi_j(t)\rangle$ change only gradually when compared to the energy gap between $\mathscr{H}_0$ and the excited states of the system, we recover the adiabatic Heisenberg picture [22]. To be more precise, note that adiabatic solutions only approximate the evolving state governed by Schrödinger's equation up to first order of $1/(T\Delta\varepsilon)$, where $\Delta\varepsilon$ is the energy gap between the ground and excited states. As an elementary example consider the adiabatic propagation through the zero-eigenvalue eigenspace $\mathscr{H}_0$ of a nonlinear Kerr medium [27] $\hat{W}(\alpha,\xi)\hat{H}_0\hat{W}^\dagger(\alpha,\xi)$, with $\hat{H}_0 = \hat{a}^\dagger\hat{a}(\hat{a}^\dagger\hat{a} - 1)$ and $\hat{W}(\alpha,\xi) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}e^{\frac{\xi^*}{2}\hat{a}^2 - \frac{\xi}{2}(\hat{a}^\dagger)^2}$ describing the combined process of coherent displacement and single-mode squeezing. The action of $\hat{a}$ onto $\mathscr{H}_0$ is governed by the connection $\hat{A}_t = \hat{\Pi}_0\hat{W}^\dagger\partial_t\hat{W}\hat{\Pi}_0$, with $\hat{\Pi}_0 = |0\rangle\langle 0| + |1\rangle\langle 1|$. Evaluating Eq. (1) leads to nonlinear equations of adiabatic motion,

$$\langle\dot{\hat{a}}\rangle_{\mathscr{H}_0} = (\dot{\alpha}^*\alpha - \dot{\alpha}\alpha^*)\langle\hat{a}\rangle_{\mathscr{H}_0} + \dot{\alpha}(\mu - \nu^*)\langle\hat{a}^\dagger\hat{a}\rangle_{\mathscr{H}_0} - \text{c.c.},$$

where $\mu = \cosh|\xi|$ and $\nu = e^{i\arg(\xi)}\sinh|\xi|$. The emergence of nonlinear equations of motion is a generic feature of the operator description of parallel transport, both adiabatic and nonadiabatic. This is due to the connection $\hat{A} = \hat{\Pi}_\psi d\hat{\Pi}_\psi$ requiring the computation of subspace projectors $\hat{\Pi}_\psi$ onto $\mathscr{H}_\psi$. Due to the generally highly nonlinear form of these projectors in terms of bosonic modes, the computation of quantum holonomies can be an extremely challenging task.

For completeness, note that the above argument extends to any eigenspace with energy $\varepsilon_n(t)$. When level crossing is neglected, i.e., if $n \neq m$, then $\varepsilon_n(t) \neq \varepsilon_m(t)$ for all $t \in [0, T]$, and as a result the degeneracy of each energy level does not change. The overall time evolution of $\hat{a}_k$ under the adiabatic assumption (long runtime T) is then determined by the composite unitary $\bigoplus_n e^{i\int_0^T \varepsilon_n(t)dt}\hat{U}_n(\gamma)$ with $\hat{U}_n(\gamma)$ being the holonomy acting on the nth eigenspace of the system Hamiltonian, and $\int_0^T \varepsilon_n(t)dt$ accounting for dynamical contributions. Note that at first glance the parallel transport condition $\langle\frac{d}{dt}\rangle_{\mathscr{K}} = 0$ might be violated in an eigenspace with $\varepsilon_n(t) \neq 0$. However, this can always be accounted for by multiplying the solutions in $\mathscr{K}$ with the dynamical phase $e^{i\int_0^T \varepsilon_n(t)dt}$. Therefore, strictly speaking, the composite unitary is not a fully geometric quantity but has dynamical contributions due to these relative (energy-dependent) phase factors.

III. LINEAR QUANTUM OPTICS

The general concepts described thus far can, in principle, be applied to any bosonic system. In the following, we will show that in a linear optical setting, that is, the Hamiltonian $\hat{H}(t)$ is bilinear in the creation and annihilation operators, certain symmetries arise that offer a deeper insight into the emergence of geometric phases. Consider a system of M bosonic modes that interact according to such a bilinear Hamiltonian. Suppose further that there is a set of orthonormal modes $\{\hat{\Psi}_j(t)\}_{j=1}^{K<M}$ whose excitations (action on $|\mathbf{0}\rangle$) span a subspace $\mathscr{H}_\psi = \{|\psi_l\rangle, l \in \mathbb{N}\}$ on which the mean energy of the Hamiltonian $\hat{H}$ vanishes. In Appendix A we show that the second-quantization formulation of the condition $\langle\frac{d}{dt}\rangle_{\mathscr{K}} = 0$ is given as $[\hat{\Psi}_j, [\hat{H}, \hat{\Psi}_k^\dagger]] = 0$. This implies $\langle\psi_l|\,\hat{H}\,|\psi_m\rangle = 0$ for any $l, m \in \mathbb{N}$, thus ensuring that the evolution is indeed of purely geometric origin. In the same way, we can view the solutions in $\mathscr{K}$ as being created by operators $\hat{\eta}_k^\dagger(t)$, $k = 1, \dots, K$, which then must satisfy the Heisenberg equation of motion. With the ansatz $\hat{\eta}_k^\dagger(t) = \sum_j \mathcal{U}_{jk}(t)\hat{\Psi}_j^\dagger(t)$ and the condition for parallel transport, this yields

$$0 = [\hat{\eta}_j, \dot{\hat{\eta}}_k^\dagger] = \sum_{l=1}^K \mathcal{U}_{lj}^*\dot{\mathcal{U}}_{lk} + \sum_{l,m=1}^K \mathcal{U}_{lj}^*\mathcal{U}_{mk}(\mathcal{A}_t)_{ml}, \qquad (2)$$

where we introduced the operator-valued connection $(\mathcal{A}_t)_{jk} = [\hat{\Psi}_k, \dot{\hat{\Psi}}_j^\dagger] = [\hat{\Psi}_k, \partial_t \hat{\Psi}_j^\dagger]$. If we now consider a cyclic evolution of the system, i.e., for $j = 1, \dots, K$ we have $\hat{\Psi}_j^\dagger(0) = \hat{\Psi}_j^\dagger(T)$ resembling a loop γ, the solution to Eq. (2) is formally given by the path-ordered integral superoperator $\mathcal{U}_\gamma = \mathcal{P}\exp \oint_\gamma \mathcal{A}$. Now, the time evolution of a mode $\hat{\eta}_k^\dagger$ is given by the mapping $\hat{\eta}_k^\dagger(T) = \mathcal{U}_\gamma[\hat{\eta}_k^\dagger(0)]$.

Strikingly, in this formulation one avoids the usage of projectors onto the relevant subspace altogether, thereby drastically simplifying the computational effort needed to determine the geometric evolution. Starting at a point where $\hat{\eta}_k^\dagger(0) = \hat{a}_k^\dagger$, it becomes evident that $\mathcal{U}_\gamma$ can be viewed as the scattering matrix of the linear optical network being restricted to purely geometric evolutions of the bosonic modes. Note that, because only $K < M$ modes are relevant to the final output of the network, there are $M - K$ remaining auxiliary modes that act as mediators for a purely geometric evolution. This result can be related to the standard formalism on geometric phases [10,11] by noting that $\mathcal{U}_\gamma[\hat{a}_k^\dagger] = \hat{U}^\dagger(\gamma)\hat{a}_k^\dagger\hat{U}(\gamma)$, with $\hat{U}(\gamma) = \mathcal{P}e^{\oint_\gamma \hat{A}}$ being the more familiar form of the holonomy. The associated connection can be obtained from $\hat{A}_\mu = \sum_{j,k}(\mathcal{A}_\mu)_{jk}\hat{a}_j^\dagger\hat{a}_k$ being bilinear in the creation and annihilation operators. In contrast to a nonlinear optical setting, here the projection onto the relevant subspace is incorporated implicitly into the connection, thus providing an elegant photon-number independent description.

A. Geometric picture of operator holonomies

From a geometric point of view, the $\hat{\eta}_k^\dagger(t)$ are the horizontal lifts of a curve γ in the Grassmann manifold $\mathscr{G}_{M,K}$ containing K-dimensional subspaces spanned by the operators $\{\hat{\Psi}_j^\dagger\}_j$ (Fig. 1). Moreover, it can be easily verified that the connection is anti-Hermitian, $(\mathcal{A})_{jk}^\dagger = -(\mathcal{A})_{kj}$, and transforms as a proper gauge potential $\mathcal{A} \mapsto G^{-1}\mathcal{A}G + G^{-1}dG$ under a unitary mixing of operators $\hat{\Psi}_j^\dagger \mapsto \sum_j G_{jk}\hat{\Psi}_j^\dagger$, $G \in \mathrm{U}(M-2)$. If we further consider the collection of all loops γ in $\mathscr{G}_{M,K}$, the set $\mathrm{Hol}(\mathcal{A}) = \{\mathcal{U}_\gamma\}_\gamma$ forms the holonomy group of the (principal fibre) bundle $\mathscr{V}_{M,K} \to \mathscr{G}_{M,K}$, where the Stiefel manifold $\mathscr{V}_{M,K}$ is made up of K-dimensional orthonormal frames $\{\hat{\Psi}_j^\dagger\}_j$. As illustrated in Fig. 1, at every point $\gamma(t)$ there is a fiber on which the Lie group $\mathrm{U}(M-2)$ acts. Further properties of $\mathrm{Hol}(\mathcal{A})$ follow from standard results on differential geometry [28].

B. Adiabatic evolution of the star graph

Consider M bosonic modes being arranged as a star graph (Fig. 2), i.e., its Hamiltonian reads

$$\hat{H}(t) = \sum_{k=1}^{M-1} \kappa_k(t)\hat{a}_k\hat{a}_M^\dagger + \kappa_k^*(t)\hat{a}_k^\dagger\hat{a}_M,$$

where the couplings $\{\kappa_k\}_k$ act as local coordinates for a $2(M-1)$-dimensional control manifold $\mathscr{M}$ which is embedded into $\mathscr{G}_{M,M-2}$ [29] (Fig. 1). The system has $M-2$ dark modes $\hat{D}_j^\dagger(t) = \kappa_{j+1}(t)\hat{a}_1^\dagger - \kappa_1(t)\hat{a}_{j+1}^\dagger$. These operators not only constitute a symmetry, that is, $[\hat{H}(t), \hat{D}_j^\dagger(t)] = 0$, but

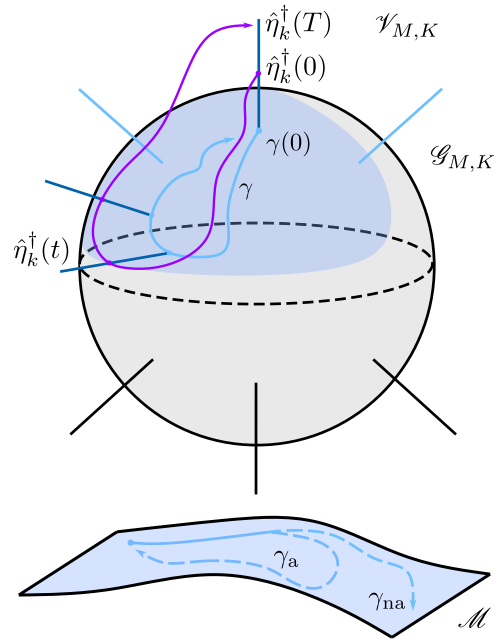

FIG. 1. The horizontal lift $\{\hat{\eta}_k^\dagger(t)\}_k$ moves along the fibers over the loop γ (dark blue spikes). The difference between $\hat{\eta}_k^\dagger(0)$ and $\hat{\eta}_k^\dagger(T)$ is the holonomy $\mathcal{U}_\gamma$. The loop can be expressed via a (closed) curve (γ_a) γ_{na} in $\mathscr{M}$ yielding the (adiabatic) nonadiabatic holonomy. The embedding $\mathscr{M}$ into $\mathscr{G}_{M,K}$ (blue shaded area) does not need to be the same for adiabatic and nonadiabatic holonomies.

obey bosonic commutation relations $[\hat{D}_j, \hat{D}_k] = [\hat{D}_j^\dagger, \hat{D}_k^\dagger] = 0$ and $[\hat{D}_j, \hat{D}_k^\dagger] = \delta_{jk}$ after being orthogonalized. Note that the demand for dark modes is no limitation at all, if we have eigenmodes with nonzero energy, the dynamical phase can be removed by rescaling the Hamiltonian, so that the condition for a purely geometric evolution is again satisfied. Note that, an onsite energy $\sigma\hat{a}_M^\dagger\hat{a}_M$ of the central mode leaves the

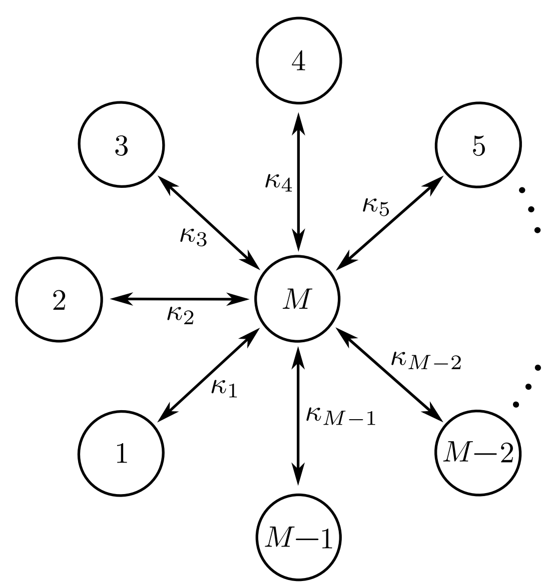

FIG. 2. Schematic representation of the star graph corresponding to the M-mode system.

degeneracy structure and dark modes unchanged but only modifies the energy gap $\Delta\varepsilon$ between dark modes and the two remaining eigenmodes, viz. $\Delta\varepsilon \mapsto \sigma/2 \pm \sqrt{\Delta\varepsilon^2 + \sigma^2/4}$. Hence, it does not pose a problem to the implementation of the system. However, a distortion of this type occurring in one of the outer modes of the star graph, i.e., $\sigma \hat{a}_k^\dagger \hat{a}_k$ for $k = 1, \ldots, M-1$, would indeed break the desired degeneracy and would have to be avoided. The above considerations make it natural to impose a second-quantization version of the adiabatic theorem [30] to which the proof can be found in Appendix B, viz.

Theorem. In the adiabatic limit, any initial operator in a subalgebra $\mathscr{A}_0$ generated from a (non)degenerate set of eigenmodes will evolve into a final operator lying also in $\mathscr{A}_0$ at every instance in time.

Any initial preparation $\hat{D}_j^\dagger(0)$ has to reside in the linear span of the modes $\{\hat{D}_j^\dagger(t)\}_j$ throughout the evolution. The dark modes evolve according to the holonomy $\mathcal{U}_\gamma$ governed by the adiabatic connection $(\mathcal{A}_\mu)_{jk} = [\hat{D}_k, \partial_\mu \hat{D}_j^\dagger]$, $\mu \in \{|\kappa_k|, \arg(\kappa_k)\}_k$ that constitutes the operator-valued counterpart of the Wilzeck-Zee connection [10].

Similar arguments to those made in Ref. [12] now reveal that the connection $\mathcal{A}$ is irreducible for the given system, and hence Hol($\mathcal{A}$) coincides with the entire unitary group U($M-2$) (a detailed proof can be found in Appendix C). More specifically, starting the holonomy at an initial point $\boldsymbol{\kappa}_0 = (0, \ldots, 0, |\kappa|)$ shows that any transformation $\sum_j (\mathcal{U}_\gamma)_{jk} \hat{D}_j^\dagger(\boldsymbol{\kappa}_0) = \sum_j (\mathcal{U}_\gamma)_{jk} \hat{a}_j^\dagger$ can be implemented holonomically by designing a suitable loop in $\mathscr{M}$. This means that, due to the composition of loops $\mathcal{U}_{\prod_j \gamma_j} = \prod_j \mathcal{U}_{\gamma_j}$, any linear optical network can be made geometrically robust by supporting it with two auxiliary modes $\hat{a}_{M-1}$ and $\hat{a}_M$, while adiabatically traversing an approximately closed path γ in $\mathscr{M}$.

Of course, this formulation can be related to the standard formalism on adiabatic holonomies [21]. Excitations of the dark modes produce zero-eigenvalue eigenstates (dark states) $|\psi_{\boldsymbol{n}}\rangle = \prod_j \frac{1}{\sqrt{n_j!}} (\hat{D}_j^\dagger)^{n_j} |\mathbf{0}\rangle$, $\boldsymbol{n} \in \mathbb{N}_0^{M-2}$, sharing an adiabatic subspace $\mathscr{H}_0$. However, these dark modes are not the only ones inducing new dark states. The Hamiltonian gives rise in addition to two nondegenerate bright modes $\hat{B}_\pm^\dagger(t) = (\sqrt{2}\varepsilon)^{-1}(\sum_j \kappa_j^*(t) \hat{a}_j^\dagger \pm \varepsilon \hat{a}_M^\dagger)$, that is, $[\hat{H}, \hat{B}_\pm^\dagger] = \pm\varepsilon \hat{B}_\pm^\dagger$, where $\varepsilon = (\sum_j |\kappa_j|^2)^{1/2}$. The total holonomy of the entire system reads

$$\mathcal{U}_{0,\gamma} \oplus e^{i\int_0^T \varepsilon(t)dt} \mathcal{U}_{+,\gamma} \oplus e^{-i\int_0^T \varepsilon(t)dt} \mathcal{U}_{-,\gamma}.$$

For photon numbers $N \geqslant 2$, there exist combinations of $\hat{B}_+^\dagger$ and $\hat{B}_-^\dagger$ producing additional dark states. The entire eigenspace $\mathscr{H}_0 = \mathrm{Span}\{|\psi_{\boldsymbol{n}}\rangle\}_{\boldsymbol{n}\in\mathbb{N}_0^{M-1}}$ can be generated from the subalgebra $\mathscr{A}_0 \otimes \mathscr{A}_B \subset \mathscr{A}$ containing sequences of eigenmodes, i.e., $\mathscr{H}_0 \cong \mathscr{A}_0 \otimes \mathscr{A}_B |\mathbf{0}\rangle$, with $\mathscr{A}_B$ containing excitations of the form $\hat{B}_+^\dagger \hat{B}_-^\dagger$. More explicitly, any element $\hat{F}$ in $\mathscr{A}_0 \otimes \mathscr{A}_B$ can be expanded as

$$\hat{F} = \sum_{\boldsymbol{n}\in\mathbb{N}_0^{M-1}} c_{n_1\ldots n_{M-1}} \frac{(\hat{D}_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(\hat{D}_2^\dagger)^{n_2}}{\sqrt{n_2!}} \cdots \frac{(\hat{B}_+^\dagger \hat{B}_-^\dagger)^{n_{M-1}}}{\sqrt{n_{M-1}!}},$$

which, by construction, produces an eigenstate with eigenvalue zero, via $\hat{F}|\mathbf{0}\rangle \in \mathscr{H}_0$.

For illustration, consider a tripod structure ($M = 4$) into which two photons are injected ($N = 2$), then there are clearly the three dark states $|\psi_{20}\rangle = \frac{1}{\sqrt{2}}(\hat{D}_1^\dagger)^2 |\mathbf{0}\rangle$, $|\psi_{11}\rangle = \hat{D}_1^\dagger \hat{D}_2^\dagger |\mathbf{0}\rangle$, $|\psi_{02}\rangle = \frac{1}{\sqrt{2}}(\hat{D}_2^\dagger)^2 |\mathbf{0}\rangle$. Moreover, because of $[\hat{H}, \hat{B}_\pm^\dagger] = \pm\varepsilon \hat{B}_\pm^\dagger$, the positive and negative eigenenergies cancel one another out in the case of simultaneous excitation of $\hat{B}_+$ and $\hat{B}_-$. Therefore, $|\psi_{+-}\rangle = \hat{B}_+^\dagger \hat{B}_-^\dagger |\mathbf{0}\rangle$ is another dark state which, however, only attains a (scalar) Berry phase while evolving adiabatically. Thus, despite the fact that $\{|\psi_j\rangle\}_j$ span a common eigenspace, $|\psi_{+-}\rangle$ evolves independently. In particular, the corresponding adiabatic evolution in that eigenspace has a block structure in which $|\psi_{+-}\rangle$ does not couple to the other eigenstates [21,32]. It can be concluded that demanding the eigenmodes (rather than the eigenstates) to evolve adiabatically explains (in contrast to the original formulation [30]) why there are eigenstates in $\mathscr{H}_0$ that do not couple to the other eigenstates in $\mathscr{H}_0$. This phenomenon was observed in Refs. [31,32] but remained, to the best of our knowledge, unexplained until now.

In order to clarify this point further, consider another benchmark Hamiltonian $\hat{H}(t)$. For simplicity, we assume the corresponding eigenmodes to be nondegenerate, that is, the modes $\hat{\Psi}_k$ belong to mutually different energies ε_k. We then have the spectral decomposition $\hat{H} = \sum_k \varepsilon_k \hat{\Psi}_k^\dagger \hat{\Psi}_k$, where $\hat{\Psi}_k^\dagger \hat{\Psi}_k$ acts as a number operator for the kth eigenmode. It can be readily checked that $\prod_k \frac{1}{\sqrt{n_k!}} (\hat{\Psi}_k^\dagger)^{n_k} |\mathbf{0}\rangle$ is an N-photon eigenstate with energy $\sum_k \varepsilon_k n_k$ such that $\sum_k n_k = N$. Interestingly, if the eigenvalues are in such a structure that $N\varepsilon_1 = \sum_{k\neq 1} \varepsilon_k n_k$, then the eigenstates $\frac{1}{\sqrt{N!}}(\hat{\Psi}_1^\dagger)^N |\mathbf{0}\rangle$ and $\prod_{k\neq 1} \frac{1}{\sqrt{n_k!}} (\hat{\Psi}_k^\dagger)^{n_k} |\mathbf{0}\rangle$ both have the same eigenvalue, even though the eigenmodes of the system were nondegenerate. The original formulation of the adiabatic theorem [30] tells us only that (under a slow change of physical parameters) these eigenstates will not couple to states with different eigenvalue. However, the second-quantization formulation additionally predicts that the states $\frac{1}{\sqrt{N!}}(\hat{\Psi}_1^\dagger)^N |\mathbf{0}\rangle$ and $\prod_{k\neq 1} \frac{1}{\sqrt{n_k!}} (\hat{\Psi}_k^\dagger)^{n_k} |\mathbf{0}\rangle$ evolve separately from one another as well, because they originate from different eigenmodes. This highlights why the second-quantization formulation is a stronger version of the adiabatic theorem. In fact, one can use the above argument to construct linear optical networks that give rise to highly degenerate subspaces [21] via a spectral decomposition with suitable eigenvalue structure. Finally, note that if only a single photon or a coherent state is injected both versions of the adiabatic theorem coincide.

C. Nonadiabatic evolution of the star graph

The construction of adiabatic holonomies can be formulated analogously for the case where the cyclic evolution is not restricted to just the eigenmodes but to a more general collection of modes for which dynamical contributions from the Hamiltonian completely disappear. We return to the linear optical setting shown in Fig. 2, where M bosonic modes are arranged as a star graph. If we assume that all couplings evolve with the same envelope, i.e., $\kappa_k(t) \propto \Omega(t)$, the evolution of dark modes is trivially $\hat{\Psi}_j(t) = \hat{D}_j(t) = \hat{D}_j(0)$ for $j = 1, \ldots, M-2$. Moreover, under these assumptions it is always possible to find another operator $\hat{B}^\dagger = \Omega^{-1}\sum_j \kappa_j^* \hat{a}_j^\dagger \in \mathscr{A}$ (that is, not an eigenmode) such that $[\hat{D}_j, \hat{B}^\dagger] = [\hat{a}_M, \hat{B}^\dagger] = 0$. The time evolution of this operator reads

$$\hat{\Psi}_{M-1}^\dagger(t) = e^{i\delta(t)}(\cos\delta(t)\hat{B}^\dagger(t) - i\sin\delta(t)\hat{a}_M^\dagger(t)), \quad (3)$$

where $\delta(t) = \int_0^t \Omega(\tau)d\tau$. One can check that $[\hat{\Psi}_j, \hat{\Psi}_k] = [\hat{\Psi}_j^\dagger, \hat{\Psi}_k^\dagger] = 0$ and $[\hat{\Psi}_j, \hat{\Psi}_k^\dagger] = \delta_{jk}$, hence the orthonormal modes $\{\hat{\Psi}_j^\dagger(t)\}_j$ create excitations in a subspace $\mathscr{H}_\psi$. Next, we demand $\delta(T) = \pi$ to ensure cyclicity, i.e., $\hat{\Psi}_j(T) = \hat{\Psi}_j(0)$ for $j = 1, \dots, M-1$. After verifying that $[\hat{\Psi}_j, [\hat{H}, \hat{\Psi}_k^\dagger]] = 0$, one can check that all conditions for a purely geometric evolution are satisfied. The only nonvanishing component of the connection is $(\mathcal{A}_t)_{M-1,M-1} = i\Omega(t)$, which corresponds to a pure gauge. Hence, the nonadiabatic quantum holonomy reads $\mathcal{U}_\gamma = \text{diag}(1, \dots, -1) \in \text{U}(M-1)$. When replacing the generating operators by the original bosonic modes via a change of gauge $\hat{\Psi}_k^\dagger(T) = \sum_j G_{jk}\hat{a}_j^\dagger$, the holonomy transforms according to $\mathcal{U}_\gamma^G(\gamma) = G^{-1}\mathcal{U}_\gamma G$. The unitary $\mathcal{U}_\gamma^G \in \text{U}(M-1)$ gives rise to noncommutative quantum holonomies. In Appendix C it is shown explicitly that suitable manipulation of the couplings $\{\kappa_k\}_k$ allows one to design a set of quantum holonomies that generate the entire unitary group $\text{U}(M-1)$. Similar to the adiabatic scenario, the holonomy $\mathcal{U}_\gamma^G$ can be viewed as a scattering matrix describing the unitary mixing of bosonic modes $\hat{a}_k^\dagger$ or, equivalently, the superoperator $\mathcal{U}_\gamma^G(\hat{a}_k^\dagger) = \hat{U}^\dagger(\gamma)\hat{a}_k^\dagger\hat{U}(\gamma)$. It thus follows that any linear optical network can be implemented by means of nonadiabatic geometric phases assuming a sufficiently large coupling space, e.g., $\mathscr{M} \cong \mathscr{G}_{M,M-1}$.

As an example we return to the tripod structure ($M = 4$). While the two dark modes evolve trivially in time $\hat{\Psi}_1^\dagger(t) = \hat{D}_1^\dagger$ and $\hat{\Psi}_2^\dagger(t) = \hat{D}_2^\dagger$ due to $[\hat{H}, \hat{D}_j^\dagger] = 0$, the mode $\hat{B}^\dagger = \Omega^{-1}(\kappa_1^*\hat{a}_1^\dagger + \kappa_2^*\hat{a}_2^\dagger + \kappa_3^*\hat{a}_3^\dagger)$ evolves into $\hat{\Psi}_3^\dagger(t)$ given by Eq. (3). These modes satisfy the condition for an all-out geometric evolution, i.e., the mixing of modes $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, and $\hat{a}_3^\dagger$ is given by a quantum holonomy. When two photons ($N = 2$) are injected into the optical setup, there are six different states $\hat{\Psi}_j^\dagger\hat{\Psi}_k^\dagger|\mathbf{0}\rangle$ with $j, k = 1, 2, 3$, spanning a subspace $\mathscr{H}_\psi$ on which a U(6) holonomy can be implemented. Interestingly, the central mode $\hat{a}_4^\dagger$ of the star graph itself satisfies $[\hat{a}_4, [\hat{H}, \hat{a}_4^\dagger]] = 0$ and evolves according to an Abelian holonomy $\hat{a}_4^\dagger(\gamma) = e^{i\pi}\hat{a}_4^\dagger(0)$. Hence, the states $\hat{a}_4^\dagger\hat{\Psi}_1^\dagger|\mathbf{0}\rangle$, $\hat{a}_4^\dagger\hat{\Psi}_2^\dagger|\mathbf{0}\rangle$, and $\hat{a}_4^\dagger\hat{\Psi}_3^\dagger|\mathbf{0}\rangle$ span another subspace $\mathscr{H}_{\psi'}$ on which holonomic U(3) transformations can be performed. This is a general feature of this operator formulation. If there are several subalgebras $\{\mathscr{A}_{\Psi,n}\}_n$, products of modes from different subalgebras will generate a combined subspace on which cyclic evolution leads to a holonomy.

Using standard results [33] we conclude that, when the photonic star graph Hamiltonian is provided with a highly entangled resource state (e.g., a cluster state [34]), universal photonic quantum computation is possible using holonomic quantum gates only. This can be done, both for adiabatic and nonadiabatic holonomies, by employing a dual-rail encoding for pairs of modes into which a single photon is injected. In addition, if a linear optical network admits additional symmetries, the geometric phase $\oint_C \mathcal{A}$ might turn into a quantized phase factor, in which case the quantum computation can even be made fully topological [35,36].

IV. EXTENSION TO A MORE GENERAL FRAMEWORK

Formally, one can construct an even larger framework. Let H be the generator of dynamics belonging to a representation of some dynamical Lie algebra $(\mathscr{A}, [\![\cdot, \cdot]\!])$. The evolution is determined by the Lie bracket via the action $[\![\cdot, H]\!]$. Demanding that the bracket vanishes on a well-defined set of solutions $\mathscr{K}$ yields a condition for parallel transport in $\mathscr{A}$. For example, let $[\![\cdot, H]\!]_\text{P}$ be the flow in the phase space Γ generated from a classical Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$ with Darboux coordinates $(\boldsymbol{q}, \boldsymbol{p})$. The relevant Lie algebra is the space of all smooth functions $C^\infty(\Gamma)$ equipped with the Poisson bracket $[\![f, g]\!]_\text{P} = \sum_j(\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p^j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p^j})$. The time evolution is then defined by the action $[\![f, H]\!]_\text{P}$ for all $f \in C^\infty(\Gamma)$. If the system varies periodically with T, then under a slow (adiabatic) change of external parameters, the explicit time dependence of H can be neglected. Hence, the equations of motion become integrable. For a system with K degrees of freedom, it follows that there exists a set of action-angle variables $\{\theta_k, J_k\}_k$ such that $[\![J_k, H]\!]_\text{P} = 0$ for $k = 1, \dots, K$, i.e., they are constants of motion [37]. Then it follows that $\langle\dot{J}_k\rangle = T^{-1}\int_0^T J_k dt \approx 0$ to satisfactory precision. By construction, $\langle\frac{d}{dt}\rangle_\mathscr{K} = 0$ for $\mathscr{K}$ containing the symmetries $\{J_k\}_k$ (they form a subalgebra). One can find simple mechanical examples. For example, in a one-dimensional system with a bounded time-dependent potential $V(q, \kappa_\mu(t))$, the relevant subspace contains only a single adiabatic invariant J. However, adiabatically traversing a closed path $\gamma : [0, T] \to \mathscr{M}$ leads to a change in the generalized coordinate θ known as Hannay's angle [38], $\Delta\theta(\gamma) = \sum_\mu \oint_\gamma \langle\partial_\mu\theta\rangle\, d\kappa_\mu$, which depends only on the area enclosed by γ, thus showing a signature of an Abelian holonomy with noncompact symmetry group $\text{GL}(1, \mathbb{R})$.

V. CONCLUSIONS

In this article, we provided a unified framework for quantum holonomies based on a holonomic Heisenberg picture. We have shown that it provides a remarkable computational advantage for bilinear bosonic Hamiltonians where the relevant geometric evolution becomes independent of any subspace projection, and thus enables a description of the holonomy independent of the overall photon number. In particular, this means that any linear optical network can be constructed using holonomies only. We have shown this explicitly for the example of a bosonic star graph Hamiltonian allowing for the generation of adiabatic and nonadiabatic quantum holonomies. Moreover, we found that a stronger version of the adiabatic theorem can be formulated, a phenomenon that occurs only in a quantum optical setting. The parallel transport condition from which these results were derived hints at a more general theory that is valid for any dynamical Lie algebra, from which the emergence of Hannay's angle follows immediately. Our article paves the way to the study of gauge symmetry by quantum optical analogies and the realization of holonomic quantum algorithms using only linear optics.

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APPENDIX A: MODE QUANTIZATION UNDER GEOMETRIC CONSTRAINTS

Consider a collection of M classical modes that interact according to a linear optical network. When neglecting any coupling to continuum modes (such as dissipative losses or scattering into the environment), the vector of amplitudes $\boldsymbol{\alpha}$ transforms according to $\boldsymbol{\alpha}(T) = \boldsymbol{U}\boldsymbol{\alpha}(0)$, where T is the time it takes to propagate through the optical setup. As such a transformation of modes must be unitary, the scattering matrix can be written as

$$\boldsymbol{U}(T) = \mathcal{T}e^{i\int_0^T \boldsymbol{\Phi}(t)dt},$$

with $\boldsymbol{\Phi}$ being a Hermitian $M\times M$ matrix. The most general Hermitian matrix must have components $(\boldsymbol{\Phi})_{jk} = \kappa_{jk} + \sigma_k\delta_{jk}$, where $\kappa_{jk} = \kappa^*_{kj}$ ($\kappa_{jj} = 0$ in this definition) and σ_k being a real number. As $\boldsymbol{\alpha}(t)$ solves the first-order differential equation $\partial_t\boldsymbol{\alpha} = i\boldsymbol{\Phi}\boldsymbol{\alpha}$, the κ_{jk} can be viewed as coupling strengths between the modes j and k, while σ_k might be viewed as a self-coupling or a propagation constant. Let us assume that the parameter configuration is such that there exist $K < M$ orthonormal modes $\boldsymbol{\Psi}_j(t) = (c_{jk}(t))_k$ satisfying the condition for a purely geometric evolution, that is, for all $j, k = 1, \ldots, K$ the relation [11]

$$(\boldsymbol{\Psi}_j^*)^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{\Psi}_k = \sum_{l,m=1}^{M} c^*_{jl}c_{km}(\boldsymbol{\Phi})_{lm} = 0 \tag{A1}$$

holds. Such configurations clearly exist, as the structure of a photonic star graph Hamiltonian is obtained for $\kappa_{jk} = \kappa_j\delta_{jM}$ and $\sigma_k = 0$ as used in the main article.

Quantization of this discrete system is carried out by promoting the basis vectors to Hilbert space operators $\hat{a}_k^\dagger$. Then we have $\boldsymbol{\Psi}_k \mapsto \hat{\Psi}_k^\dagger = \sum_j c_{jk}\hat{a}_k^\dagger \in \mathscr{A}$. Analogously, the Hamiltonian $\hat{H}$ is obtained by comparing the Heisenberg equation $\partial_t\hat{a}_k^\dagger = i[\hat{H}, \hat{a}_k^\dagger]$ to $\partial_t\hat{\boldsymbol{a}}^\dagger = i\boldsymbol{\Phi}\hat{\boldsymbol{a}}^\dagger$ leading to

$$\hat{H}(t) = \sum_{j<k}^{M}\kappa_{jk}(t)\hat{a}_j\hat{a}_k^\dagger + \kappa^*_{jk}(t)\hat{a}_j^\dagger\hat{a}_k + \sum_{j=1}^{M}\sigma_j(t)\hat{a}_j^\dagger\hat{a}_j.$$

We now show that, on the level of Hilbert space operators, Eq. (A1) is equivalently represented by the relation $[\hat{\Psi}_j, [\hat{H}, \hat{\Psi}_k^\dagger]] = 0$, thus incorporating the condition for parallel transport. Using the properties of the commutator as well as the bosonic commutation relations, we arrive at

$$\begin{aligned}[\hat{\Psi}_j, [\hat{H}, \hat{\Psi}_k^\dagger]] &= \sum_{l,m} c^*_{jl}c_{km}\Big(\sum_{n<p}(\kappa_{np}[\hat{a}_l, [\hat{a}_n\hat{a}_p^\dagger, \hat{a}_m^\dagger]] + \kappa^*_{np}[\hat{a}_l, [\hat{a}_n^\dagger\hat{a}_p, \hat{a}_m^\dagger]]) + \sum_n \sigma_n[\hat{a}_l[\hat{a}_n^\dagger\hat{a}_n, \hat{a}_m^\dagger]]\Big),\\ &= \sum_{l,m} c^*_{jl}c_{km}\Big(\sum_{n<p}(\kappa_{np}\delta_{lp}\delta_{nm} + \kappa^*_{np}\delta_{ln}\delta_{pm}) + \sigma_l\delta_{lm}\Big), = \sum_{l,m} c^*_{jl}c_{km}(\kappa_{lm} + \sigma_l\delta_{lm}),\end{aligned} \tag{A2}$$

proving the assertion.

In order to verify that the quantization procedure indeed leaves Fock states with an evolution that is without dynamical contributions, one expects that any state $|\psi_{\boldsymbol{n}}\rangle$ lying in $\mathscr{H}_\psi = \mathrm{Span}\{\prod_{j=1}^K(\hat{\Psi}_j^\dagger)^{n_j}/\sqrt{n_j!}\,|\boldsymbol{0}\rangle \mid \boldsymbol{n}\in\mathbb{N}_0^K\}$ satisfies the parallel transport condition $\langle\frac{d}{dt}\rangle_{\mathscr{K}} = 0 \;\Leftrightarrow\; \langle\psi_{\boldsymbol{n}}|\hat{H}|\psi_{\boldsymbol{m}}\rangle = 0$ for all sequences $\boldsymbol{n}, \boldsymbol{m}\in\mathbb{N}_0^K$. To convince oneself that this is indeed the case, we first notice that, if both sequences differ in their total photon number, $\sum_j(\boldsymbol{n})_j \neq \sum_j(\boldsymbol{m})_j$, then $\langle\psi_{\boldsymbol{n}}|\hat{H}|\psi_{\boldsymbol{m}}\rangle = 0$ follows immediately, because $\hat{H}$ does not alter the total number of photons. Second, the claim is obviously true for a single photon, as $\langle\psi_{\boldsymbol{n}}|\hat{H}|\psi_{\boldsymbol{m}}\rangle = \langle\boldsymbol{0}|\,\hat{\Psi}_k\hat{H}\hat{\Psi}_j^\dagger\,|\boldsymbol{0}\rangle = \langle\boldsymbol{0}|\,[\hat{\Psi}_k, [\hat{H}, \hat{\Psi}_j^\dagger]]\,|\boldsymbol{0}\rangle = 0$ (we made use of $\hat{H}\,|\boldsymbol{0}\rangle = 0$) corresponds to the initially assumed parallel transport condition. For two photons, note that

$$\langle\boldsymbol{0}|\,\hat{\Psi}_j\hat{\Psi}_k\hat{H}\hat{\Psi}_l^\dagger\hat{\Psi}_m^\dagger\,|\boldsymbol{0}\rangle = \langle\boldsymbol{0}|\,[\hat{\Psi}_j\hat{\Psi}_k, [\hat{H}, \hat{\Psi}_l^\dagger\hat{\Psi}_m^\dagger]]\,|\boldsymbol{0}\rangle\,.$$

A direct calculation reveals that

$$\begin{aligned}&\langle\boldsymbol{0}|\,[\hat{\Psi}_j\hat{\Psi}_k, [\hat{H}, \hat{\Psi}_l^\dagger\hat{\Psi}_m^\dagger]]\,|\boldsymbol{0}\rangle\\ &\quad= \langle\boldsymbol{0}|\,\hat{\Psi}_j[\hat{\Psi}_k, [\hat{H}, \hat{\Psi}_l^\dagger]\hat{\Psi}_m^\dagger + \hat{\Psi}_l^\dagger[\hat{H}, \hat{\Psi}_m^\dagger]]\,|\boldsymbol{0}\rangle\,,\\ &= \delta_{km}\,\langle\boldsymbol{0}|\,\hat{\Psi}_j\hat{H}\hat{\Psi}_l^\dagger\,|\boldsymbol{0}\rangle + \delta_{kl}\,\langle\boldsymbol{0}|\,\hat{\Psi}_j\hat{H}\hat{\Psi}_m^\dagger\,|\boldsymbol{0}\rangle = 0,\end{aligned}$$

where we made use of orthogonality relation $[\hat{\Psi}_j, \hat{\Psi}_k^\dagger] = \delta_{jk}$ as well as $\langle\boldsymbol{0}|\,\hat{\Psi}_k\hat{H}\hat{\Psi}_j^\dagger\,|\boldsymbol{0}\rangle = 0$ for all $j, k = 1, \ldots, K$. One can continue the argument for higher photon numbers, so that the remainder of the proof follows by induction [39].

APPENDIX B: PROOF OF THE STRONG ADIABATIC THEOREM

Even though the adiabatic propagation of photon-number states, subject to a bilinear Hamiltonian $\hat{H}(t)$, does not violate the original formulation of the adiabatic theorem [30], it has become clear that a stronger version can be formulated.

Theorem. In the adiabatic limit, any initial operator in a subalgebra $\mathscr{A}_0(0)$ generated from a (non)degenerate set of eigenmodes will evolve into a final operator lying also in $\mathscr{A}_0(t)$ at every instance in time t.

Proof. Consider $\hat{H}(t)$ to be the quantum system of interest, giving rise to (possibly) degenerate eigenmodes $\hat{\Psi}_{nj}(t)$ with eigenvalue $\varepsilon_n(t)$, i.e., $[\hat{H}, \hat{\Psi}_{nj}^\dagger] = \varepsilon_n\hat{\Psi}_{nj}^\dagger$ at every instance t. We make the ansatz,

$$\hat{\eta}^\dagger(t) = \sum_{n,j} c_{nj}(t)\hat{\Psi}_{nj}^\dagger(t), \tag{B1}$$

for the most general bosonic mode of the time-dependent system. When comparing the explicit time derivative of Eq. (B1) with the Heisenberg equation of motion $\dot{\hat{\eta}}^\dagger = i[\hat{H}, \hat{\eta}^\dagger] = i\sum_{n,j} \varepsilon_n \hat{\Psi}^\dagger_{nj}$, one arrives at

$$\sum_{n,j} (\dot{c}_{nj}(t)\hat{\Psi}^\dagger_{nj}(t) + c_{nj}(t)\partial_t \hat{\Psi}^\dagger_{nj}(t)) = 0, \tag{B2}$$

where we made use of the Heisenberg equation for the eigenmodes $\dot{\hat{\Psi}}^\dagger_{nj} = i\varepsilon_n \hat{\Psi}^\dagger_{nj} + \partial_t \hat{\Psi}^\dagger_{nj}$. Select the mth energy level with K-fold degenerate eigenmodes $\{\hat{\Psi}_{mk}(t)\}_{k=1}^K$, and contract Eq. (B2) with $[\hat{\Psi}_{mk}, \cdot\,]$. Further, using bosonic commutation relations $[\hat{\Psi}_{mk}, \hat{\Psi}^\dagger_{nj}] = \delta_{mn}\delta_{kj}$ leads to

$$\dot{c}_{mk} = -\sum_{n,j} c_{nj}[\hat{\Psi}_{mk}, \partial_t \hat{\Psi}^\dagger_{nj}]. \tag{B3}$$

Next, we apply ∂_t to the generalized eigenvalue problem which yields

$$[\dot{\hat{H}}, \hat{\Psi}^\dagger_{nj}] + [\hat{H}, \partial_t \hat{\Psi}^\dagger_{nj}] = \dot{\varepsilon}_n \hat{\Psi}^\dagger_{nj} + \varepsilon_n \partial_t \hat{\Psi}^\dagger_{nj},$$

where we noticed that $\partial_t \hat{H} = \dot{\hat{H}}$. Contracting this result with $\hat{\Psi}_{mk}$ for $m \neq n$ leaves one with

$$\varepsilon_n[\hat{\Psi}_{mk}, \partial_t \hat{\Psi}^\dagger_{nj}] = [\hat{\Psi}_{mk}, [\dot{\hat{H}}, \hat{\Psi}^\dagger_{nj}]] + [\hat{\Psi}_{mk}, [\hat{H}, \partial_t \hat{\Psi}^\dagger_{nj}]]. \tag{B4}$$

Using the Jacobi identity it is easy to show that $[\hat{\Psi}_{mk}, [\hat{H}, \partial_t \hat{\Psi}^\dagger_{nj}]] = \varepsilon_m[\hat{\Psi}_{mk}, \partial_t \hat{\Psi}^\dagger_{nj}]$. With this result, Eq. (B4) can be rewritten in the compact form,

$$[\hat{\Psi}_{mk}, \partial_t \hat{\Psi}^\dagger_{nj}] = \frac{[\hat{\Psi}_{mk}, [\dot{\hat{H}}, \hat{\Psi}^\dagger_{nj}]]}{\varepsilon_n - \varepsilon_m}.$$

Inserting the above result into Eq. (B3) one obtains

$$\dot{c}_{mk} = -\sum_j c_{mj}(\mathcal{A}_t^{(m)})_{jk} - \sum_{n\neq m}\sum_j c_{nj}\frac{[\hat{\Psi}_{mk}, [\dot{\hat{H}}, \hat{\Psi}^\dagger_{nj}]]}{\varepsilon_n - \varepsilon_m}, \tag{B5}$$

with the $K \times K$ matrix $\mathcal{A}^{(m)}$ being the local connection one-form for adiabatic parallel transport in the mth energy level. Its components were defined as $(\mathcal{A}_t^{(m)})_{jk} = [\hat{\Psi}_{mk}, \partial_t \hat{\Psi}^\dagger_{mj}]$.

An evolution is said to be adiabatic if the Hamiltonian $\hat{H}$ changes slowly enough over time $t \in [0, T]$, such that its explicit time dependence can be neglected in the evolution governed by Eq. (B5). This is clearly the case when

$$\max_{0\leqslant t\leqslant T} \|[\hat{\Psi}_{mk}, [\dot{\hat{H}}, \hat{\Psi}^\dagger_{nj}]]\| \ll \min_{0\leqslant t\leqslant T} |\varepsilon_n - \varepsilon_m|, \tag{B6}$$

giving a validity condition for the adiabatic propagation. On the left-hand side of Eq. (B6), we maximize with respect to the induced operator norm on $\mathscr{A}$. We further observe that in this adiabatic limit the evolution of the components $c_{mk}(t)$ is governed by the system of first-order differential equations $\dot{\boldsymbol{c}} = \mathcal{A}_t \boldsymbol{c}$, with $\boldsymbol{c} = (c_{mk})_{k=1}^K$. In this limit, it becomes evident that the dynamical equations for c_{mk} and c_{nk} decouple for $m \neq n$. This means that any initial mode $\hat{\eta}^\dagger(0) \in \mathscr{A}_0(0)$ will evolve according to $\hat{\eta}^\dagger(T) = \mathcal{T} e^{\int_0^T \mathcal{A}_t dt}\hat{\eta}^\dagger(0)$ ($\mathcal{T}$ being time ordering) lying in $\mathscr{A}_0(T)$. Here, $\mathscr{A}_0(t)$ denotes the subalgebra of $\mathscr{A}$ containing analytic functions of eigenmodes $\{\hat{\Psi}^\dagger_{mk}(t)\}_{k=1}^K$ at time t. The decoupling of equations for $c_{mk}(t)$ implies further that any operator function $\hat{F}(\hat{\Psi}_{mk}, \hat{\Psi}^\dagger_{mk})$, depending solely on the eigenmodes of the mth level, will reside inside $\mathscr{A}_0(t)$ for all $t \in [0, T]$. ■

APPENDIX C: IRREDUCIBILITY OF THE CONNECTION

Here we show that, for a photonic star graph structure with the Hamiltonian,

$$\hat{H}(t) = \sum_{k=1}^{M-1} \kappa_k(t)\hat{a}_k\hat{a}^\dagger_M + \kappa_k^*(t)\hat{a}^\dagger_k\hat{a}_M, \tag{C1}$$

the associated connection $\mathcal{A}$, mediating the parallel transport of a bosonic mode, is irreducible. A practical consequence of this statement is that it is possible to create any linear optical network by means of holonomies only. We give separate proofs for adiabatic and nonadiabatic connections, respectively.

1. Adiabatic case

The Hamiltonian in Eq. (C1) possesses $M-2$ (not yet orthogonal) dark modes $\hat{D}^\dagger_j(t) = \kappa_{j+1}(t)\hat{a}^\dagger_1 - \kappa_1(t)\hat{a}^\dagger_{j+1}$, i.e., $[\hat{H}, \hat{D}^\dagger_j] = 0$ for $j = 1, \ldots, M-2$ generating a subalgebra $\mathscr{A}_0$. After orthogonalization, these modes satisfy canonical commutation relations $[\hat{D}_j, \hat{D}_k] = [\hat{D}^\dagger_j, \hat{D}^\dagger_k] = 0$, and $[\hat{D}_j, \hat{D}^\dagger_k] = \delta_{jk}$. Under the (strong) adiabatic assumption, any mode from $\mathscr{A}_0(0)$ has to be mapped onto an operator in the linear span of $\{\hat{D}^\dagger_j(T)\}_j$ under time evolution according to the holonomy $\mathcal{U}_\gamma = \mathcal{T}\exp\int_0^T \mathcal{A}_t dt$. Here, $\mathcal{T}$ is the time-ordering symbol and $(\mathcal{A}_t)_{kj} = [\hat{D}_j, \partial_t \hat{D}^\dagger_k]$ is the local connection one-form. Next, let us concentrate on loops of the form,

$$\kappa_1 = \kappa\cos\theta\sin\vartheta\, e^{i\varphi},$$
$$\kappa_2 = \kappa\sin\theta\sin\vartheta\, e^{i\varphi},$$
$$\kappa_3 = \kappa\cos\vartheta,$$
$$\kappa_4 = \cdots = \kappa_{M-1} = 0,$$

where $\theta \in [0, \pi]$ and $\vartheta, \varphi \in [0, 2\pi)$.

Due to normalization, the degree of freedom in $\kappa > 0$ can be omitted and the remaining coordinates $\{\theta, \vartheta, \varphi\}$ parametrize a three-dimensional submanifold of $\mathscr{M}$. A simple calculation reveals that the fibers over this submanifold are spanned by the dark modes:

$$\hat{D}^\dagger_1 = \sin\theta\hat{a}^\dagger_1 - \cos\theta\hat{a}^\dagger_2,$$
$$\hat{D}^\dagger_2 = \cos\vartheta\cos\theta\hat{a}^\dagger_1 + \cos\vartheta\sin\theta\hat{a}^\dagger_2 - \sin\vartheta e^{i\varphi}\hat{a}^\dagger_3,$$
$$\hat{D}^\dagger_3 = \hat{a}^\dagger_4,$$
$$\vdots$$
$$\hat{D}^\dagger_{M-2} = \hat{a}^\dagger_{M-1}.$$

The components of the local connection one-form are then computed as

$$\mathcal{A}_\theta = \begin{bmatrix} 0 & \cos\vartheta \\ -\cos\vartheta & 0 \end{bmatrix}, \quad \mathcal{A}_\vartheta = 0, \quad \mathcal{A}_\varphi = \begin{bmatrix} 0 & 0 \\ 0 & i\sin^2\vartheta \end{bmatrix}.$$

Note that we have indeed a non-Abelian gauge potential at our disposal, i.e., $[\mathcal{A}_\theta, \mathcal{A}_\varphi] \neq 0$. Now, path ordering in the relevant submanifold can be satisfied by traversing a plaquette □ along the coordinate lines. To be specific, let us choose the loop $\gamma(\square)$ as depicted in Fig. 3. A direct

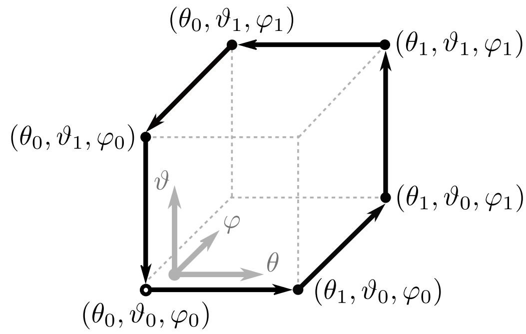

FIG. 3. Sequence of six steps forming the loop $\gamma(\Box)$ in the parameter space $\mathscr{M}$. The hollow dot denotes the starting point $(\theta_0, \vartheta_0, \varphi_0)$.

integration along the Wilson lines yields the holonomy as a path-ordered product of matrix exponentials,

$$\begin{aligned}\mathcal{U}_{\gamma(\Box)} &= \exp\left(\int_{\varphi_1}^{\varphi_0} \mathcal{A}_\varphi\big|_{\vartheta=\vartheta_1} d\varphi\right)\exp\left(\int_{\theta_1}^{\theta_0} \mathcal{A}_\theta\big|_{\vartheta=\vartheta_1} d\theta\right)\\ &\quad\times \exp\left(\int_{\varphi_0}^{\varphi_1} \mathcal{A}_\varphi\big|_{\vartheta=\vartheta_0} d\varphi\right)\exp\left(\int_{\theta_0}^{\theta_1} \mathcal{A}_\theta\big|_{\vartheta=\vartheta_0} d\theta\right),\\ &= \begin{bmatrix}1 & 0\\ 0 & e^{-i\sin^2\vartheta_1\Delta\varphi}\end{bmatrix}\begin{bmatrix}\cos\phi_1 & -\sin\phi_1\\ \sin\phi_1 & \cos\phi_1\end{bmatrix}\\ &\quad\times\begin{bmatrix}1 & 0\\ 0 & e^{i\sin^2\vartheta_0\Delta\varphi}\end{bmatrix}\begin{bmatrix}\cos\phi_0 & \sin\phi_0\\ -\sin\phi_0 & \cos\phi_0\end{bmatrix},\end{aligned} \tag{C2}$$

where the second equality can be verified by inserting the definitions $\Delta\varphi = \varphi_1 - \varphi_0$ and $\phi_i = \cos\vartheta_i(\theta_1 - \theta_0)$ for $i = 0, 1$. The transformation (C2) is nothing other than the decomposition of a general 2×2 unitary matrix, and any element of U(2) can be implemented by traversing a corresponding plaquette. Physically, this means that any optical two-mode transformation between $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ [starting at $(\theta_0, \vartheta_0, \varphi_0) = (\pi, \pi, 0)$] can be performed by traversing a closed loop in the submanifold $(\theta, \vartheta, \varphi)$. Due to the symmetry of the star graph structure, there is no preferred pair of outer waveguides $(\hat{a}_j^\dagger, \hat{a}_k^\dagger)_{j\neq k}$ for $j, k = 1, \dots, M-2$. Hence, traversing closed loops in the corresponding submanifolds of $\mathscr{M}$ generates any U(2) transformation between two arbitrary modes. From the point of view of differential geometry, this just corresponds to the statement that the holonomy group is independent of the chosen starting point [28]. It is well known that any $U(M-2)$ mixing of bosonic modes $\hat{D}_j^\dagger(0) = \hat{a}_j^\dagger$ $(j = 1, \dots, M-2)$ can be obtained from a sequence of U(2) transformations acting on a pair of modes [26]. This shows that the connection $\mathcal{A}$ is irreducible and any Reck-Zeilinger-like scheme can be implemented by adiabatic quantum holonomies utilizing the star graph structure. A particular resource-efficient way to implement such $U(M-2)$ transformations utilizes real-valued couplings κ_j for $j = 1, \dots, M-2$ which are comparatively easier to design than complex parameters. It is only necessary to have one complex coupling κ_{M-1} in order to have an irreducible connection. This is due to the reason that in the star graph, ideally, all outer modes are interchangable. This means that a single ancilla mode can be used to mediate a unitary mixing of any pair of modes $(\hat{a}_j^\dagger, \hat{a}_k^\dagger)_{j\neq k}$ for $j, k = 1, \dots, M-2$.

2. Nonadiabatic case

The proof in the previous section is readily extended to apply to the nonadiabatic connection one-form associated with a subspace of the star graph structure. This subspace is derived from the subalgebra $\mathscr{A}_\Psi$ generated by a set of $M-1$ orthonormal modes $\{\hat{\Psi}_j(t)\}_j$ satisfying the parallel transport condition,

$$[\hat{\Psi}_j(t), [\hat{H}(t), \hat{\Psi}_k^\dagger(t)]] = 0, \tag{C3}$$

for all $j, k = 1, \dots, M-1$ and at every instance t in an interval $[0, T]$. Certainly all dark modes satisfy the condition (C3), because $[\hat{H}(t), \hat{D}_j^\dagger(t)] = 0$ and we choose $\hat{\Psi}_j(t) = \hat{D}_j(t)$ for $j = 1, \dots, M-2$. By the argument of basis completion, it is always possible to find another operator $\hat{B}$ such that $[\hat{D}_j(t), \hat{B}^\dagger(t)] = [\hat{a}_M(t), \hat{B}^\dagger(t)] = 0$. In order to further simplify the search for the remaining mode $\hat{\Psi}_{M-1}(t)$ (this will not be an eigenoperator), we consider that all couplings evolve with the same envelope, i.e., $\kappa_j(t) = \Omega(t)g_j$, with $\Omega(t)$ being a real-valued, piecewise continuously differentiable function of time and $\{g_j\}_j$ being constant weights such that $\sum_j |g_j|^2 = 1$. First, this implies $\hat{\Psi}_j(t) = \hat{D}_j(0)$ for $j = 1, \dots, M-2$. Second, the time evolution of the operator $\hat{B}^\dagger = \sum_{j=1}^{M-1} g_j^* \hat{a}_j^\dagger$, when subjected to the Hamiltonian $\hat{H}(t) = \Omega(t)\hat{h}$, can be obtained from the series expansion,

$$\begin{aligned}\hat{U}^\dagger(t)\hat{B}^\dagger\hat{U}(t) &= \hat{B}^\dagger + \sum_{n=1}^{\infty}\frac{(-i\delta)^n}{n!}\underbrace{[\hat{h}, [\hat{h}, \dots [\hat{h}, \hat{B}^\dagger]]]}_{n-\text{times}},\\ &= \hat{B}^\dagger - i\delta[\hat{h}, \hat{B}^\dagger] + \frac{(-i\delta)^2}{2}[\hat{h}, [\hat{h}, \hat{B}^\dagger]] \mp \cdots,\end{aligned}$$

where we defined the shorthand $\delta(t) = \int_0^t \Omega(\tau)d\tau$. Making use of $[\hat{h}, \hat{B}^\dagger] = \hat{a}_M^\dagger$ and $[\hat{h}, \hat{a}_M^\dagger] = \hat{B}^\dagger$, we then get

$$\begin{aligned}\hat{\Psi}_{M-1}^\dagger(t) &= e^{i\delta(t)}\hat{U}^\dagger(t)\hat{B}^\dagger(t)\hat{U}(t),\\ &= e^{i\delta(t)}(\cos\delta(t)\hat{B}^\dagger(t) - i\sin\delta(t)\hat{a}_M^\dagger(t)),\end{aligned}$$

where the global phase factor $e^{i\delta(t)}$ has been inserted. We see that in the nonadiabatic scenario it is also possible that the central mode $\hat{a}_M$ can participate throughout the evolution. Finally, one can check that the entire set $\{\hat{\Psi}_j(t)\}_j$ satisfies Eq. (C3), thus ensuring a purely geometric evolution of bosonic modes. Note that, under the condition $\delta(T) = \pi$, the generating modes return to their initial form after period T, viz. $\hat{\Psi}_j(0) = \hat{\Psi}_j(T)$ for all j. The connection $(\mathcal{A})_{kj} = [\hat{\Psi}_j, d\hat{\Psi}_k^\dagger]$, responsible for describing nonadiabatic parallel transport, has only a single nonvanishing component, that is, $(\mathcal{A}_t)_{M-1,M-1} = i\Omega(t)$, which corresponds to a pure gauge. This has a geometric interpretation. If the connection corresponds to a pure gauge, we have a vanishing curvature [28]. Nonetheless, one can still find nontrivial holonomies, as the vanishing curvature is attributed to the fact that we chose a single curve $\kappa_j(t) \propto \Omega(t)$ to generate the holonomy. The one-dimensional space represented by the curve always looks locally like a straight line having no curvature [one can attach an $(M-1)$-dimensional Cartesian vielbein along the path].

From the explicit form of $\mathcal{A}$ we obtain the nonadiabatic holonomy $\mathcal{U}_\gamma = \mathrm{diag}(1, \dots, -1) \in U(M-1)$. Looking at the diagonal form of $\mathcal{U}_\gamma$, one might ask if any useful transformations can be obtained at all. This is indeed the case, and it becomes evident when transforming back to the original bosonic modes $\hat{a}_i$ $(i = 1, \ldots, M-1)$ via a change of gauge $G(g_j) \in \mathrm{U}(M-1)$, i.e., $\mathcal{U}_\gamma^G = G^\dagger \mathcal{U}_\gamma G$. In order to prove that any element in $\mathrm{U}(M-1)$ can be obtained from a suitable sequence of loops $\prod_j \gamma_j$ in $\mathscr{G}_{M,M-1}$, we first set

$$\kappa_1(t) = \Omega(t)\sin(\theta/2)e^{i\varphi},$$
$$\kappa_2(t) = \Omega(t)\cos(\theta/2),$$
$$\kappa_3(t) = \cdots = \kappa_{M-1}(t) = 0,$$

where (θ, φ) are constant parameter angles determining the unitary of choice. In this case, the relevant operators are

$$\hat{\Psi}_1^\dagger(t) = \sin(\theta/2)e^{i\varphi}\hat{a}_2^\dagger - \cos(\theta/2)\hat{a}_1^\dagger,$$
$$\hat{\Psi}_2^\dagger(t) = \hat{a}_3^\dagger,$$
$$\vdots$$
$$\hat{\Psi}_{M-2}^\dagger(t) = \hat{a}_{M-2}^\dagger,$$
$$\hat{\Psi}_{M-1}^\dagger(t) = e^{i\delta(t)}(\cos\delta(t)\hat{B}^\dagger - i\sin\delta(t)\hat{a}_M^\dagger),$$

where

$$\hat{B}^\dagger = \sin(\theta/2)e^{-i\varphi}\hat{a}_1^\dagger + \cos(\theta/2)\hat{a}_2^\dagger,$$

for the given configuration. The general case can be constructed along similar lines but is not necessary because of the following argument. Under cyclic evolution $\delta(T) = \pi$ the (operator-valued) holonomy becomes

$$\begin{pmatrix} \hat{a}_1^\dagger(T) \\ \hat{a}_2^\dagger(T) \end{pmatrix} = \begin{pmatrix} \cos\theta & -e^{-i\varphi}\sin\theta \\ -e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1^\dagger(0) \\ \hat{a}_2^\dagger(0) \end{pmatrix}. \quad \text{(C4)}$$

As was noted in Ref. [7] for a fermionic system, any matrix $\mathcal{U}_\gamma \in \mathrm{U}(2)$ can be realized via a composition of two suitable loops γ_1 and γ_2 in the Grassmann manifold $\mathscr{G}_{M,2}$. Note that, in contrast to an adiabatic evolution, here the loops γ_1 and γ_2 might correspond to open (or even trivial) paths in $\mathscr{M}$. Finally, following the same argument as for the adiabatic case, the holonomy group $\mathrm{Hol}(\mathcal{A})$ is independent of the chosen submanifold in the coupling space $\mathscr{M}$. Hence, the U(2) transformations can be applied to any pair of modes $(\hat{a}_i, \hat{a}_j)_{j\neq k}$ for all $j, k = 1, \ldots M-1$. The fact that we can construct any element of $\mathrm{U}(M-1)$ in a fully holonomic fashion is now a direct consequence of the argument by Reck *et al.* [26].

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